Thermal induced motion of functionally graded beams subjected to surface heating

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Abstract Thin beam of the functionally graded (FG) type subjected to a step heat input on one surface and insulated or exposed to convective heat loss on the opposite surface is under consideration for the evaluation of thermal induced motion. The dynamic displacement and dynamic thermal moment of the beam are analysed when the temperature gradient is independent of the beam displacement. The power law index dictates the metal–ceramic distribution across thickness of the beam and its effect on the thermal vibration of the beam is examined. The article discusses, in depth, the influence of various factors such as length to thickness ratio of beam, heat transfer boundary conditions, physical boundary conditions, and metal–ceramic combination on the thermal oscillations of FG beam. It is found that attenuation of the amplitude of static thermal deflection and superimposed thermal oscillations is a strong function of the metal–ceramic combination for the FG beam.

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1. Introduction

Structural elements subjected to high temperature, severe temperature gradient, and uneven heating rates are unavoidable during the operation of gas turbines, nuclear reactors, castings, forgings, radiant burners, pipes in heat exchangers, artillery barrels, etc. Sharp temperature gradients in the structural elements arise due to sudden exposure to very large amount of heat which is observed during launching of rocket, space craft structural components subjected to radiant solar heat (Yu [1] and Thornton et al. [2]), sudden load fluctuations on nuclear reactor, heat exchangers (Hong et al. [3]), friction generated heat in rotors of turbogenerator (Jevti et al. [4]) and during heat treatment of cast and forged components, etc. Thermal induced motion and their control are important to avoid catastrophic malfunctioning of various precision instruments and accessories in satellites. Pioneering research by Boley [5], Boley and Barber [6] and Boley [7] on thermally induced vibrations of beam provides a detailed inter-relation of time dependent temperature variation on the structural vibrations. Boley [5] considered the problem of lateral vibration of a thin beam with simple supports and subjected to rapid heat flux on the upper surface and thermally insulated condition on the bottom. Boley [5] proved that the time dependent thermal moment acts as a force-
ing function to induce a structural deformation and for thicker beams the inertia effect can be neglected and the dynamic solution approaches the static solution. After establishment of this original work in 1956, other researchers were motivated to probe the thermally induced vibrations in other structures and/or with different thermal/mechanical boundary conditions. Considering the presence and absence of the inertial effects Boley and Barber [6] revealed that the occurrence of thermally induced vibrations depends only on a non-dimensional parameter $B$ which is the ratio of the thermal response time of the structure to the structural response time. Further, it is shown that the existence of thermally induced vibrations depends only on $B$ and as this parameter increases, inertia forces disappear. Manolis and Beskos [8] have worked on the problem attempted by Boley [5] using Laplace Transform and discussed the effects of axial load, internal viscoelastic damping and external viscous damping on thermal vibrations of simply supported beam subjected to rapid heating. Apart from structures which are subjected to radiant/surface heating, structures undergoing internal heating may also experience thermally induced vibrations [9,10]. Bladino and Thornton [9] have carried out a detailed study on the thermally induced vibration caused by internal heating. The analysis showed that the natural frequency of the beam was more important than the heating rate in determining whether vibrations occur. The steady-state vibration amplitude is reached when the internal heating is balanced by convection from the beam surface. Malik et al. [10] reported their findings related to the effect of boundary conditions on thermally induced vibrations of isotropic beam subjected to internal heating and convective heat loss. It was observed that the boundary conditions influence the magnitude of dynamic displacement and dynamic thermal moment. A sustained thermally induced motion is observed with progress of time when the temperature gradient being evaluated is dependent on the forced convection generated due to beam motion. Finite element analysis on the transient behaviour of aluminium and graphite epoxy plates subjected to an instantaneously imposed heat flux has been reported by Chang et al. [11].

During the last two decades, materials with gradation in properties are being developed and one such material is functionally graded material (FGM), typically like the one that caters to high temperature, good wear resistance, in conjunction with high strength and toughness. FGMs were proposed in Japan during 1984–1985 for the space plan project. FGMs are a class of composites that have a continuous variation of material properties from one surface to another. The smooth transition between metallic and ceramic components reduces thermal stresses, residual stresses and stress concentration factors found in laminated composites. With the developments in manufacturing processes, special varying gradients can be achieved to suit various goals of engineering applications as presented by Sobczak and Drenchev [12]. These materials can be fabricated by varying the percentage content of two or more materials such that the new materials have the desired property gradation in spatial directions. FGMs have gained widespread applicability as thermal-barrier, wear and corrosion resistant coatings. Yet another application of FGM is thin-walled members such as plates and shells, which are used in reactor vessels, turbines and other machine parts that are susceptible to instabilities due to buckling load and large amplitude deflections, or excessive stresses induced by thermal or combined thermo-mechanical loading.

In view of the above developments vast research on the static and dynamic analysis of functionally graded material structures such as beams and plates has been attempted. Thai and Vo Thue [13], and Sina et al. [14] have analytically investigated the effects of boundary conditions, volume fraction and shear deformation on natural frequencies, and mode shapes on the bending and free vibration of FGM beams. Simsek et al. [15] analysed the influence of volume fraction index, material properties, length scale parameter, aspect ratio and Poisson’s effect on the static bending behaviour of FGM beams and showed that the deflections of the microbeam by the classical beam theory are always larger than those by the modified couple stress theory. Under the influence of a moving harmonic load, Simsek and Kocaturk [16] analysed free vibration characteristics and dynamic behaviour of a FG simply supported beam. The system of equations of motion was derived using Lagrange’s equation under the assumptions of Euler–Bernoulli beam theory. It was observed that the effects of different material distribution, velocity of the moving harmonic load and excitation frequency play very important role on the dynamic behaviour of the FGM beam. Malekzadeh and Shojaee [17] showed that under the influence of moving heat source, the amplitude of centre deflection decreases by increasing the velocity of heat source and convective heat transfer coefficient. Wattanasakulpong et al. [18] analysed the linear thermal buckling and vibration characteristics of thick FG beams and found that the fundamental frequency decreases with the increase in temperature and tends towards minimum point closing to zero at the critical temperature while in the post-buckling region, the fundamental frequency increases with the increase of temperature. The elasticity solutions of a transversely loaded Euler–Bernoulli FGM beam were obtained by Sankar [19] assuming exponential variation of Young’s modulus and the stress concentration effects were studied by loading the beam on softer and harder side. Sankar and Tseng [20] showed that the thermoelastic properties of beam can be tailored to reduce the thermal residual stresses for a given temperature distribution which can be accomplished by varying the thermoelastic constants in a manner opposite to the graduation of temperature through the thickness. As far as thermal vibrations of plates are concerned, Shen [21] revealed that the width-to-thickness ratio of a plate, plate aspect ratio as well as the volume fraction distribution have a significant effect while control voltage has a minor effect on the thermal bending response of FGM plates. Sohn and Kim [22] analysed static and dynamic stabilities of FG panels under supersonic air flows considering temperature and volume fraction changes and showed that FG panels are more stable than the isotropic metal panel, in terms of thermal post-buckling characteristics. Yang et al. [23] have investigated the large amplitude vibration of initially stressed FGM laminated rectangular plates with thermo-electro-mechanical loading and demonstrated that the linear and nonlinear vibration behaviour of the pre-stressed laminated plates is greatly influenced by the various factors such as vibration amplitude and material composition. The thermal shock
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strength of a FGM was evaluated theoretically by Wang et al. [24] for a plate containing a surface crack wherein it is found that FGM with high metal content exhibits significant resistance to crack growth from the ceramic side to the metal side. Reddy and Chin [25] have analysed numerically the thermomechanical behaviour of FGMs under abrupt thermal loading conditions for different volume fractions of the constituents and combinations of different constituent materials. Recently, Alshorbagy et al. [26] have studied the effect of slenderness ratio, material distribution and boundary conditions on the free vibrational characteristics of a FG beam by using finite element method. The equation of motion of FG beam has been derived using Euler–Bernoulli beam theory and virtual work principle. Considering wide material variations and applications of FGMs, it is important to study the static and dynamic characteristics of FG structures, such as beams, plates and shells.

While literature on the vibrations induced by rapid heating in isotropic homogeneous and composite structures is enormous, a survey reveals that investigation on thermally induced vibrations of FG structures with combined structural and thermal boundary conditions is unattempted. The present study deals with such phenomenon in beams made of FGMs. The aim of this article was to have detailed investigation on the thermal induced vibration of a FG beam when subjected to a step thermal load for various structural boundary conditions. The study considers one surface of the beam subjected to step heating with opposite surface insulated or subjected to convective heat loss. A thermal moment arising from the temperature gradient across the thickness of the beam is the source of forcing function for the structure. Dynamic response of the beam is obtained using Newmark’s method. In order to have a more accurate estimation of the structural response, the thermal oscillations of the FG beam are discussed for two types of metal–ceramic combination, wherein material properties are considered to be dependent on temperature. A comparative study is presented to ensure the efficiency and accuracy of the established procedure. Parametric studies are conducted to examine the influences of the various involved parameters.

2. Formulation of governing equations

2.1. FGM beam

Fig. 1 illustrates the configuration of a conventional metal–ceramic FGM beam subjected to step heating. In a FGM beam with the volume fraction of metal or ceramic based on power law distribution, the following expressions hold for the evaluation of effective material property \( P_{\text{eff}} \) of FGM, such as Young’s Modulus \( E \), mass density \( \rho \), Poisson’s ratio \( \nu \), coefficient of thermal expansion \( \alpha \), thermal conductivity \( k \) and specific heat \( c_{p} \).

\[
P_{\text{eff}}(z) = P_{m}V_{m}(z) + P_{c}V_{c}(z)
\]

where subscripts \( c \) and \( m \) denote the properties at the top and bottom surfaces of the beam respectively corresponding to either pure ceramic or pure metal. Variation of material properties is considered in thickness direction i.e. along \( z \) axis. The metal volume fraction \( V_{m} \) is determined based on simple rule of mixture i.e. \( V_{m} = 1 - V_{c} \).

Volume fraction of ceramic \( V_{c} \) is given as follows:

\[
V_{c} = \left( \frac{z}{h/2} \right)^{n}
\]

and accordingly, the top surface, \( z = h/2 \), of the beam is pure ceramic whereas bottom surface is pure metal and for different values of \( n \), one can obtain different volume fraction of ceramic. \( n \) is the power law index that takes on values greater than or equal to zero and \( h \) is the thickness of the beam. Eq. (1) is based on the fact that FGM structure is considered as a laminate of multiple perfectly bonded layers of isotropic material with each layer having a composition of \( V_{c} \) and \( V_{m} \) being different from the adjacent layers and so on.

2.2. Governing equations for thermal vibration of FGM beams

2.2.1. Transverse vibration of beam subjected to external heat source

Consider a differential element \( dx \), of a thin beam under the action of mechanical and thermal load. \( T_{b} \) and \( T_{h} \) correspond to temperature on top and bottom surfaces respectively, \( F \) is the shear force and \( p \) is the load intensity. Based on the Euler–Bernoulli’s thin beam theory, the curvature \( (\partial^{2} w/\partial x^{2}) \) of a uniform beam under the simultaneous action of external forces and heat input can be expressed as [5],

\[
M + M_{T} = EI \left( \frac{\partial^{2} w}{\partial x^{2}} \right)
\]

where \( M \) is the internal bending moment, \( M_{T} \) is the thermal moment which is due to temperature gradient across the thickness of the beam, \( w \) is the transverse deflection in the \( z \) direction and \( I \) the moment of inertia of beam cross section. For the beam subjected to heat flux on one side and insulated on the other side the thermal moment acts as a forcing function which is given as

\[
M_{T} = \int_{A} E \cdot z \cdot \Delta T \cdot z \cdot dA
\]

In the above equation, \( \Delta T \) is the change in temperature and \( A \) is the cross sectional area. The thermal moment is calculated at uniform intervals across the thickness from the top to bottom surfaces of the beam and it is summed up in order to obtain the total thermal moment across the section. There is no temperature variation along the length of the beam, i.e. temperature is independent of \( x \), hence, \( M_{T} = M_{T}(t) \). Differentiating Eq. (3) twice with respect to \( x \), the resulting expression is,

\[
\frac{\partial^{2} M}{\partial x^{2}} + \frac{\partial M_{T}}{\partial x} = \frac{\partial^{2}}{\partial x^{2}} \left( EI \left( \frac{\partial^{2} w}{\partial x^{2}} \right) \right)
\]

According to D’Alembert’s principle the applied loads will be equated to the inertia force with a negative sign as follows:

\[
\frac{\partial^{2} M}{\partial x^{2}} = -\rho A \left( \frac{\partial^{2} w}{\partial t^{2}} \right)
\]

Using Eq. (6) in Eq. (5) will result in the governing equation for deflection of a beam in the transverse direction in the presence of thermal moment as

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\[
\frac{\partial^2 M_T}{\partial z^2} = \frac{\partial^2}{\partial x^2} \left( EI \left( \frac{\partial^2 w}{\partial x^2} \right) \right) + \rho A \left( \frac{\partial^2 w}{\partial t^2} \right) \tag{7}
\]

The significance of the term \( \frac{\partial^2 M_T}{\partial z^2} \) in Eq. (7), is seen in situations when there is a moving heat source, a point heat source or local heat source, local heat generation and the likes [17].

2.2.2. Transient heat transfer governing equations

Consider a FGM beam subjected to heat flux and internal heat generation. The equation governing the conduction of heat in one dimensional co-ordinate is given by,

\[
k \left( \frac{\partial^2 T}{\partial x^2} \right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \tag{8}
\]

where \( \dot{q} \) is the internal heat generation rate per unit volume. The thermal conduction equation, Eq. (8), must be solved for prescribed boundary and initial conditions. The initial condition specifies the temperature distribution at time zero, i.e. \( T(z,0) = T_{\infty} \), where, \( T_{\infty} \) is the free air stream temperature and that of the beam. The presence of convective heat transfer on the bottom surface of the beam at \( z = -h/2 \) is represented by a boundary condition given as

\[
-k \frac{\partial T}{\partial x} = h_c (T - T_{\infty}) \tag{9}
\]

where \( h_c \) is convective heat transfer coefficient.

2.3. Finite element equations for temperature distribution across beam cross section

Using linear Lagrange interpolation function, the finite element equation for evaluating temperature across the thickness of the beam when the beam is exposed to sudden heating on one surface and insulated on other surface is as follows:

\[
k A_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \frac{\rho c_p A_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{bmatrix} = \dot{q} A_e \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{10}
\]

and for the beam exposed to sudden heating on one side and uniform convection on the other surface of the beam is

\[
\begin{bmatrix} k A_e & 0 & 0 \\ 0 & k A_e & 0 \\ 0 & 0 & h_c A_e \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} 0 \frac{\rho c_p A_e}{6} \frac{\rho c_p A_e}{6} \end{bmatrix} \begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} = \dot{q} A_e \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + h_c A_e T_{\infty} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}
\]

In Eq. (11) the second matrix on LHS and second vector on RHS are contribution from convection and will be taken into consideration only for last element, and \( T_1 \) and \( T_2 \) are the nodal temperatures. The global finite element equation for time dependent temperature distribution has the following form:

\[
K_{\text{comb}} T + K_{\text{cap}} T = F_Q \tag{12}
\]

\( K_{\text{comb}} = \) conduction and/or convection matrix, \( K_{\text{cap}} = \) capacitance matrix and \( F_Q = \) force vector. Eq. (12) must be solved for the variation of temperature in space and time domains to obtain the temperature distribution across the thickness of the beam. A step heat input \( Q \) constant along \( x \), is assumed to be applied over the surface \( z = h/2 \), while the surface at \( z = -h/2 \) is insulated. The close form solution for temperature distribution \( T \), across the thickness of the isotropic beam, as given by Carslaw and Jaeger [27] is

\[
T = \left( \frac{h Q}{k} \right) \left[ \tau + \frac{1}{2} \frac{z + 1}{h} \right] - \frac{1}{6} \sum_{j=1}^{\infty} \left( -1 \right)^j \frac{e^{-r^2 j^2}}{j^2} \cos j \left( \frac{z}{h} + \frac{1}{2} \right) \tag{13}
\]

provided that the beam is initially at \( T = 0 \). Here \( k \) is the thermal conductivity, and \( \tau \) is non-dimensional time parameter defined by

\[
\tau = \frac{k \theta}{h^2} \tag{14}
\]

where \( \kappa \) is the thermal diffusivity given as

\[
\kappa = \frac{k}{\rho c_p} \tag{15}
\]
length \( l \) along \( x \) axis wherein numbers with suffix ‘s’ denote node numbers and numbers in brackets denote element numbers. The theoretical model based on finite element considers the beam as one-dimensional problem. The weak form of the governing equation Eq. (7) is obtained by multiplying the governing equation by weight function \( N^7 \) [28].

\[
EI \left( \int_0^{x} \frac{\partial^2 w}{\partial x^2} \, dx \right) - \int_0^{x} \frac{\partial^2 w}{\partial x \partial t} \, dx + \int_0^{x} \frac{\partial^2 w}{\partial x^2} \, dx + \int_0^{x} M_t \, \frac{\partial N^7}{\partial x} \, dx = 0
\]

where the first term refers to shear force, the second term refers to moment due to temperature variation across the thickness, the third term will give the stiffness matrix, the fourth term will give the mass matrix, the fifth term will give the shear force and the last term will be zero as there is no change in thermal moment along the length of the beam. The first, second and the fifth terms of Eq. (16) contribute to the generalised force vector. The significance of the last term is seen in situations when there is a moving heat source, a point heat source or local heat source, local heat generation and the likes [17]. Hermite interpolation functions \( N \) are used to develop the various finite element matrices. After obtaining the time dependent temperature distribution \( \Delta T \) across the beam thickness by solving Eq. (10) or Eq. (11), the generalised force vector \( F_t \) is evaluated consisting of thermal moment \( M_T \) obtained by solving Eq. (4), i.e.,

\[
F_t = \left[ \begin{array}{c} 0 \\ M_T \\ 0 \\ -M_T \end{array} \right] = \frac{1}{h} \left[ \begin{array}{c} 0 \\ \frac{E_b h \Delta T (h^2_{11} - h^2_{11})}{2} \\ 0 \\ \frac{-E_b h \Delta T (h^2_{11} - h^2_{11})}{2} \end{array} \right]
\]

Referring to Eq. (17), \( M_T \) denotes the static thermal moment, alternately denoted as \( M_{Tr} \). To evaluate \( M_{Tr} \), the static equation \( Kw = F_t \) solved, where \( K \) is the stiffness matrix and \( F_t \) is the generalised force vector or more specifically thermal force vector. The displacement thus obtained at time \( t \) is termed as the static displacement \( w_{SS} \). Newmark’s method is used to solve the second order equation of motion involving the time dependent forcing function:

\[
M\ddot{w} + Kw = F_t
\]

where \( M \) is the mass matrix. The displacement obtained by solving the above equation is termed as dynamic displacement \( w_{dyn} \). From the global dynamic displacement vector, the displacement vector for the central element of the beam is extracted to calculate the dynamic force vector (\( F_{TD} \)) at the centre of the beam which entirely consists of thermal moment termed as dynamic thermal moment. Therefore,

\[
(F_{TD})^e = (K)^e(w_{dyn})^e
\]

where superscript \( e \) refers to elemental solution. From the Eqn. (19) dynamic thermal moment at the nodes of the beam element is obtained. Hence, the dynamic thermal moment at the centre of the beam is given as,

\[
M_{Tdyn} = M_{Tdyn}^{mode} \pm M_{Tr}
\]

where \( M_{Tdyn}^{mode} \) is dynamic thermal moment and \( M_{Tdyn}^{mode} \) is the dynamic thermal moment evaluated at particular node. The positive or negative sign for \( M_{Tdyn} \) depends on the node on which thermal moment is evaluated, positive for RHS node and negative for LHS node.

3. Numerical results and discussion

A FORTRAN computer program is written based on the above formulation to study the dynamic response of FG beams. Numerical analyses are carried out for thin functionally graded beams with various structural boundary conditions such as simple support (SS), clamped-simple support (CS) and clamped-free (CF) to analyse the dynamic response and dynamic thermal moment. The beam is subjected to step heating with heat transfer boundary condition as insulation and convective heat loss on the other surface. The slenderness ratios of the beam considered for the study are 165 and 100. Length of the beam is 0.254 m and has unit width. The evaluation of the temperature distribution across the depth of the beam has been verified with the close form solution given by Carslaw and Jaeger [27], see also Boley [5]. The finite element analysis results of the dynamic response of aluminium–zirconia FGM beam with \( n = 400 \) (beam is fully metallic) subjected to thermal boundary conditions have been verified with the non-dimensional results for aluminium beam reported by Boley [5] and Manolis and Beskos [8] for the simply supported beam. The fundamental non-dimensional free vibration frequencies of the beam are verified with the frequencies reported by Simsek and Kocaturk [16] and Alshorbagy et al. [26].

3.1. Verification of simply supported FGM beam subjected to step heating

Studies are carried out for the simply supported FGM beam (Fig. 1) subjected to heat source on top surface and insulated on the bottom surface. The thermo-physical data for the Aluminium–Zirconia FGM beam are as follows: \( k_m = 201.87 \) W/mK, \( \alpha_m = 22.0 \times 10^{-6} ^\circ \text{C}^{-1} \), \( \rho_m = 2700 \) kg/m$^3$, \( \epsilon_{pm} = 869.38 \) J/kg K, \( E_m = 73.5 \times 10^9 \) Pa, \( G_m = 36.0 \times 10^6 \) Pa for Aluminium and \( k_e = 2.09 \) W/mK, \( \alpha_e = 10.0 \times 10^{-6} \) ^\circ \text{C}^{-1} \), \( \rho_e = 3000 \) kg/m$^3$, \( \epsilon_{pe} = 530 \) J/kg K, \( E_e = 151 \times 10^6 \) Pa for Zirconia. Geometric details of the beam are \( L = 0.254 \) m, and by setting non-dimensional parameter, \( B = 1 \), as referred from Boley [5], the thickness of the beam is \( h = 0.001544 \) m, hence, \( L/h = 165 \). The parameter \( B \) is the square root of the ratio of the characteristic time of heat transfer problem, to characteristic time of the vibration problem (i.e. proportional to the natural period of vibration), Boley [5].

\[
B = \frac{h}{L \sqrt{\varepsilon (pA/E)}} = \left( \frac{h h c A L c p}{L c p \ S} \right)^{1/2}
\]

where \( S \) is the slenderness ratio and

\[
ce = \sqrt{E/\rho}
\]

is the velocity of propagation of longitudinal waves. Dynamic response is obtained for time, \( t = 0.04 \) s. When the power law index is set to \( n = 400 \), FG beam reduces to pure aluminium. Heat flux on the top surface is, \( Q = 1.63 \times 10^6 \) W/m$^2$. The properties are assumed to be independent of the temperature and results are evaluated considering ambient temperature of zero (0) degree Celsius.
3.2. Verification of free vibration characteristics of simply supported FGM beam

The following non-dimensional quantities are introduced here as referred from Simsek and Kocaturk [16], $E_{\text{ratio}} = \frac{E_0}{E_1}$, $\rho_{\text{ratio}} = \frac{\rho_0}{\rho_1}$, $\lambda^2 = \frac{\omega L^4}{\rho U^2}$, where, subscripts “$L$” and “$U$” refer to lower and upper surface of the beam respectively. The bottom surface of the beam is pure steel, whereas the top surface of the beam is pure alumina. A set of linear homogeneous equations (frequency equation) can be expressed as

$$Kw - \lambda^2 Mw = 0 \quad (23)$$

The non-dimensional frequencies $\lambda$ are found from the condition that the determinant of the system of equations given by Eq. (23) must vanish. The parameters of the beam used for this study are as follows: $h = 0.4 \text{ m}$, $h = 1.0 \text{ m}$ and $L = 20 \text{ m}$.

In Table 1, the first non-dimensional frequencies of the FG beam are given for different values of Young’s modulus ratio, $E_{\text{ratio}}$, and the power-law exponent $n$ for $L/h = 20$. The material properties of steel are $E = 210 \text{ GPa}$ and $\rho = 7800 \text{ kg/m}^3$ and for alumina, $E = 390 \text{ GPa}$ and $\rho = 3960 \text{ kg/m}^3$. In these calculations, the mass ratio of the top and the bottom material is taken as constant ($\rho_{\text{ratio}} = 1$). It is observed that, the natural frequencies increase with the increase in power exponent (when $E_{\text{ratio}} < 1$), and decrease with an increase in power exponent (when $E_{\text{ratio}} > 1$). It is seen from the table that $E_{\text{ratio}}$ is more effective on the dimensionless frequencies for small values of $n$ than for large values of $n$. As the values of $n$ increase, the
for fully metal beam. Non-dimensionalizing is not carried out for the subsequent plots since effective material properties for FGM beam vary with each layer across the thickness.

Fig. 6 shows the dynamic mid span deflection for a simply supported FGM beam of $L/h$ ratio of 165 which corresponds to $B = 1$. It can be seen from Fig. 6(a) that the dynamic displacement increases initially from time $t = 0.0$ and as time progresses beam oscillates with constant amplitude about the static thermal deflection. The influence of the volume fraction of ceramic or metal is also being observed. The metal rich (i.e. $n = 1–400$) beam undergoes smaller magnitude of static deflection and the deflections become steady within 25 ms whereas beams with increasing ceramic percentage, (i.e. $n = 0.5$, 0.1, 0.05 and 0.0) exhibit higher value of static deflection, increasing gradually with time and deflections are steady after 500 ms. Studies were also carried out for aluminium–zirconia beam with thermo-mechanical properties dependent on temperature since they are used in high temperature environments. The properties were expressed as functions of temperature as reported in the literature [25]. It is observed from Fig. 6(b) that the metal rich beam exhibits thermal oscillations about the thermal static deflections. As in the case of temperature independent properties, the thermal deflections remain unchanged after 500 ms. The beams with increasing ceramic percentage exhibit higher value of thermal static deflection which increases gradually with time. Experimental investigations are required to be carried out to check the validity of the theoretical results of Al–ZrO$_2$ FG beam with temperature dependent material properties.

3.4. Effect of thermo-physical properties of material and physical boundary conditions on thermally induced vibrations of beams subjected to step heating

The analysis considers FG beam with $L/h = 100$. The material considered for the study is gradation of Stainless Steel and Aluminium Oxide (SUS304–Al$_2$O$_3$). Thermo-physical properties for material are referred from Reddy and Chin [25] and are considered to be dependent on the temperature distribution across the thickness of the beam. The study has been carried out for a FG beam with end supports such as SS, CS and CF. The beam is subjected to step heating on top surface ($Q = 1.63 \times 10^5 \text{W/m}^2$) and insulated on the bottom surface.

Fig. 7 shows the dynamic mid span deflection and Fig. 8 shows dynamic mid span thermal moment for a simply supported SUS304–Al$_2$O$_3$ FG beam for various volume fractions of ceramic. It is seen from Fig. 7 that the dynamic displacement increases initially from time $t = 0.0$ and as time progresses, the beam oscillates with constant amplitude about static displacement. The ceramic rich beam ($n = 0.0$, 0.05, 0.1 and 0.5) undergoes smaller magnitude of static deflections and deflections become steady within 100 ms whereas metal rich beams ($n = 400.0$, 100.0, and 10.0) exhibit higher value of static deflection, increasing gradually with time and deflection becomes steady after 300 ms. These results for simply supported SUS304–Al$_2$O$_3$ FG beam of $L/h = 100$ when compared with results for Al–ZrO$_2$ FG beam of $L/h = 165$ (refer to Fig. 6a) show the influence of the metal–ceramic combination on the mean thermal deflection. The mean thermal deflection for metal rich beam for Al–ZrO$_2$ FG beam is less as compared to SUS304–Al$_2$O$_3$ FG beam and vice versa for ceramic rich
The major influencing factors for such a behaviour of these FG material combinations are the modulus of elasticity, density and thermal conductivity of the material which influence the stiffness, mass and thermal moment causing vibrations. It is also observed from Fig. 7 that the amplitude of the thermal oscillations for metal rich beam is more as compared to ceramic rich beam. Frequency of thermal oscillation about the mean static thermal deflection is less for ceramic rich beam and temperature induced oscillations occur about the static thermal deflection. The same phenomenon holds good for the cases of CS and CF beams as shown in

<table>
<thead>
<tr>
<th>$E_{\text{ratio}}$</th>
<th>n = 0.0</th>
<th>n = 0.1</th>
<th>n = 0.2</th>
<th>n = 0.5</th>
<th>n = 1.0</th>
<th>n = 2.0</th>
<th>n = 5.0</th>
<th>n = 10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2.121</td>
<td>2.390</td>
<td>2.500</td>
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<td>Alshorbagy et al. [26]</td>
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Figs. 9 and 11 respectively. From Figs. 7 and 9 it is clearly seen that the displacements calculated at the mid-span for SS beam are four times those for CS beam. For SS beam, Fig. 8, the dynamic thermal moment shows oscillatory trend about zero but for CS beam, Fig. 10, the dynamic thermal moment first increases with time and shows the oscillatory trend about a mean thermal moment other than zero depending on the volume fraction of ceramic. This is because one end is fixed-clamped and the thermal moment at this end is zero but the thermal moment present at the simply supported end puts the beam into oscillatory state about a mean value. The plots in Fig. 11 for CF beam revealed that the mean displacements of the CF beam at the free end are four times those for SS beam and frequency of thermal vibration is less as compared to other two. Due to free expansion, as shown in Fig. 12, the dynamic thermal moment for the CF beam at the free end is negligible as compared to the dynamic thermal moment in case of SS and CS as shown in Figs. 8 and 10 respectively.

3.5. Effect of natural convection on thermally induced vibrations of FGM beam subjected to surface heating

Studies are also carried out on the simply supported SUS304–Al2O3 FG beam of $L/h = 100$ subjected to sudden heating on top surface and bottom surface subjected to natural convection as shown in Fig. 13. Temperature variations across the thickness of the beam, as shown in Fig. 14 and calculated from...
Eq. (11), are extracted when a natural convective heat transfer coefficient of 100 W/m$^2$ K is maintained. The temperature distribution across the thickness is similar as in the case of SS beam subjected to step heating on top surface and insulated on bottom surface. Due to presence of the convection on bottom surface, the surface temperature on the bottom layer is less as compared to insulated beam and varies exponentially with time as shown in Fig. 15. The corresponding dynamic mid-span deflection and thermal moment are illustrated in Figs. 16 and 17 respectively, and have similar characteristics of a beam subjected to step heating on one side and insulation on opposite side as examined in Figs. 7 and 8.

Fig. 18 shows the comparison of the dynamic mid span deflection for $n = 100$ (insulated v/s convection).

Eq. (11), are extracted when a natural convective heat transfer coefficient of 100 W/m$^2$ K is maintained. The temperature distribution across the thickness is similar as in the case of SS beam subjected to step heating on top surface and insulated on bottom surface. Due to presence of the convection on bottom surface, the surface temperature on the bottom layer is less as compared to insulated beam and varies exponentially with time as shown in Fig. 15. The corresponding dynamic mid-span deflection and thermal moment are illustrated in Figs. 16 and 17 respectively, and have similar characteristics of a beam subjected to step heating on one side and insulation on opposite side as examined in Figs. 7 and 8.

Fig. 18 shows the comparison of the dynamic mid span deflection for the simply supported SUS304-Al$_2$O$_3$ FG beam for power law index of $n = 100$, which means that beam is 99.99% metal rich. The top surface of the beam is subjected to step heat source and bottom surface being (i) insulated and (ii) subjected to natural convective heat transfer coefficient of 100 W/m$^2$ K. It is observed from the plot that during the initial time of the response, the difference in mid span deflection...
for both the cases is minimal. As the time progresses the beams experience larger static deflection as well as thermal oscillation. The mid span deflection for beam with bottom surface insulated is more than the one with convective heat loss due to higher value of thermal moment driving the vibrations. The difference in the static thermal deflection for the two cases is 1 mm close to the steady state at 300 ms.

4. Conclusions

A theoretical analysis was presented on the thermally induced vibrations of FGM beams subjected to step heat input on top surface. The bottom surface was insulated or subjected to convection heat loss. The method has been verified for fully metallic aluminium beam (n = 400) with the results given by Boley [5]. Fundamental non-dimensional frequencies have been verified with frequencies reported by Simsek and Kocaturk [16] and Alshorbagy et al. [26]. Two functionally graded material combinations, aluminium–zirconia and stainless steel–aluminium oxide with temperature dependent and temperature independent material properties were considered for the study. Some parametric studies are performed to examine the effect of various influential parameters. After accomplishing some parametric studies the following general conclusions may be cited:

(1) Static thermal deflection and the associated thermal oscillations are dependent strongly on the elastic modulus of the FGM beam. FGM beam having low elastic modulus undergoes smaller magnitude of static thermal deflection with thermal oscillation superimposed and attains steady state faster.

(2) The amplitude of the thermal oscillations for metal rich beam is higher as compared to ceramic rich beam; frequency of oscillations is less for metal rich beam and temperature induced oscillations occur with respect to the mean position.

(3) Mid-span deflection for SS beam is four times of CS, and free end deflections for CF beam are four times of SS beam.

(4) Metal–ceramic combination and power law index of the FGM beam greatly affect the vibration characteristics of the beam under thermal load.

(5) The mid span deflection for beam with bottom surface insulated is more than the one with convective heat loss as the surface temperature on the bottom layer is less as compared to insulated beam and the difference varies exponentially with time.

References

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