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Oscillation criteria for third-order neutral differential equations with continuously distributed delay

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ABSTRACT

The purpose of this paper is to study the oscillation of a certain class of third-order neutral differential equations with continuously distributed delay. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

1. Introduction

In this paper, we are concerned with oscillatory behavior of third order neutral differential equation with continuously distributed delay

$$\left[r(t)\left[x(t)+\int_a^b p(t,\mu)x[\tau(t,\mu)]\mathrm{d}\mu\right]''\right]' + \int_c^d q(t,\xi)f(x[\sigma(t,\xi)])\mathrm{d}\xi = 0, \quad t \ge t_0. \tag{1.1}$$

Throughout this paper, we will assume the following hypotheses:

(H_1) $r(t) \in C^1([t_0,\infty),(0,\infty))$, $\int^\infty \frac{1}{r(t)}\mathrm{d}t = \infty$;

(H_2) $p(t,\mu) \in C([t_0,\infty)\times[a,b],R)$, $0 \le p(t) \equiv \int_a^b p(t,\mu)\mathrm{d}\mu \le P < 1$;

(H_3) $\tau(t,\mu) \in C([t_0,\infty)\times[a,b],R)$ is not a decreasing function for μ, and such that

$$\tau(t,\mu) \le t \quad \text{and} \quad \lim_{t\to\infty}\min_{\mu\in[a,b]}\tau(t,\mu) = \infty;$$

(H_4) $q(t,\xi) \in C([t_0,\infty)\times[c,d],(0,\infty))$;

(H_5) $\sigma(t,\xi) \in C([t_0,\infty)\times[c,d],R)$ is not a decreasing function for ξ, and such that

$$\sigma(t,\xi) \le t \quad \text{and} \quad \lim_{t\to\infty}\min_{\xi\in[c,d]}\sigma(t,\xi) = \infty;$$

(H_6) $f(x) \in C(R,R)$, $\frac{f(x)}{x} \ge \delta > 0$, $x \ne 0$.

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Define the function by

$$z(t) = x(t) + \int_a^b p(t, \mu)x[\tau(t, \mu)]\mathrm{d}\mu. \tag{1.2}$$

By a solution of (1.1), we mean a nontrivial function $x(t)$ satisfying (1.1) which has the properties $x(t) \in C^1([T_x, \infty), R)$ for $T_x \geq t_0$, and $r(t)z''(t) \in C^1[T_x, \infty)$. Our attention is restricted to those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution x of Eq. (1.1) is said to be oscillatory on $[T_x, \infty)$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. In recent years, there has been much research activity concerning the oscillation theory and applications of differential equations; see [1–3]. Especially, the study content of oscillatory criteria of second order differential equations is very rich. In contrast, the study of oscillatory criteria of three order differential equations is relatively less, but most of them are about delay equation; there are few results dealing with the oscillation of the solutions of three-order neutral differential equations with continuously distributed delay in [4–9]. In recent years, [10] considered the second order neutral delay differential equation with continuous distributed

$$\big(r(t)\,[x(t) + p(t)x(t-\tau)]'\big)' + \int_a^b q(t, \xi)x[g(t, \xi)]\mathrm{d}\sigma(\xi) = 0, \tag{1.3}$$

and obtained oscillatory criteria of Philos-type of (1.3). Our aim in this paper is to give oscillatory criteria of Philos-type of (1.1). Our results improve the results established in [10], but also supply the oscillatory theorems of three order delay differential equations in [4–9].

2. Several lemmas

Lemma 2.1. *Let $x(t)$ be a positive solution of (1.1), and $z(t)$ is defined as in (1.2). Then $z(t)$ has only one of the following two properties:*

(I) $z(t) > 0, z'(t) > 0, z''(t) > 0$;
(II) $z(t) > 0, z'(t) < 0, z''(t) > 0$,

where $t \geq t_1$, t_1 sufficiently large.

Proof. Let $x(t)$ be a positive solution of (1.1) on $[t_0, \infty)$, so that $z(t) > x(t) > 0$, and

$$\big[r(t)z''(t)\big]' = -\int_c^d q(t, \xi)f(x[\sigma(t, \xi)])\mathrm{d}\xi < 0.$$

Then $r(t)z''(t)$ is a decreasing function and therefore eventually of one sign, so $z''(t)$ is either eventually positive or eventually negative on $t \geq t_1 \geq t_0$. We assert that $z''(t) > 0$ on $t \geq t_1 \geq t_0$. Otherwise, assume that $z''(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)z''(t) \leq -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z'(t) \leq z'(t_1) - M\int_{t_1}^t \frac{1}{r(s)}\mathrm{d}s.$$

Let $t \to \infty$. Then from (H_1), we have $z'(t) \to -\infty$, and therefore eventually $z'(t) < 0$. Since $z''(t) < 0$ and $z'(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, $z(t)$ has only one of the two properties (I) and (II). This completes the proof. □

Lemma 2.2. *Let $x(t)$ be a positive solution of (1.1), correspondingly $z(t)$ has the property* (II). *Assume that*

$$\int_{t_0}^\infty \int_v^\infty \left[\frac{1}{r(u)}\int_u^\infty \int_c^d q(s, \xi)\mathrm{d}\xi\mathrm{d}s\right]\mathrm{d}u\mathrm{d}v = \infty. \tag{2.1}$$

Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a positive solution of (1.1). Since $z(t)$ has the property (II), then there exists finite limit $\lim_{t\to\infty} z(t) = l$.

We assert that $l = 0$. Assume that $l > 0$, then we have $l + \varepsilon > z(t) > l$ for all $\varepsilon > 0$. Choosing $\varepsilon < \frac{l(1-P)}{P}$, we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t, \mu)x[\tau(t, \mu)]\mathrm{d}\mu \\ &> l - \int_a^b p(t, \mu)z[\tau(t, \mu)]\mathrm{d}\mu \geq l - p(t)z[\tau(t, a)] \\ &\geq l - P(l + \varepsilon) = K(l + \varepsilon) > Kz(t), \end{aligned} \tag{2.2}$$

where $K = \frac{l-P(l+\varepsilon)}{l+\varepsilon} > 0$. Using ($H_6$) and (2.2), we find from (1.1) that

$$\big(r(t)z''(t)\big)' \leq -K\delta \int_c^d q(t,\xi)z[\sigma(t,\xi)]\mathrm{d}\xi.$$

Note that $z(t)$ has the property (II) and (H_5), we have

$$\big(r(t)z''(t)\big)' \leq -K\delta z[\sigma(t,d)] \int_c^d q(t,\xi)\mathrm{d}\xi \equiv -q_1(t)z[\sigma_1(t)], \tag{2.3}$$

where $q_1(t) = K\delta \int_c^d q(t,\xi)\mathrm{d}\xi$, $\sigma_1(t) = \sigma(t,d)$. Integrating inequality (2.3) from t_1 to t, we obtain

$$r(t)z''(t) \geq \int_t^\infty q_1(s)z[\sigma_1(s)]\mathrm{d}s.$$

Using $z[\sigma_1(s)] \geq l$, we obtain

$$z''(t) \geq \frac{l}{r(t)} \int_t^\infty q_1(s)\mathrm{d}s. \tag{2.4}$$

Integrating inequality (2.4) from t to ∞, we have

$$-z'(t) \geq l \int_t^\infty \left[\frac{l}{r(u)} \int_u^\infty q_1(s)\mathrm{d}s\right]\mathrm{d}u.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geq l \int_{t_1}^\infty \int_v^\infty \left[\frac{l}{r(u)} \int_u^\infty q_1(s)\mathrm{d}s\right]\mathrm{d}u\mathrm{d}v.$$

Because (2.3) and the last inequality contradict (2.1), we have $l = 0$. And since $0 \leq x(t) \leq z(t)$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3 ([11]). *Let $u(t) > 0, u'(t) > 0, u''(t) \leq 0, t \geq t_0$. Then for every content $\alpha \in (0,1)$ there exists $T_\alpha \geq t_0$ such that*

$$u(\sigma(t)) \geq \alpha \frac{\sigma(t)}{t} u(t), \quad t \geq T_\alpha.$$

Lemma 2.4 ([12]). *Let $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) \leq 0, t \geq T_\alpha$. Then there exists $\beta \in (0,1)$ and $T_\beta \geq T_\alpha$ such that*

$$z(t) \geq \beta t z'(t), \quad t \geq T_\beta.$$

3. Main results

In this section, we obtain new oscillatory criteria for (1.1) by using the generalized Riccati transformation and integral averaging technique of Philos-type [13]. Let

$$D = \{(t,s) : t \geq s \geq t_0\}; \qquad D_0 = \{(t,s) : t > s \geq t_0\}.$$

A function $H \in \mathrm{C}^1(D, \mathrm{R})$ is said to belong to X class ($H \in X$) if it satisfies

(i) $H(t,t) = 0, t \geq t_0$; $H(t,s) > 0, (t,s) \in D_0$;

(ii) $\frac{\partial H(t,s)}{\partial s} \leq 0$, there exist $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ and $h \in \mathrm{C}(D_0, \mathrm{R})$ such that

$$\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = -h(t,s)\sqrt{H(t,s)}. \tag{3.1}$$

The following is the main results of this paper.

Theorem 3.1. *Assume that (2.1) holds and there exist $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ and $H \in X$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)Q(s) - \frac{1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s = \infty, \tag{3.2}$$

where

$$Q(s) = \alpha\beta\delta(1-P)\frac{\rho(t)}{t}\sigma^2(t,c) \int_c^d q(t,\xi)\mathrm{d}\xi. \tag{3.3}$$

Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. Assume that (1.1) has a nonoscillatory solution x. Without loss generality we may assume that $x(t) > 0, t \geq t_1$, $x[\tau(t,\mu)] > 0$, $(t,\mu) \in [t_1,\infty) \times [a,b]$, $x[\sigma(t,\xi)] > 0$, $(t,\xi) \in [t_1,\infty) \times [c,d]$. $z(t)$ is defined as in (1.2). By Lemma 2.1, we have that $z(t)$ has the property (I) or the property (II).

When $z(t)$ has the property (I), we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\mu)x[\tau(t,\mu)]\mathrm{d}\mu \geq z(t) - \int_a^b p(t,\mu)z[\tau(t,\mu)]\mathrm{d}\mu \\ &\geq z(t) - z[\tau(t,b)]\int_a^b p(t,\mu)\mathrm{d}\mu \geq \left(1 - \int_a^b p(t,\mu)\mathrm{d}\mu\right) z(t) \geq (1-P)z(t). \end{aligned} \tag{3.4}$$

Using (H_5) and (H_6), we have

$$\begin{aligned} \left(r(t)z''(t)\right)' &\leq -\delta(1-P)\int_c^d q(t,\xi)z[\sigma(t,\xi)]\mathrm{d}\xi \\ &\leq -\delta(1-P)z[\sigma(t,c)]\int_c^d q(t,\xi)\mathrm{d}\xi \equiv -q_2(t)z[\sigma_2(t)], \end{aligned} \tag{3.5}$$

where

$$q_2(t) = \delta(1-P)\int_c^d q(t,\xi)\mathrm{d}\xi, \qquad \sigma_2(t) = \sigma(t,c). \tag{3.6}$$

Let

$$W(t) = \rho(t)\frac{r(t)z''(t)}{z'(t)}, \quad t \geq t_1. \tag{3.7}$$

Then

$$W'(t) \leq -\frac{\rho(t)q_2(t)z(\sigma_2(t))}{z'(t)} + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{W^2(t)}{\rho(t)r(t)}. \tag{3.8}$$

Choosing $u(t) = z'(t)$ in Lemma 2.3, we obtain

$$\frac{1}{z'(t)} \geq \frac{\alpha\sigma_2(t)}{tz'[\sigma_2(t)]}, \quad t \geq T_\alpha \geq t_1. \tag{3.9}$$

Using Lemma 2.4, we get

$$z[\sigma_2(t)] \geq \beta\sigma_2(t)z'[\sigma_2(t)], \quad t \geq T_\beta \geq T_\alpha. \tag{3.10}$$

Combining with (3.8)–(3.10), we obtain

$$W'(t) \leq -Q(t) + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{W^2(t)}{\rho(t)r(t)}, \quad t \geq T_\beta, \tag{3.11}$$

where $Q(t)$ is defined as in (3.3). Let

$$A(t) = \frac{\rho'(t)}{\rho(t)}, \qquad B(t) = \frac{1}{\rho(t)r(t)}, \tag{3.12}$$

we have for $t \geq t_2 \geq T_\beta$, that

$$\begin{aligned} \int_{t_2}^t H(t,s)Q(s)\mathrm{d}s &\leq \int_{t_2}^t H(t,s)\left[-W'(s) + A(s)W(s) - B(s)W^2(s)\right]\mathrm{d}s \\ &= H(t,t_2)W(t_2) - \int_{t_2}^t \left[h(t,s)\sqrt{H(t,s)}W(s) + H(t,s)B(s)W^2(s)\right]\mathrm{d}s \\ &= H(t,t_2)W(t_2) - \int_{t_2}^t \left[\sqrt{H(t,s)B(s)}W(s) + \frac{h(t,s)}{2\sqrt{B(s)}}\right]^2\mathrm{d}s + \int_{t_2}^t \frac{h^2(t,s)}{4B(s)}\mathrm{d}s. \end{aligned} \tag{3.13}$$

Hence

$$\frac{1}{H(t,t_2)}\int_{t_2}^t \left[H(t,s)Q(s) - \frac{1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s \leq W(t_2).$$

The last inequality contradicts (3.2).

If $z(t)$ has the property (II). Since (2.1) holds, then the conditions in Lemma 2.2 are satisfied. Hence $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 3.2. *Assume that other conditions of Theorem 3.1 are satisfied except condition (3.2). Also let*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty \tag{3.14}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)h^2(t,s)\mathrm{d}s < \infty \tag{3.15}$$

hold. If there exists $\psi \in \mathrm{C}([t_0,\infty),\mathrm{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\psi_+^2(s)}{\rho(s)r(s)}\mathrm{d}s = \infty, \tag{3.16}$$

where $\psi_+(t) = \max\{\psi(t), 0\}$, *and such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)Q(s) - \frac{1}{4}\rho(s)r(s)h^2(t,s)\right]\mathrm{d}s \geq \psi(T). \tag{3.17}$$

Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. We proceed as in the proof of Theorem 3.1. If $z(t)$ has the property (I), we obtain (3.13). Hence

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)Q(s) - \frac{h^2(t,s)}{4B(s)}\right]\mathrm{d}s\\ &\quad\leq W(t_2) - \liminf_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[\sqrt{H(t,s)B(s)}W(s) + \frac{h(t,s)}{2\sqrt{B(s)}}\right]^2\mathrm{d}s.\end{aligned}$$

From (3.17), we have

$$W(t_2) \geq \psi(t_2) + \liminf_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[\sqrt{H(t,s)B(s)}W(s) + \frac{h(t,s)}{2\sqrt{B(s)}}\right]^2\mathrm{d}s.$$

Thus

$$\liminf_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[\sqrt{H(t,s)B(s)}W(s) + \frac{h(t,s)}{2\sqrt{B(s)}}\right]^2\mathrm{d}s < \infty. \tag{3.18}$$

Now define the function by

$$\begin{aligned}U(t) &= \frac{1}{H(t,t_2)}\int_{t_2}^{t}H(t,s)B(s)W^2(s)\mathrm{d}s,\\ V(t) &= \frac{1}{H(t,t_2)}\int_{t_2}^{t}\sqrt{H(t,s)}h(t,s)W(s)\mathrm{d}s.\end{aligned}$$

From (3.18), we find

$$\liminf_{t\to\infty}[U(t)+V(t)] < \infty.$$

The rest of the proof is similar to the proof of corresponding theorem in [8,9], and hence it is omitted. If $z(t)$ has the property (II), we obtain $\lim_{t\to\infty}x(t) = 0$ by Lemma 2.2. This completes the proof. □

Theorem 3.3. *Assume that other conditions of Theorem 3.2 are satisfied except (3.15). Also let*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)Q(s)\mathrm{d}s < \infty.$$

Then every solution $x(t)$ of Eq. (1.1) is either oscillatory or converges to zero.

Proof. We proceed as in the proof of Theorem 3.2 and hence it is omitted. □

Remark. If let $H(t,s) = (t-s)^n$, then Philos-type is simplified to Kamenev-type. Function H can also have other choices, such as $H(t,s) = \left(\ln\frac{t}{s}\right)^n$ or $\left(e^t - e^s\right)^n$. Generally choose $H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n$, where n is an integer and $n > 1$, and $\theta \in C([t_0,\infty), R^+)$ such that

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{du}{\theta(u)} = \infty.$$

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