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Probability Metric Space on MV-Algebras

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Abstract

By means of the probability measure theory of MV-algebras we introduce the concepts of probability truth degrees of the elements of MV-algebra and probability similarity degrees between elements, and then define therefrom three probability metric spaces on MV-algebra, and get some good results.

Keywords-MV-algebra; probability truth degree; probability similarity degree; probability metric space;

1. Introduction

MV-algebra [1] were introduced by Chang.C.C in 1958 as the algebraic counterpart of propositional Łukasiewicz logic. An MV-algebra is a many-valued generalization of a Boolean algebra, MV-algebras can be viewed as algebraic generalizations of certain collections of fuzzy sets, so-called Łukasiewicz tribes, as well. Currently, MV-algebra has many applications and many algebras is closely linked with the MV-algebra. States on MV-algebras were investigated by Mundici in [2] as [0,1]-valued additive functionals on formulas in Łukasiewicz propositional logic with the intention to capture the notion of average truth degree of a formula. In [3] a probability theory on MV-algebras is systematically developed. In [4,5] the conditional probability on MValgebra and σ MV-algebra is systematically developed also. In recent years, a focus of attention is study of grated of mathematical logic in the background of uncertainty reasoning. Based on the measure theory Wang [6] introduced the truth degree of elements in MV-algebras, for established the probability on MV-algebra provided an effective method. In the present paper, by means of the probability measure theory we introduce the concepts of probability truth degrees of the elements in MV-algebra and probability similarity degrees between elements, and then define therefrom three probability metric spaces on MV-algebra, and get some good results.

The paper is structured as follows. Basic notions and results on MV-algebra are repeated in section 2, the concepts of probability truth degree of elements and probability similarity degrees between elements

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on MV-algebras are introduced in section 3, three probability metric spaces on MV-algebra are introduced and some good results are get.

2. Basic notions of MV-algebra

Definition 2.1 (Chang C.C [1]) An MV-algebra is an algebra (M, $\oplus$, $\neg$, 0), where M is a non-empty set, $\oplus$ is binary operation having 0 as the neutral element, $\neg$ is a unary operation, satisfying the following properties:

(i) (M, $\oplus$, 0) is a commutative semi-group,

(ii) $a\oplus\neg 0 = \neg 0$,

(iii) $\neg(\neg a) = a$,

(iv) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$, $\forall a, b \in M$.

Let (M, $\oplus$, $\neg$, 0) be an MV-algebra, the constant 1 is defined as follows: $1 = \neg 0$, a partial order $\leq$ can be introduced on M by defining $a \leq b$ if and only if $\neg a \oplus b = 1$, then (M, $\leq$) is a allocation lattice. Binary operations $\vee$, $\wedge$ are defined as follows: $a\vee b = \neg(\neg a\oplus b) \oplus b$, $a \wedge b = \neg(\neg a\vee\neg b)$.

Proposition 2.1 (Chang C.C [1]) Let (M, $\oplus$, $\neg$, 0) be an MV-algebra, binary operations $\odot$, $\rightarrow$ are defined as follows:

$$x \odot y = \neg(\neg x \oplus \neg y), \quad x \rightarrow y = \neg x \oplus y, \tag{1}$$

then

(i) $x \odot y \leq z$ iff $x \leq y \rightarrow z$,

(ii) $1 \rightarrow x = x$, $x \rightarrow 0 = \neg x$, $x \odot 0 = 0$,

(iii) $x \rightarrow y = 1$ iff $x \leq y$,

(iv) $x \rightarrow y = \neg y \rightarrow \neg x$,

(v) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(vi) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(vii) $x \odot (y \wedge z) = (x \odot y)\wedge(x \odot z)$,

(viii) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Example 2.1 Any Boolean algebra B is a MV-algebra, where $0 = \varnothing$; $1 = X$, $\oplus$ is the union $\cup$.

Example 2.2 The real unit interval [0,1] equipped with the Łukasiewicz operation $a \oplus b = \min(a + b, 1)$ and the standard complement $\neg a = 1 - a$, where $a, b \in [0, 1]$, is a σ-complete MV-algebra called the standard MV-algebra.

Definition 2.2(Wang and Zhou [6]) Let M be MV-algebra, $[0, 1]_{MV}$ standard MV-algebra, the homomorphism υ: M $\rightarrow$ [0, 1] is called a assignment of M, i.e., $\upsilon(x\oplus y) = \upsilon(x)\oplus\upsilon(y) = (\upsilon(x)+\upsilon(y))\wedge 1$, $\upsilon(\neg x)=1-\upsilon(x)$. the set of all assignments of M is denote Ω

$\forall\upsilon\in\Omega$, υ satisfying all operations of M, i.e.,

$\upsilon(x \odot y) = \upsilon(x) \odot \upsilon(y) = (\upsilon(x) + \upsilon(y) - 1) \vee 0$,

$\upsilon(x\rightarrow y) = \upsilon(x)\rightarrow\upsilon(y) = (1-\upsilon(x) + \upsilon(y)) \wedge 1$,

$\upsilon(x\vee y) = \max\{\upsilon(x), \upsilon(y)\}$,

$\upsilon(x\wedge y) = \min\{\upsilon(x), \upsilon(y)\}$.

Proposition 2.2 (Wang and Zhou [6]) Let M be MV-algebra, defining $\overline{x} : \Omega \rightarrow [0, 1]$ is following

$$\overline{x}(v) = v(x), \tag{2}$$

Let $\overline{M} = \left\{\overline{x} \middle| x \in M\right\}$, and defining

$$\overline{x} \oplus \overline{y} = \overline{x \oplus y}, \neg\left(\overline{x}\right) = \overline{\neg x},$$

Then ($\overline{M}, \oplus, \neg, \overline{0}$) is an MV-algebra.

3. Probability truth degree of elements on MV-algebra

Let M be MV-algebra, Ω be the set of all assignments of M. Suppose *A* be σ-algebra on Ω, μ be probability measure on Ω, then (Ω, A, μ) is a probability space[7]. ∀x∈M, let $\overline{x}(v) = v(x)$, υ∈Ω, then $\overline{x}$ is μ-integrable function[7].

Definition 3.1 (Wang and Zhou [6]) Defining *T*: M → [0, 1] is following:

$$T(x) = \int_{\Omega} \overline{x}(v)\, d\mu, x \in M, \tag{3}$$

$T(x)$ is called μ-truth degree of element x, in short, truth degree of x.

Proposition 3.1 (Wang and Zhou [6]) (i) $0 \le T(x) \le 1$, x ∈ M,
(ii) $T(0) = 0$, $T(1) = 1$,
(iii) $T(\neg x) = 1 - T(x)$,
(iv) If x ≤ y, then $T(x) \le T(y)$, x, y ∈ M.
Proposition 3.2 (Wang and Zhou [6]) (i) $T(x \vee y) = T(x) + T(y) - T(x \wedge y)$,
(ii) $T(x \oplus y) = T(x) + T(y) - T(x \odot y)$,
(iii) If E = $\{v \in \Omega \mid \overline{x}(v) \le \overline{y}(v)\}$ is μ-measurable, then $T(x) + T(x \to y) = T(y) + T(y \to x)$.
Proposition 3.3 (Wang and Zhou [6]) Let x, y ∈ M, α,β ∈[0, 1], then
(i) if $T(x) \ge \alpha$, $T(x \to y) \ge \beta$, then $T(y) \ge \alpha + \beta - 1$,
(ii) if $T(x \to y) \ge \alpha$, $T(y \to z) \ge \beta$, then $T(x \to z) \ge \alpha + \beta - 1$.
Corollary 3.1 Let x, y, z ∈ M, then
(i) $T(x \to y) \le T(x) \to T(y)$,
(ii) $T(x \to y) \odot T(y \to z) \le T(x \to z)$,
(iii) $T(x \to y) \le T(y \to z) \to T(x \to z)$.
Definition 3.2 Let x, y ∈ M, defining $\eta : M \times M \to [0, 1]$ is following:
(i) $\eta_1(x, y) = T((x \to y) \wedge (y \to x))$ is called the first μ-similarity degree,
(ii) $\eta_2(x, y) = T(x \to y) \wedge T(y \to x)$ is called the second μ-similarity degree,
(iii) $\eta_3(x, y) = (T(x) \to T(y)) \wedge (T(y) \to T(x))$ is called the third μ-similarity degree,
η_1, η_2, η_3 uniformly expressed as η, so called similarity degree.
If $\eta_i(x, y) = 1$, (i = 1, 2, 3) then x and y is called i-similarity, denote $x \sim_i y$.
Lemma 3.1 Let a, b, c, d ∈ $[0; 1]_{MV}$, then
$(a \wedge c) \odot (b \wedge d) \le (a \odot b) \wedge (c \odot d)$.
Theorem 3.1 Let x, y, z ∈ M, then
(i) $\eta(x, x) = 1$,
(ii) $\eta(x, y) = \eta(y, x)$,
(iii) $\eta(\neg x, \neg y) = \eta(x, y)$,
(iv) $\eta(x, z) \ge \eta(x, y) + \eta(y, z) - 1$.
Proof (i) and (ii) is clearly established.
(iii) ∀x, y∈M, by proposition 1(iv) we obtain $\overline{x \to y} = \overline{\neg y \to \neg x}$, so

$$\begin{aligned}
\eta_1(\neg x, \neg y) &= \tau((\neg x \to \neg y) \wedge (\neg y \to \neg x)) \\
&= \int_\Omega \overline{(\neg x \to \neg y) \wedge (\neg y \to \neg x)} d\mu \\
&= \int_\Omega (\overline{y \to x} \wedge \overline{x \to y}) d\mu \\
&= \tau((y \to x) \wedge (x \to y)) = \eta_1(x, y). \\
\eta_2(\neg x, \neg y) &= \tau(\neg x \to \neg y) \wedge \tau(\neg y \to \neg x) \\
&= \int_\Omega \overline{\neg x \to \neg y} d\mu \wedge \int_\Omega \overline{\neg y \to \neg x} d\mu \\
&= \int_\Omega \overline{y \to x} d\mu \wedge \int_\Omega \overline{x \to y} d\mu \\
&= \tau(y \to x) \wedge \tau(x \to y) = \eta_2(x, y). \\
\eta_3(\neg x, \neg y) &= (\tau(\neg x) \to \tau(\neg y)) \wedge (\tau(\neg y) \to \tau(\neg x)) \\
&= (\tau(y) \to \tau(x)) \wedge (\tau(x) \to \tau(y)) \\
&= \eta_3(x, y).
\end{aligned}$$

(iv) For η_1 see [6] proposition 7(iv).

For η_2, from corollary 3.1(ii) we can obtain $T(x \to z) \geq T(x \to y) \odot T(y \to z)$ and $T(z \to x) \geq T(z \to y) \odot T(y \to x) = T(y \to x) \odot T(z \to y)$. From lemma 3.1,

$$\begin{aligned}
& \tau(x \to z) \wedge \tau(z \to x) \\
\geq\ & (\tau(x \to y) \odot \tau(y \to z)) \wedge (\tau(y \to x) \odot \tau(z \to y)) \\
\geq\ & (\tau(x \to y) \wedge \tau(y \to x)) \odot (\tau(y \to z) \wedge \tau(z \to y))
\end{aligned}$$

So $\eta_2(x, z) \geq \eta_2(x, y) + \eta_2(y, z) - 1$.

For η_3, from proposition 2.1(viii) we obtain $T(x) \to T(z) \geq (T(x) \to T(y)) \odot (T(y) \to T(z))$ and $T(z) \to T(x) \geq (T(z) \to T(y)) \odot (T(y) \to T(x))$, similar to η_2, we can proved.

Theorem 3.2 Let $x, y \in M$, then

(i) $\eta_1(x, y) = \int_\Omega \left(1 - \left|\bar{x}(v) - \bar{y}(v)\right|\right) d\mu$,

$\eta_3(x, y) = 1 - \left|T(x) - T(y)\right|$,

(ii) $\eta_1(x, y) \leq \eta_2(x, y) \leq \eta_3(x, y)$.

Proof (i) Because 1-a+b and 1-b+a not both greater than 1 for $a, b \in [0, 1]_{MV}$, thus

$$\begin{aligned}
\eta_1(x, y) &= \int_\Omega (1 - \bar{x}(v) + \bar{y}(v)) \wedge (1 - \bar{y}(v) + \bar{x}(v)) d\mu \\
&= \int_\Omega (1 - |\bar{x}(v) - \bar{y}(v)|) d\mu. \\
\eta_3(x, y) &= (1 - \tau(x) + \tau(y)) \wedge (1 - \tau(y) + \tau(x)) \\
&= 1 - |\tau(x) - \tau(y)|.
\end{aligned}$$

(ii) Because $\int_\Omega (\bar{x}(v) \to \bar{y}(v)) \wedge (\bar{y}(v) \to \bar{x}(v)) d\mu \leq \int_\Omega (\bar{x}(v) \to \bar{y}(v)) d\mu$ and $\int_\Omega (\bar{x}(v) \to \bar{y}(v)) \wedge (\bar{y}(v) \to \bar{x}(v)) d\mu \leq \int_\Omega (\bar{y}(v) \to \bar{x}(v)) d\mu$, thus $\int_\Omega (\bar{x}(v) \to \bar{y}(v)) \wedge (\bar{y}(v) \to \bar{x}(v)) d\mu \leq \int_\Omega (\bar{x}(v) \to \bar{y}(v)) d\mu \wedge \int_\Omega (\bar{y}(v) \to \bar{x}(v)) d\mu$, i.e., $\eta_1(x,y) \leq \eta_2(x,y)$.

From corollary 3.1(i), we see that $\tau(x \to y) \leq \tau(x) \to \tau(y)$ and $\tau(y \to x) \leq \tau(y) \to \tau(x)$, so $\eta_2(x,y) = \tau(x \to y) \wedge \tau(y \to x) \leq (\tau(x) \to \tau(y)) \wedge (\tau(y) \to \tau(x)) = \eta_3(x,y)$.

4. Probability metrics of MV-algebra

Definition 4.1 Let x, y $\in$ M, defining

$$\rho(x,y) = 1 - \eta(x,y), \tag{4}$$

$$\rho_i(x,y) = 1 - \eta_i(x,y), i = 1,2,3. \tag{5}$$

Theorem 4.1 $\rho_i : M \times M \to [0,1]$ is metric of MV-algebra M(i=1,2,3).

Proof Let x, y, z $\in$ M, from theorem 3.1 we see that $\rho(x,x) = 1 - \eta(x,x) = 0$, and $\rho(x,y) = 1 - \eta(x,y) = 1 - \eta(y,x) = \rho(y,x)$.

$$\rho(x,z) = 1 - \eta(x,z) \leq 1 - (\eta(x,y) + \eta(y,z) - 1) = (1 - \eta(x,y)) + 1 - \eta(y,z) = \rho(x,y) + \rho(y,z)$$

Definition 4.2 (M, ρ_i) is called probability metric space of MV-algebra (i=1,2,3).

Theorem 4.2 Let x, y $\in$ M, then

$$\begin{aligned} \rho_1(x,y) &= \int_\Omega |\bar{x}(v) - \bar{y}(v)| d\mu, \\ \rho_3(x,y) &= |\tau(x) - \tau(y)| = \left| \int_\Omega (\bar{x}(v) - \bar{y}(v)) d\mu \right|. \end{aligned}$$

Theorem 4.3 Let x, y $\in$ M, then

(i) $\rho_1(x,y) \geq \rho_2(x,y) \geq \rho_3(x,y)$,

(2) $\rho_2(x,y) = \frac{1}{2}(\rho_1(x,y) + \rho_3(x,y))$.

Proof (i) From theorem 3.2(ii), we can proved.

(ii)

$$\begin{aligned} \rho_2(x,y) &= 1 - \eta_2(x,y) \\ &= 1 - \tau(x \to y) \wedge (\tau(y \to x)) \\ &= (1 - \tau(x \to y)) \vee (1 - \tau(y \to x)) \\ &= (1 - \int_\Omega |1 - \bar{x} + \bar{x} \wedge \bar{y}| d\mu) \vee \\ &\quad (1 - \int_\Omega |1 - \bar{y} + \bar{y} \wedge \bar{x}| d\mu) \\ &= \int_\Omega |\bar{x} - \bar{x} \wedge \bar{y}| d\mu \vee \int_\Omega |\bar{y} - \bar{y} \wedge \bar{x}| d\mu \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\Big(\int_\Omega |\bar{x} - \bar{x}\wedge\bar{y}|d\mu + \int_\Omega |\bar{y} - \bar{y}\wedge\bar{x}|d\mu \\
&\quad +\Big|\int_\Omega |\bar{x} - \bar{x}\wedge\bar{y}|d\mu - \int_\Omega |\bar{y} - \bar{y}\wedge\bar{x}|d\mu\Big|\Big) \\
&= \frac{1}{2}\Big(\int_\Omega (\bar{x}+\bar{y})d\mu - 2\int_\Omega (\bar{x}\wedge\bar{y})d\mu \\
&\quad +\Big|\int_\Omega (\bar{x}-\bar{y})d\mu\Big|\Big) \\
&= \frac{1}{2}\Big(\int_\Omega (\bar{x}+\bar{y})d\mu - 2\int_\Omega (\bar{x}+\bar{y}-|\bar{x}-\bar{y}|)d\mu \\
&\quad +\Big|\int_\Omega (\bar{x}-\bar{y})d\mu\Big|\Big) \\
&= \frac{1}{2}\Big(\int_\Omega |\bar{x}-\bar{y}|d\mu + \Big|\int_\Omega (\bar{x}-\bar{y})d\mu\Big|\Big) \\
&= \frac{1}{2}(\rho_1(x,y)+\rho_3(x,y))
\end{aligned}$$

Theorem 4.4 (Wang and Zhou [6]) On probability metric space of MV-algebra (M, ρ_1), operations $\neg$, $\oplus$, $\odot$, $\vee$, $\wedge$ and $\to$ are continuous.

Lemma 3.2 Let $x, y, u, w \in M$, $\alpha, \beta\in[0, 1]$, if $\eta_2(x, u)\geq\alpha$, $\eta_2(y, w)\geq\beta$, then $\eta_2(x\to y, u \to w)\geq\alpha+ \beta-1$.

Proof Let $[0, 1]_{MV}$ be standard MV-algebra, $a, b, c \in [0, 1]$, then $a \to b \leq (b \to c) \to (a \to c)$ and $b \to a \leq (a \to c) \to (b \to c)$. Thus $\forall v\in\Omega$, we see that

$$\bar{x} \to \bar{u} \leq ((\bar{x} \to \bar{y}) \to (\bar{u} \to \bar{y})),$$

And

$$\bar{u} \to \bar{x} \leq ((\bar{u} \to \bar{y}) \to (\bar{x} \to \bar{y})),$$

From proposition 3.1,

$$\tau(x \to u) \leq \tau((x \to y) \to (u \to y)),$$

$$\tau(u \to x) \leq \tau((u \to y) \to (x \to y)),$$

so $\tau(x \to u) \wedge \tau(u \to x) \leq \tau((x \to y) \to (u \to y)) \wedge \tau((u \to y) \to (x \to y))$, i.e., $\eta_2(x \to y, u \to y) \geq \eta_2(x,y) \geq \alpha$.

Similarly, $\eta_2(u \to y, u \to w) \geq \eta_2(y,w) \geq \beta$, thus

$$\begin{aligned}
&\eta_2(x \to y, u \to w) \\
\geq\ & \eta_2(x \to y, u \to y) + \eta_2(u \to y, u \to w) - 1 \\
\geq\ & \eta_2(x,u) + \eta_2(y,w) - 1 \\
\geq\ & \alpha + \beta - 1.
\end{aligned}$$

Theorem 4.5 Suppose (M, ρ_2) be probability metric space of MV-algebra, then operations $\neg$, $\oplus$, $\odot$, $\vee$, $\wedge$ and $\to$ on (M, ρ_2) are continuous.

Proof First, from theorem 3.1 we see that $\rho_2(\neg x, \neg y) = 1 - \eta_2(\neg x, \neg y) = 1 - \eta_2(x, y) = \rho_2(x, y)$, therefore $\neg$ is continuous.

Second, let $\rho_2(x, u) \le \varepsilon$, $\rho_2(y, w) \le \varepsilon$, then $\eta_2(x, u) \ge 1-\varepsilon$, $\eta_2(y, w) \ge 1-\varepsilon$, so from lemma 3.2 we have

$$\eta_2(x \to y, u \to w) \ge (1-\epsilon) + (1-\epsilon) - 1 = 1 - 2\epsilon.$$

i.e., $\rho_2(x \to y, u \to w) \le 2\varepsilon$, so implication operation $\to$ is continuous.

Because $x \oplus y = \neg x \to y$, $x \odot y = \neg(x \to \neg y)$, $x \vee y = (x \to y) \to y$, $x \wedge y = \neg(\neg x \vee \neg y)$, therefore $\oplus$, $\odot$, $\vee$, $\wedge$ are all continuous.

Theorem 4.6 (Wang and Zhou [6]) If (M, ρ_1) be complete metric space, then M is a countably complete lattice.

Theorem 4.7 If (M, ρ_2) be complete metric space, then M is also a countably complete lattice.

Proof Suppose (M, ρ_2) be complete metric space, $\Delta = \{x_1, x_2, \cdots\} \subset M$. Let $y_n = V_{i=1}^{n} x_i$, then $y_1, y_2, \cdots$ is an increasing sequence. From proposition 3.1 we see that

$T(y_1), T(y_2), \cdots$ is an increasing sequence in [0,1], so is a Cauchy sequence. And

$$\bar{y}_n \to \bar{y}_m = 1,\ \bar{y}_m \to \bar{y}_n = 1 - \bar{y}_m + \bar{y}_n,\ n \le m.$$

Therefore, if $n \le m$ then

$$\begin{aligned}
\rho_2(y_m, y_n) &= 1 - \eta_2(y_m, y_n) \\
&= 1 - \tau(y_m \to y_n) \wedge \tau(y_n \to y_m) \\
&= 1 - \tau(y_m \to y_n) \\
&= 1 - \int_\Omega (1 - \bar{y}_m + \bar{y}_n) d\mu \\
&= \int_\Omega \bar{y}_m d\mu - \int_\Omega \bar{y}_n d\mu = \tau(y_m) - \tau(y_n).
\end{aligned}$$

From $T(y_1), T(y_2), \cdots$ is a Cauchy sequence, we see that $y_1, y_2, \cdots$ is a Cauchy sequence in complete metric space (M, ρ_2), so it converges to y, where y is a point of M.

We fix an n, then $y_n = y_n \wedge y_{n+k}$, from the continuity of operation see that

$$y_n = \lim_{k\to\infty} (y_n \wedge y_{n+k}) = y_n \wedge (\lim_{k\to\infty} y_{n+k}) = y_n \wedge y.$$

So $y_n \le y$, i.e., y is a upper bound of $\Sigma = \{y_1, y_2, \cdots\}$. Taking any a upper bound of Σ, then $y_n = y_n \wedge z$, so

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} (y_n \wedge z) = (\lim_{n\to\infty} y_n) \wedge z = y \wedge z.$$

Thus $y \le z$, i.e., $y = \sup\Sigma \in M$. It is clearly $\sup\Delta = \sup\Sigma \in M$.

Similarly, we can prove $\inf \Delta \in M$. Therefore M is countable complete.

Theorem 4.8 Suppose (M, ρ_3) be probability metric space of MV-algebra, then operations $\neg$ is continuous, but operation $\oplus$, $\odot$, $\vee$, $\wedge$ and $\to$ are not continuous all.

5. Conclusion

In this paper, we introduced three probability metric space on MV-algebras, studied the propositions of probability metric spaces on MV-algebras, get some good results. We can see that the method of one paper can be extended to other algebras, so the method of one paper is meaningful.

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