On Algebras of Toeplitz Matrices

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ABSTRACT

Necessary and sufficient conditions for Toeplitz and block Toeplitz matrices to have Toeplitz inverse are given. Maximal algebras of scalar Toeplitz matrices are fully described.

1. INTRODUCTION

Huang and Cline [6] obtained a simple criterion for a nonsingular scalar Toeplitz matrix $R = (r_{p-q})_{p,q=0}^n$ with $r_0 \neq 0$ to have a Toeplitz inverse. Greville [4] pointed out that their result remains true also when $r_0 = 0$. The result is as follows:

**THEOREM 1.1.** Let $R = (r_{p-q})_{p,q=0}^n$ be a nonsingular Toeplitz matrix. Then $R^{-1}$ is a Toeplitz matrix if and only if there exist complex numbers $a, b, t_1, t_2, \ldots, t_n$ such that $R$ admits the representation

$$R = \begin{pmatrix}
  r_0 & bt_n & \cdots & bt_1 \\
  at_1 & r_0 & \cdots & bt_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  at_n & at_{n-1} & \cdots & r_0
\end{pmatrix}. \tag{1.1}$$

The matrix in (1.1) is known as a generalized circulant (see [3]).
In this paper we prove an equivalent condition which will be generalized to the case of block Toeplitz matrices.

**Theorem 1.2.** Let $R = (r_{p-q})_{p,q=0}^n$ be a nonsingular Toeplitz matrix. Then $R^{-1}$ is a Toeplitz matrix if and only if there exist complex numbers $z_0, z_1, \ldots, z_n$ such that

$$r_{p-1-n} = r_p z_0 - z_p r_0 \quad (p = 1, 2, \ldots, n) \quad (1.2)$$

and

$$r_p z_q = z_p r_q \quad (p, q = 1, 2, \ldots, n). \quad (1.3)$$

This result leads to the following generalization of Theorem 1.1.

**Theorem 1.3.** Let $R = (r_{p-q})_{p,q=0}^n$ be a Toeplitz matrix. Then all the following statements are equivalent:

(a) There exist numbers $a, b, t_1, \ldots, t_n$ such that either $a = 1$ or $a = 0$, $b = 1$ for which $R$ is of the form (1.1).

(b) The matrix $R^2$ is a Toeplitz matrix.

(c) All positive powers of $R$ are Toeplitz matrices.

(d) The matrix $R$ belongs to an algebra of Toeplitz matrices.

In the case of a nonsingular matrix $R$ the following statement is equivalent to each of the previous ones:

(e) The matrix $R^{-1}$ is a Toeplitz matrix.

Theorem 1.3 enables us to give a full description of all maximal algebras of scalar Toeplitz matrices.

For any complex number $\alpha$, let us denote by $\Pi_\alpha$ the set of all Toeplitz matrices $R$ of the form (1.1), where $a = 1$, $b = \alpha$ and $r_0, t_1, \ldots, t_n$ are any complex numbers. Let us denote by $\Pi_\infty$ the set of all upper triangular Toeplitz matrices [i.e. all Toeplitz matrices $R$ of the form (1.1) where $a = 0$, $b = 1$ and $r_0, t_1, \ldots, t_n$ are any complex numbers].

The following theorem gives a one-to-one correspondence between all the maximal algebras of scalar Toeplitz matrices and all the numbers in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

**Theorem 1.4.** The set $\{\Pi_\alpha\}_{\alpha \in \mathbb{C}_\infty}$ consists of all the maximal algebras of scalar Toeplitz matrices.
Let us consider the generalizations of the above results to the case of block matrices in which all the entries of the matrices are $m \times m$ matrices.

Kailath and Koltracht [7] generalized the theorem of Huang and Cline [6] in the following way:

**Theorem 1.5.** A nonsingular block Toeplitz matrix $R = (r_{p-q})_{p,q=0}^n$ such that $r_0$ is invertible has a block Toeplitz inverse if and only if

$$r_p(r_0)^{-1}r_{q-1-n} = r_{p-1-n}(r_0)^{-1}r_q \quad (p, q = 1, 2, \ldots, n). \quad (1.4)$$

This paper contains a generalization of Theorem 1.2 to the block case which has no restriction on $r_0$.

**Theorem 1.6.** A nonsingular block Toeplitz matrix $R = (r_{p-q})_{p,q=0}^n$ has a block Toeplitz inverse if and only if there exist $m \times m$ matrices $z_0, z_1, \ldots, z_n$ and $w_0, w_1, \ldots, w_n$ such that

$$r_0w_n = z_0r_0, \quad (1.5)$$

$$r_{p-1-n} = z_0r_p - r_0w_{n-p} = r_pw_n - z_pr_0 \quad (p = 1, 2, \ldots, n), \quad (1.6)$$

and

$$r_pw_{n-q} = z_pr_q \quad (p, q = 1, 2, \ldots, n). \quad (1.7)$$

We shall use this result to prove the following theorem, which has a similar structure to Theorem 1.5.

**Theorem 1.7.** A nonsingular block Toeplitz matrix $R = (r_{p-q})_{p,q=0}^n$ such that $r_{q_0}$ is invertible for an integer $q_0 > 0$ has a block Toeplitz inverse if and only if

$$r_0(r_{q_0})^{-1}r_{q_0-1-n} = r_{q_0-1-n}(r_{q_0})^{-1}r_0 \quad (1.8)$$

and

$$r_{p-1-n} = r_p(r_{q_0})^{-1}r_{q_0-1-n} - r_{q_0-1-n}(r_{q_0})^{-1}r_p \quad (p = 1, 2, \ldots, n). \quad (1.9)$$

Unlike the scalar case, there are examples of block Toeplitz matrices $R$ such that $R^{-1}$ is a block Toeplitz matrix but $R^2$ is not. We shall prove the
following theorem, which gives necessary and sufficient conditions on a block Toeplitz matrix to have block Toeplitz positive powers.

**Theorem 1.8.** Let \( R = (r_{p, q})_{p, q=0}^n \) be a block Toeplitz matrix, and let \( k \) be any integer \( k > 1 \). The matrices \( R^2, R^3, \ldots, R^k \) are all block Toeplitz matrices if and only if

\[
r_p(r_0)^j r_{q-1-n} - r_{p-1-n}(r_0)^j r_q \quad (p, q = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, k - 2).
\]

(1.10)

The generalization of Theorem 1.4 to the block case still remains an open question.

2. **NOTATION AND PRELIMINARIES**

The following notation will be used in both the scalar and the block cases. For example, the numeral 1 denotes either the unit number or the identity matrix \( I_n \).

Let \( \{ e^{(p)} \}_{p=0}^n \) and \( \{ f^{(p)} \}_{p=0}^n \) be the following sets of columns and rows:

\[
e^{(p)} = \text{col}(\delta_{p, q} 1)^n_{q=0}, \quad f^{(p)} = \text{row}(\delta_{p, q} 1)^n_{q=0} \quad (p = 0, 1, \ldots, n).
\]

The notation

\[
L(g_0, \ldots, g_n) = \begin{pmatrix}
g_0 & 0 & \cdots & 0 \\
g_1 & g_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
g_n & g_{n-1} & \cdots & g_0
\end{pmatrix},
\]

\[
U(g_0, \ldots, g_n) = \begin{pmatrix}
g_0 & g_1 & \cdots & g_n \\
0 & g_0 & \cdots & g_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & g_0
\end{pmatrix}
\]
and
\[
C_d(g_0, \ldots, g_n) = \begin{pmatrix}
g_0 & g_1 & \cdots & g_{n-1} & g_n \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]

\[
C_r(g_0, \ldots, g_n) = \begin{pmatrix}
0 & 0 & \cdots & 0 & g_0 \\
1 & 0 & \cdots & 0 & g_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & g_n
\end{pmatrix}
\]

will be used in the sequel.

For any matrix \(A(a_{p,q})_{p,q=0}^{n} \), let \(A^T = (a_{q,p})_{p,q=0}^{n} \) denote the transposed matrix of \(A\). Define

\[
C_d(g_0, \ldots, g_n) = C_r(g_0, \ldots, g_n)^T, \quad C_r(g_0, \ldots, g_n) = C_u(g_0, \ldots, g_n)^T.
\]

For any complex number \(\alpha\), let \(S_\alpha\) denote the matrix \(S_\alpha = C_u(0,0,\ldots,0,\alpha)\) or equivalently \(S_\alpha = C_r(\alpha,0,\ldots,0)\). It is clear that \(S_0\) is the lower shift matrix, i.e. \(S_0 = L(0,1,0,\ldots,0)\). Let \(S_\infty\) denote the upper shift matrix, i.e. \(S_\infty = U(0,1,0,\ldots,0)\). Finally, let \(J\) denote the matrix

\[
J = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

It is clear that \(J^2 = I\), and that if \(A\) is a scalar Toeplitz matrix then \(JA^TJ = A\).

We shall use the following result, which is valid in both the scalar and the block cases. The proof is quite simple and therefore is omitted.

**Proposition 2.1.** Let \(A = (a_{p,q})_{p,q=0}^{n}\) and \(B = (b_{p,q})_{p,q=0}^{n}\) be two (block) Toeplitz matrices. The product \(AB\) is a (block) Toeplitz matrix if and only if

\[
a_p b_{q-1-n} = a_{p-1-n} b_q \quad (p, q = 1, 2, \ldots, n).
\]
Following the results of Heinig and Rost [5], a criterion for a nonsingular matrix to have a Toeplitz inverse has been given in [1]. The generalization of this result to the block case has been given in [2]. These results led to Theorems 1.2 and 1.6. For a shorter proof, we shall use the following equivalent result, which can be derived from an earlier work of Lerer and Tismenetsky [8, 9].

**Theorem 2.2.** A nonsingular block matrix $R$ has a block Toeplitz inverse if and only if there exist block vectors $z = \text{col}(z_p)_{p=0}^{n}$ and $w = \text{row}(w_p)_{p=0}^{n}$ such that

$$RC_u(w_0, w_1, \ldots, w_n) = C_r(z_0, z_1, \ldots, z_n)R.$$ (2.1)

The proof of this theorem is straightforward, and therefore it is omitted as well.

### 3. Algebras of Scalar Toeplitz Matrices

The following theorem is the scalar version of Theorem 2.2. We shall omit its proof, which is simple.

**Theorem 3.1.** A nonsingular matrix $R$ has a Toeplitz inverse if and only if there exists a vector $z = \text{col}(z_p)_{p=0}^{n}$ such that

$$RC_u(z_n, z_{n-1}, \ldots, z_0) = C_r(z_0, z_1, \ldots, z_n)R.$$ (3.1)

Theorem 1.2 follows immediately from Theorem 3.1 by explicit calculations using (3.1) and the assumption that $R$ itself is a Toeplitz matrix.

We shall use the following necessary and sufficient condition on a matrix $R$ to commute with $S_\alpha$ and omit its straightforward computational proof.

**Lemma 3.2.** For any $\alpha$ in $\mathbb{C}_\infty$, a matrix $R$ commutes with $S_\alpha$ if and only if $R \in \Pi_\alpha$.

The following equivalent version of Theorem 1.1 will be used in the course of the proof of Theorem 1.3.

**Theorem 3.3.** Let $R = (r_{p,q})_{p,q=0}^{n}$ be a nonsingular Toeplitz matrix. The matrix $R^{-1}$ is a Toeplitz matrix if and only if there exists $\alpha$ in $\mathbb{C}_\infty$ such that $R \in \Pi_\alpha$. 
Proof. First let us assume that \( R \in \Pi_\alpha \) for a certain \( \alpha \in \mathbb{C}_\infty \). Therefore, it follows from Lemma 3.2 that

\[
RS_\alpha = S_\alpha R. \tag{3.2}
\]

If \( \alpha \neq \infty \), then (3.2) implies both (1.2) and (1.3) with \( z_0 = \alpha, \ z_1 = z_2 = \cdots = z_n = 0 \).

In the case of \( \alpha = \infty \), then \( R \) is an upper triangular Toeplitz matrix \( R = U(r_0, r_1, \ldots, r_n) \). The invertibility of \( R \) implies \( r_0 \neq 0 \), and therefore (1.2) and (1.3) hold for \( z_0 = 0 \) and \( z_p = -r_{n+1-p}(r_0)^{-1} \) (\( p = 1, \ldots, n \)).

Conversely, assume that \( R^{-1} \) is a Toeplitz matrix. It follows from Theorem 1.2 that there exists a vector \( z = \text{col}(z_p)_{p=0}^n \) such that (1.2) and (1.3) hold. It follows from (1.2) that

\[
r_{p-1-n}r_q - r_pr_{q-1-n} = (r_pr_q - r_pr_q)z_0 - (z_pr_q - r_pr_{q-1-n})r_0 \quad (p, q = 1, \ldots, n),
\]

which, according to (1.3), implies

\[
r_{p-1-n}r_q = r_pr_{q-1-n} \quad (p, q = 1, 2, \ldots, n). \tag{3.3}
\]

If \( r_q = 0 \) for all \( q = 1, 2, \ldots, n \), then \( R \in \Pi_{\infty} \). Otherwise, there exists an integer \( q_0 > 0 \) such that \( r_{q_0} \neq 0 \). Consequently, it follows from (3.3) that

\[
r_{p-1-n} = r_p(r_{q_0})^{-1}r_{q_0-1-n} \quad (p = 1, 2, \ldots, n),
\]

which implies (3.2) for \( \alpha = (r_{q_0})^{-1}r_{q_0-1-n} \). Finally, it is clear from Lemma 3.2 that \( R \in \Pi_\alpha \) for that \( \alpha \).

The following corollary might be deduced from Lemma 3.2 and the course of the proof of Theorem 3.3.

**Corollary 3.4.** Let \( R = (r_{p-q})_{p, q=0}^n \) be a Toeplitz matrix. There exists a number \( \alpha \in \mathbb{C}_{\infty} \) such that \( R \in \Pi_\alpha \) if and only if

\[
r_{p-1-n}r_q = r_pr_{q-1-n} \quad (p, q = 1, 2, \ldots, n).
\]

Most of the details of the proof of Theorem 1.3 have been just presented. Let us complete the proof by collecting all the necessary details and putting them in order.
Proof of Theorem 1.3. Note that (a) means that \( R \in \Pi_\alpha \) for a certain \( \alpha \in \mathbb{C}_\infty \).

It is clear that (d) implies (c) and that (c) implies (b), while (b) is equivalent to (a) by Corollary 3.4 and Proposition 2.1 applied to \( A = B = R \).

Lemma 3.2 shows that \( \Pi_\alpha \) is an algebra of Toeplitz matrices. Indeed, if \( A_1 \) and \( A_2 \) are in \( \Pi_\alpha \), then both their sum and their product commute with \( S_\alpha \), and therefore the sum and the product are in \( \Pi_\alpha \) too. Thus (a) is equivalent to (d).

Finally, it is clear from Theorem 3.3 that if \( R \) is nonsingular, then (e) is equivalent to (a).

Note that in statement (b) of Theorem 1.3 the fact that \( R^2 \) is a Toeplitz matrix cannot be replaced by higher powers of \( R \). The following \( 3 \times 3 \) Toeplitz matrix \( R \) is such that \( R^3 \) is a Toeplitz matrix but neither \( R^2 \) nor \( R^{-1} \) is:

\[
R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R^3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},
\]

while

\[
R^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.
\]

Let us conclude this section by proving Theorem 1.4.

Proof of Theorem 1.4. First let us show that for any \( \alpha \in \mathbb{C}_\infty \) the algebra of Toeplitz matrices \( \Pi_\alpha \) is maximal. Indeed, \( \Pi_\alpha \) contains the matrix \( S_\alpha \), and it is clear from Proposition 2.1 that if \( R \) is a Toeplitz matrix, then \( RS_\alpha \) is a Toeplitz matrix if and only if \( R \in \Pi_\alpha \).

Conversely, let \( \Pi \) be a maximal algebra of Toeplitz matrices. It is clear that \( \Pi \) does not contain only diagonal Toeplitz matrices, as in that case it would be a proper subalgebra of each \( \Pi_\alpha \). Let \( R \) be any nondiagonal Toeplitz matrix in \( \Pi \). It is clear from Theorem 1.3 and Lemma 3.2 that \( R \) commutes with a certain \( S_\alpha \). Moreover, for any Toeplitz matrix \( A \in \Pi \) it is clear that \( AR \) is a Toeplitz matrix, and Proposition 2.1 implies

\[
a_{p-1-n} r_q = a_p r_{q-1-n}, \quad (p, q = 1, 2, \ldots, n).
\]

Therefore, as \( R \) is nondiagonal and commutes with \( S_\alpha \), this relation implies
the commutativity of $A$ and $S$. Thus, by Lemma 3.2, $A \in S$. Consequently, $\Pi$ is a subalgebra of $\Pi$, but being a maximal algebra of Toeplitz matrices, $\Pi = \Pi$. Note that the uniqueness of $\alpha$ is provided by the nondiagonality of $R$. 

4. FUNCTIONS OF BLOCK TOEPLITZ MATRICES

First let us point out that Theorem 1.6 follows immediately from Theorem 2.2. Indeed, (1.5)–(1.7) can be obtained by straightforward computations on (2.1) using the fact that $R$ is a block Toeplitz matrix.

The following theorem is a generalization of Theorem 1.7, and it stresses the role of generalized block circulants in the analysis of block Toeplitz algebras.

**Theorem 4.1.** Let $R = (r_{p-q}, p, q = 0, n)$ be a nonsingular block Toeplitz matrix such that

$$\text{rank} \text{col}(r_p)^{n-1} = \text{rank} \text{row}(r_p)^{n-1} = m. \quad (4.1)$$

The matrix $R^{-1}$ is a block Toeplitz matrix if and only if there exist $m \times m$ matrices $c$ and $d$ such that

$$r_{p-1-n} = cr_p = r_p d \quad (p = 1, 2, \ldots, n) \quad (4.2)$$

and

$$cr_0 = r_0 d. \quad (4.3)$$

**Proof.** First assume that there exist such matrices $c$ and $d$. Define $z_0 = c, z_1 = \cdots = z_n = 0$ and $w_0 = w_1 = \cdots = w_{n-1} = 0, w_n = d$. It is clear that (1.5)–(1.7) follow from (4.2) and (4.3). Therefore, it follows from Theorem 1.6 that $R^{-1}$ is a block Toeplitz matrix. Note that (4.1) is not required in this direction.

Conversely, assume that $R^{-1}$ is a block Toeplitz matrix. According to Theorem 1.6 there exist block vectors $z = \text{col}(z_p)^{n-0}$ and $w = \text{row}(w_p)^{n-0}$ such that

$$\text{col}(r_{p-1-n})^{n-1} = \text{col}(r_p)^{n-1} w_n - \text{col}(z_p) r_0, \quad (4.4)$$

$$\text{row}(r_{p-1-n})^{n-1} = z_0 \text{row}(r_p)^{n-1} - r_0 \text{row}(w_{n-p})^{n-1}, \quad (4.5)$$

$$r_0 w_n = z_0 r_0. \quad (4.6)$$
and
\[
\text{col}(r_p)_{p=1}^n \text{row}(w_{n-p})_{p=1}^n = \text{col}(z_p)_{p=1}^n \text{row}(r_p)_{p=1}^n.
\] (4.7)

It follows from (4.1) that there exist block vectors \(u = \text{row}(u_p)_{p=1}^n\) and \(v = \text{col}(v_p)_{p=1}^n\) such that
\[
\text{row}(u_p)_{p=1}^n \text{col}(r_p)_{p=1}^n = \text{row}(r_p)_{p=1}^n \text{col}(v_p)_{p=1}^n = I_m.
\] (4.8)

Therefore, (4.7) implies
\[
\text{col}(z_p)_{p=1}^n = \text{col}(r_p)_{p=1}^n \left( \sum_{q=1}^n w_{n-q}v_q \right),
\] (4.9)

and
\[
\text{row}(w_{n-p})_{p=1}^n = \left( \sum_{q=1}^n u_qz_q \right) \text{row}(r_p)_{p=1}^n.
\] (4.10)

Define \(c = z_0 - r_0 \sum_{q=1}^n u_qz_q\) and \(d = w_n - (\sum_{q=1}^n w_{n-q}v_q) r_0\). Then it is clear from (4.4) and (4.9) that \(r_{p-1-n} = r_p d\), while (4.5) and (4.10) imply \(r_{p-1-n} = cr_p\) for all \(p = 1, 2, \ldots, n\). Consequently, (4.2) holds.

Finally, it follows from the definition of \(c\) that
\[
cr_0 = z_0 - r_0 \text{row}(u_p)_{p=1}^n \text{col}(z_p)_{p=1}^n r_0,
\]
which, according to (4.6) and (4.9), is equal to
\[
r_0 w_n - r_0 \text{row}(u_p)_{p=1}^n \left[ \text{col}(r_p)_{p=1}^n \text{row}(w_{n-p})_{p=1}^n \text{col}(v_p)_{p=1}^n \right] r_0.
\]

Thus, by (4.8),
\[
cr_0 = r_0 \left[ w_n - \text{row}(w_{n-p})_{p=1}^n \text{col}(v_p)_{p=1}^n r_0 \right],
\]
which is equivalent to (4.3)

It is clear that Theorem 1.7 follows from Theorem 4.1, as, in particular, \(r_{q_0-1-n} - n = cr_{q_0} = r_{q_0} d\) and therefore \(c = r_{q_0-1-n}(r_{q_0})^{-1}\) and \(d = (r_{q_0})^{-1}r_{q_0-1-n}\).

Let us show how Theorem 1.5 can be obtained from Theorem 1.6.
Proof of Theorem 1.5. First assume that (1.4) holds. Then it is clear that (1.5)–(1.7) are fulfilled, taking $z_0 = w_n = 0$ and $w_{n-p} = - (r_0)^{-1} r_{p-1-n}, z_p = - r_{p-1-n}(r_0)^{-1}$ for $p = 1, 2, \ldots, n$.

Conversely, assume that $R^{-1}$ is a block Toeplitz matrix. Then it follows from Theorem 1.6 that there exist block vectors $z = \text{col}(z_p)_{p=0}^{n}$ and $w = \text{row}(w_p)_{p=0}^{n}$ such that (1.5)–(1.7) hold. It follows from (1.6) that

$$
r_p(r_0)^{-1} r_{q-1-n} - r_{p-1-n}(r_0)^{-1} r_q = [r_p(r_0)^{-1} z_0 r_q - r_p w_{n-q}] - [r_p w_n(r_0)^{-1} r_q - z_p r_q]
$$

for $(p, q = 1, 2, \ldots, n)$.

or, using (1.7),

$$
r_p(r_0)^{-1} r_{q-1-n} - r_{p-1-n}(r_0)^{-1} r_q = r_p \left[ (r_0)^{-1} z_0 - w_n(r_0)^{-1} \right] r_q
$$

for $(p, q = 1, 2, \ldots, n)$. (4.11)

Note that (1.5) implies $w_n(r_0)^{-1} = (r_0)^{-1} z_0$. Therefore, (4.11) gives (1.4). □

The following example is of a $2 \times 2$ block Toeplitz matrix with $2 \times 2$ matrices as its entries. In this example, the inverse is a block Toeplitz matrix while the square is not.

$$
R = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}, \quad 2R^{-1} = \begin{pmatrix}
1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1
\end{pmatrix},
$$

$$
R^2 = \begin{pmatrix}
1 & 2 & 1 & 0 \\
1 & 2 & 2 & 1 \\
2 & 4 & 2 & 2 \\
0 & 2 & 1 & 1
\end{pmatrix}.
$$

In order to simplify the proof of Theorem 1.8 we shall need the following notation. For given $(r_p)_{p=-n}^{n}$ let us define $\sigma(p, q, 0) = r_{p-q}$, and for any $j \geq 1$ define

$$
\sigma(p, q, j) = \sum_{s_1=0}^{n} \sum_{s_2=0}^{n} \cdots \sum_{s_{j-1}=0}^{n} \sum_{s_j=1}^{n} r_{p-s_1} r_{s_1-s_2} \cdots r_{s_{j-1}-s_j} r_{s_j-q}.
$$
The following lemma will be used in the course of the proof of Theorem 1.8.

**Lemma 4.2.** For given \( \{ r_p \}_{p=-n}^{n} \), the condition

\[
   r_p(r_0)^j = r_{p-1-n}(r_0)^j \quad (p, q = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, k-2)
\]

implies

\[
   r_{p-1-n} r_{t_1} r_{t_2} \cdots r_{t_i} r_q = r_p r_{t_1} r_{t_2} \cdots r_{t_i} r_{q-1-n}
\]

\( (p, q = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, k-1) \),

where \( t_1, t_2, \ldots, t_i \) are any numbers \(-n \leq t_1, \ldots, t_i \leq n\), not all 0.

**Proof.** We shall prove this lemma by induction on \( i \). For \( i = 1 \) then either \( t_1 < 0 \), for which (4.12) implies that

\[
   r_{p-1-n} r_{t_1} r_q = r_{p-1-n} r_{t_1+1-n} r_q - n
\]

or \( t_1 > 0 \), for which

\[
   r_{p-1-n} r_{t_1} r_q = r_p r_{t_1} r_{q-1-n}
\]

For the induction let us pass from \( i \) to \( i + 1 \). Let \( t_i \) be the first positive (or the last negative) index among \( t_1, \ldots, t_{i+1} \). In case \( t_1 = t_2 = \cdots t_{i-1} = 0 \) we may use (4.12), and otherwise we may use induction, to obtain

\[
   r_{p-1-n} r_{t_1} \cdots r_{t_i} r_q = r_p r_{t_1} \cdots r_{t_i-1-n} \cdots r_q,
\]

which is equal to \( r_p r_{t_1} \cdots r_{t_i} r_{q-1-n}. \)

**Proof of Theorem 1.8.** We prove the theorem by induction and obtain in addition that if \( R^k \) is a block Toeplitz matrix \( R^k = (r_{p, q})_{p,q=0}^{n} \), then

\[
   r_p^{(k)} = \sum_{j=0}^{k-1} \sigma(p, 0, j)(r_0)^{k-j} \quad (p = 0, 1, \ldots, n) \quad (4.13)
\]

and

\[
   r_{p-1-n}^{(k)} = \sum_{j=0}^{k-1} \sigma(p, n+1, j)(r_0)^{k-j} \quad (p = 1, 2, \ldots, n). \quad (4.14)
\]
First consider the case \( k = 2 \). It is clear from Proposition 2.1 that \( R^2 \) is a block Toeplitz matrix if and only if

\[
r_{p}r_{q-1-n} = r_{p-1-n}r_{q} \quad (p, q = 1, 2, \ldots, n),
\]

which is the equivalent of (1.10) in this case.

Moreover, if \( R^2 \) is a block Toeplitz matrix then it is clear that

\[
r_{p}^{(2)} = \sum_{s=0}^{n} r_{p-s}r_{s} = r_{p}r_{0} + \sum_{s=1}^{n} r_{p-s}r_{s} \quad (p = 0, 1, \ldots, n), \quad (4.15)
\]

while

\[
r_{p-1-n}^{(2)} = \sum_{s=0}^{n} r_{p-1-s}r_{s-n} = r_{p-1-n}r_{0} + \sum_{s_{1}=s+1}^{n} r_{p-s_{1}}r_{s_{1}} \quad (p = 1, \ldots, n). \quad (4.16)
\]

But (4.15) and (4.16) are equivalent to (4.13) and (4.14) for this case of \( k = 2 \).

Assume now that (1.10), (4.13), and (4.14) hold for some integer \( k \geq 2 \). By this, we assume that \( R, R^2, \ldots, R^k \) are all block Toeplitz matrices. Applying Proposition 2.1 to \( A = R^k \) and \( B = R \), it is clear that \( R^{k+1} \) is a block Toeplitz matrix if and only if

\[
r_{p}^{(k)}r_{q-1-n} = r_{p}^{(k)}r_{q} \quad (p, q = 1, 2, \ldots, n). \quad (4.17)
\]

Combining (4.13) and (4.14) to (4.17), it is clear that \( R^{k+1} \) is a block Toeplitz matrix if and only if

\[
\sum_{j=0}^{k-1} \sigma(p, 0, j)(r_0)^{k-1-j}r_{q-1-n} = \sum_{j=0}^{k-1} \sigma(p, n+1, j)(r_0)^{k-1-j}r_{q} \quad (p, q = 1, \ldots, n). \quad (4.18)
\]

It is clear from (1.10) that

\[
\sigma(p, 0, j)(r_0)^{k-1-j}r_{q-1-n} = \sigma(p, n+1, j)(r_0)^{k-1-j}r_{q} \quad (p, q = 1, \ldots, n, \quad j = 1, \ldots, k-1),
\]
and therefore, (4.18) implies that $R^{k+1}$ is a block Toeplitz matrix if and only if (1.10) holds with $j$ running all over $j = 0, 1, \ldots, k - 1$.

Moreover, $r_p^{(k+1)} = \sum_{s=0}^{n} r_{p-s} r_s^{(k)}$ for any $p = 0, 1, \ldots, n$, and therefore (4.13) implies

$$r_p^{(k+1)} = \sum_{s=0}^{n} r_{p-s} \sum_{j=0}^{k-1} \sigma(s, 0, j)(r_0)^{k-1-j} \quad (p = 0, 1, \ldots, n),$$

which is the equivalent of (4.13) for $k + 1$.

Similarly, $r_{p-1-n}^{(k+1)} = \sum_{s=0}^{n} r_{p-1-s} r_s^{(k)}$ for any $p = 1, \ldots, n$. Therefore, (4.14) implies

$$r_{p-1-n}^{(k+1)} = \sum_{s=0}^{n} r_{p-1-s} \sum_{j=0}^{k-1} \sigma(s, n+1, j)(r_0)^{k-1-j} \quad (p = 1, 2, \ldots, n).$$

Distinguishing the case of $s = n$ and attaching an index $s_1 = s + 1$ for $s < n$, it is clear from Lemma 4.2 that this is the equivalent of (4.14) for $k + 1$.

It is known that for any $l \geq m$, $(r_0)^l$ is a linear combination of $r_0, (r_0)^2, \ldots, (r_0)^{m-1}$. Thus if (1.10) holds for $k = m + 1$, then all the positive powers of $R$ are block Toeplitz matrices.

Let us conclude with the following result, which can be derived from the previous ones. It might give a hint to the structure of algebras of block Toeplitz matrices.

**Theorem 4.3.** Let $R = (r_{p-q})_{p, q=0}^{n}$ be a block Toeplitz matrix. If

$$\text{rank col}(r_p)^n_{p=1} = \text{rank row}(r_p)^n_{p=1} = m,$$  \hspace{1cm} (4.19)

then the following statements are equivalent:

(a) Both $R^2$ and $R^3$ are block Toeplitz matrices.

(b) All positive powers of $R$ are block Toeplitz matrices.

(c) There exists an $m \times m$ matrix $c$ such that

$$r_{p-1-n} = cr_p = r_p c \quad (p = 1, 2, \ldots, n) \hspace{1cm} (4.20)$$

and

$$cr_0 = r_0 c. \hspace{1cm} (4.21)$$
Proof. First assume that there exists such a matrix \( c \). Clearly (1.10) holds for any \( j \geq 0 \), and therefore all the positive powers of \( R \) are block Toeplitz matrices. In particular, both \( R^2 \) and \( R^3 \) are block Toeplitz matrices. Thus, we have proved that (c) implies (b) and (a). Note that (4.19) is not required in this direction.

It is clear that (b) implies (a), and therefore it remains to prove that (a) implies (c). Assuming that (a) holds, it follows from Theorem 1.8 that both

\[
\text{col}(r_p)_p=1^n \text{row}(r_{p-1-n})_p=1^n = \text{col}(r_{p-1-n})_p=1^n \text{row}(r_p)_p=1^n \quad (4.22)
\]

and

\[
\text{col}(r_p)_p=1^n \text{row}(r_{p-1-n})_p=1^n = \text{col}(r_{p-1-n})_p=1^n \text{row}(r_p)_p=1^n \quad (4.23)
\]

It follows from (4.19) that there exist block vectors \( u = \text{row}(u_p)_p=1^n \) and \( v = \text{col}(v_p)_p=1^n \) such that

\[
\text{row}(u_p)_p=1^n \text{col}(r_p)_p=1^n = \text{row}(r_p)_p=1^n \text{col}(v_p)_p=1^n = I_m. \quad (4.24)
\]

Therefore, (4.22) combined with (4.24) implies both

\[
r_{p-1-n} = \left( \sum_{q=1}^{n} u_q r_{q-1-n} \right) r_p \quad (p = 1, 2, \ldots, n),
\]

and

\[
r_{p-1-n} = r_p \left( \sum_{q=1}^{n} r_{p-1-n} w_q \right) \quad (p = 1, 2, \ldots, n).
\]

But

\[
\sum_{p=1}^{n} u_p r_{p-1-n} = \sum_{p=1}^{n} u_p r_p \sum_{q=1}^{n} r_{q-1-n} w_q = \sum_{q=1}^{n} r_{q-1-n} w_q,
\]

and therefore (4.20) holds for \( c = \sum_{p=1}^{n} u_p r_{p-1-n} \). Accordingly, by (4.23), it is clear that

\[
\sum_{p=1}^{n} u_p r_{p-1-n} r_0 = r_0 \sum_{p=1}^{n} r_{p-1-n} w_p.
\]

Thus (4.21) is fulfilled. \( \blacksquare \)
Many thanks to Professor I.C. Gohberg for useful remarks and fruitful encouraging guidance.

REFERENCES


Received 30 July 1986; revised 18 January 1987