Domain theoretic models of topological spaces

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Abstract

A model of a space X is simply a continuous dcpo D and a homeomorphism $\phi : X \to \max D$, where $\max D$ is given its inherited Scott topology. We show that a space has a coherent model iff it has a Scott domain model and investigate the topological structure of spaces which have G_δ models.

1 Introduction

Why would someone ever ask "Which spaces have domain theoretic models?" Let us begin with an example.

Example 1.1 Let $f : X \to X$ be a contraction on a compact metric space X. Define $\mu : UX \to [0,\infty)^*$ by $\mu k = \operatorname{diam} k$, where UX is the ω-continuous Scott domain of compact subsets of X ordered under reverse inclusion, and $[0,\infty)^*$ is the set of nonnegative reals in their dual order. Observe that μ is Scott continuous and that $\mu k = 0$ iff $k \in \max UX$. The contraction f extends to a Scott continuous mapping $f : UX \to UX$, whose least fixed point is $r = \bigsqcup f^{(n)}(\bot)$, where $\bot = X$. However,

$$\mu r = \bigsqcup \mu f^{(n)}(\bot) = 0,$$

so $r \in \max UX$. That is, $r = \{x\}$, for some $x \in X$, which is the unique fixed point of the contraction we began with.

The very same argument is applied to the formal ball model in [2] to give a domain theoretic proof of the Banach contraction mapping theorem for any complete metric space. As a matter of fact, careful examination of the argument above reveals that it may be carried out on any domain theoretic model of a metric space X, provided that the model allows the extension

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of contraction mappings, and that it admits a function like μ. Of course, this is but one example of a recurring theme in domain theory today. There are many other instances, where not only has an argument or construction been carried out domain theoretically, but in addition, the domain theoretic characterization has led to the extension or sharpening of various ideas and results (e.g. Edalat's weakly hyperbolic IFS's, integration, etc). The ability to do this usually depends only on the fact that a space admits a domain theoretic model. The rest is just domain theory.

So what do we know about the spaces which admit domain theoretic models? Lawson [11] has proven that a space is Polish iff it has an ω-continuous model where the relative Scott and Lawson topologies agree at the top. This is by far the most progress which has been made on the question. One of the aspects of his work that is so interesting is the idea that in such a domain, max D is always a countable intersection of *Scott* open sets, i.e., such domains are G_δ models. We will spend time considering these. Of particular relevance to our example above is that a function like μ exists on a domain iff the maximal elements are a G_δ. We will also consider the topological structure of spaces with G_δ models. One surprising result is that such spaces are *always* first countable and Baire.

In the algebraic setting, there is the work of Flagg and Kopperman [6]. Here we learn that a space is complete, separable ultrametric iff it can be modelled with an ω-algebraic Scott domain iff it can be modelled with an ω-algebraic dcpo where the Scott and Lawson topologies agree at the top. The most striking feature of this work is that the ω-algebraic dcpo's comprise a class of domains in which Scott domains, coherent domains, and domains where the Scott and Lawson topologies agree on max D (*coherence at the top*), all model the same class of spaces. We wonder of course whether or not this result holds in general. We suspect that it does, which is one reason we use the phrase "coherent at the top." We will not answer this question entirely, but we will show that coherent domains, Scott domains (and hence FS-domains) all model the same class of spaces.

2 Models Coherent at The Top

A **domain** is a continuous dcpo D. The maximal elements of a domain are denoted max D. This is also called the top of a domain. A domain is **coherent** if its Lawson topology is compact. A **Scott domain** is a continuous dcpo D (with $\perp$) in which suprema of consistent pairs exist. Note that our definition differs from the traditional one in that we do **not** assume Scott domains algebraic. For basic definitions, consult [1].

Definition 2.1 A **model** of a topological space X is a continuous dcpo D and a homeomorphism $\phi : X \to \max D$, where max D carries its inherited Scott topology from D. A model is a G_δ **model** if in addition max D is the intersection of countably many Scott open sets.

Definition 2.2 A continuous dcpo D is **coherent at the top** exactly when the inherited Scott and Lawson topologies on $\max D$ coincide.

Remark 2.3 A separable metric space X is *Polish* if its topology may be induced by at least one complete metric. X is a *complete, separable, ultrametric space* if there is an ultrametric yielding the topology which is at the same time complete. A Tychonoff space X is *Čech-complete* if it is a G_δ in its Stone-Čech compactification βX. (The latter is only needed in the proof of Theorem 4.4.)

In the ω-continuous case, the spaces represented by domains coherent at the top were classified by J.D. Lawson.

Theorem 2.4 (Lawson [11]) *A space X is Polish iff $\exists$ ω-continuous dcpo D which is coherent at the top such that $X \simeq \max D$.*

An important fact which arises in his proof is that $\max D$ is a G_δ in D w.r.t the Scott topology. Much rests on this fact when considering the probabilistic powerdomain of D (see [3]) or more traditionally the topological structure of $\max D$. We will see a few examples of the latter in Section 4.

Proposition 2.5 (Lawson [11]) *For any ω-continuous dcpo D which is coherent at the top, $\max D$ is the intersection of countably many Scott open sets. That is, all such domains D are G_δ models of $\max D$.*

As mentioned earlier, it has already been proven that Scott domains and domains coherent at the top (hence coherent domains) model the same class of spaces *provided* the models used are ω-algebraic.

Theorem 2.6 (Flagg and Kopperman [6]) *For a topological space X, the following are equivalent:*

(i) X is a complete separable ultrametric space.

(ii) X has an ω-algebraic model which is a Scott domain.

(iii) X has an ω-algebraic model which is coherent at the top.

Lawson's theorem tells us that any space at the top of an ω-algebraic Scott domain must be Polish. Ideally, the addition of algebraicity to the model should mean that the spaces at the top are now exactly Polish spaces which are also zero-dimensional. In 1982, in fact, Scott [12] remarked that the top of an ω-algebraic Scott domain was zero-dimensional and that it could be "conveniently embedded into the real line." We now give what we feel is a more intuitive characterization of the spaces at the tops of ω-algebraic Scott domains.

Theorem 2.7 *For a topological space X, the following are equivalent:*

(i) X is Polish and zero-dimensional.

(ii) X is a complete separable ultrametric space.

(iii) X is a G_δ subset of the real line which does not contain an interval.

Proof. (i) $\Rightarrow$ (ii): First embed X in the Cantor set (see 1.3.16 of [4]). The closure of the image yields a zero-dimensional, metrizable compactification of X, which we will call $\sigma(X)$. Since $\sigma(X)$ is a zero-dimensional, separable metric space, section 2 of [7] guarantees that there is an ultrametric d which induces the topology of $\sigma(X)$. This ultrametric is complete as a result of the compactness of $\sigma(X)$. Now, because X is complete with respect to *some* metric, and it also resides as a dense subset of the compact Hausdorff space $\sigma(X)$, X is a G_δ in $\sigma(X)$. The Alexandroff result holds for complete ultrametric spaces as pointed out in [6], that is, not only is there some metric relative to which X is complete, but because X is a G_δ in a complete *ultrametric* space, we can choose an *ultrametric* relative to which X is complete. Then this proves that X is a complete, separable, ultrametric space.
(ii) $\Rightarrow$ (iii) Any zero-dimensional, separable metric space can be embedded in the Cantor set, and hence in the real line. Zero-dimensionality implies that the space contains no (nontrivial) interval, and since X is complete, it is a G_δ in its closure: a compact subset of the Cantor set. Then X is a G_δ in $\mathbb{R}$ — an instance of the *absolute* G_δ property that all complete metrics possess.
(iii) $\Rightarrow$ (i) A subset of the real line is zero-dimensional iff it does not contain a (nontrivial) interval. A G_δ subset of $\mathbb{R}$ is Polish. $\square$

It is difficult to imagine spaces with coherent models which are at the same time nowhere locally compact. Unfortunately, they do exist. The most popular example seems to be the domain of partial functions on the naturals, a classic ω-algebraic Scott domain, which provides a model of the irrationals. In 1928, however, Urysohn and Alexandroff provided the following characterization of them.

Theorem 2.8 (Urysohn-Alexandroff [5]) *The only Polish space which is zero-dimensional and has no nonempty compact open sets is* $\mathbb{R} \setminus \mathbb{Q}$.

Consequently, there is only 1 nowhere locally compact space which can be modelled with an ω-algebraic Scott domain.

Corollary 2.9 *If D is an ω-algebraic dcpo which is coherent at the top, then either* $\max D \simeq \mathbb{R} \setminus \mathbb{Q}$ *or there is a point where* $\max D$ *is locally compact.*

3 Coherent Domains and Scott Domains

In the last section, we saw that ω-algebraic Scott domains and ω-algebraic coherent domains model the very same spaces: G_δ subsets of the real line which do not contain an interval. In this section, we prove that Scott domains and coherent domains *always* model the same class of spaces.

Proposition 3.1 *Every Scott domain is coherent.*

The next example shows that coherent domains which are not Scott domains are very easy to find.

Example 3.2 Let Disc [1] denote the collection of closed discs of the plane and the plane itself ordered under reverse inclusion. Disc is easily seen to be an ω-continuous dcpo which provides a model of the plane. Disc is coherent because it is an FS-domain. However, the intersection of discs is not always a disc, so it is not a Scott domain.

Theorem 3.3 (Hofmann-Mislove [9]) *For any continuous dcpo D,*

$$\kappa_D = \{ \, \emptyset \neq K \subseteq D : K = \uparrow K \text{ Scott compact} \, \},$$

ordered under reverse inclusion, is itself a continuous domain whose approximation relation is given by

$$A \ll B \text{ iff } (\, \exists \, U \in \sigma_D \,) \; B \subseteq U \subseteq A.$$

It was shown in [1] that κ_D is isomorphic to the **Smyth Powerdomain** of D. As it turns out, then, the Smyth powerdomain provides us with an example of a domain theoretic construction which preserves the object being modelled.

Theorem 3.4 *For any continuous dcpo D, κ_D is a model of* max D.

Proof. First, $K \in \max \kappa_D$ if and only if $K = \{m\}$, for a unique $m \in \max D$. This establishes that

$$\phi : \max D \to \max \kappa_D$$
$$\phi(m) = \{m\}$$

is a bijection. Now a straightforward argument using the approximation relations proves that the map is both continuous and open. □

Proposition 3.5 *The spaces which can be modelled with Scott domains are precisely the spaces which can be modelled with coherent domains.*

Proof. A domain D is coherent iff κ_D is a Scott domain. By the result above, κ_D is a model of max D. □

In a recent paper [10], Jung and Sünderhauf remark that it is presently unknown as to which spaces can be modelled with FS-domains. By the work above, we can give a partial solution to this problem.

Corollary 3.6 *A topological space X can be modelled by an FS-domain iff it can be modelled by a coherent domain iff it can be modelled by a Scott domain.*

Proof. FS-domains are coherent and every Scott domain is an FS-domain.□

Finally, observe that we cannot take an arbitrary model and use κ_D to construct one which is coherent at the top.

Proposition 3.7 *If κ_D is coherent at the top, D is also coherent at the top.*

Proof. The homeomorphism $\phi : \max D \to \max \kappa_D$ satisfies

$$\phi^{-1}(\, \uparrow U(x) \cap \max \kappa_D \,) = \uparrow x \cap \max D,$$

where $U(x) = \uparrow x \in \kappa_D$. □

4 G_δ models

As remarked earlier, Lawson has shown that G_δ models exist in abundance. It is well-known that G_δ subsets are useful in proving theorems of a topological nature. However, when working in the realm of domain theoretic models, they play a much larger role.

Theorem 4.1 *For a subset X of a continuous dcpo D, the following are equivalent:*

(i) X is a G_δ in D.

(ii) $\exists$ Scott continuous map $\mu : D \to [0,\infty)^*$ with $\mu x = 0$ iff $x \in X$.

Proof. (i) $\Rightarrow$ (ii) First write $X = \bigcap U_n$ as the intersection of a descending family of countably many Scott open sets. Define $n : D \to \mathbb{N} \cup \{\infty\}$ by

$$n(x) = \begin{cases} \sup\{n : x \in U_n, n \geq 1\} & \text{if } x \in U_1 \\ 0 & \text{if } x \in D \setminus U_1 \end{cases}$$

Observe that $n(x) = \infty$ iff $x \in X$. Now define μ as

$$\mu : D \to [0,\infty)^*$$
$$\mu x = \frac{1}{2^{n(x)}}$$

This is the desired mapping. □

In our opening example, the function μ is used to measure the progress of a computation: it provides an *a priori* estimate of the error in computing a fixed point r. The fact that such measuring devices and G_δ subsets are equivalent tells us that a G_δ subset of a continuous dcpo is actually a *computational* notion.

Lemma 4.2 *For a subset X of a continuous dcpo D, the following are equivalent:*

(i) X is an upper set which is T_1 in its relative Scott topology.

(ii) $X \subseteq \max D$.

Proof. ($\Rightarrow$) The intersection of all open sets in X containing $x \in X$ is an upper set. Since X is T_1, this intersection is exactly $\{x\}$. □

Proposition 4.3 *For a continuous dcpo D and a point $x \in D$, the following are equivalent:*

(i) D is first countable at x.

(ii) x is the limit of a sequence of approximations.

Proof. (i) $\Rightarrow$ (ii) Choose a sequence (a_n) of approximations of x, one in each member U_n of the countable basis $\{U_n\}$ at x. Use the directedness of $\Downarrow x$ to construct an *increasing* sequence (x_n) with $x_n \ll x$ and $a_n \sqsubseteq x_n$. Since $\bigsqcup x_n \in U_n$, for all $n \in \mathbb{N}$, and $\{U_n\}$ is a base at x, we must have $\bigsqcup x_n = x$. □

Example 4.4 Let X be a compact Hausdorff space which is not first countable at some point Ω. Then the space $X \times \mathbb{R}$ is a locally compact Hausdorff space which is not first countable at the points $\{\Omega\} \times \mathbb{R}$. It is a dense G_δ in $\beta(X \times \mathbb{R})$.

The point of the example above is that a G_δ subset of even a compact Hausdorff space can lack first countability at many points. Because of this, the next theorem is very surprising: it is a topological characteristic of continuous domains which *does not* necessarily hold for locally compact Hausdorff spaces, and so it is one which *cannot* be derived from a more general result on locally compact sober spaces.

Theorem 4.5 *A continuous dcpo D is first countable at every point of a G_δ subset X provided X is T_1 in its relative Scott topology.*

Proof. We know that there exists a Scott continuous map $\mu : D \to [0,\infty)^*$ with $\ker \mu = X$. Using the directedness of $\Downarrow x$, we can construct an increasing sequence (x_n) with $x_n \ll x$ and $\mu x_n \leq \frac{1}{n}$. Applying the continuity of μ reveals that $\mu(\bigsqcup x_n) = 0$, which means that $\bigsqcup x_n = x$ by maximality. □

Theorem 4.6 *If D is a G_δ model of a space X, then X is a first countable, T_1 Baire space.*

Proof. The continuous dcpo D is a locally compact sober space w.r.t. the Scott topology, and so it is a Baire space [8]. $X = \max D$ is a dense G_δ in a Baire space. Then X too must be Baire. First countability of X is inherited from D. □

Corollary 4.7 *There is no G_δ model of the rationals.*

Proof. The rationals are not a Baire space. □

Corollary 4.8 (The Baire Category Theorem) *Every complete metric space is Baire.*

Proof. Use the formal ball model of Edalat and Heckmann [2], which provides a G_δ model for any complete metric space. The result now follows immediately from Theorem 4.3. □

Another interesting property of spaces with G_δ models is that local compactness may be detected domain theoretically.

Corollary 4.9 *If X is a Hausdorff space with a G_δ model, then X is a k-space. Consequently, X is locally compact iff UX is a continuous dcpo.*

Proof. UX is the dcpo of compact subsets of X, ordered under reverse inclusion. If X is a Hausdorff k-space, then the continuity of UX implies local compactness of X. This is due to J.D. Lawson [9]. First countable Hausdorff spaces are k-spaces, i.e., spaces determined by their compact sets. □

Finally, if we consider *coherent* G_δ models, we can say even more.

Theorem 4.10 *If D is a coherent G_δ model of a space X, then X is a first countable, Čech-complete space. In particular,*

X is metrizable iff X is completely metrizable.

Proof. Since D is coherent at the top, we may think of X as having the Lawson topology. Since D is compact Hausdorff in the Lawson topology, X is Tychonoff. The Lawson closure of X in D is a compactification of X. Since X is a G_δ in D, it is a G_δ in its closure. Then X is a G_δ in all its compactifications. This proves that X is Čech-complete. Lastly, a metric space is Čech-complete iff it is completely metrizable. □

Remark 4.11 There is a corollary to this result which may be of interest here. Suppose that X is a metric space with a coherent model which admits a proof of the Banach contraction mapping theorem, the way we did in example 1.1, or as in [2]. Then X is necessarily complete. That is, we cannot use a coherent model to generalize the Banach contraction mapping theorem.

5 Further Research

There are several important questions which need to be answered. Some of my favorites are as follows:

(i) Does Theorem 4.4 generalize, i.e., if X is a metric space with a G_δ model, then is X necessarily completely metrizable? To answer this negatively, we must find a G_δ model of a Baire metric space which is not completely metrizable.

(ii) Is max D a G_δ in D for *every* ω-continuous dcpo D? To answer this negatively, we need to find a model of a space which is not a Baire space. As we have seen, an ω-continuous dcpo D with max $D \simeq \mathbb{Q}$ will do the trick (if it exists).

(iii) Is it true that a space X has a coherent at the top model iff it is Čech-complete? Also, is a space completely metrizable iff it has a coherent at the top G_δ model? (These seem to be natural generalizations of Lawson's theorem.)

(iv) Can every Polish space be modelled with an ω-continuous, *coherent*, dcpo? Note that, if so, it immediately now follows that any Polish space can also be modelled with an ω-continuous Scott domain.

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