Probability Metric Space on MV-Algebras

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Abstract

By means of the probability measure theory of MV-algebras we introduce the concepts of probability truth degrees of the elements of MV-algebra and probability similarity degrees between elements, and then define therefrom three probability metric spaces on MV-algebra, and get some good results.

1. Introduction

MV-algebra [1] were introduced by Chang.C.C in 1958 as the algebraic counterpart of propositional Łukasiewicz logic. An MV-algebra is a many-valued generalization of a Boolean algebra, MV-algebras can be viewed as algebraic generalizations of certain collections of fuzzy sets, so-called Łukasiewicz tribes, as well. Currently, MV-algebra has many applications and many algebras is closely linked with the MV-algebra. States on MV-algebras were investigated by Mundici in [2] as [0,1]-valued additive functionals on formulas in Łukasiewicz propositional logic with the intention to capture the notion of average truth degree of a formula. In [3] a probability theory on MV-algebras is systematically developed. In [4,5] the conditional probability on MV-algebra and σ MV-algebra is systematically developed also. In recent years, a focus of attention is study of grated of mathematical logic in the background of uncertainty reasoning. Based on the measure theory Wang [6] introduced the truth degree of elements in MV-algebras, for established the probability on MV-algebra provided an effective method. In the present paper, by means of the probability measure theory we introduce the concepts of probability truth degrees of the elements in MV-algebra and probability similarity degrees between elements, and then define therefrom three probability metric spaces on MV-algebra, and get some good results.

The paper is structured as follows. Basic notions and results on MV-algebra are repeated in section 2, the concepts of probability truth degree of elements and probability similarity degrees between elements
on MV-algebras are introduced in section 3, three probability metric spaces on MV-algebra are introduced and some good results are get.

2. Basic notions of MV-algebra

Definition 2.1 (Chang C.C [1]) An MV-algebra is an algebra \((M, \oplus, \ominus, 0)\), where \(M\) is a non-empty set, \(\oplus\) is binary operation having 0 as the neutral element, \(\ominus\) is a unary operation, satisfying the following properties:

(i) \((M, \oplus, 0)\) is a commutative semi-group,
(ii) \(a \ominus 0 = -a\),
(iii) \(\ominus (\ominus a) = a\),
(iv) \(\ominus (\ominus a \oplus b) \oplus b = (\ominus b \oplus a) \ominus a\), \(\forall a, b \in M\).

Let \((M, \oplus, \ominus, 0)\) be an MV-algebra, the constant 1 is defined as follows: 1 = \(\ominus 0\), a partial order \(\preceq\) can be introduced on \(M\) by defining \(a \preceq b\) if and only if \(\ominus a \oplus b = 1\), then \((M, \preceq)\) is an allocation lattice. Binary operations \(\odot, \oslash\) are defined as follows: \(a \odot b = \ominus (\ominus a \ominus b) \ominus b\), \(a \oslash b = \ominus (\ominus a \ominus b) \ominus a\).

Proposition 2.1 (Chang C.C [1]) Let \((M, \oplus, \ominus, 0)\) be an MV-algebra, binary operations \(\odot, \oslash\) are defined as follows:

(i) \(x \odot y = -(-x \oplus -y)\), \(x \oslash y = -x \ominus y\),
(ii) \(1 \odot x = x\), \(x \odot 0 = -x\), \(x \oslash 0 = 0\),
(iii) \(x \odot y = 1\) iff \(x \preceq y\),
(iv) \(x \odot y = -y \oslash (\ominus x)\),
(v) \(x \odot (y \oslash z) = (x \odot y) \oslash (x \odot z)\),
(vi) \(x \oslash (y \odot z) = (x \oslash y) \odot (x \oslash z)\),
(vii) \(x \oslash (y \oslash z) = (x \oslash y) \oslash (x \oslash z)\),
(viii) \(x \oslash (y \odot z) = (x \oslash y) \odot (x \oslash z)\).

Example 2.1 Any Boolean algebra \(B\) is a MV-algebra, where 0 = \(\varnothing\); 1 = \(B\) is the union \(\cup\).

Example 2.2 The real unit interval \([0,1]\) equipped with the Łukasiewicz operation \(a \oplus b = \min(a + b, 1)\) and the standard complement \(\ominus a = 1 - a\), \(\forall a, b \in [0,1]\), is a \(\sigma\)-complete MV-algebra called the standard MV-algebra.

Definition 2.2 (Wang and Zhou [6]) Let \(M\) be MV-algebra, \([0,1]_{MV}\) standard MV-algebra, the homomorphism \(\lambda : M \rightarrow [0,1]_{MV}\) standard MV-algebra, the homomorphism \(v : M \rightarrow [0,1]\) is called an assignment of \(M\), i.e., \(v(x \odot y) = v(x) \odot v(y) = (v(x) + v(y)) \ominus 1\), \(v(-x) = v(x) - v(x)\), the set of all assignments of \(M\) is denoted by \(\Omega\).

Proposition 2.2 (Wang and Zhou [6]) Let \(M\) be MV-algebra, defining \(\overline{x} : \Omega \rightarrow [0,1]\) is following

\[ \overline{x}(v) = v(x), \]

(2)

Let \(\overline{M} = \{\overline{x} \mid x \in M\}\), and defining

\[ \overline{x} \ominus \overline{y} = \overline{x \ominus y}, \overline{\ominus x} = -\overline{x}, \]
Then \((\overline{M}, \oplus, \neg, \overline{0})\) is an MV-algebra.

3. Probability truth degree of elements on MV-algebra

Let \(M\) be MV-algebra, \(\Omega\) be the set of all assignments of \(M\). Suppose \(\sigma\)-algebra on \(\Omega\), \(\mu\) be probability measure on \(\Omega\), then \((\Omega, \sigma, \mu)\) is a probability space\(^7\). \(\forall x \in M\), let \(\overline{x}(v) = \nu(x), v \in \Omega\), then \(\overline{x}\) is \(\mu\)-integrable function\(^7\).

Definition 3.1 (Wang and Zhou \([6]\)) Defining \(T: M \rightarrow [0, 1]\) is following:
\[
T(x) = \int_{\Omega} \overline{x}(v) d\mu, x \in M,
\]
\[(3)\]
\(T(x)\) is called \(\mu\)-truth degree of element \(x\), in short, truth degree of \(x\).

Proposition 3.1 (Wang and Zhou \([6]\)) (i) \(0 \leq T(x) \leq 1\), \(x \in M\),
(ii) \(T(0) = 0, T(1) = 1\),
(iii) \(T(\neg x) = 1 - T(x)\),
(iv) If \(x \leq y\), then \(T(x) \leq T(y)\), \(x, y \in M\).

Proposition 3.2 (Wang and Zhou \([6]\)) (i) \(T(x \ovee y) = T(x) + T(y) - T(x \land y)\),
(ii) \(T(x \oplus y) = T(x) + T(y) - T(x \lor y)\),
(iii) If \(E = \{v \in \Omega | \overline{x}(v) \leq \overline{y}(v)\}\) is \(\mu\)-measurable, then \(T(x) + T(x \rightarrow y) = T(y) + T(y \rightarrow x)\).

Proposition 3.3 (Wang and Zhou \([6]\)) Let \(x, y \in M\), \(\alpha, \beta \in [0, 1]\), then
(i) if \(T(x) \geq \alpha, T(x \rightarrow y) \geq \beta\), then \(T(y) \geq \alpha + \beta - 1\),
(ii) if \(T(x \rightarrow y) \geq \alpha, T(y \rightarrow z) \geq \beta\), then \(T(x \rightarrow z) \geq \alpha + \beta - 1\).

Corollary 3.1 Let \(x, y, z \in M\), then
(i) \(T(x \rightarrow y) \leq T(x) \rightarrow T(y)\),
(ii) \(T(x \rightarrow y) \lor T(y \rightarrow z) \leq T(x \rightarrow z)\),
(iii) \(T(x \rightarrow y) \leq T(y \rightarrow z) \rightarrow T(x \rightarrow z)\).

Definition 3.2 Let \(x, y \in M\), defining \(\eta: M \times M \rightarrow [0, 1]\) is following:
(i) \(\eta_1(x, y) = T((x \rightarrow y) \land (y \rightarrow x))\) is called the first \(\mu\)-similarity degree,
(ii) \(\eta_2(x, y) = T(x \rightarrow y) \land T(y \rightarrow x)\) is called the second \(\mu\)-similarity degree,
(iii) \(\eta_3(x, y) = (T(x) \rightarrow T(y)) \land (T(y) \rightarrow T(x))\) is called the third \(\mu\)-similarity degree,
\(\eta_1, \eta_2, \eta_3\) uniformly expressed as \(\eta\), so called similarity degree.

If \(\eta_i(x, y) = 1, (i = 1, 2, 3)\) then \(x\) and \(y\) is called i-similarity, denote \(x \sim_i y\).

Lemma 3.1 Let \(a, b, c, d \in [0; 1]^\text{MV}\), then
\((a \land c) \lor (b \land d) \leq (a \lor b) \land (c \lor d)\).

Theorem 3.1 Let \(x, y, z \in M\), then
(i) \(\eta(x, x) = 1\),
(ii) \(\eta(x, y) = \eta(y, x)\),
(iii) \(\eta(\neg x, \neg y) = \eta(x, y)\),
(iv) \(\eta(x, z) \geq \eta(x, y) + \eta(y, z) - 1\).

Proof (i) and (ii) is clearly established.
(iii) \(\forall x, y \in M\), by proposition 1(iv) we obtain \(\overline{x \rightarrow y} = \overline{\neg y \rightarrow \neg x} \), so
(iv) For \( \eta_1 \) see [6] proposition 7(iv).

For \( \eta_2 \), from corollary 3.1(ii) we can obtain
\[ T(x \to z) \geq T(x \to y) \odot T(y \to z) \text{ and } T(z \to x) \geq T(z \to y) \odot T(y \to x) = T(y \to x) \odot T(z \to y). \]
From lemma 3.1,
\[ T(x \to z) \land T(z \to x) \geq (T(x \to y) \odot T(y \to z)) \land (T(y \to x) \odot T(z \to y)) \]
\[ \geq (T(x \to y) \land T(y \to x)) \odot (T(y \to z) \land T(z \to y)) \]

So \( \eta_2(x, z) \geq \eta_2(x, y) + \eta_2(y, z) - 1. \)

For \( \eta_3 \), from proposition 2.1(viii) we obtain \( T(x) \to T(z) \geq (T(x) \to T(y)) \odot (T(y) \to T(z)) \) and \( T(z) \to T(x) \geq (T(z) \to T(y)) \odot (T(y) \to T(x)) \), similar to \( \eta_2 \), we can proved.

**Theorem 3.2** Let \( x, y \in M \), then

(i) \( \eta_1(x, y) = \int_{\Omega} (1 - |x(v) - y(v)|) d\mu, \)

\[ \eta_3(x, y) = 1 - |T(x) - T(y)|, \]

(ii) \( \eta_1(x, y) \leq \eta_2(x, y) \leq \eta_3(x, y). \)

**Proof** (i) Because 1-a+b and 1-b+a not both greater than 1 for \( a, b \in [0, 1]_{MV} \), thus
\[ \eta_1(x, y) = \int_{\Omega} (1 - x(v) + y(v)) \land (1 - y(v) + x(v)) d\mu \]
\[ = \int_{\Omega} (1 - |x(v) - y(v)|) d\mu. \]

\[ \eta_3(x, y) = (1 - \tau(x) + \tau(y)) \land (1 - \tau(y) + \tau(x)) \]
\[ = 1 - |\tau(x) - \tau(y)|. \]
4. Probability metrics of MV-algebra

Definition 4.1 Let $x, y \in M$, defining

$$\rho(x, y) = 1 - \eta(x, y), \quad \rho_i(x, y) = 1 - \eta_i(x, y), i = 1, 2, 3. \quad (4)$$

Theorem 4.1 \( \rho: M \times M \to [0, 1] \) is metric of MV-algebra \( M(i=1,2,3) \).

Proof Let \( x, y, z \in M \), from theorem 3.1 we see that \( \rho(x, x) = 1 - \eta(x, x) = 0 \), and

\[
\rho(x, y) = 1 - \eta(x, y) = 1 - \eta(y, x) = \rho(y, x).
\]

\[
\rho(x, z) = 1 - \eta(x, z) \leq 1 - (\eta(x, y) + \eta(y, z) - 1) = (1 - \eta(x, y)) + 1 - \eta(y, z) = \rho(x, y) + \rho(y, z).
\]

Definition 4.2 \((M, \rho_i)\) is called probability metric space of MV-algebra \((i=1,2,3)\).

Theorem 4.2 Let \( x, y \in M \), then

Theorem 4.3 Let \( x, y \in M \), then

\( (i) \ \ \rho_1(x, y) = \rho_2(x, y) = \rho_3(x, y) \),

\( (ii) \ \ \rho_2(x, y) = \frac{1}{2} (\rho_1(x, y) + \rho_3(x, y)) \).

Proof (i) From theorem 3.2(ii), we can proved.

(ii) Because

\[
\int_{\Omega} (\bar{x}(v) \to \bar{y}(v)) \land (\bar{y}(v) \to \bar{x}(v))d\mu \leq \int_{\Omega} (\bar{x}(v) \to \bar{y}(v))d\mu \text{ and } \int_{\Omega} (\bar{x}(v) \to \bar{y}(v)) \land (\bar{y}(v) \to \bar{x}(v))d\mu \leq \int_{\Omega} (\bar{y}(v) \to \bar{x}(v))d\mu,
\]

thus

\[
\int_{\Omega} (\bar{x}(v) \to \bar{y}(v)) \land (\bar{y}(v) \to \bar{x}(v))d\mu \leq \int_{\Omega} (\bar{x}(v) \to \bar{y}(v))d\mu \land \int_{\Omega} (\bar{y}(v) \to \bar{x}(v))d\mu, \text{ i.e., } \eta_1(x, y) \leq \eta_2(x, y).
\]

From corollary 3.1(i), we see that \( \tau(x \to y) \leq \tau(x) \to \tau(y) \) and \( \tau(y \to x) \leq \tau(y) \to \tau(x) \), so \( \eta_2(x, y) = \tau(x \to y) \land \tau(y \to x) \leq (\tau(x) \to \tau(y)) \land (\tau(y) \to \tau(x)) = \eta_3(x, y) \).
Theorem 4.4 (Wang and Zhou [6]) On probability metric space of MV-algebra (M, $\rho_1$), operations $\neg$, $\oplus$, $\ominus$, $\vee$, $\wedge$ and $\rightarrow$ are continuous.

Lemma 3.2 Let $x, y, u, w \in M, \alpha, \beta \in [0, 1]$, if $\eta_2(x, u) \geq \alpha, \eta_2(y, w) \geq \beta$, then $\eta_2(x \rightarrow y, u \rightarrow w) \geq \alpha + \beta - 1$.

Proof Let $[0, 1]_{\text{MV}}$ be standard MV-algebra, $a, b, c \in [0, 1]$, then $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$ and $b \rightarrow a \leq (a \rightarrow c) \rightarrow (b \rightarrow c)$. Thus $\forall \forall \in \Omega$, we see that

$$\bar{x} \rightarrow \bar{u} \leq ((\bar{x} \rightarrow \bar{y}) \rightarrow (\bar{u} \rightarrow \bar{y})),$$

And

$$\bar{u} \rightarrow \bar{x} \leq ((\bar{u} \rightarrow \bar{y}) \rightarrow (\bar{x} \rightarrow \bar{y})).$$

From proposition 3.1,

$$\tau(x \rightarrow u) \leq \tau((x \rightarrow y) \rightarrow (u \rightarrow y)),$$

$$\tau(u \rightarrow x) \leq \tau((u \rightarrow y) \rightarrow (x \rightarrow y)),$$

so $\tau(x \rightarrow u) \land \tau(u \rightarrow x) \leq \tau((x \rightarrow y) \rightarrow (u \rightarrow y)) \land \tau((u \rightarrow y) \rightarrow (x \rightarrow y))$, i.e., $\eta_2(x \rightarrow y, u \rightarrow y) \geq \eta_2(x, y) \geq \alpha$.

Similarly, $\eta_2(u \rightarrow y, u \rightarrow w) \geq \eta_2(y, w) \geq \beta$, thus

$$\eta_2(x \rightarrow y, u \rightarrow w) \geq \eta_2(x \rightarrow y, u \rightarrow y) + \eta_2(u \rightarrow y, u \rightarrow w) - 1$$

$$\geq \eta_2(x, u) + \eta_2(y, w) - 1$$

$$\geq \alpha + \beta - 1.$$

Theorem 4.5 Suppose $(M, \rho_2)$ be probability metric space of MV-algebra, then operations $\neg$, $\oplus$, $\ominus$, $\vee$, $\wedge$ and $\rightarrow$ on $(M, \rho_2)$ are continuous.
Proof First, from theorem 3.1 we see that ρ₂(−x, −y) = 1 − η₂(−x, −y) = 1 − η₂(x, y) = ρ₂(x, y), therefore − is continuous.

Second, let ρ₂(x, u) ≤ ε, ρ₂(y, w) ≤ ε, then η₂(x, u) ≥ 1−ε, η₂(y, w) ≥ 1−ε, so from lemma 3.2 we have
η₂(x → y, u → w) ≥ (1 − ε) + (1 − ε) − 1 = 1 − 2ε.

i.e., ρ₂(x → y, u → w) ≤ 2ε, so implication operation → is continuous.

Because x ⊕ y = −x → y, x ⊙ y = −(x → −y), x ∨ y = (x → y) → y, x ∧ y = −(−x ∨ −y), therefore ⊕, ⊙, ⊢, ∧ are all continuous.

Theorem 4.6 (Wang and Zhou [6]) If (M, ρ₁) be complete metric space, then M is a countably complete lattice.

Theorem 4.7 If (M, ρ₂) be complete metric space, then M is also a countably complete lattice.

Proof Suppose (M, ρ₂) be complete metric space, Δ={x₁,x₂,⋯}⊂M. Let yₙ = Vⁿ⁻¹x₁, then y₁, y₂, ⋯ is an increasing sequence. From proposition 3.1 we see that
T(y₁), T(y₂),⋯ is an increasing sequence in [0,1], so is a Cauchy sequence. And
yₙ → yᵐ = 1, yᵐ → yₙ = 1 − yᵐ + yₙ, n ≤ m.

Therefore, if n ≤ m then
ρ₂(yₙ, yₙ) = 1 − η₂(yₙ, yₙ) = 1 − 1 - (yₙ → yₙ) ∧ (yₙ → yₙ) = 1 - 1 - (yₙ → yₙ) = 1 - ∫₀⁻¹ (1 - yᵐ + yₙ)dμ = ∫₀⁻¹ yᵐdμ - ∫₀⁻¹ yₙdμ = τ(yₙ) - τ(yₙ).

From T(y₁), T(y₂),⋯ is a Cauchy sequence, we see that y₁, y₂, ⋯ is a Cauchy sequence in complete metric space (M, ρ₂), so it converges to y, where y is a point of M.

We fix an n, then yₙ = yₙ ∧ yₙ⁺k, from the continuity of operation see that
yₙ = limₖ→∞ (yₙ ∧ yₙ⁺k) = yₙ ∧ (limₖ→∞ yₙ⁺k) = yₙ ∧ y.

So yₙ ≤ y, i.e., y is a upper bound of ∑ = {y₁, y₂, ⋯}. Taking any a upper bound of ∑, then yₙ = yₙ ∧ z, so
y = limₙ→∞ yₙ = limₙ→∞ (yₙ ∧ z) = (limₙ→∞ yₙ) ∧ z = y ∧ z.

Thus y ≤ z, i.e., y = sup∑ ∈ M. It is clearly supΔ = sup∑ ∈ M.

Similarly, we can prove inf Δ ∈ M. Therefore M is countable complete.

Theorem 4.8 Suppose (M, ρ₃) be probability metric space of MV-algebra, then operations − is continuous, but operation ⊕, ⊙, ∨, ∧ and → are not continuous all.

5. Conclusion

In this paper, we introduced three probability metric space on MV-algebras, studied the propositions of probability metric spaces on MV-algebras, get some good results. We can see that the method of one paper can be extended to other algebras, so the method of one paper is meaningful.
References