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CONVENIENT CATEGORIES OF TOPOLOGICAL ALGEBRAS, AND THEIR DUALITY THEORY

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Introduction

Concrete algebraic structures with a topology have long arisen in mathematical practice, leading to the notion of a topological space with algebraic operations making the underlying set an algebra for the type under consideration. Classes of such objects (together with continuous maps respecting the algebraic structure) form categories which, understandably, do not share some important properties of their purely algebraic analogues. Specially, *their relation with the base category* ***S*** *of sets is not satisfactory* (e.g., they are not monadic (i.e., tripleable) with respect to the natural forgetful functors). This is essentially due to the fact that taking forgetful functors into ***S*** is forgetting too much. Of importance is also the fact that the set of morphisms between any two such algebras carries a topology which is inherited from the topologies of the algebras, and which is not taken into account (it is ignored). That is, the ubiquitous "always at our disposal, no need to be defined" representable functors do not retain any topological information.

The category of topological spaces is actually the natural *base* category (that is, the place where the forgetful and representable functors land) for a categorical approach to the study of classes of topologized algebraic structures. However, this category is not "set-like" enough to make such an approach possible. Categories which, like ***S***, have enough structure to serve as base categories have been recognized by category theorists during the sixties, when the concept of *closed category* was developed. The category ***K*** of compactly generated Hausdorff topological spaces is such a convenient (closed) category.

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The study of enriched category theory has reached a level of development which puts at our disposal enriched versions of most of the important machinery of ordinary category theory. If the base category V is good enough, the V-world is as good as the set-based world. Although not completely, this is very much the case with $\boldsymbol{K}$. *Working in the* $\boldsymbol{K}$*-world allows us to deal with topologized algebraic structures in a purely algebraic way.* The continuity of the functions is always guaranteed, the topology in the constructions does not require ad-hoc definition. The topological information is carried automatically due to the closed structure of $\boldsymbol{K}$. The (left) adjoints to the representable functors, currently called *tensors* and *cotensors*, can not (in the $\boldsymbol{K}$-world) be obtained as colimits and limits. Thus, they provide a (categorical) characterization for certain constructions which is not available (or even possible) in an ordinary set based approach. The algebraically defined categories of groups in $\boldsymbol{K}$, modules in $\boldsymbol{K}$ over a ring in $\boldsymbol{K}$, algebras in $\boldsymbol{K}$ over a ring in $\boldsymbol{K}$, etc. etc., are all $\boldsymbol{K}$-monadic (i.e., $\boldsymbol{K}$-tripleable) with respect to the natural "forget-the-algebraic-structure" forgetful $\boldsymbol{K}$-functors into $\boldsymbol{K}$, in exactly the same way that the analogous categories in the set based world.

We introduce here a systematic treatment of categories of (complex) topological algebras considered as categories based in the category $\boldsymbol{K}$ *of compactly generated Hausdorff spaces.* This leads to the definition of K-topological algebras (i.e., the concept of associative algebra over the field of complex numbers ***relativized*** to the $\boldsymbol{K}$-world). Roughly, a K-topological algebra turns out to be a complex algebra with a topology making the operations continuous when restricted to compact sets. This is a broad class of algebras, containing *all* algebras with jointly continuous product, and also many interesting algebras with discontinuous, separately continuous, product (cf. Examples 1.3 and 1.4).

Cotensors realize the (classical) construction of algebras of continuous functions, and play an important role in duality theory, as illustrated by the following simple formulation of the main result in Gelfand Theory: "The complex numbers are a $\boldsymbol{K}$-codense $\boldsymbol{K}$-cogenerator of the $\boldsymbol{K}$-category of commutative C^*-algebras with identity".

In Section 1 we introduce our basic definitions and show how K-topological algebras (and subclasses like algebras with involution, Banach algebras, Fréchet algebras, C^*-algebras, etc.) form $\boldsymbol{K}$-categories, in Section 2 we establish some properties of these $\boldsymbol{K}$-categories, and in Section 3 we show that any complete locally m-convex commutative algebra with identity, and with an involution such that for a defining family of seminorms $\{p\}$ the identity $p(xx^*) = p(x)^2$ holds, is the algebra of *all* continuous complex valued functions, with the uniform convergence on compact sets, on a certain topological space (topologically as well as algebraically, if considered qua K-topological algebras) (cf. Theorem 3.13). This is done by interpreting functional representation within the general framework of an (enriched) duality machinery. This machinery is basically the interplay of the cotensors with the contravariant representable functor determined by the complex numbers (which realize the classical construction of the spectrum of an algebra), in what could be called an iterated double dualization process.

Let us remark that the words "topology", "topologized", "continuous", etc., could have been omitted (except in the examples, of course) immediately after Section 0, where we collect some known basic properties of $\boldsymbol{K}$. We did not do so in order to remind the reader (and ourselves) that we are dealing with topological spaces and continuous functions.

Finally, we point out that other categorical approaches to standard Functional Analysis theories have been exploited, for instance, in the recent papers [25, 29, 30].

§ 0. The category *K*

We will denote by $\boldsymbol{K}$ the full subcategory of topological spaces whose objects are the *compactly generated Hausdorff* spaces, that is: $X \in \boldsymbol{K}$, if and only if X is a Hausdorff space and $X = \xrightarrow[K \subset X]{\text{colim}} K$, K running over all the compact subsets of X. This means that X is topologically equal to the topological colimit of its compact subsets; in other words, a map $X \to Z$ into any other topological space Z is continuous if and only if it is continuous on each compact subset of X. Following [16], we will call such a space Kelley space. Given any Hausdorff space Z is clear that Z, ***as a set*** is equal to the colimit of its compact subsets. The colimit topology defines a Kelley space denoted KeZ, and the inclusions $K \to Z$ of the compact subsets of Z determine a (unique, continuous) map Ke$Z = \xrightarrow[K \subset Z]{\text{colim}} K \to Z$. Z is a Kelley space if and only if Z = KeZ. KeZ has the same underlying set as Z, and its topology is the finest among those having the same compact subsets as the given topology of Z. Given any topological space H, a map Ke$Z \to H$ is continuous if and only if $Z \to H$ is continuous on compact subsets. Also: given any Kelley space X, a map $X \to Z$ is continuous if and only if $X \to$ KeZ is continuous. Denoting by $\mathbf{Top}_2$ the category of Hausdorff spaces and continuous maps, for any $Z \in \mathbf{Top}_2$, the assignment $Z \rightsquigarrow$ KeZ is then a functor $\mathbf{Top}_2 \xrightarrow{\text{Ke}} \boldsymbol{K}$ (Ke$f = f$, for any $Z \xrightarrow{f} Z'$) which provides a right adjoint (coreflexion) to the full inclusion $\boldsymbol{K} \to \mathbf{Top}_2$. We call this functor the *K-ation functor*, and for $Z \in \mathbf{Top}_2$, KeZ will be the *K-ation of Z*

We will give below a list of properties of the category $\boldsymbol{K}$. The reader is referred to [16] for quick proofs of the results below. For a more extensive treatment and additional results, he can use [31], [35], [37], and for a treatment with a categorical flavor, [32] and [33].

The categorical language and terminology used here is by now standard in articles written in English on this side of the Atlantic. The basic categorical concepts can be found in [26]. For the notion of closed category and related subjects there is a condensed presentation in [10]. A more extensive presentation is given in [6]. On the other hand, [14] is a complete, exhaustive and meticulous reference article. [7], [10] and [23] consider further developments of the subject.

If X and Y are Kelley spaces, the set of all continuous functions from X to Y will be denoted by $\boldsymbol{K}_0(X, Y)$. Thus, $\boldsymbol{K}_0(X, Y) = \mathbf{Top}_2(X, Y)$.

0.0. *If $X \in \mathbf{K}$, then $C \subset Z$ is closed if and only if $C \cap K$ is closed in X for all K compact in X. For $Z \in \mathbf{Top}_2$, the family $\{C \subset Z : C \cap K$ is closed in Z for all K compact in $Z\}$ is a basis of closed sets for* Ke*Z*.

0.1. *Metrizable spaces are Kelley spaces.*

0.2. *Locally compact Hausdorff spaces are Kelley spaces.*

0.3. *Any Hausdorff quotient of a Kelley space is a Kelley space.*

0.4. *A closed subset of a Kelley space is a Kelley space with the induced topology.*

0.5. *An open subset of a Kelley space is a Kelley space with the induced topology.*

0.6. Definition. *Given a continuous monomorphism $X \xrightarrow{i} Y$ between Kelley spaces, i is a full injection if given any other Kelley space V, a function $V \to X$ is continuous if and only if the composite $V \to X \xrightarrow{i} Y$ is continuous.* This is equivalent to saying that the topology of X is the K-ation of the inverse image under i of the topology of Y. A topological subspace is a full injection, but the converse does not hold.

0.7. *$\mathbf{K}$ is a complete category.* That is: $\mathbf{K}$ has all (small) inverse limits (limits). If $\Lambda \to \mathbf{K}$ ($\lambda \rightsquigarrow X_\lambda$) is a functor, then $\varprojlim_\lambda X_\lambda \in \mathbf{K}$ is the limit space of the X_λ with the K-ation of the limit topology (which is automatically Hausdorff).

0.8. *$\mathbf{K}$ is a cocomplete category.* That is: $\mathbf{K}$ has all (small) direct limits (colimits). If $\Lambda \to \mathbf{K}$ ($\lambda \rightsquigarrow X_\lambda$) is a functor, $\underset{\overrightarrow{\lambda}}{\mathrm{colim}}\, X_\lambda \in \mathbf{K}$ is the largest Hausdorff quotient of the colimit space of the X_λ with the colimit topology (which is automatically compactly-generated).

0.9. *Given two Kelley spaces X and Y we will denote by $X \boxtimes Y$ the product of X and Y in $\mathbf{K}$ (the existence of which follows from 0.7).* We have $X \boxtimes Y = \mathrm{Ke}(X \times Y)$, where $X \times Y$ denotes the ordinary cartesian product. If $\tau : X \boxtimes Y \to Y \boxtimes X$ is defined by $\tau(x, y) = (y, x)$, then τ is an isomorphism, or $X \boxtimes Y \overset{\tau}{\approx} Y \boxtimes X$. The Kelley space consisting of a single point $\{*\}$ will be denoted by 1. It is a *terminal object* of $\mathbf{K}$ and $X \boxtimes 1 \approx 1 \boxtimes X \approx X$ for all $X \in \mathbf{K}$.

0.10. *Given two Kelley spaces X and Y, $X \boxtimes Y = \underset{\overrightarrow{K, K'}}{\mathrm{colim}}\, K \times K'$ where $K \subset X$, $K' \subset Y$ range over all the compact subsets of X and Y.*

0.11. *If X is locally compact (whence $X \in \mathbf{K}$, see 0.2), then $X \boxtimes Y = X \times Y$ for all $Y \in \mathbf{K}$* and in fact this property characterizes locally compact spaces.

0.12. *For all* $X \in \mathbf{K}$, *the functor* $\mathbf{K} \xrightarrow{-\boxtimes X} \mathbf{K}$, $Y \rightsquigarrow Y \boxtimes X$ *has a right adjoint* $\mathbf{K} \xrightarrow{\mathbf{K}(X,\ )} \mathbf{K}$, $V \rightsquigarrow \mathbf{K}(X, V)$ *where* $\mathbf{K}(X, V)$ *is the space of all continuous functions* $X \to V$ *with the K-ation of the compact-open topology.* Thus $\mathbf{K}_0(Y \boxtimes X, V) \overset{\omega_0}{\approx} \mathbf{K}_0(Y, \mathbf{K}(X, V))$ with $\omega_0 \circ \omega_0 = \mathrm{id}$ (we denote this bijection with the same letter in both directions). We will also write:

$$\frac{Y \boxtimes X \to V}{Y \to \mathbf{K}(X, V)}\ \omega_0 .$$

0.13. It follows (categorically) that the *above bijection is actually a (natural) homeomorphism* $\mathbf{K}(Y \boxtimes X, V) \overset{\omega}{\approx} \mathbf{K}(Y, \mathbf{K}(X, V))$. *There are also homeomorphisms* $\mathbf{K}(Y, \mathbf{K}(X, V)) \overset{\sigma}{\approx} \mathbf{K}(X, \mathbf{K}(Y, V))$ *and the (obviously defined) maps in the following list are well defined and continuous:*

$$Y \to \mathbf{K}(\mathbf{K}(X, V), V)$$

$$Y \to \mathbf{K}(X, Y \boxtimes X)$$

$$\mathbf{K}(X, Y) \boxtimes X \to Y$$

$$\mathbf{K}(X, Y) \boxtimes \mathbf{K}(Y, V) \to \mathbf{K}(X, V)$$

etc.

0.14. Proposition. *If* $X \xrightarrow{i} Y$ *is a full injection (see Def. 0.6) then* $\mathbf{K}(V, X) \xrightarrow{\mathbf{K}(V, i)} \mathbf{K}(V, Y)$ *is also a full injection for all* $V \in \mathbf{K}$.

Proof. Let $W \in \mathbf{K}$ be any Kelley space and consider a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & \mathbf{K}(V, X) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \mathbf{K}(V, i)} \\ & & \mathbf{K}(V, Y) \end{array}$$

where g is continuous. The proposition will follow if we prove that f is necessarily continuous. Consider now the diagram

$$\begin{array}{ccc} W \boxtimes V & \xrightarrow{\ h\ } & X \\ & {\scriptstyle \omega_0(g)}\searrow & \downarrow{\scriptstyle i} \\ & & Y \end{array}$$

where ω_0 is the bijection of 0.12 and h is the function $(x, v) \rightsquigarrow f(x)(v)$. The diagram clearly commutes, and hence h is continuous. By naturality of ω_0 the diagram

$$\begin{array}{ccc} W & \xrightarrow{\omega_0(h)} & K(V,X) \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle K(V,f)} & \\ & K(V,Y) & \end{array}$$

also commutes. It follows that $f = \omega_0(h)$ and therefore f is continuous, as desired. Q.E.D.

We summarize: the category $\boldsymbol{K}$ is a *symmetrical monoidal closed category* (0.9 and 0.12) with a tensor product given by the (categorical) product (0.9). That is: $\boldsymbol{K}$ is a *cartesian closed category*. Furthermore, $\boldsymbol{K}$ is a *complete and cocomplete category* (0.7 and 0.8'). Finally, it is easy to observe that $1 \in \boldsymbol{K}$ is a *generator* and that $\boldsymbol{K}$ is *well-powered* (i.e., the class of subobjects of any fixed object of $\boldsymbol{K}$ is a set).

Observe that $\boldsymbol{K}$ is equivalent to the category of *all* Hausdorff spaces and all functions that are continuous on compact subsets, between them. More precisely, denoting by $\boldsymbol{K}\mathbf{Top}_2$ this category, the inclusion $\boldsymbol{K} \to \boldsymbol{K}\mathbf{Top}_2$ is still full and the K-ation functor Ke is also a functor $\boldsymbol{K}\mathbf{Top}_2 \xrightarrow{\mathrm{Ke}} \boldsymbol{K}$. The map $\mathrm{Ke}Z \to Z$ (for $Z \in \boldsymbol{K}\mathbf{Top}_2$) is an isomorphism in $\boldsymbol{K}\mathbf{Top}_2$ and therefore Ke (together with the inclusion) is an equivalence of categories. Thus, *the choice between $\boldsymbol{K}$ and $\boldsymbol{K}\mathbf{Top}_2$ is just a matter of personal taste.*

§ 1. Categories of topological algebras

The field of complex numbers with the ordinary (metric) topology is a Kelley space, that we will denote by C.

1.1. Definition. *By a K-topological algebra we will understand an algebra over C in $\boldsymbol{K}$. Specifically, a K-topological algebra consists of the following:*

A Kelley space X (that is, an object of $\boldsymbol{K}$) together with

1.1.1. *maps (in $\boldsymbol{K}$)* $\quad X \boxtimes X \xrightarrow{+} X$

$$1 \xrightarrow{0} X$$

I

$$X \xrightarrow{-} X$$

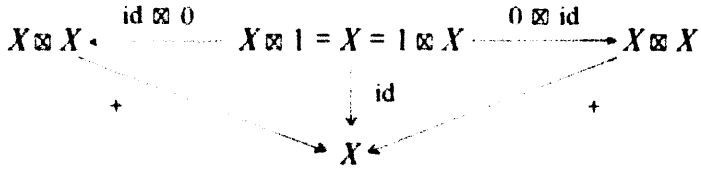
such that

(i) *+ is associative and commutative. That is, the diagrams*

$$\begin{array}{ccc} X \boxtimes X \boxtimes X & \xrightarrow{\mathrm{id} \boxtimes +} & X \boxtimes X \\ {\scriptstyle + \boxtimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle +} \\ X \boxtimes X & \xrightarrow{+} & X \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} X \boxtimes X & \xrightarrow{\tau} & X \boxtimes X \\ & {\scriptstyle +}\searrow & \downarrow{\scriptstyle +} \\ & & X \end{array}$$

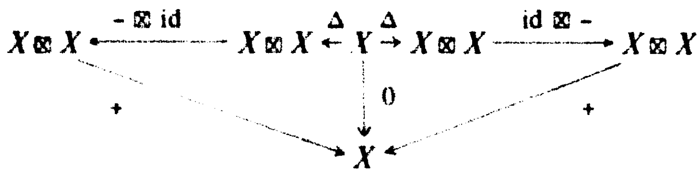
commute (where τ is defined as in 0.9).

(ii) *0 is a unit for +. That is, the diagram*



commutes.

(iii) *– is an inverse for + with respect to 0. That is, the diagram*

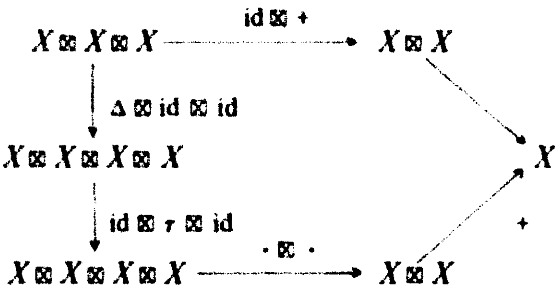


commutes (where $\Delta(x) = (x, x)$ *and* $X \xrightarrow{0} X = X \to 1 \xrightarrow{0} X$*).*

1.1.2. *a map (in **K**)* $X \boxtimes X \xrightarrow{\cdot} X$ *such that*

(i) $\cdot$ *is associative (in the same sense as +; see 1.1.1.(i)).*

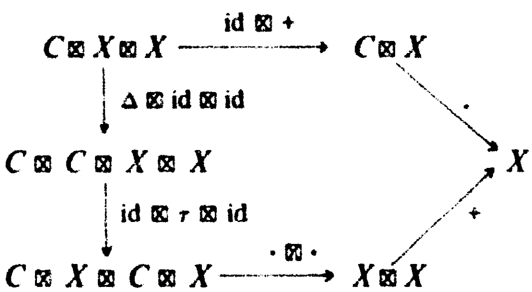
(ii) $\cdot$ *is distributive with respect to +. That is, the diagram*



(and the corresponding one expressing distributivity on the right side) commute.

1.1.3. *a map (in **K**)* $C \boxtimes X \xrightarrow{\cdot} X$ *such that*

(i) $C \boxtimes X \xrightarrow{\cdot} X$ *is distributive with respect to + and to the sum of complex numbers (also denoted by* $C \boxtimes C \xrightarrow{+} C$*). That is, the diagrams*



and

$$\begin{array}{ccc}
C\boxtimes C\boxtimes X & \xrightarrow{+\boxtimes \mathrm{id}} & C\boxtimes X \\
\downarrow {\scriptstyle \mathrm{id}\boxtimes\mathrm{id}\boxtimes\Delta} & & \searrow {\scriptstyle \cdot} \\
C\boxtimes C\boxtimes X\boxtimes X & & X \\
\downarrow {\scriptstyle \mathrm{id}\boxtimes\tau\boxtimes\mathrm{id}} & & \nearrow {\scriptstyle +} \\
C\boxtimes X\boxtimes C\boxtimes X & \xrightarrow{\cdot\boxtimes\cdot} & X\boxtimes X
\end{array}$$

commute.

(ii) *$C\boxtimes X \xrightarrow{\cdot} X$ is associative with respect to · and the product of complex numbers (also denoted $C\boxtimes C \xrightarrow{\cdot} C$). That is, the diagrams*

$$\begin{array}{ccc}
C\boxtimes C\boxtimes X & \xrightarrow{\mathrm{id}\boxtimes\cdot} & C\boxtimes X \\
\downarrow {\scriptstyle \cdot\boxtimes\mathrm{id}} & & \downarrow {\scriptstyle \cdot} \\
C\boxtimes X & \xrightarrow{\cdot} & X
\end{array}
\quad \textit{and} \quad
\begin{array}{ccc}
C\boxtimes X\boxtimes X & \xrightarrow{\mathrm{id}\boxtimes\cdot} & C\boxtimes X \\
\downarrow {\scriptstyle \cdot\boxtimes\mathrm{id}} & & \downarrow {\scriptstyle \cdot} \\
C\boxtimes X & \xrightarrow{\cdot} & X
\end{array}$$

commute.

(iii) *The action of $1 \xrightarrow{1} C$ is the identity. That is, the diagram*

$$\begin{array}{ccc}
X = 1\boxtimes X & \xrightarrow{1\boxtimes\mathrm{id}} & C\boxtimes X \\
& \searrow {\scriptstyle \mathrm{id}} & \downarrow {\scriptstyle \cdot} \\
& & X
\end{array}$$

commutes.

We will denote such a K-topological algebra by $A = (X, +, ., .)$ and its underlying Kelley space by $X = |A|$.

It is clear that the complex numbers with the ordinary topology and algebraic operations form a K-topological algebra, which, by abuse of language, we will also denote by C.

Part 1.1.1 in the above definition expresses the fact that a K-topological algebra is *an Abelian group* in $\boldsymbol{K}$ (in the sense of [15] exercise 2.c or [13]), or a *weak group* in the sense of [34]. The continuous sum $X \boxtimes X \xrightarrow{+} X$ is in general only continuous on compact subsets when considered as a map $X \times X \xrightarrow{+} X$ and hence the topology of X will not (in general) be a group topology. This notion was introduced by Spanier [34] to obtain some results in algebraic topology (exploiting the clear fact that for arbitrary $Z \in \mathbf{Top}_2$, the identity map Ke$Z \to Z$ is always a homotopy equivalence). A *different* notion relating group structures and functions continuous on compact subsets has also been considered: Noble in [31, Chap. V] defines a "*k-group*" as being a group X with a group topology behaving within the category of topological groups as compactly generated spaces behave within the category of all Hausdorff topological spaces. More precisely, a *morphism of groups $X \to G$ into any other topological group is continuous if and only if it is continuous over compact subsets.* The topology of a k-group, however, need not be compactly generated (cf. [31] corollaries to Th. 5.2).

The product (1.1.2 above) of a K-topological algebra will be, in general, continuous only on compact subsets as a map $X \times X \to X$. On the other hand, the product by scalars (1.1.3) is continuous $C \times X \to X$, because C being locally compact, we have $C \boxtimes X = C \times X$ (0.11).

We see then that a K-topological algebra is simply an algebra over the complex numbers with a Hausdorff compactly generated topology which makes the sum and the product continuous when each variable is restricted to a compact subset, and the product by scalars globally continuous.

Any Hausdorff topological algebra with continuous multiplication (in the sense of [17], for instance) determines canonically a K-topological algebra consisting of the same underlying set with the K-ation of the given topology and the same algebraic operations. This is clear since the functor Ke being a right adjoint, preserves limits, and therefore for any Hausdorff spaces Y, Z there is an isomorphism $\mathrm{Ke}(Y \times Z) \approx \mathrm{Ke}Y \boxtimes \mathrm{Ke}Z$. Observe that different topological algebras may determine the same K-topological algebra.

Definition 1.1 is categorical and could have been given in any category with finite products and a terminal object (the empty product). If considered in a category equivalent to $\boldsymbol{K}$, Def. 1.1 would yield the concept of a mathematical object which is categorically indistinguishable from the concept of K topological algebra. Therefore (cf. end of §0) we can think on K-topological algebras as being a Hausdorff space with a structure of complex algebra in which sum, product and product by scalars are continuous only on compact subsets. A morphism is then a linear multiplicative function which is continuous on compact sets. In this approach, any topological algebra in the classical sense with a product continuous on compact sets *is* a K-topological algebra. With this interpretation, however, algebraically isomorphic topological algebras with the same compact subsets are considered equal. Observe that metrizable algebras (in particular, Fréchet algebras, normed algebras) satisfy directly our definition of K topological algebras, since metrizable spaces are in $\boldsymbol{K}$ (see 0.1).

It may be interesting to observe that the K-ation of a topological algebra may fail to be a topological algebra. We have the following:

1.2. Example. Let A be an arbitrary complex vector space of dimension larger than $\aleph_0$, and define a (locally convex, see [22]) topology on A by means of the semi norms $p(a) = |f(a)|$ where f ranges over the set of all linear maps $f: A \to C$. If the product on A is defined by $ab = 0$ for all $a, b \in A$, clearly A is a topological (locally m-convex, see [27]) algebra (observe that the continuity of the product is obvious, and therefore we actually don't need to know that A is locally m-convex). It follows from [22] (p. 53, Ex. H), that the compact subsets of A are finite dimensional, whence the topology of $\mathrm{Ke}A$ can be described by: $O \subset \mathrm{Ke}A$ is open if and only if for every finite dimensional subspace $F \subset A$ it follows that $O \cap F$ is open in F (for the only Hausdorff linear topology of F), i.e., KeA has the *finite topology* of [20]. But then it follows from Th. 1 in [20] that $\mathrm{Ke}A \times \mathrm{Ke}A \xrightarrow{+} \mathrm{Ke}A$ is not continuous, and therefore KeA is *not* a topological algebra (although its product *is* continuous).

Observe that, as in the example above, we can always make a K-topological vector space (namely, an object $X \in K$ together with 1.1.1 and 1.1.3 of Def. 1.1 above) into a K-topological algebra by defining the product ($ab = 0$) as $(X \boxtimes X \to X) = (X \boxtimes X \to 1 \xrightarrow{0} X)$.

The concept of K-topological algebra includes many types of algebras which fail to be topological algebras, and which have been studied under the somewhat artificial concept of topological algebra with partially continuous operations. The following examples also show that K-topological algebras abound in traditional fields such as von Neumann algebras and convolution algebras. The text resumes on page 298.

1.3. Example. Let $\mathcal{H}$ be a complex Hilbert space with inner product $(x \mid y)$ and norm $\|x\| = (x \mid x)^{1/2}$. We denote by $\mathfrak{B}(\mathcal{H})$ the set of all linear bounded operators $T: \mathcal{H} \to \mathcal{H}$. If $T \in \mathfrak{B}(\mathcal{H})$ we denote $\|T\| = \mathrm{Sup}\{\|Tx\|; x \in \mathcal{H}, \|x\| \leq 1\}$, and $T^* \in \mathfrak{B}(\mathcal{H})$ will be the adjoint of T, characterized by the identity $(Tx \mid y) = (x \mid T^*y)$ for all $x, y \in \mathcal{H}$. We define $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \xrightarrow{+} \mathfrak{B}(\mathcal{H})$, $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \xrightarrow{\cdot} \mathfrak{B}(\mathcal{H})$ and $C \times \mathfrak{B}(\mathcal{H}) \xrightarrow{\cdot} \mathfrak{B}(\mathcal{H})$ by $(S+T)x = Sx + Tx$, $(ST)x = S(Tx)$ and $(\lambda T)x = \lambda Tx$. It is clear that $\|\alpha S + \beta T\| \leq |\alpha|\,\|S\| + |\beta|\,\|T\|$, $\|ST\| \leq \|S\|\,\|T\|$, $\|T^*\| = \|T\|$; $(\alpha S + \beta T)^* = \bar{\alpha} S^* + \bar{\beta} T^*$ and $(ST)^* = T^* S^*$ for all $S, T \in \mathfrak{B}(\mathcal{H})$ and α, β complex ($\bar{\alpha}, \bar{\beta}$ are the conjugates of α, β). In particular $\mathfrak{B}(\mathcal{H})$ is an *algebra over the complex numbers with an involution* $T \rightsquigarrow T^*$. If $\dim \mathcal{H} = n < +\infty$ then $\mathfrak{B}(\mathcal{H})$ is isomorphic to the algebra of $n \times n$ complex matrices. For general facts concerning Hilbert spaces, we refer to [8] or [11].

We will consider now several topologies on $\mathfrak{B}(\mathcal{H})$ that have been extensively used in von Neumann algebras (cf. [8]).

The *uniform topology* on $\mathfrak{B}(\mathcal{H})$ is the topology induced by the norm $\|T\| = \mathrm{Sup}\{\|Tx\|; x \in \mathcal{H}, \|x\| \leq 1\}$. With this norm, $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra with identity $I = \mathrm{id}_{\mathcal{H}}$ (cf. [9]).

The *strong topology* on $\mathfrak{B}(\mathcal{H})$ is determined by the following notion of convergence: if $\{T_\alpha\}_{\alpha \in A}$ is a net in $\mathfrak{B}(\mathcal{H})$ and $T \in \mathfrak{B}(\mathcal{H})$, then $T_\alpha \to T$ in the strong topology (or *strongly*) if for each $x \in \mathcal{H}$, we have $\|T_\alpha x - Tx\| \to 0$ with $\alpha \in A$. If $S_\alpha \to S$, $T_\beta \to T$ strongly, it is clear that $\|(S_\alpha + T_\beta)x - (S+T)x\| \leq \|(S_\alpha - S)x\| + \|(T_\beta - T)x\| \to 0$ so that $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \xrightarrow{+} \mathfrak{B}(\mathcal{H})$ is continuous for this topology. Also, if $\lambda_\gamma \to \lambda$ in C (= complex numbers), the $\|\lambda_\gamma T_\beta x - \lambda T x\| \leq \|T(\lambda - \lambda)x\| + |\lambda_\gamma|\,\|T_\beta x - Tx\| \to 0$ and $C \times \mathfrak{B}(\mathcal{H})$ is also strongly continuous.

The *ultrastrong topology* on $\mathfrak{B}(\mathcal{H})$ is determined by the following notion of convergence. Let $X = \{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$ with the property

$$(*) \qquad \sum_{k=1}^{\infty} \|x_k\|^2 < +\infty$$

and for $T \in \mathfrak{B}(\mathcal{H})$ define $p_X(T) = [\Sigma_{k=1}^{\infty} \|Tx_k\|^2]^{1/2}$ (observe that $\Sigma_{k=1}^{\infty} \|Tx_k\|^2 \leqslant \|T\|^2 \; \Sigma_{k=1}^{\infty} \|x_k\|^2$ is finite). A net $\{T_\alpha\}_{\alpha \in A}$ in $\mathfrak{B}(\mathcal{H})$ is convergent to $T \in \mathfrak{B}(\mathcal{H})$ in the ultrastrong topology (or *ultrastrongly convergent*) if $p_X(T_\alpha - T) \to 0$ for each $X = \{x_k\}_{k=1}^{\infty}$ satisfying (*). Since it can be easily verified that $p_X(S+T) \leqslant p_X(S) + p_X(T)$ and $p_X(\lambda T) = |\lambda| \, p_X(T)$, it follows that $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \stackrel{+}{\to} \mathfrak{B}(\mathcal{H})$ and $C \times \mathfrak{B}(\mathcal{H}) \stackrel{\cdot}{\to} \mathfrak{B}(\mathcal{H})$ are ultrastrongly continuous.

The *weak topology* on $\mathfrak{B}(\mathcal{H})$ is determined by the following notions of convergence: a set $\{T_\alpha\}_{\alpha \in A}$ converges in the weak topology (or *weakly*) to T if $(T_\alpha x | y) \to (Tx | y)$ for all $x, y \in \mathcal{H}$. Again if $S_\alpha \to S$, $T_\beta \to T$ weakly, then $((S_\alpha + T_\beta)x | y) = (S_\alpha x | y) + (T_\beta x | y)$ converges to $(Sx | y) + (Tx | y) = ((S+T)x | y)$ and if $\lambda_\gamma \to \lambda$ in C, then $|(\lambda_\gamma T_\alpha x | y) - (\lambda T x | y)| \to 0$; thus $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H})$ and $C \times \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ are weakly continuous.

The *ultraweak topology* on $\mathfrak{B}(\mathcal{H})$ is determined by the following notion of convergence. If $X = \{x_k\}_{k=1}^{\infty}$ and $Y = \{y_k\}_{k=1}^{\infty}$ satisfy (*) above and $T \in \mathfrak{B}(\mathcal{H})$, set $p_{X,Y}(T) = \Sigma_{k=1}^{\infty} |(Tx_k | y_k)|$ (observe that $\Sigma_{k=1}^{\infty} |(Tx_k | y_k)| \leqslant \|T\| \; \Sigma_{k=1}^{\infty} \|x_k\| \|y_k\| \leqslant \|T\| \, (\Sigma_{k=1}^{\infty} \|x_k\|^2)^{1/2} (\Sigma_{k=1}^{\infty} \|y_k\|^2)^{1/2} < +\infty$). Then, a set $\{T_\alpha\}_{\alpha \in A}$ converges in the ultraweak topology (or ultraweakly) to T if $p_{X,Y}(T_\alpha - T) \to 0$ for all sequences X, Y satisfying (*). We have again $p_{X,Y}(S+T) \leqslant p_{X,Y}(S) + p_{X,Y}(T)$ and $p_{X,Y}(\lambda T) = |\lambda| \, p_{X,Y}(T)$ and therefore $\mathfrak{B}(\mathcal{H}) \times \mathfrak{B}(\mathcal{H}) \stackrel{+}{\to} \mathfrak{B}(\mathcal{H})$ and $C \times \mathfrak{B}(\mathcal{H}) \stackrel{\cdot}{\to} \mathfrak{B}(\mathcal{H})$ are ultraweakly continuous.

It is easy to see that the uniform topology is the strongest and the weak topology the weakest, and in fact we have:

$$\text{uniform} \to \text{ultrastrong} \begin{matrix} \nearrow \text{strong} \searrow \\ \searrow \text{ultraweak} \nearrow \end{matrix} \text{weak} .$$

If $\dim H = \infty$, all these topologies are different [8]; if $\dim H < \infty$, they all coincide.

A *-subalgebra A of $\mathfrak{B}(\mathcal{H})$ (that is, a subset $A \subset \mathfrak{B}(\mathcal{H})$ such that whenever $S, T \in A$, and $\lambda \in C$ we have $S \pm T \in A$, $\lambda T \in A$, $T^* \in A$ and $I \in A$, where $I = \mathrm{id}_{\mathcal{H}}$) is by definition a *von Neumann algebra* provided it is closed in the weak topology. As a matter of fact, it can be proved (see [8] I.3.4, Th. 2) that a *-subalgebra of $\mathfrak{B}(\mathcal{H})$ is weakly closed if and only if it is closed in either the ultraweak, strong or ultrastrong topologies (but not the uniform topology). Given a von Neumann algebra A, we will denote by A_u, A_s, A_{us}, A_w and A_{uw} the algebra A together with the topologies uniform, strong, ultrastrong, weak and ultraweak, respectively.

Assume now that $K \subset A$ is compact in any of the above topologies. Then, necessarily, K is weakly compact. This implies that for $x, y \in \mathcal{H}$, the function $T \rightsquigarrow (Tx | y)$ is (continuous, hence) bounded on K. It follows from the principle of uniform boundedness [9] (or using the elementary proof in [18]) that $\mathrm{Sup}\,\{\|T\|: T \in K\} < +\infty$, that is to say, K is *norm bounded* in $\mathfrak{B}(\mathcal{H})$. With this remark in mind, we can now prove:

1.3.1. $A \times A \stackrel{\cdot}{\to} A$ *is continuous on compact sets for each of the topologies: uniform, strong and ultrastrong.*

In other words, $S, T \rightsquigarrow ST$ is continuous as a map $\mathrm{Ke}(\mathsf{A}_u \times \mathsf{A}_u) \to \mathsf{A}_u$, $\mathrm{Ke}(\mathsf{A}_s \times \mathsf{A}_s) \to \mathsf{A}_s$ and $\mathrm{Ke}(\mathsf{A}_{us} \times \mathsf{A}_{us}) \to \mathsf{A}_{us}$ and therefore: $\mathrm{Ke}\mathsf{A}_u = \mathsf{A}_u$, $\mathrm{Ke}\mathsf{A}_s$ *and* $\mathrm{Ke}\mathsf{A}_{us}$ *are K-topological algebras.*

Consider the case of A_u. The product is continuous everywhere due to the inequality $\|ST\| \leq \|S\|\,\|T\|$; moreover, this topology is metrizable (being determined by a single norm) and therefore (from 0.1), $\mathrm{Ke}\mathsf{A}_u = \mathsf{A}_u$. Observe now that (cf. [8]):

1.3.2. *If $r > 0$ and $B_r \subset \mathfrak{B}(\mathcal{H})$ is the ball $B_r = \{T \in \mathfrak{B}(\mathcal{H});\ \|T\| \leq r\}$ then the strong and ultrastrong topologies agree on B_r.* This can be seen as follows. Let $\{T_\alpha\}_{\alpha \in A}$ be a net in B_r converging strongly to $T \in B_r$. For $X = \{x_k\}_{k=1}^\infty$ satisfying (*) above we have $\sum_{k=1}^\infty \|(T_\alpha - T)x_k\|^2 = \sum_{k=1}^N \|(T_\alpha - T)x_k\|^2 + \sum_{k=N+1}^\infty \|(T_\alpha - T)x_k\|^2 \leq \sum_{k=1}^N \|(T_\alpha - T)x_k\|^2 + 2r \sum_{k=N+1}^\infty \|x_k\|^2$, so that $\limsup_{\alpha \in A} \sum_{k=1}^\infty \|(T_\alpha - T)x_k\|^2 \leq \limsup_{\alpha \in A} \sum_{k=1}^N \|(T_\alpha - T)x_k\|^2 + 2r \sum_{k=N+1}^\infty \|x_k\|^2$. But $\lim_{\alpha \in A} \sum_{k=1}^N \|(T_\alpha - T)x_k\|^2 = 0$ hence $\limsup_{\alpha \in A} \sum_{k=1}^\infty \|(T_\alpha - T)x_k\|^2 \leq 2r \sum_{k=N+1}^\infty \|x_k\|^2$ for all $N = 1, 2, \ldots$, and this means that $p_X(T_\alpha - T) \to 0$; thus the strong and ultrastrong topologies agree on B_r.

The first consequence of this fact is that it is enough to prove the continuity of the product on strongly compact sets. This is done as follows: let $\{S_\alpha\}_{\alpha \in A}$ and $\{T_\beta\}_{\beta \in B}$ be nets such that $S_\alpha \to S$, $T_\beta \to T$ strongly and $\|S_\alpha\| \leq r$, $\|T_\beta\| \leq r$ for all α, β and some $r > 0$. Assume $X = \{x_k\}_{k=1}^\infty$ satisfies (*) above. Then $p_X(S_\alpha T_\beta - ST) \leq p_X(S_\alpha(T_\beta - T)) + p_X((S_\alpha - S)T)$. Now, for all $U, V \in \mathfrak{B}(\mathcal{H})$ it is easy to see that $p_X(UV) \leq \|U\|\, p_X(V)$ and $p_X(UV) \leq \|V\|\, p_X(U)$. Then $p_X(S_\alpha T_\beta - ST) \leq \|S_\alpha\|\, p_X(T_\beta - T) + \|T\|\, p_X(S_\alpha - S) \leq r p_X(T_\beta - T) + \|T\|\, p_X(S_\alpha - S) \to 0$ with $\alpha \in A$, $\beta \in B$. Therefore, $S, T \rightsquigarrow ST$ is strongly (eq., ultrastrongly) continuous as a map $B_r \times B_r \to B_{r^2}$, and our claim follows.

The second consequence of the agreement on normed bounded sets of the strong and ultrastrong topologies is that they have the same K-ation: for any von Neumann algebra A, $\mathrm{Ke}\mathsf{A}_s = \mathrm{Ke}\mathsf{A}_{us}$; this means that A_s and A_{us} are the same K-topological algebra.

It should be remarked that $\mathrm{Ke}\mathsf{A}_s = \mathrm{Ke}\mathsf{A}_{us}$ is a genuine K-topological algebra in the sense that it is not obtained as the K-ation of a topological algebra with continuous product. In fact:

1.3.3. *If $\mathsf{A} = \mathfrak{B}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$, the product is not continuous as a map $\mathsf{A}_w \times \mathsf{A}_w \to \mathsf{A}_w$, $\mathsf{A}_s \times \mathsf{A}_s \to \mathsf{A}_s$, $\mathsf{A}_{uw} \times \mathsf{A}_{uw} \to \mathsf{A}_{uw}$ or $\mathsf{A}_{us} \times \mathsf{A}_{us} \to \mathsf{A}_{us}$* (cf. [36] or [8] I.3, Ex. 2). This can be seen as follows: Assume $\mathcal{H}$ is separable. If $X = \{x_k\}_{k=1}^\infty$ satisfies (*) above with $c = \sum_{k=1}^\infty \|x_k\|^2$, define $B = \{T \in \mathfrak{B}(\mathcal{H});\ p_X(T) \leq 1\}$. Let now $\{e_l\}_{l=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. For all N, M positive integers we have $\sum_{l=1}^N (\sum_{k=1}^M |(x_k \mid e_l)|^2) \leq \sum_{k=1}^M (\sum_{l=1}^\infty |(x_k \mid e_l)|^2) = \sum_{k=1}^M \|x_k\|^2 \leq c$ and therefore $\sum_{l=1}^\infty (\sum_{k=1}^\infty |(x_k \mid e_l)|^2) \leq c$ which shows that $\lim_{l \to \infty} \sum_{k=1}^\infty |(x_k \mid e_l)|^2 = 0$. Choose $\{d_l\}_{l=1}^\infty$ a sequence of nonnegative reals such that $d_l^2 \sum_{k=1}^\infty |(x_k \mid e_l)|^2 \leq 1$ for $l = 1, 2, \ldots$

and $\lim_{l\to\infty} d_l = \infty$. Let $T_n \in \mathfrak{B}(\mathcal{H})$, $n = 1, 2, \ldots$ be defined by $T_n e_l = \delta_{nl} d_n e_n$ (δ_{nl} = Kronecker's delta). Clearly $\Sigma_{k=1}^{\infty} \| T_k x_n \|^2 = \Sigma_{k=1}^{\infty} \| \Sigma_{l=1}^{\infty} (x_k | e_l) T_n e_l \|^2 = \Sigma_{k=1}^{\infty} |(x_k | e_n)|^2 d_n^2 \leq 1$, i.e., $T_n \in B$ and $\| T_n \| = d_n \to \infty$. This means that $\mathrm{Sup}\{\| T \|; T \in B\} = \infty$, and therefore by the uniform boundedness principle (see [11]) there is a $z \in \mathcal{H}$ such that $\mathrm{Sup}\{\|Tz\|; T \in B\} = \infty$, with (necessarily) $z \neq 0$. Now we will prove that, *for* $x, y \in \mathcal{H}$ *both different from zero, no choice of* $X = \{x_k\}_{k=1}^{\infty}$ *will make true that if* $S, T \in B$ *then* $|(STx | y)| \leq 1$. In fact, if z satisfies

$$(**) \qquad \mathrm{Sup}\{\| Tz \|; T \in B\} = \infty,$$

define $S \in \mathfrak{B}(\mathcal{H})$ by $Su = \lambda(u | x) z$ where $\lambda \in C$ and $|\lambda| > 0$ is small enough in order that $p_X(S) \leq 1$ or $S \in B$. Then choose $T \in B$ with $|\lambda| \, \|y\| \, \|TSx\| = \|y\| \, |\lambda|^2 \|x\|^2 \|Tz\| > 1$ (use (**) above) and $U \in \mathfrak{B}(\mathcal{H})$ unitary and such that $U^* y = \mu TSx$ where $\mu \in C$ satisfies $|\mu| \, \|TSx\| = \|y\|$. Then $UT, S \in B$ and $|(UTSx | y)| = |(TSx | U^* y)| = |\lambda \mu| \, \|TSx\|^2 > 1$, as claimed. This shows that $S, T \rightsquigarrow ST$ is discontinuous at $S = T = 0 \in \mathfrak{B}(\mathcal{H})$ as a map $\mathfrak{B}(\mathcal{H})_{us} \times \mathfrak{B}(\mathcal{H})_{us} \to \mathfrak{B}(\mathcal{H})_w$ which implies that the product is discontinuous in all four the weak, strong, ultraweak, or ultrastrong topologies, as claimed in 1.3.3.

An argument similar to the one given above (1.3.2) shows that $\mathrm{Ke}A_w = \mathrm{Ke}A_{uw}$. But in general, the product is *not* continuous on weakly compact sets, in fact not even sequentially continuous, as the following example shows: take $\mathcal{H} = l^2(\boldsymbol{Z})$ where $\boldsymbol{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the integer and let $U \in \mathfrak{B}(\mathcal{H})$ be the shift defined by $U\{c_k\} = \{d_k\}$ where $d_k = c_{k+1}$, $k = 0, \pm 1, \pm 2, \ldots$. Then $U^n \to 0$, $U^{-n} \to 0$ weakly as $n \to \infty$. However, $U^n U^{-n}$ = Identity.

We see then that some weak von Neumann algebras are K-topological algebras (the finite dimensional algebras, for instance) while some others ($\mathfrak{B}(\mathcal{H})_w$, $\dim H = \infty$) are not. The following important algebra is another example of the former, as we shall see. Let $\boldsymbol{S}$ be a (fixed) set, $\mathcal{H} = l^2(\boldsymbol{S})$ the Hilbert space of all complex functions $x: \boldsymbol{S} \to C$ such that $\Sigma\{|x(s)|^2, s \in \boldsymbol{S}\} < +\infty$ (with operations defined pointwise and inner product $(x | y) = \Sigma\{x(s)\overline{y(s)}; s \in \boldsymbol{S}\}$). Let $l^\infty(\boldsymbol{S}) \subset \mathfrak{B}(\mathcal{H})$ be the algebra consisting of all the operators on $l^2(\boldsymbol{S})$ expressible as $Tx = dx$ for some $d: \boldsymbol{S} \to C$ satisfying $\mathrm{Sup}\{|d(s)|; s \in \boldsymbol{S}\} < +\infty$ and where dx is the function $(dx)(s) = d(s)x(s)$. It is not hard to see (cf. [8] I.7) that $l^\infty(\boldsymbol{S})$ is a von Neumann algebra which is ***Abelian*** (i.e. $ST = TS$ for $S, T \in l^\infty(\boldsymbol{S})$). We shall identify $T \in l^\infty(\boldsymbol{S})$ with $d: \boldsymbol{S} \to C$ when $Tx = dx$ for all $x \in l^2(\boldsymbol{S})$. It is clear that $\| T \| = \mathrm{Sup}\{|d(s)|; s \in \boldsymbol{S}\}$, $T^* = \bar{d}$ the conjugate function, and $(dd')(s) = d(s) d'(s)$. It follows from ([8] III.6, Prop. 5, 6 and 7) that $l^\infty(\boldsymbol{S})_w = l^\infty(\boldsymbol{S})_{uw}$ and $l^\infty(\boldsymbol{S})_s = l^\infty(\boldsymbol{S})_{us}$ (in fact, with the notation of [8], $C_A = 1$ when $A = l^\infty(\boldsymbol{S})$). In other words, the weak (resp., strong) topology coincides on $l^\infty(\boldsymbol{S})$ with the ultraweak (resp., ultrastrong). Actually one can prove in a very elementary way that the weak and ultraweak topologies on $l^\infty(\boldsymbol{S})$ both agree with the weak * topology of $l^\infty(\boldsymbol{S})$ as the Banach dual of $l^1(\boldsymbol{S})$ (cf. [11]), which means that on $l^\infty(\boldsymbol{S})$ the weak or ultraweak convergence is determined by the seminorms $x \rightsquigarrow |\Sigma\{x(s)z(s); s \in \boldsymbol{S}\}|$ where $z \in l^1(\boldsymbol{S})$, i.e., $\Sigma\{|z(s)|; s \in \boldsymbol{S}\} < +\infty$. With this in mind, one can prove independently of w = us, s = us, that

1.3.4. *If $K \subset l^\infty(S)$, the following are equivalent:*

(i) *K is strongly (= ultrastrongly) compact,*

(ii) *K is ultraweakly compact,*

(iii) *K is strongly closed and norm bounded,*

(iv) *K is ultraweakly closed and norm bounded,*

(v) *K is weakly closed and norm bounded.*

Moreover, if K satisfies any of the above, the weak, ultraweak, strong (and ultrastrong) topologies coincide on K.

We will indicate a proof of 1.3.4 based on the plan: (iii) $\Leftrightarrow$ (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iv) and (ii) $\Rightarrow$ (v) $\Rightarrow$ (i). The following remarks are elementary: first, the equivalence (ii) $\Leftrightarrow$ (iv) is the Alaoglu-Bourbaki Theorem [11]; second, we observed above (last paragraph before 1.3.1) that weakly (hence ultraweakly, strongly or ultrastrongly) compact implies norm bounded, whence (i) $\Rightarrow$ (iii). On normed bounded sets, the strong and ultrastrong topologies agree (1.3.2 above) and therefore the strong topology is stronger on these than the ultraweak, which proves (i) $\Rightarrow$ (ii). For similar reasons, (ii) $\Rightarrow$ (v). The implications (iii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i), together with the second half of 1.3.4 follow from the following: if $\{d_\alpha\}_{\alpha\in A}$ is a net, $\|d_\alpha\| \leqslant r$ and $d_\alpha(s) \to 0$ for each $s\in S$, then $d_\alpha \to 0$ strongly. In fact, for each $x \in l^2(S)$ and a finite $F\subset S$, we have $\|d_\alpha x\|^2 \leqslant \Sigma_{s\in F} |d_\alpha(s)\, x(s)|^2 + r\Sigma_{s\notin F} |x(s)|^2$, whence $\limsup_{\alpha\in A} \|d_\alpha x\|^2 \leqslant \Sigma_{s\notin F} |x(s)|^2$ and since F is arbitrary, $\lim_{\alpha\in A} \|d_\alpha x\|^2 = 0$ as claimed.

Another way of writing 1.3.4 is the following:

1.3.5. *For any set S,*

$$\mathrm{K}e\, l^\infty(S)_\mathrm{w} = \mathrm{K}e\, l^\infty(S)_\mathrm{uw} = \mathrm{K}e\, l^\infty(S)_\mathrm{s} = \mathrm{K}e\, l^\infty(S)_\mathrm{us}\,.$$

It is interesting to observe that *the product $S, T \rightsquigarrow ST$ is not weakly (= ultraweakly) or strongly (= ultrastrongly) continuous, even when restricted to $l^\infty(S)$,* when S is infinite. In fact, we can assume that $S \supset \{1, 2, \ldots\} = N$. Let $h\in l^2(S)$ be defined by $h(n) = 1/n$, $n\in N$, $h = 0$ elsewhere. Consider $V = \{d\in l^\infty(S);\ \|dh\| \leqslant 1\}$. V is a strong neighborhood of $0\in l^\infty(S)$. We will see that for *no* strong neighborhood W of $0 \in l^\infty(S)$ it will be true that $d^2 \in V$ if $d\in W$. Clearly, W can be assumed to be of the form $W = \{d;\ \Sigma_{k=1}^\infty \|dx_k\|^2 < \epsilon\}$ for some sequence $\{x_k\}_{k=1}^\infty$ in l^2 satisfying ((*) above): $\Sigma\, \|x_k\|^2 < +\infty$. Since $\Sigma_{k,n=1}^\infty |x_k(n)|^2 < \infty$ we conclude that $\{\Sigma_{k=1}^\infty |x_k(n)|^2\}_{n=1}^\infty$ is summable (this is the same argument used in the proof of 1.3.3). Therefore for some positive integer m we have $\epsilon^{-1}\,\Sigma_{k=1}^\infty |x_k(n)|^2 < m^{-1}$. Choose λ real such that $\epsilon^{-1}\,\Sigma_{k=1}^\infty |x_k(n)|^2 < \lambda^{-2} < m^{-1}$ and define $d_0\in l^\infty(S)$ by $d_0(m) = \lambda$, $d_0 = 0$ elsewhere. Then $\Sigma_{k=1}^\infty \|d_0 x_k\|^2 = \Sigma_{k=1}^\infty \lambda^2 |x_k(n)|^2 = \lambda^2\, \Sigma_{k=1}^\infty |x_k(n)|^2 < \epsilon$ and $\|d_0^2 h\| = \lambda^2 m^{-1} > 1$ so that $d_0 \in W$ and $d_0^2 \notin V$, as claimed. This shows that $d \rightsquigarrow d^2$ is not continuous as a map $l^\infty(S)_\mathrm{us} \to l^\infty(S)_\mathrm{s}$. A similar argument shows that $d \rightsquigarrow d^2$ is not continuous as a map $l^\infty(S)_\mathrm{uw} \to l^\infty(S)_\mathrm{w}$, and therefore the product in $l^\infty(S)$ is not continuous for any of the topologies weak, ultraweak, strong, ultrastrong.

Actually, this proof yields a bit more: if $P = \{d \in l^\infty(S),\ d(s) \geq 0\}$ and P_w ($= P_{uw}$), P_s ($= P_{us}$) denote P with the relative weak (= ultraweak) and strong (= ultrastrong) topologies, respectively and $\varphi: P \to P$ is the map $\varphi(d) = d^2$, then we just saw the proof of the first half of:

1.3.6. *If S is infinite:*
(i) *φ is not continuous as a map $P_w \to P_w$, $P_s \to P_s$ ($P_{uw} \to P_{uw}$, $P_{us} \to P_{us}$)*
(ii) *$P_s \overset{\varphi}{\to} P_w = P_{us} \overset{\varphi}{\to} P_{uw}$ is a homeomorphism.*

It is clear that 1.3.6 (ii) follows from $\Sigma d^2(s)|x(s)y(s)| = \Sigma(d(s)|x(s)y(s)|^{1/2})^2$ for $x, y \in l^2(S)$.

Observe that if we assume that $\mathrm{Ke}P_w = P_w$, then $P_s \overset{\varphi}{\to} P_w = \mathrm{Ke}P_w = \mathrm{Ke}P_s \overset{\mathrm{id}}{\to} P_s$ would be continuous contradicting 1.3.6 (i) when S is infinite. Thus:

1.3.7. *If S is infinite, then* $\mathrm{Ke}\, l^\infty(S)_w \neq l^\infty(S)_w$.

In fact, it can be proved that when S is countable, $\mathrm{Ke}\, l^\infty(S)_w$ coincides with the relative product topology of $l^\infty(S) \subset \prod_S C$. This completes our Example 1.3.

1.4. Example. For general results concerning topological groups we refer the reader to [2] and [3]. Let G be a Hausdorff locally compact group and denote by $U(G)$ the algebra of all bounded uniformly continuous functions on G with values in the complex numbers C. The norm $\|f\|_\infty = \mathrm{Sup}\{|f(s)|; s \in G\}$ makes $U(G)$ a Banach algebra (in fact a C^*-algebra since it is easy to see that $U(G)$ is a closed $*$-subalgebra of the C^*-algebra $B(G)$ of all bounded complex valued function on G with the norm $\|f\|_\infty$). For $s \in G$ and $f \in U(G)$, the function $\gamma(s)f$ is defined by $[\gamma(s)f](t) = f(s^{-1}t)$. Clearly $\gamma(s)f \in U(G)$ and $\|\gamma(s)f\|_\infty = \|f\|_\infty$. Thus G acts (isometrically) on $U(G)$ by $s \rightsquigarrow \gamma(s)$. If $S \subset U(G)$ is any subset, we denote by $\Lambda(S)$ the linear subspace of $U(G)$ spanned by S. If $f \in U(G)$, then $[f] \subset U(G)$ denotes the subspace $[f] = \Lambda\{\gamma(f)f; s \in G\}$ generated by the (left) translates of f. We shall abbreviate $\dim[f] = \dim_C[f]$.

1.4.1. $\dim[f]$ *is finite for all $f \in U(G)$ if and only if G is finite.*

Proof. It is clear that if G is finite, $\dim_C U(G) = \mathrm{card}(G) < +\infty$ and therefore $\dim[f] \leq \mathrm{card}(G)$ for all $f \in U(G)$.

Assume now that $\dim[f] < +\infty$ for all $f \in U(G)$ and define $U_n = \{f \in U(G); \dim[f] \leq n\}$ for $n = 1, 2, \ldots$. Clearly $U(G) = \bigcup_{n=1}^\infty U_n$. We claim that each U_n is closed in $U(G)$. In fact, assume $f_j \to f$ (as $j \to +\infty$) in $U(G)$ and $f_j \in U_n$ for $j = 1, 2, \ldots$. Then for any choice of $n+1$ elements $s_0, s_1, \ldots, s_n$ of G the functions $\gamma(s_0)f_j, \ldots, \gamma(s_n)f_j$ are linearly dependent and therefore there are complex numbers $\alpha_0^j, \alpha_1^j, \ldots, \alpha_n^j$ such that $\Sigma_{k=0}^n \alpha_k^j \gamma(s_k)f_j = 0$ and $\Sigma_{k=0}^n |\alpha_k^j| = 1$ for each $j = 1, 2, \ldots$. By passing to an appropriate subsequence we can assume that $\alpha_k^j \to \alpha_k$ as $j \to \infty$ for each $k = 0, 1, \ldots, n$ and therefore $\Sigma|\alpha_k| = 1$ also. But clearly from $f_j \to f$ and $\alpha_k^j \to \alpha_k$

(as $j \to \infty$) we obtain $\alpha_k^j \gamma(s_k) f_j \to \alpha_k \gamma(s_k) f$ in $U(G)$ and therefore $\Sigma_{k=0}^{n} \alpha_k \gamma(s_k) f = \lim_{j \to \infty} \Sigma_{k=0}^{n} \alpha_k^j \gamma(s_k) f_j = 0$ which shows that any $n+1$ translates of f are linearly dependent or in other words, that $f \in U_n$. (Observe that we have actually proved that U_n is closed in $U(G)$ for *any* linear, translation invariant topology on $U(G)$.) Since $U(G)$ under $\|f\|_\infty$ is a complete metric space, from Baire's Theorem (see [21]) follows that there exist $n_0, f_0 \in U_{n_0}$ and $\epsilon > 0$ such that if $h \in U(G)$ and $\|h\|_\infty \leqslant \epsilon$, then $h - f_0 \in U_{n_0}$, i.e., $\dim[h - f_0] \leqslant n_0$. Observe now that for $f, f' \in U(G)$, λ a nonzero scalar, always $\dim[f+f'] \leqslant \dim[f] + \dim[f']$ and $\dim[\lambda f] = \dim[f]$. Hence, for $f \in U(G)$ and λ small enough (so that $\lambda \|f\|_\infty < \epsilon$) we have $\dim[f] = \dim[\lambda f] \leqslant \dim[\lambda f - f_0] + \dim[f_0] \leqslant 2n_0$. Thus $U(F) = U_{2n_0}$. But now if $s_1, \ldots, s_m$ are distinct elements of G, there is a compact neighborhood of V of the identity of G such that the sets $s_j V$, $j = 1, \ldots, m$ are pairwise disjoint. Let $f \in U(G)$ be a function with support in $s_1 V$ and satisfying $f(s_1) = 1$. Clearly $\dim[f] \geqslant m$. It follows that $m \leqslant 2n_0$ and therefore G cannot have more than $2n_0$ different elements, or, G is finite.

We recall that if f is a function on a locally compact group G and μ is a measure on G, the *convolution* $\mu * f$ is the function

$$(\mu * f)(x) = \int f(s^{-1}x)\, d\mu(s)$$

and in particular, if ϵ_t is the point mass measure at $t \in G$ with total mass $+1$, then

$$(\dagger) \qquad \epsilon_t * f = \gamma(t) f;$$

if μ and ν are measures on G, $\mu * \nu$ is the measure satisfying

$$\int f(x)\, d(\mu * \nu)(x) = \iint_{G \times G} f(sx)\, d\mu(s)\, d\nu(x)$$

for each f, say, continuous with compact support. Fubini's Theorem applies to show that

$$(\dagger\dagger) \qquad (\mu * \nu)(f) = \nu(\check{\mu} * f)$$

where for any function h and measure β, we write $\beta(h) = \int h(x)\, d\beta(x)$ and $\check{\beta}(h) = \int h(x^{-1})\, d\beta(x)$. A table of sufficient conditions for the existence of $\mu * f$ and $\mu * \nu$ can be found in the last page of [3]. One of these is the following: *if μ and ν are bounded then $\mu * \nu$ exists and is bounded* (we recall that a measure β is *bounded* if

$$\|\beta\| = \mathrm{Sup}\{|\beta(f)| : f \in K(G),\ |f(s)| \leqslant 1 \text{ for all } s \in G\}$$

is finite: $\|\beta\| < +\infty$, where we denote by $K(G)$ the space of continuous functions with compact support). It is not hard to prove that $\mu * \nu$ has desirable properties and in particular that the set of bounded measures under ordinary sum and convolution is an algebra which we will denote by $M^1(G)$. The *vague topology* (denoted T_1) on $M^1(G)$ is the topology corresponding to the simple convergence on $K(G)$, i.e., $\mu_\alpha \to \mu$ *vaguely* if $\mu_\alpha(f) \to \mu(f)$ for each $f \in K(G)$. If G is compact then all measures are bounded and $M^1(G)$ can be identified with the dual of the Banach space $K(G)$ under the norm $\|f\| = \text{Sup}\{|f(s)|\, s\in G\}$, the vague topology coinciding with the w^*-topology. In particular, the vaguely relatively compact sets coincide with the norm-bounded sets of $M^1(G)$. Denote by M_I the algebra $M^1(G)$ endowed with the vague topology T_I.

1.4.2. *Let G be a compact group. The convolution $M_I \times M_I \xrightarrow{*} M_I$ is continuous if and only if G is finite.*

Proof. If G is finite, $M^1(G)$ is finite dimensional and therefore any bilinear map $M_I \times M_I \to M_I$ is continuous. Assume now that G is infinite. First, let us observe that a neighborhood base of $0\in M_I$ for the vague topology is provided by the sets $V = \{\mu\in M_I;\, |\mu(f_j)| \leqslant \epsilon, j = 1, 2, ..., n\}$ where $\epsilon > 0$ and $f_1, f_2, ..., f_n \in K(G)$. Let now $f\in K(G)$ be such that $\dim[f] = \infty$ (cf. 1.4.1 above). We are going to show that for any choice of functions $f_1, ..., f_n, g_1, ..., g_m$ in $K(G)$, there are measures μ, ν such that $\mu(f_j) = 0, \nu(g_k) = 0$ for all $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant m$ and yet $(\mu*\nu)(f) \neq 0$. This will show that $\mu, \nu \rightsquigarrow \mu*\nu$ is not continuous at 0. According to (+) above, $\{\mu*f, \mu\in M_I\} \supset [\gamma(s)f; s\in G]$ and in particular the linear map $\mu \rightsquigarrow \mu*f$, $M_I \to K(G)$ has infinite dimensional range. Clearly the subspace $N \subset M_I$ of all measures satisfying $\mu(f_j) = 0$, $j = 1, 2, ..., n$ has finite codimension in M_I, and therefore the linear map $\mu \rightsquigarrow \mu*f$ restricted to $N \to K(G)$ also has infinite dimensional range, which we denote by $R \subset K(G)$. Hence the subspace $[g_1, ..., g_m]$ generated by $g_1, ..., g_m$ can not contain R. It follows then from the Hahn-Banach Theorem [11] that there is an element $\nu \in (K(G))' = M_I$ vanishing on $[g_1, ..., g_m]$ and such that $\nu(h) \neq 0$ for some $h\in R$. But then necessarily $h = \check{\mu}*f$ for some μ and (cf. (++) above) $(\mu*\nu)(f) = \nu(\check{\mu}*f) \neq 0$, as desired.

1.4.3. *Let G be a compact group. Then the convolution* $\text{Ke}M_I \boxtimes \text{Ke}M_I \xrightarrow{*} \text{Ke}M_I$ *is continuous. In other words,* $(M_I, +, *)$ *is a* ***K-topological algebra.***

Proof. In fact, assume $\mu_\alpha \to 0$, $\nu_\beta \to 0$ vaguely and $\|\mu_\alpha\| \leqslant L$, $\|\nu_\beta\| \leqslant L$ for some L and all α, β. Let $f\in K(G)$ and define $f_\alpha = \check{\mu}_\alpha * f$, or $f_\alpha(x) = \int f(sx)\, d\mu_\alpha(s)$. It is clear that $f_\alpha(x) \to 0$ for each $x\in G$. We shall prove that the family $\{f_\alpha\}$ is equi-uniformly continuous on G. First, for each neighborhood V of the identity $e\in G$, define $V^G = \text{closure} \cup \{xVx^{-1}\}$. Clearly $V \subset V^G$ and V^G is compact. Assume now $z\in V^G$ for all V. Then one can pick $x_V\in G$, $y_V\in V$ such that $x_V y_V x_V^{-1} \in zV$ for each V, for that, in particular, $x_V y_V x_V^{-1} \to z$ following the filter $\{V\}$. G being compact, there is a subnet $\{x_U\}$ of $\{x_V\}$ such that $x_U \to x$ for some $x\in G$. Hence $x_U^{-1} \to x^{-1}$ and since $y_V \in V$, clearly $y_U \to e$. Thus $z = \lim_U x_U y_U x_U^{-1} = xex^{-1} = e$. We conclude that $\cap V^G = \{e\}$. It follows easily that the family $\{V^G\}$ is also a neighborhood base of e. We go back now to the equi-uniform continuity of $\{f_\alpha\}$. Assume $\epsilon > 0$ and choose V such that if $xy^{-1} \in V^G$, then $|f(x) - f(y)| \leqslant \epsilon/L$. This is always possible because f is continuous and G is compact. Assume now that $xy^{-1} \in V$. Then

$|f_\alpha(x) - f_\alpha(y)| \leqslant \int |f(sx) - f(sy)|\, \mathrm{d}\, |\mu_\alpha|(s) \leqslant L\ \mathrm{Sup}\{|f(sx) - f(sy)|; s \in G\}$. But clearly $(sx)(sy)^{-1} = sxy^{-1}s^{-1} \in sVs^{-1} \subset V^G$, so that $\mathrm{Sup}\{|f(sx) - f(sy)|; s \in G\} \leqslant \epsilon/L$ and therefore $|f_\alpha(x) - f_\alpha(y)| \leqslant \epsilon$ as desired. Finally, it is easy to see that if $f_\alpha(x) \to 0$ for each x and the family is equicontinuous, then $f_\alpha \to 0$ uniformly on G. Hence $|\nu_\beta(f_\alpha)| \leqslant \|f_\alpha\|\, \|\nu_\beta\| \leqslant \|f_\alpha\|\, L \to 0$ and $(\mu_\alpha * \nu_\beta)(f) = \nu_\beta(f_\alpha) \to 0$ as desired. Q.E.D.

There are several interesting variations on this theme. For instance, in the case of a locally compact group G, one can consider the following topologies on $M^1(G)$:

- T_{II}. the weak topology of $M^1(G)$ as a Banach space under the norm $\|\beta\|$ defined above.
- T_{III}. the topology for which $\mu_\alpha \to \mu$ if and only if

  $$\int f(x)\, \mathrm{d}\mu_\alpha(x) \to \int f(x)\, \mathrm{d}\mu(x) \qquad \text{for all } f\colon G \to C$$

  continuous and bounded.
- T_{IV} the topology for which $\mu_\alpha \to \mu$ if and only if $\mu_\alpha(U) \to \mu(U)$ for each open set $U \subset G$.

The convolution product is not continuous on either of the above topologies. Yet, it is continuous on compact sets in all cases. In other words, if M_{II}, M_{III} and M_{IV} denotes the algebra $M^1(G)$ with the topologies T_{II}, T_{III} and T_{IV} respectively, then $\mathrm{Ke}M_q$ is a K-topological algebra, $\mathrm{I} \leqslant q \leqslant \mathrm{IV}$ (cf. [3] Chap. VIII, §3, Ex. 11). In fact $\mathrm{Ke}M_{\mathrm{III}} = \mathrm{Ke}M_{\mathrm{IV}}$ although $M_{\mathrm{III}} \neq M_{\mathrm{IV}}$. This completes our Example 1.4.

By a morphism of K-topological algebras we will understand a continuous function which is linear and multiplicative. Specifically:

1.5. Definition. *Given two K-topological algebras A, B, a morphism $A \xrightarrow{\varphi} B$ is a map $|A| \xrightarrow{\varphi} |B|$ in **K** such that the diagrams*

$$\begin{array}{ccc} |A| \boxtimes |A| & \xrightarrow{\varphi \boxtimes \varphi} & |B| \boxtimes |B| \\ \downarrow \cdot \ \downarrow + & & \downarrow \cdot \ \downarrow + \\ |A| & \xrightarrow{\varphi} & |B| \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} C \boxtimes |A| & \xrightarrow{\mathrm{id} \boxtimes \varphi} & C \boxtimes |B| \\ \downarrow \cdot & & \downarrow \cdot \\ |A| & \xrightarrow{\varphi} & |B| \end{array}$$

commute.

The class of K-topological algebras with the above morphisms between them form a category that we will denote $\boldsymbol{A}$. Given $A, B \in \boldsymbol{A}$, $\boldsymbol{A}_0(A, B)$ will denote the set of morphisms from A to B. Clearly $\boldsymbol{A}_0(A, B) \subset \boldsymbol{K}_0(|A|, |B|)$ and we have a functor $\boldsymbol{A} \xrightarrow{|\ |} \boldsymbol{K}$, the *"underlying Kelley space"* functor. If $A \xrightarrow{\varphi} B$ in $\boldsymbol{A}$, then $|\varphi| = \varphi$.

1.6. Proposition. *$\boldsymbol{A}$ is a $\boldsymbol{K}$-category in such a way that $\boldsymbol{A} \xrightarrow{|\ |} \boldsymbol{K}$ is a $\boldsymbol{K}$-functor. Furthermore, $\boldsymbol{A}(A, B) \xrightarrow{|\ |} \boldsymbol{K}(|A|, |B|)$ is a full injection (cf. 0.6).*

Proof. Define $\boldsymbol{A}(A,B)$ to be the K-ation of $\boldsymbol{A}_0(A,B)$ (considered as a subspace of $\boldsymbol{K}(|A|,|B|)$. The proof then is completely straightforward. For example, the composition $\boldsymbol{A}(A,B) \boxtimes \boldsymbol{A}(B,D) \to \boldsymbol{A}(A,D)$ in $\boldsymbol{K}$ is defined in the diagram

$$
\begin{array}{ccc}
\boldsymbol{A}(A,B) \boxtimes \boldsymbol{A}(B,D) & \xrightarrow{\ \gamma\ } & \boldsymbol{A}(A,D) \\
\Big\downarrow {\scriptstyle |\,|\,\boxtimes\,|\,|} & & \Big\downarrow {\scriptstyle |\,|} \\
\boldsymbol{K}(|A|,|B|) \boxtimes \boldsymbol{K}(|B|,|D|) & \xrightarrow{\ k\ } & \boldsymbol{K}(|A|,|D|)\,.
\end{array}
$$

Since k is continuous (see 0.13), it follows that γ is continuous (use the fact that $\boldsymbol{A}(A,D) \overset{|\,|}{\to} \boldsymbol{K}(|A|,|D|)$ is a full injection), that is, $\gamma \in \boldsymbol{K}$. The commutativity of the diagram above is precisely one of the conditions of $\boldsymbol{K}$-functoriality. Etc. ... Q.E.D.

Observe that $\boldsymbol{A}_0(A,B)$ is a closed subset of $\boldsymbol{K}(|A|,|B|)$, and hence, the topology of $\boldsymbol{A}(A,B)$ is actually the relative topology (see 0.4).

An identity for the product in a K-topological algebra is a map $1 \overset{e}{\to} |A|$ *in* $\boldsymbol{K}$ *such that the diagrams:*

$$
\begin{array}{ccccc}
|A| \boxtimes |A| & \xleftarrow{\ \mathrm{id} \boxtimes e\ } & |A| \boxtimes 1 = |A| = 1 \boxtimes |A| & \xrightarrow{\ e \boxtimes \mathrm{id}\ } & |A| \boxtimes |A| \\
& \searrow & \Big\downarrow {\scriptstyle \mathrm{id}} & \swarrow & \\
& & |A| & &
\end{array}
$$

commutes. If the product of A has a unit, we will say that A is an *algebra with identity*. Given two algebras with identity, a morphism in $\boldsymbol{A}$ which preserves the identity in the sense that the diagram

$$
\begin{array}{ccc}
|A| & \longrightarrow & |B| \\
{\scriptstyle e}\nwarrow & & \nearrow{\scriptstyle e} \\
& 1 &
\end{array}
$$

commutes, will be called a *morphism of algebras with identity*. Algebras with identity and morphisms of algebra with identity form a (not full) subcategory of $\boldsymbol{A}$ that will be denoted by $\boldsymbol{A}'$, and we have $\boldsymbol{A}'_0(A,B) \subset \boldsymbol{A}_0(A,B)$. Proposition 1.6 holds similarly. In general, we have:

1.7. Proposition. *Let* $\boldsymbol{C}$ *be a subcategory of* $\boldsymbol{A}$ *(i.e., a class of K-topological algebras with certain morphisms of K-topological algebras between them, containing all the identities and closed under composition). Then* $\boldsymbol{C}$ *is a* $\boldsymbol{K}$*-category and* $\boldsymbol{C} \overset{|\,|}{\to} \boldsymbol{K}$ *is a* $\boldsymbol{K}$*-functor such that for all* $A,B \in \boldsymbol{C}$, $\boldsymbol{C}(A,B) \overset{|\,|}{\to} \boldsymbol{K}(|A|,|B|)$ *is a full injection.*

Proof. Similar to the proof of Proposition 1.6. Q.E.D.

We deduce from this that some standard classes, for example, commutative K-topological algebras, normed (or Banach) algebras, K-topological algebras with involution and morphisms preserving the involution, C^*-algebras, locally multiplicative K-topological algebras, Fréchet algebras, etc., etc., are all K-categories.

§ 2. Categorical properties of $\mathbf{A}$ and $\mathbf{A}'$

In this section we will show that $\mathbf{A}$ and $\mathbf{A}'$ are K-complete K-categories (cf. [10]). This property furnishes the basic (and only!) tool needed for the duality theory developed in § 3. A second important property to be established is the existence of the free K-topological algebra over a Kelley space. Furthermore, we will show that $\mathbf{A}$ and $\mathbf{A}'$ are K-monadic (or synonymously, K-tripleable) over K, and that both $\mathbf{A}$ and $\mathbf{A}'$ are also K-cocomplete. These facts will be exploited later on.

2.1. Proposition. *$\mathbf{A}$ and $\mathbf{A}'$ are cotensored K-categories. Furthermore, the "underlying Kelley space" K-functors preserve cotensors (strictly).*

Proof. The above statement just means that all the representable functors $\mathbf{A}^{\mathrm{op}} \xrightarrow{\mathbf{A}(-,A)} \mathbf{K}$ have a K-left adjoint. In order to prove it, it will be enough to show that for all $A \in \mathbf{A}$ and $X \in \mathbf{K}$, the cotensor of A with X exists, or in other words, there is an object $\bar{\mathbf{A}}(X, A) \in \mathbf{A}$ and a K-natural isomorphism

$$\mathbf{A}(-, \bar{\mathbf{A}}(X, A)) \overset{\sigma}{\approx} \mathbf{K}(X, \mathbf{A}(-, A)).$$

Define $|\bar{\mathbf{A}}(X, A)| = \mathbf{K}(X, |A|)$ with operations:

$$\mathbf{K}(X, |A|) \boxtimes \mathbf{K}(X, |A|) \approx \mathbf{K}(X, |A| \boxtimes |A|) \underset{\mathbf{K}(X, \text{"}\cdot\text{"})}{\overset{\mathbf{K}(X, \text{"}+\text{"})}{\rightrightarrows}} \mathbf{K}(X, |A|),$$

$$\mathbf{K}(X, |A|) \xrightarrow{\mathbf{K}(X, \text{"}-\text{"})} \mathbf{K}(X, |A|),$$

$$\cfrac{C \boxtimes \mathbf{K}(X, |A|) \longrightarrow \mathbf{K}(X |A|)}{\left(\cfrac{C \to \mathbf{K}(|A|, |A|)}{C \boxtimes |A| \to |A|}\ \omega_0\right) \xrightarrow{\mathbf{K}(X, -)} \mathbf{K}(\mathbf{K}(X, |A|), \mathbf{K}(X, |A|))}\ \omega_0 .$$

and

$$\cfrac{1 \to \mathbf{K}(X, |A|)}{1 \boxtimes X \to 1 \xrightarrow{0} |A|}\ \omega_0 .$$

A routine diagram decomposition process shows that these operations make all diagrams in Definition 1.1 commutative, and therefore $\bar{A}(X, A) \in \mathbf{A}$. It can be checked that the above definitions produce the standard point-wise operations on functions. The advantage of this presentation resides not in the algebraic properties to be checked, but in the fact that the continuity is automatically guaranteed.

Given any other $B \in \mathbf{A}$, consider the diagram

$$\begin{array}{ccc} \mathbf{A}(B, \bar{A}(X, A)) & & \mathbf{K}(X, \mathbf{A}(B, A)) \\ \downarrow{\scriptstyle |\ |} & & \downarrow{\scriptstyle \mathbf{K}(X, |\ |)} \\ \mathbf{K}(|B|, \mathbf{K}(X, |A|)) & \overset{\sigma}{\approx} & \mathbf{K}(X, \mathbf{K}(|B|, |A|)) \end{array} .$$

It is easy to see that the isomorphism σ restricted to the upper level provides a bijection. Since both vertical arrows are full injections (see 0.14), this bijection is bicontinuous, i.e., an isomorphism in **K**. The **K**-naturality follows now from the **K**-naturality in the lower level and the fact that $|\ |$ is a **K**-faithful **K**-functor. Finally, the commutativity of the diagram above [completed with $\mathbf{A}(B, \bar{A}(X, A)) \to \mathbf{K}(X, \mathbf{A}(B, A))$] means that $|\ |$ preserves (strictly) the cotensor just constructed.

For **A'** we manipulate similarly: if $A \in \mathbf{A}'$, define $\bar{A}'(X, A) = \bar{A}(X, A)$ with the identity

$$\frac{1 \boxtimes X \to 1 \xrightarrow{e} |A|}{1 \to \mathbf{K}(X, |A|)} \quad \omega_0 .$$

The proof follows the same lines as in the case of **A**. Q.E.D.

2.2. Proposition.

(a) *Given any functor* $\Gamma \xrightarrow{\Gamma} \mathbf{A}$ *such that the composite* $\Gamma \xrightarrow{\Gamma} \mathbf{A} \xrightarrow{|\ |} \mathbf{K}$ *has a limit* ($= \varprojlim$), *then* $\Gamma \xrightarrow{\Gamma} \mathbf{A}$ *has also a limit which is strictly preserved by* $\mathbf{A} \xrightarrow{|\ |} \mathbf{K}$. *Furthermore, given any* $A \in \mathbf{A}$, *the limit of* Γ *is also preserved under* $\mathbf{A} \xrightarrow{\bar{A}(A, -)} \mathbf{K}$.

(b) *Similar to* (a) *with* **A** *replaced by* **A'**

Proof. Define $|\lim \Gamma_\lambda| = \lim |\Gamma_\lambda|$ with operations:

$$\begin{array}{ccc} \varprojlim_\lambda |\Gamma_\lambda| \boxtimes \varprojlim_\lambda |\Gamma_\lambda| & \dashrightarrow & \varprojlim_\lambda |\Gamma_\lambda| \\ \downarrow{\scriptstyle P_\lambda \boxtimes P_\lambda} & & \downarrow{\scriptstyle P_\lambda} \\ |\Gamma_\lambda| \boxtimes |\Gamma_\lambda| & \overset{+}{\underset{\cdot}{\rightrightarrows}} & |\Gamma_\lambda| \end{array} , \qquad \begin{array}{ccc} \varprojlim_\lambda |\Gamma_\lambda| & \dashrightarrow & \varprojlim_\lambda |\Gamma_\lambda| \\ \downarrow{\scriptstyle P_\lambda} & & \downarrow{\scriptstyle P_\lambda} \\ |\Gamma_\lambda| & \xrightarrow{-} & |\Gamma_\lambda| \end{array} ,$$

$$\begin{array}{ccc} C \boxtimes \varprojlim_\lambda |\Gamma_\lambda| & \dashrightarrow & \varprojlim_\lambda |\Gamma_\lambda| \\ \downarrow{\scriptstyle \mathrm{id} \boxtimes P_\lambda} & & \downarrow{\scriptstyle P_\lambda} \\ C \boxtimes |\Gamma_\lambda| & \xrightarrow{\cdot} & |\Gamma_\lambda| \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & \dashrightarrow & \varprojlim_\lambda |\Gamma_\lambda| \\ & {\scriptstyle 0} \searrow \quad \swarrow {\scriptstyle P_\lambda} & \\ & |\Gamma_\lambda| & \end{array} .$$

The fact that for any $\lambda \xrightarrow{f} \mu$ in $\mathbf{\Gamma}$, $|\Gamma_\lambda| \xrightarrow{\Gamma(f)} |\Gamma_\lambda|$ is a morphism of K-topological algebras, is all what it is needed in order to check the commutativities required for the existence of the dotted arrows. It is indeed a straightforward matter that the axioms (Definition 1.1) are satisfied. In order to see that the resulting K-topological algebra is actually the $\varprojlim$ of Γ and that it is preserved by the representable functors, consider the diagram:

$$\begin{array}{ccc} \mathbf{A}(A, \varprojlim_\lambda \Gamma_\lambda) & & \varprojlim_\lambda \mathbf{A}(A, \Gamma_\lambda) \\ \Big\downarrow {\scriptstyle |\;|} & & \Big\downarrow {\scriptstyle \varprojlim_\lambda |\;|} \\ \mathbf{K}(|A|, \varprojlim_\lambda |\Gamma_\lambda|) & \approx & \varprojlim_\lambda \mathbf{K}(|A|, |\Gamma_\lambda|) \end{array} .$$

The homeomorphism at the lower level (which is in fact a homeomorphism because $\mathbf{K}(|A|, -)$ has a left adjoint) induces by restriction a bijection in the upper level which is continuous in both directions because both vertical arrows are full injections (it can be checked easily that a $\varprojlim$ of full injections is a full injection). Similar arguments apply to $\mathbf{A}'$. Q.E.D.

Since $\mathbf{K}$ is a complete category (see 0.7), it follows from the Proposition above that $\mathbf{A}$ and $\mathbf{A}'$ are also complete. The fact that the limits in $\mathbf{A}$ and $\mathbf{A}'$ are preserved by the representables into $\mathbf{K}$ means (see [10] for definitions) that they are ***K**-limits.* This together with Proposition 2.1 amount to saying that ***A** and **A**′ are **K**-complete **K**-categories.* It also follows from the proofs of Prop. 2.1 and 2.2 that the inclusion $\mathbf{A}' \to \mathbf{A}$ is a $\mathbf{K}$-functor which preserves limits and cotensors.

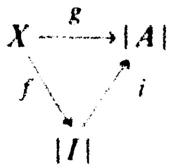
We proceed to prove now some other facts promised at the beginning of this section.

Let X be a Kelley space and A a K-topological algebra. We say that *X generates A (via f)* if there is a continuous function $X \xrightarrow{f} |A|$ (i.e., a morphism in $\mathbf{K}$) such that the set-theoretical image of f algebraically generates A (or, no proper purely algebraic subalgebra of A contains the image of f). *The class of all K-topological algebras generated by any given Kelley space X is a set.* In fact, there is only a set of surjective functions with domain X. For each of them, there is only a set of algebras algebraically generated, and finally, for each of those there is only a set of possible topologies.

A similar definition and conclusion are clear in the case of K-topological algebras with identity.

2.3. Proposition. *The $\mathbf{K}$-functors $\mathbf{A} \xrightarrow{|\;|} \mathbf{K}$ and $\mathbf{A}' \xrightarrow{|\;|} \mathbf{K}$ have left adjoints $\mathbf{K} \xrightarrow{F} \mathbf{A}$ and $\mathbf{K} \xrightarrow{F'} \mathbf{A}'$. Furthermore, F and F' are **K**-functors and **K**-left adjoints.*

Proof. Since $\mathbf{A}$ is well-powered and $\mathbf{A} \xrightarrow{|\;|} \mathbf{K}$ preserves limits, by the Adjoint Functor Theorem [15] it is enough to obtain, for any given $X \in \mathbf{K}$, a solution set. But the set of K-topological algebras generated by X furnishes a solution. In fact, let $X \xrightarrow{g} |A|$ be any map in $\boldsymbol{K}$ and let $I \in \boldsymbol{A}$ be the algebraic subalgebra of A generated by the set-theoretical image of g endowed with the K-ation of the relative topology corresponding to $I \subset A$. It is clear that I is a K-topological algebra and the inclusion $I \xrightarrow{i} A$ is a morphism in $\boldsymbol{A}$. The map $X \xrightarrow{g} |A|$ has a factorization



Since i is a full injection, f is continuous and it is clear that X generates I via f, therefore the set of K-topological algebras generated by X is a solution set, as claimed. Thus, $\boldsymbol{A} \xrightarrow{|\,|} \boldsymbol{K}$ has a left adjoint $\boldsymbol{K} \xrightarrow{F} \boldsymbol{A}$. Since $\boldsymbol{A}$ is cotensored and $|\,|$ preserves cotensors, the last part of 2.3 follows as an application of the criterion given in [23] 4.1, p. 173. The corresponding results for $\boldsymbol{A}' \xrightarrow{|\,|} \boldsymbol{K}$ are obtained in the same way.

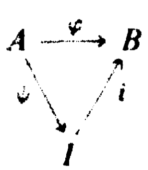
Q.E.D.

A description of FX, $X \in \boldsymbol{K}$ can be given as follows: let $V(X)$ be the free complex vector space over X (which can be pictured as the space of all functions $a: X \to \mathbb{C}$ such that $a(x) \neq 0$ holds only for finitely many $x \in X$). A topology on $V(X)$ is determined by the convergence $a_\alpha \to a$ if and only if for each $A \in \boldsymbol{A}$ and $X \xrightarrow{\varphi} |A|$, φ a morphism in $\boldsymbol{K}$, we have $\Sigma\{a_\alpha(x)\varphi(x); x \in X\} \to \Sigma\{a(x)\varphi(x), x \in X\}$. This topology can be lifted to the tensor algebra $T[V(X)] = V(X) \oplus (V(X) \otimes_C V(X)) \oplus \ldots$ and $F(X)$ = largest Hausdorff quotient of $\mathrm{Ke}T[V(X)]$. Similarly, $F'A$ is an extension of FX by C with trivial action (cf. Proposition 2.10).

2.4. Proposition. *The K-functors* $\boldsymbol{A} \xrightarrow{|\,|} \boldsymbol{K}$ *and* $\boldsymbol{A}' \xrightarrow{|\,|} \boldsymbol{K}$ *are (strictly)* ***K****-monadic. More specifically,* **A** *and* **A**' *are (****K****-isomorphic to) the* ***K****-categories of algebras over the* ***K****-monads determined in* **K** *by the pairs of* ***K****-adjoint functors* $F \dashv_{\boldsymbol{K}} |\,|$ *and* $F' \dashv_{\boldsymbol{K}} |\,|$.

Proof. This result is an easy application of the enriched version of Beck's Tripleability Theorem (cf. [10] Theorem II. 2.1). There is no difficulty in checking the hypotheses for the "underlying Kelley space" $\boldsymbol{K}$-functors $\boldsymbol{A} \xrightarrow{|\,|} \boldsymbol{K}$ and $\boldsymbol{A}' \xrightarrow{|\,|} \boldsymbol{K}$. Q.E.D.

2.5. Remark. *Given any map* $A \xrightarrow{\varphi} B$ *in* **A** *(resp.* **A**'*)* φ *can be factored in* **A** *(resp.* **A**'*)*



where ψ *is a surjective function and* i *is a full injection.*

Proof. Define I to be the set-theoretic image of φ with the K-ation of the relative topology. It is easy to check that $I \in \boldsymbol{A}$ and clearly φ factors as $A \xrightarrow{\psi} I \xrightarrow{i} B$. Finally, ψ is continuous because i is a full injection. Q.E.D.

2.6. Proposition. *The $\boldsymbol{K}$-categories $\boldsymbol{A}$ and $\boldsymbol{A}'$ have all coequalizers.*

Proof. Let $A \xrightarrow{\varphi} B$, $A \xrightarrow{\psi} B$ be any pair of maps in $\boldsymbol{A}$. From 2.5 follows that there is a solution set for the coequalizer of φ and ψ (namely the set of all $B \xrightarrow{d} D$ in $\boldsymbol{A}$ which are surjective functions and such that $d\varphi = d\psi$) and therefore the coequalizer of φ and ψ, $B \xrightarrow{h} H$ does exist. In fact, form the category $\boldsymbol{\Gamma}$ whose objects are maps $B \xrightarrow{d} D$ as above and whose arrows $d \xrightarrow{l} d'$ are maps $D \xrightarrow{l} D'$ in $\boldsymbol{A}$ such that $d' = ld$. $\boldsymbol{\Gamma}$ is a small category and there is a functor $\boldsymbol{\Gamma} \xrightarrow{\Gamma} \boldsymbol{A}$, $\Gamma(B \xrightarrow{d} D) = D$, $\Gamma l = l$. Since $\boldsymbol{A}$ is complete, the (inverse) limit of Γ exists. Thus, H is this limit with $B \xrightarrow{h} H$ defined as follows

$$\begin{array}{ccc} B & \xrightarrow{\exists !\, h} & H \\ & {\scriptstyle d}\searrow \quad \swarrow{\scriptstyle P_d} & \\ & D & \end{array}$$

where P_d is the projection corresponding to $(B \xrightarrow{d} D) \in \boldsymbol{\Gamma}$. The same is done in the case of $\boldsymbol{A}'$. Q.E.D.

Since $\boldsymbol{K}$ is a cocomplete category (see 0.8) and the $\boldsymbol{K}$-functors $\boldsymbol{A} \xrightarrow{|\,|} \boldsymbol{K}$ and $\boldsymbol{A}' \xrightarrow{|\,|} \boldsymbol{K}$ are $\boldsymbol{K}$-monadic, and in particular, monadic (= tripleable), it follows from Prop. 2.6 and a well known result of Linton [24] that $\boldsymbol{A}$ and $\boldsymbol{A}'$ have all (small) colimits. We state this:

2.7. Proposition. *The $\boldsymbol{K}$-categories $\boldsymbol{A}$ and $\boldsymbol{A}'$ have all (small) colimits. Furthermore, they are preserved by the (contravariant) representables into $\boldsymbol{K}$ and therefore they are $\boldsymbol{K}$-colimits.*

Proof. It only remains to be seen that the representables $\boldsymbol{A}^{\mathrm{op}} \xrightarrow{\boldsymbol{A}(-,A)} \boldsymbol{K}$ preserve colimits. But this is clear since $\boldsymbol{A}$ is cotensored and therefore the $\boldsymbol{K}$-functors $\boldsymbol{A}(-,A)$, have left adjoints. Q.E.D.

The statement above reads: given any functor $\boldsymbol{\Gamma} \xrightarrow{\Gamma} \boldsymbol{A}$ where $\boldsymbol{\Gamma}$ is small, then $\operatorname{colim}_{\overrightarrow{\lambda}} \Gamma_\lambda$ exists in $\boldsymbol{A}$ and for any K-topological algebra $A \in \boldsymbol{A}$ there is a homeomorphism of the Kelley spaces:

$$\boldsymbol{A}(\operatorname*{colim}_{\overrightarrow{\lambda}} \Gamma_\lambda, A) \approx \lim_{\overleftarrow{\lambda}} \boldsymbol{A}(\Gamma_\lambda, A)\,.$$

Let us observe now that, as in the case of $\boldsymbol{K}$ (see 0.14), the representables of $\boldsymbol{A}$ or any $\boldsymbol{K}$-category of K-topological algebras (Prop. 1.7) preserve full injections. More precisely:

2.8. Remark. Given $I \xrightarrow{i} B$ in $\boldsymbol{A}$ such that $|I| \xrightarrow{i} |B|$ is a full injection (in $\boldsymbol{K}$), then for all $A \in \boldsymbol{A}$, $\boldsymbol{A}(A, I) \xrightarrow{\boldsymbol{A}(A,i)} \boldsymbol{A}(A, B)$ is also a full injection (in $\boldsymbol{K}$).

Proof. In the diagram

$$\begin{array}{ccc} \boldsymbol{A}(A,I) & \xrightarrow{\boldsymbol{A}(A,i)} & \boldsymbol{A}(A,B) \\ \downarrow & & \downarrow \\ \boldsymbol{K}(|A|,|I|) & \xrightarrow{\boldsymbol{K}(|A|,i)} & \boldsymbol{K}(|A|,|B|) \end{array}$$

the two vertical arrows and the lower level arrow are full injections, whence the upper level arrow is also a full injection. Q.E.D.

2.9. Proposition. *$\boldsymbol{A}$ and $\boldsymbol{A}'$ are tensored $\boldsymbol{K}$-categories.*

Proof. The meaning of this statement is that all representables $\boldsymbol{A} \xrightarrow{\boldsymbol{A}(A,-)} \boldsymbol{K}$ have a $\boldsymbol{K}$-left adjoint. But these functors preserve limits (Prop. 2.2) and $\boldsymbol{A}$ is well powered (and complete), so that, by the Adjoint Functor Theorem there will exist left adjoints provided that for any given $X \in \boldsymbol{K}$ there is a solution set. Let $A \in \boldsymbol{A}$ and $X \in \boldsymbol{K}$ be fixed objects. Denote by $A \xrightarrow{i_x} \coprod_X A$ the coproduct in $\boldsymbol{A}$ of A repeated as a factor once for each point of X. Given any $B \in \boldsymbol{A}$ and $X \xrightarrow{f} \boldsymbol{A}(A, B)$ in $\boldsymbol{K}$, let $\coprod_X A \xrightarrow{\omega_0(f)} B$ be the map (in $\boldsymbol{A}$) defined by the diagram

$$\begin{array}{ccc} \coprod_X A & \overset{\exists !}{\dashrightarrow} & B \\ {\scriptstyle i_x}\nwarrow & & \nearrow{\scriptstyle f(x)} \\ & A & \end{array} \qquad x \in X.$$

It is clear that the correspondence ω_0 is one-to-one. Let now $S = \{ X \xrightarrow{h} \boldsymbol{A}(A,H);\ H \in \boldsymbol{A},\ h \in \boldsymbol{K},\ \omega_0(h) \text{ is onto} \}$. Since there is only a set of surjective functions $\coprod_X A \to H$, S is a set. In order to see that S is a solution set we proceed as follows. Let $X \xrightarrow{f} \boldsymbol{A}(A,B)$ and consider the factorization of $\omega_0(f)$ described in Remark 2.5:

$$\begin{array}{ccc} \coprod_X A & \xrightarrow{\omega_0(f)} & B \\ {\scriptstyle g}\searrow & & \nearrow{\scriptstyle i} \\ & H & \end{array}$$

Define $X \xrightarrow{h} \boldsymbol{A}(A, H)$ by $h(x) = (A \xrightarrow{i_x} \coprod_X A \xrightarrow{g} H)$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & A(A,B) \\ & {}_h\searrow & \nearrow_{A(A,i)} \\ & A(A,H) & \end{array}$$

clearly commutes, and therefore, from Remark 2.8 follows that h is continuous, or $h \in \boldsymbol{K}$. Finally, it is obvious that $\omega_0(h)\, i_x = g i_x$ (for all $x \in X$) and therefore $\omega_0(h) = g$. Hence $\omega_0(h)$ is onto, that is, $h \in S$. This completes the proof of the existence of a left adjoint for $\boldsymbol{A}(A, -)$. Since the representables preserve cotensors (always), it follows as in Prop. 2.3 that this left adjoint is necessarily a $\boldsymbol{K}$-functor and a $\boldsymbol{K}$-left adjoint. This proof can be adapted without difficulty to $\boldsymbol{A}'$. Q.E.D.

We finish this section by expressing $\boldsymbol{K}$-functorially the standard procedure of adding an identity to an algebra possibly lacking it.

2.10. Proposition. *The $\boldsymbol{K}$-inclusion $\boldsymbol{A}' \to \boldsymbol{A}$ has a $\boldsymbol{K}$-left adjoint $\boldsymbol{A} \xrightarrow{\widetilde{(-)}} \boldsymbol{A}'$. That is, given any K-topological algebra A there is a K-topological algebra with identity $\tilde{A}$ and a natural homeomorphism $\boldsymbol{A}(A,B) \approx \boldsymbol{A}'(\tilde{A},B)$ (for all K-topological algebras with identity B). At the level of sets, we have a bijection*

$$\frac{A \longrightarrow B}{\tilde{A} \longrightarrow B}$$

between continuous linear multiplicative functions $A \to B$ and continuous linear multiplicative functions $\tilde{A} \to B$ which preserve the identity.

Proof. This result follows, for example, from Prop. 2.4 and 2.6 and Theorem A.1 in the Appendix of [10]. It is only necessary to observe that the inclusion $\boldsymbol{A}' \to \boldsymbol{A}$ commutes with the "underlying Kelley space" $\boldsymbol{K}$-functors.

§ 3. Gelfand $\boldsymbol{K}$-monads, the duality determined by C

Given a Kelley space $X \in \boldsymbol{K}$, consider the cotensor $\bar{A}(X,C) = \bar{A}'(X,C)$ (equality occurs since C belongs to both $\boldsymbol{A}$ and $\boldsymbol{A}'$ and $\boldsymbol{A}' \xrightarrow{i} \boldsymbol{A}$ preserves cotensors). According to the definitions given (§2) this is nothing but the long-considered algebra of all complex valued functions on X, endowed with the K-ation of the compact-open topology, in other words, the K-ation of the topology of uniform convergence on compact sets of X. In particular, $\bar{A}(X,C)$ is the K-ation of a complete locally m-convex algebra (cf. [27], Appendix D). If K is a compact space, then $\bar{A}(K,C)$ is just the commutative C^*-algebra of all continuous complex valued functions on K with the supremum norm, which is already a k-space (0.1). Since the $\boldsymbol{K}$-functor

$K^{op} \xrightarrow{A(-,C)} A$ preserves limits, we have $A(X,C) = \varprojlim_{K \subset X} A(K,C)$ for all $X \in K$ (where K stands for an arbitrary compact subset of X). Thus, $A(X,C)$ is always a filtered inverse limit of commutative C^*-algebras (recall however that this limit is taken qua K-topological algebras). These limits are easily characterized: we write now the "K-ation" of a well known result:

3.1. Proposition. *Given a K-topological algebra $A \in A$, the following are equivalent:*

(a) *A is a limit (in A) of commutative C^*-algebras with identity;*

(b) *A is commutative with identity and there is a family of algebra seminorms $\{p\}$ which defines the topology of A (in the sense that $|A|$ is the K-ation of the locally m-convex algebra $(A, \{p\})$). Furthermore the locally m-convex algebra $(A, \{p\})$ is complete and there is an involution * satisfying $p(a^*a) = p(a)^2$ for all $a \in A$ and all p.*

Proof. (a) ⇒ (b) is straightforward. For (b) ⇒ (a), observe that $A = \varprojlim_p A_p$ where A_p is the completion of the quotient of A by the null set of p; A_p is a C^*-algebra and $(A, \{p\}) = \varprojlim_p A_p$ (lim in the category of locally convex algebras). For more details see [17], [19], [27]. Q.E.D.

We remark that in the same fashion, the equivalence of the following statements about a K-topological algebra A can be established:

(a′) *A is a limit (in A) of Banach algebras;*

(b′) *there is a family $\{p\}$ of algebra seminorms which defines the topology of A (in the above sense) and such that the locally m-convex algebra $(A, \{p\})$ is complete.*

As above, we have now $A = \varprojlim A_p$.

Let us remark that given an algebra over the complex numbers (in the purely algebraic sense) non-equivalent families of algebra seminorms may determine the same K-topological algebra (by non-equivalent families of seminorms we mean to understand that the induced locally convex topologies do not coincide). For example, given a locally m-convex algebra A whose topology is determined by a family $\{p\}$ of seminorms, we can enlarge $\{p\}$ by adding in any set of (in particular, all) seminorms continuous on compact subsets of A. It is clear that the locally convex topologies may disagree, and yet, the K-topological algebras determined by these two families necessarily coincide. Also, starting from an $A \in A$ determined by a *complete* locally m-convex algebra, the process of adding seminorms $\{q\}$ just described leads to a new locally m-convex algebra B that may not be complete, in which case A, although still determined by B may not coincide with the enlarged limit $\varprojlim B_q$. We describe now some of these phenomena.

3.2. Example. Let X be the locally compact space of ordinals $X = [1,\Omega)$ with the order topology, where Ω is the smallest uncountable ordinal, A^{b} the locally m-convex algebra of all continuous complex valued functions on X with the topology of uniform convergence on compact sets. We have $\bar{A}(X,C) = \mathrm{Ke}A^{\mathrm{b}}$. It is well known that every function $f \in A^{\mathrm{b}}$ is constant on a tail $[\alpha, \Omega)$ for some α (which depends of course on f) and therefore, f has an extension f^0 to $\beta X = [1, \Omega]$. The correspondence $f \rightsquigarrow f^0$ is an algebraic isomorphism (onto) between A^{b} and $\bar{A}(\beta X, C)$. Denote now by $L \in A(\bar{A}(\beta X, C), C)$ the continuous multiplicative linear map $L(f) = f(\Omega)$. It is clear that L induces a multiplicative linear map (also denoted by L) from A^{b} to C.

3.2.1. *$L \in A(\bar{A}(X,C),C)$, but L is not continuous on A^{b}.*

Proof. First, for each $\alpha < \Omega$ denote by p_α the seminorm $p_\alpha(f) = \mathrm{Sup}\{|f(\gamma)|;\ 1 \leqslant \gamma \leqslant \alpha\}$. It is easy to see that the family $\{p_\alpha\}$, $1 \leqslant \alpha < \Omega$ determines the topology of A^{b}. Now, let $g_\alpha \in A^{\mathrm{b}}$ be the function defined by $g_\alpha(\gamma) = 0$ if $1 \leqslant \gamma \leqslant \alpha$, $g_\alpha(\gamma) = 1$ if $\gamma \geqslant \alpha + 1$, where $1 \leqslant \alpha < \Omega$. It is clear that $p_\alpha(g_\sigma) = 0$ if $\sigma > \alpha$, and therefore $g_\sigma \to 0$ (as $\sigma \to \Omega$) in A^{b} (0 is the zero function). However, $L(g_\sigma) = g_\sigma^0(\Omega) = 1$ which shows that L *is not continuous* on A^{b}. In order to prove the first part, we observe that $L \in A(\bar{A}(X,C),C)$ just means that the restrictions of L to the compact subsets of A^{b} are continuous. This follows from the fact that if $H \subset A^{\mathrm{b}}$ is a compact subset, then there is an ordinal $\delta < \Omega$ such that *all* functions $f \in H$ are constant on $[\delta, \Omega)$ ([4] Chap. IV, §4, Ex. 17). For the sake of completeness, we sketch here the proof. First, there is a constant $M > 0$ such that $|f(\sigma)| \leqslant M$ for all $f \in H$ and $\sigma \in X$, since otherwise there would exist ordinals $\sigma_n < \Omega$, $n = 1, 2, \ldots$ and functions $f_n \in H$ with $|f_n(\sigma_n)| \geqslant n$; but necessarily $\sigma = \mathrm{Sup}\,\sigma_n < \Omega$ and $p_\sigma(f)$ is bounded on H (compact), which contradicts the above conclusion. Now define for $\sigma \in X$ the number $s(\sigma) = \mathrm{Sup}\{|f(\sigma') - f(\sigma'')|;\ f \in H,\ \sigma \leqslant \sigma',\ \sigma \leqslant \sigma''\}$. Clearly $s(\sigma) \leqslant 2M$ and $\sigma(s)$ decreases as $\sigma \to \Omega$; this means that s is eventually constant: $s(\sigma) = u$ for all $\sigma \geqslant \sigma_0$ in X. If $u > 0$, there exist sequences $\{\sigma_n\}$ and $\{\tau_n\}$ in X and $\{f_n\}$ in H such that $\sigma_n \leqslant \tau_n \leqslant \sigma_{n+1} \leqslant \tau_{n+1}$, $n = 1, 2, \ldots$ and $|f_n(\sigma_n) - f_n(\tau_n)| \geqslant \frac{1}{2}u$; clearly both $\{\sigma_n\}$ and $\{\tau_n\}$ converge to $\bar{\sigma} = \mathrm{Sup}\{\sigma_n\} = \mathrm{Sup}\{\tau_n\}$, and $\{f_n\}$ has a convergent subset $\{f_{n'}\}$ with limit $f \in H$. This leads to the contradiction $f(\bar{\sigma}) = \lim f_{n'}(\sigma_n) \neq \lim f_{n'}(\tau_n) = f(\bar{\sigma})$. Thus, $u = 0$ and $f(\sigma') = f(\sigma'')$ for all $f \in H$ and $\sigma' \geqslant \sigma_0$, $\sigma'' \geqslant \sigma_0$, as desired. From here it follows trivially that the restriction of L to H is continuous, as claimed. Q.E.D.

Observe that $q_L(f) = |L(f)|$ is an algebra seminorm on A^{b}, continuous on compact subsets. Thus the locally m-convex algebra B defined by the family $\{p_\sigma\} \cup \{q_L\}$ on A^{b} satisfies $\mathrm{Ke}A^{\mathrm{b}} = \mathrm{Ke}B$. However A^{b} (with $\{p_\sigma\}$) is complete, and B (with $\{p_\sigma\}, q_L$) *is not:* the net $\{g_\sigma\}_{1 \leqslant \sigma < \Omega}$ defined above is a Cauchy net for p_σ and q_L, but does not converge in B; as is easily seen. This shows in particular that $B \neq A^{\mathrm{b}}$ and therefore all the statements made in the last paragraph before 3.2 have now been justified. This example will be continued later (see 3.6 below).

Given a K-topological algebra $A \in \mathbf{A}$, $\mathbf{A}(A,C)$ is the set of all continuous linear multiplicative functionals on A with the K-ation of its compact-open topology. If $A \in \mathbf{A}'$, then $\mathbf{A}'(A,C)$ consists only of those functionals that preserve the identity. In the case where A is determined by (i.e., is obtained as the K-ation of) a locally m-convex algebra B, the above spaces differ from what has been classically called the "spectrum" or "carrier space" of B ([5], [9], [17]) in two different ways. On one hand, they contain more points (namely, all those characters [= linear multiplicative functionals] which are continuous on compact sets, but not continuous on B); on the other hand, the topology of $\mathbf{A}(A,C)$ (and $\mathbf{A}'(A,C)$) is *not* the customary topology for spectra, namely, the topology of simple convergence on the elements of B (or even the K-ation of it). These facts support a statement asserting that $\mathbf{A}(A,C)$ and $\mathbf{A}'(A,C)$ differ substantially from the traditionally considered spectra. And yet, if A $(=B)$ is a commutative C^*-algebra with identity, then $\mathbf{A}'(A,C)$ coincides with the spectrum of A, set-wise and even topologically. In fact, the only thing to be verified is that pointwise convergence of characters coincides with uniform convergence on compact sets, an obvious fact since in this case the characters are equicontinuous. In particular $\mathbf{A}'(A,C)$ is a compact space; $\mathbf{A}(A,C)$ is also compact and in fact is obtained from $\mathbf{A}'(A,C)$ by adding one isolated point: the zero functional. It follows from these considerations that $\mathbf{A}'(A,C)$ is a legitimate generalization of the spectrum of A when A is C^*, and therefore $\mathbf{A}'(A,C)$, for $A \in \mathbf{A}$ should play a similar role than the spectrum concerning, for instance, the existence of idempotents, representations, etc., and in particular the Gelfand Theory of C^*-algebras. The rest of the section is devoted to developing a generalization of the latter.

Consider the K-category $\mathbf{A}$ and the pair of $\mathbf{K}$-adjoint functors

$$\mathbf{A} \underset{\bar{\mathbf{A}}(-,C)}{\overset{\mathbf{A}(-,C)}{\rightleftarrows}} \mathbf{K}^{\mathrm{op}} .$$

The corresponding $\mathbf{K}$-monad in $\mathbf{A}$ will be called the *Gelfand* ***K****-monad*, denoted $\mathbf{T} = (T, \eta, \mu)$, where $\mathbf{A} \xrightarrow{T} \mathbf{A}$, $TA = \bar{\mathbf{A}}(\mathbf{A}(A,C),C)$ for any given K-topological algebra $A \in \mathbf{A}$. The unit $\mathrm{id} \xrightarrow{\eta} T$ is the *Gelfand transformation* (sometimes called *Fourier transformation*, or even *Fourier-Gelfand transformation*): $A \xrightarrow{\eta A} \bar{\mathbf{A}}(\mathbf{A}(A,C),C)$. If $a \in A$, we introduce the notation $\eta A(a) = \hat{a}$, where $\hat{a}(\psi) = \psi(a)$ for any $A \xrightarrow{\psi} C$ in $\mathbf{A}$. The multiplication of the monad $TT \xrightarrow{\mu} T$, $\bar{\mathbf{A}}(\mathbf{A}(\bar{\mathbf{A}}(\mathbf{A}(A,C),C),C),C) \xrightarrow{\mu A} \bar{\mathbf{A}}(\mathbf{A}(A,C),C)$ has the following action: given $\mathbf{A}(\bar{\mathbf{A}}(\mathbf{A}(A,C),C),C) \xrightarrow{\varphi} C$ and $A \xrightarrow{\psi} C$, then $\mu A(\varphi)(\psi) = \varphi(\hat{\psi})$ where $\bar{\mathbf{A}}(\mathbf{A}(A,C),C) \xrightarrow{\hat{\psi}} C$ is defined on $\mathbf{A}(A,C) \xrightarrow{f} C$ by $\hat{\psi}(f) = f(\psi)$. (Recall that, as for any $\mathbf{K}$-monad determined by a pair of $\mathbf{K}$-adjoint functors, $\mu A = \bar{\mathbf{A}}(\epsilon \mathbf{A}(A,C),C)$, where ϵ is the counit (in $\mathbf{K}^{\mathrm{op}}$). Given $X \in \mathbf{K}$, $x \in X$, then $X \xrightarrow{\epsilon X} \mathbf{A}(\bar{\mathbf{A}}(X,C),C)$ is $\epsilon X(x) = \hat{x}$, where $\hat{x}(f) = f(x)$ for any $X \xrightarrow{f} C$ in $\mathbf{K}$.) The reader can verify easily that this description of the unit (Gelfand transformation) and multiplication of the Gelfand $\mathbf{K}$-monad actually describes the unit and multiplication as intrinsically obtained from the definition of the monad: it suffices to go back to the cotensoring isomorphism σ of Prop. 2.1, and observe that it is a restriction of the isomorphism σ provided by the *closed Cartesian structure of* $\boldsymbol{K}$ (see 0.13). The continuity of the Gelfand transformation is automatically guaranteed, since it is a map in $\boldsymbol{A}$. Analogously, to the pair

$$\boldsymbol{A}' \underset{\bar{A}'(-,C)}{\overset{A'(-,C)}{\leftrightarrows}} \boldsymbol{K}^{\mathrm{op}}$$

there corresponds a $\boldsymbol{K}$-monad in $\boldsymbol{A}'$, that we also call the Gelfand $\boldsymbol{K}$-monad, and denote by $\boldsymbol{T}' = (T', \eta', \mu')$.

In order to indicate the overall picture, we take up space to review some past examples.

3.3. Example. If A is a commutative C^*-algebra with identity, then $T'A = A$. We will consider this in further detail below (see Prop. 3.7).

3.4. Example. Assume $X \in \boldsymbol{K}$ is completely regular (see [21]) and σ-compact, that is, X has a countable covering by compact subsets. Then, the locally m-convex algebra A^{b} of all continuous complex functions with the compact open topology is metrizable, and therefore $\bar{A}(X,C) = A^{\mathrm{b}}$. Assume now that $\varphi \in A(A^{\mathrm{b}}, C)$. Clearly for some compact subset $K \subset X$ we have $|\varphi(f)| \leqslant cp_K(f)$ where $p_K(f) = \mathrm{Sup}\{|f(k)|;\ k \in K\}$. Let $R \subset \bar{A}(K,C)$ be the algebra of restrictions to K of functions in A^{b} (i.e., R is the image of $\bar{A}(X,C) \xrightarrow{\bar{A}(i,C)} \bar{A}(K,C)$ where $K \xrightarrow{i} X$ is the inclusion. φ induces an element of $A(R,C)$. Let $J \subset R$ be the kernel of ψ: $g \in J$ if and only if $\psi(g) = 0$. If for every $x \in K$ there is a $g_x \in J$ such that $g_x(x) \neq 0$, then $h_x = |g_x|^2 = \bar{g}_x g_x \in J$ and h_x is real valued and satisfies $h_x(x) > 0$. A compactness argument shows that we can find $x_1, \ldots, x_n \in K$ with $h = \Sigma h_{x_i} > 0$ everywhere on K. But this means that h has an inverse in R and yet $\psi(h) = 0$. Thus for some $x \in K$ we have $g(x) = 0$ for all $g \in J$. Since J is maximal, the converse follows, and therefore for arbitrary g, $\varphi(g - g(x)1) = 0$, so that $\varphi(g) = g(x)$. This proves that $A'(A^{\mathrm{b}}, C)$ can be identified at the level of sets to X. However, if X is completely regular (and $X \in \boldsymbol{K}$), the topologies also agree and therefore $A'(A^{\mathrm{b}}, C) = X$. It follows that for these X, $T'\bar{A}(X,C) = \bar{A}(X,C)$. Observe that this equality follows from the immediate result $\bar{A}(X,C) = A^{\mathrm{b}}$ and Cor. 3.10. However, we have proved something stronger, namely that $X = A'(\bar{A}(X,C), C)$ as topological spaces.

3.5. Example. Let S be a set and $A = \mathrm{Ke}l^\infty(S)_{\mathrm{w}}$ (notation as in Ex. 1.3). From 1.3.4 follows easily that the family of functions in A with finite support (i.e., vanishing off a finite set) are dense in A. Then, if $\varphi \in A'(A,C)$ and φ is not zero, we must have $\varphi(a_s) \neq 0$ for some $s \in S$, where $a_s \in A$ is the function $a_s(t) = 0$ if $t \neq s$, $a_s(s) = 1$. Since $a_s^2 = a_s$, clearly $\varphi(a_s) = 1$. If $a \in A$ is any element, we have $a_s(a - a(s)1) = 0$ so that $\varphi(a - a(s)1) = 1$, and therefore $\varphi(a) = a(s)$. This shows that $A'(A,C) = S$ whence $T'A = T'\,\mathrm{Ke}l^\infty(S)_{\mathrm{w}} = \Pi_S\, C \neq A$ (in A). In fact, $T'T'A = T'A$. This follows, in case of S countable, from Ex. 3.4 above. In general it can be seen that $\prod_S C$ satisfies the conditions of Prop. 3.9 (see [31] Th. 5.2) and therefore Cor. 3.10 applies.

3.6. Example. This is a continuation of Example 3.2. X denotes again the space $X = [1,\Omega) \in \boldsymbol{K}$. Since $A = \boldsymbol{A}(X, C)$ is algebraically isomorphic (via $f \rightsquigarrow f^0$) to the C^*-algebra $\boldsymbol{A}(\beta X, C)$, it follows that the linear multiplicative functionals φ on A are the following: $\varphi = \varphi_\sigma$ defined as $\varphi_\sigma(f) = f(\sigma)$ for some $\sigma \in X$, which are obviously continuous, or $\varphi = L$, which according to 3.2.1 is also continuous. Thus $\boldsymbol{A}'(\boldsymbol{A}(X,C),C) = \beta X$. Since βX is compact, $B = \boldsymbol{A}(\beta X, C)$ is a C^*-algebra, and therefore $\boldsymbol{A}(B, C) = \beta X$ again. In other words, we have $T'T'\bar{\boldsymbol{A}}'([1,\Omega), C) = T'\bar{\boldsymbol{A}}'([1,\Omega),C) = \bar{\boldsymbol{A}}'([1,\Omega], C) = \bar{\boldsymbol{A}}'([1,\Omega),C)$. This last equality follows since both sides are algebraically isomorphic (via $f \rightsquigarrow f^0$), and this forces $\eta' A$ and $\boldsymbol{A}'(\epsilon X, C)$ to be mutually inverses of each other. We have then that $\mathrm{Ke}A^b$ is the C^*-algebra $\boldsymbol{A}'([1,\Omega], C)$, which indicates how substantially the K-ation functor changes the topology of a locally m-convex algebra. To add the semi-norm q_I is an intermediate step, that, although it makes every functional which is continuous over compact subsets be continuous, still does not constitute a Kelley topology. (Cf. [31] corollaries to Th. 5.2 for similar phenomena.)

Observe that the Gelfand $\boldsymbol{K}$-monad is the *codensity K-monad* of the $\boldsymbol{K}$-functor $\boldsymbol{I} \xrightarrow{C} \boldsymbol{A}$ (cf. [10] p. 83), where $\boldsymbol{I}$ is the $\boldsymbol{K}$-category consisting of one single object $1 \in \boldsymbol{I}$ and $\boldsymbol{I}(1,1) = 1 \in \boldsymbol{K}$. A $\boldsymbol{K}$-functor $\boldsymbol{I} \to \boldsymbol{A}$ is characterized completely by one object of $\boldsymbol{A}$, and vice versa. This holds, of course, for all $\boldsymbol{K}$-categories, and not only for $\boldsymbol{A}$. Similarly, the Gelfand $\boldsymbol{K}$-monad in $\boldsymbol{A}'$ is the codensity $\boldsymbol{K}$-monad of $\boldsymbol{I} \xrightarrow{C} \boldsymbol{A}'$

Let now $\boldsymbol{C}^*$ denote the $\boldsymbol{K}$-category of commutative C^*-algebras with identity (see Prop. 1.2). Of course, $C \in \boldsymbol{C}^*$. From the remarks made immediately after Example 3.2, we know that for any $A \in \boldsymbol{C}^*$, $\boldsymbol{C}^*(A, C)$ is a compact space. On the other hand, given any compact space $K \in \boldsymbol{K}$, $\boldsymbol{A}'(K, C)$ is a commutative C^*-algebra with identity, and therefore, since $\boldsymbol{C}^* \xrightarrow{i} \boldsymbol{A}'$ is a $\boldsymbol{K}$-full subcategory, $C^*(K, C) \overset{\text{def}}{=} \boldsymbol{A}'(K, C)$ is a cotensor of C with K in $\boldsymbol{C}^*$, so that $\boldsymbol{C}^*$ has, at least, cotensors of C with compact spaces. It follows that the codensity $\boldsymbol{K}$-monad of $\boldsymbol{I} \xrightarrow{C} \boldsymbol{C}^*$ exists, which simply means that we also have a Gelfand $\boldsymbol{K}$ monad in $\boldsymbol{C}^*$. It is clear that it is the restriction of $\boldsymbol{T}'$ to $\boldsymbol{C}^*$. The classical Gelfand duality (cf. for instance [5] or [28]) says that this $\boldsymbol{K}$-monad is isomorphic to the identity $\boldsymbol{C}^* \xrightarrow{\mathrm{id}} \boldsymbol{C}^*$ (recall that our $\boldsymbol{A}(A, C)$ coincides with the spectrum of $A \in \boldsymbol{C}^*$, as observed above). Thus, in the language of [10]:

3.7. Proposition. *The algebra C of complex numbers is a K-codense cogenerator of the K-category $\boldsymbol{C}^*$ of commutative C^*-algebras with identity.* Q.E.D.

In fact, for the Gelfand $\boldsymbol{K}$-monad $\boldsymbol{T}'$ in $\boldsymbol{A}'$ we also have $T'A \approx A$, via $\eta' A$, for all commutative C^*-algebras A, but it is clear that $\boldsymbol{T}'$ is not isomorphic to the identity on all of $\boldsymbol{A}'$.

Consider now the $\boldsymbol{K}$-category of $\boldsymbol{T}'$-algebras, that is, objects $A \in \boldsymbol{A}'$ provided with a $\boldsymbol{T}'$-algebra structure $T'A \xrightarrow{\alpha} A$, where $\alpha \circ \eta' A = \mathrm{id}$ and $T\alpha \circ \alpha = \mu' A \circ \alpha$, and maps: morphisms $A \xrightarrow{g} B$ in $\boldsymbol{A}'$ making the diagram

$$\begin{array}{ccc} T'A & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle T'\varphi} & & \downarrow{\scriptstyle \varphi} \\ T'B & \xrightarrow{\beta} & B \end{array}$$

commutative. Prop. 1.2 shows how $\boldsymbol{T}'$-algebras form a $\boldsymbol{K}$-category. (For a historical account and further references concerning these concepts see [12]; the enriched version used here is considered, for instance and among other places, in [6] and [10].) Denote this $\boldsymbol{K}$-category by $\boldsymbol{A}'^{T'}$. For any Kelley space $X \in \boldsymbol{K}$, $\bar{A}'(X,C)$ is a $\boldsymbol{T}'$-algebra with

$$T'\bar{A}'(X,C) = \boldsymbol{A}'(\boldsymbol{A}'(\bar{A}'(X,C),C),C) \xrightarrow{\bar{A}'(\epsilon X,C)=\alpha} \bar{A}'(X,C).$$

This is a general fact [10]. Given any C^*-algebra with identity A, the inverse of $\eta' A$ is a structure of T'-algebra on A, which is *trivial*: We will say that a $\boldsymbol{T}'$-algebra $T'A \xrightarrow{\alpha} A$ is trivial if α is a (two sided) inverse of $\eta' A$. Very naturally there arises the conjecture that all $\boldsymbol{T}'$-algebras are trivial. This is equivalent to saying that T' is idempotent $T'T' = \iota'$ (i.e. μ' is an isomorphism). It is also equivalent to the fact that the $\boldsymbol{K}$-functor $\boldsymbol{A}'^{T'} \xrightarrow{U^{T'}} \boldsymbol{A}'$, $U^{T'}(T'A \xrightarrow{\alpha} A) = A$ be $\boldsymbol{K}$-full-and-faithful, and, finally, equivalent to the (apparently weaker) fact that for any $\boldsymbol{T}'$-algebra, $T'A \xrightarrow{\alpha} A$, all maps into the complex numbers $A \to C$ in $\boldsymbol{A}'$ be morphism of $\boldsymbol{T}'$-algebras (see [10] Prop. II.4.6, p. 103). In fact, the conjecture would be true (in general) provided that is true for all $\boldsymbol{T}'$-algebras of the form $\bar{A}'(X,C)$ with $\alpha = \bar{A}'(\epsilon X, C)$, $X \in \boldsymbol{K}$. Recall that this is the case if X is completely regular and σ-compact (see Example 3.4 above). If there is an affirmative answer, a simple categorical $\boldsymbol{K}$-duality could be obtained as we shall see in Prop. 3.9. All the concrete $\boldsymbol{T}'$-algebras we have found are indeed trivial, but we have not succeeded, however, in proving this conjecture in general. Due to this unpleasant situation, we are forced to resort to a considerably more sophisticated machinery, developed in [1], in order to go on. The enriched version needed here, is actually to be found in [10].

The $\boldsymbol{K}$-duality produced below will give, as a byproduct the result that every $\boldsymbol{K}$-topological algebra A satisfying the equivalent conditions of Prop. 3.1 is of the form $A = \bar{A}(X,C) = \bar{A}'(X,C)$ for some $X \in \boldsymbol{K}$. If our conjecture above is true, then necessarily $X = \boldsymbol{A}'(A,C)$. Before going to the general case, we will describe some sufficient conditions for this to be so.

3.8. Proposition. *If $A \in \boldsymbol{A}'$ satisfies the equivalent conditions in Prop. 3.1, then A has a canonical structure of $\boldsymbol{T}'$-algebra.*

Proof. We know that $A = \varprojlim_p A_p$ and that for each p, $A_p \in \boldsymbol{C}^*$. Therefore each A_p is a trivial $\boldsymbol{T}'$-algebra. Hence any morphism $A_p \to A_q$ in $\boldsymbol{A}'$ is a morphism of $\boldsymbol{T}'$-algebras; we can therefore take the limit $\varprojlim_p A_p$ *in* $\boldsymbol{A}'^{T'}$ (cf. [10] Propositions II.4.5 and

II.4.8). The $\boldsymbol{T}'$-algebra so obtained can also be described by

$$(1)\qquad \begin{array}{ccc} T'A & \overset{\exists !\ \alpha}{\dashrightarrow} & A = \lim\limits_{\overleftarrow{p}} A_p \\ {\scriptstyle T'\pi_p}\big\downarrow & (1) & \big\downarrow{\scriptstyle \pi_p} \\ T'A_p & \underset{(\eta' A_p)^{-1}}{\overset{\approx}{\longrightarrow}} & A_p \end{array}$$

Q.E.D.

3.9. Proposition. *Let $A \in \boldsymbol{A}'$ satisfy the equivalent conditions in Prop. 3.1 and assume furthermore that all morphisms $A \overset{\varphi}{\to} C$ in $\boldsymbol{A}$ are continuous in the locally m-convex topology defined by $\{p\}$. Then, the canonical $\boldsymbol{T}'$-algebra structure of A (Prop. 3.8) is trivial.*

Proof. Since $A \overset{\varphi}{\to} C$ is $\{p\}$-continuous we can assume, after replacing the family of seminorms by an equivalent family, if necessary, that φ is one of the projections $A \overset{\pi_p}{\to} A_p$ with $A_p = C$. Then, by definition (see diagram (1) above), φ is a morphism of $\boldsymbol{T}'$-algebras. Thus, any morphism $A \overset{\varphi}{\to} C$ in $\boldsymbol{A}$ is a morphism of $\boldsymbol{T}'$-algebras, and Prop. II.4.6 of [10] applies to complete the proof. Q.E.D.

3.10. Corollary. $A \approx T'A\ (= \bar{\boldsymbol{A}}'(\boldsymbol{A}'(A,C),C))$ *in* $\boldsymbol{A}$. Q.E.D.

This corollary applies notably to the case of Fréchet algebras (see [27]), that is, to algebras satisfying (b) in Prop. 3.1 with a *countable* family $\{p\}$ and to arbitrary products of such algebras (cf. [31] Th. 5.2).

It is clear that K-topological algebras of the form $\bar{\boldsymbol{A}}'(X,C)$ for a general $X \in \boldsymbol{K}$ will not satisfy the assumptions in Prop. 3.9 (see 3.2.1 in Example 3.2), and this calls for a different approach. But before we describe it, let us observe the following.

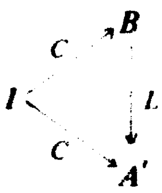
3.11. Remark. *If $A \in \boldsymbol{A}$ has a structure of $\boldsymbol{T}$-algebra α, then A has an identity, i.e., $A \in \boldsymbol{A}'$. If both $A, B \in \boldsymbol{A}$ are $\boldsymbol{T}$-algebras, any morphism of $\boldsymbol{T}$-algebras $A \to B$ preserves the identity.*

Proof. The identity of A is $\alpha(e)$ where e is the identity of $TA = \boldsymbol{A}(\boldsymbol{A}(A,C),C)$; if $A \overset{\varphi}{\to} B$ in $\boldsymbol{A}$, $T(\varphi)$ always preserve the identity. Q.E.D.

Consider then the $\boldsymbol{K}$-functor $\boldsymbol{I} \overset{C}{\to} \boldsymbol{A}'$ and its codensity $\boldsymbol{K}$-monad $\boldsymbol{T}'$, that is, the Gelfand $\boldsymbol{K}$-monad. The following informal considerations are stated and proved in detail in [10], on pp. 135ff. under the heading "Second Relative V-Completion". The Gelfand $\boldsymbol{K}$-monad $\boldsymbol{T}'$ determines the $\boldsymbol{K}$-category $\boldsymbol{A}'^{\boldsymbol{T}'}$ of $\boldsymbol{T}'$-algebras, and $C \in \boldsymbol{A}'^{\boldsymbol{T}'}$, whence, we have a functor $\boldsymbol{I} \overset{C}{\to} \boldsymbol{A}'^{\boldsymbol{T}'}$. $\boldsymbol{A}'^{\boldsymbol{T}'}$ is a cotensored $\boldsymbol{K}$-category and then we have a codensity $\boldsymbol{K}$-monad in $\boldsymbol{A}'^{\boldsymbol{T}'}$. The cotensors in $\boldsymbol{A}'^{\boldsymbol{T}'}$ are (strictly) preserved by $\boldsymbol{A}'^{\boldsymbol{T}'} \overset{U^{\boldsymbol{T}'}}{\longrightarrow} \boldsymbol{A}'$, that is they are nothing but a $\boldsymbol{T}'$-algebra structure on $\bar{\boldsymbol{A}}'(X,C)$, for $X \in \boldsymbol{K}$ (see the second paragraph after Prop. 3.7). The codensity $\boldsymbol{K}$-monad in $\boldsymbol{A}'^{\boldsymbol{T}'}$ determines its own $\boldsymbol{K}$-category of algebras, which in turn, gives rise to a new $\boldsymbol{K}$-monad, and the whole process repeats itself again. After going up in this way an infinite number of times (once for each natural number), we have the (inverse) limit of the chain of $\boldsymbol{K}$-categories thus obtained. This category, due to the completeness of $\boldsymbol{K}$, is also a $\boldsymbol{K}$-category, which happens to be cotensored and "to contain" the object C (because C is coherently contained in each of the $\boldsymbol{K}$-categories in the chain). But then we have its codensity $\boldsymbol{K}$-monad, and this chain process starts again: and in this way we go through all the ordinals. *The limit of the (large!) chain just described (exists and) is also a* $\boldsymbol{K}$*-category*, call it $\boldsymbol{B}$, which is cotensored and such that C is a $\boldsymbol{K}$-codense cogenerator for $\boldsymbol{B}$: this means that the process stops. It is possible (and sometimes handy) to think on the objects of $\boldsymbol{B}$ as being those $\boldsymbol{K}$-topological algebras $A \in \boldsymbol{A}'$ which can be lifted all the way up, i.e., which admit a structure of algebra at every level in the chain. More accurately, in view of the possibility of different liftings, they should be considered as $\boldsymbol{K}$-topological algebras *together* with a structure of algebra at every level. *If a* $\boldsymbol{K}$*-topological algebra* $A \in \boldsymbol{A}'$ *is a trivial* $\boldsymbol{T}'$ *algebra (e.g., a commutative C*-algebra) then it admits a unique lifting all the way up.* This is essentially due to the fact that the inverse of $\eta' A$ provides a (forced) lifting into $\boldsymbol{A}'^{\boldsymbol{T}'}$ and that for the object so determined, the $\boldsymbol{K}$-monad in $\boldsymbol{A}'^{\boldsymbol{T}'}$ is also trivial. This phenomenon is preserved in the steps corresponding to limit ordinals, too. We can summarize as follows:

3.12. Theorem. *There is a* $\boldsymbol{K}$*-complete* $\boldsymbol{K}$*-category* $\boldsymbol{B}$ *and a* $\boldsymbol{K}$*-faithful* $\boldsymbol{K}$*-functor* $\boldsymbol{B} \xrightarrow{L} \boldsymbol{A}'$ *which (strictly) preserves cotensors and* $\boldsymbol{K}$*-limits. Furthermore:*

(a) *there is a unique object* $C \in \boldsymbol{B}$ *such that* $LC = C$; *the diagram*



commutes;

(b) C *is a* $\boldsymbol{K}$*-codense cogenerator of* $\boldsymbol{B}$*, that is, for all* $B \in \boldsymbol{B}$, $B \approx \bar{\boldsymbol{B}}(\boldsymbol{B}(B,C),C)$;

(c) *given any* $A \in \boldsymbol{A}'$ *such that* $T'A \approx A$ *via* $\eta' A$*, there is a unique object* $B \in \boldsymbol{B}$ *such that* $LB = A$; *and moreover, for any other* $B' \in \boldsymbol{B}$, $\boldsymbol{B}(B,B') \approx \boldsymbol{A}'(A, LB')$ *via* L.

The *proof* is to be found in [10]. Q.E.D.

We have the following corollary:

3.13. Theorem. *A* $\boldsymbol{K}$*-topological algebra* $A \in \boldsymbol{A}'$ *is of the form* $\bar{\boldsymbol{A}}'(X,C)$ *for some k-space* X *if and only if it satisfies the equivalent conditions in Prop. 3.1.*

Proof. We keep the notations of Prop. 3.1. The considerations made before Prop. 3.1

justify the "if" part. Assume now that $A = \varprojlim_p A_p$. From (c) in 3.12 there are (unique) $B_p \in \boldsymbol{B}$ such that $LB_p = A_p$, and we can take the limit $B = \varprojlim_p B_p$ in $\boldsymbol{B}$ (see Prop. 3.8). Since L preserves (strictly) limits, $LB = A$. On the other hand, it follows from (b) in 3.12 that $B \approx \bar{\boldsymbol{B}}(\boldsymbol{B}(B,C),C)$, and since L (strictly) preserves cotensors, we have $A = LB \approx L\bar{\boldsymbol{B}}(\boldsymbol{B}(B,C),C) = \bar{\boldsymbol{A}}'(\boldsymbol{B}(B,C),LC) = \bar{\boldsymbol{A}}'(\boldsymbol{B}(B,C),C)$. Thus, $A \approx \bar{\boldsymbol{A}}'(\boldsymbol{B}(B,C),C)$, and the proof is complete. Q.E.D.

Similar statements establishing functional representations for topological algebras can be found in [27] Theorem 8.4 and [19] Theorem 5. The result in the last corollary gives an isomorphism with an algebra of continuous complex functions in its natural Kelley topology, so that, in a sense, it can not be improved. However, if we adopt the customary standpoint of considering an algebra satisfying (a) and (b) in Prop. 3.1 qua locally m-convex algebra rather than as an element of $\boldsymbol{A}'$, the bijection $A \approx \bar{\boldsymbol{A}}'(X,C)$ is no longer a homeomorphism for the locally m-convex topologies on these algebras, but only continuous as $\bar{\boldsymbol{A}}'(X,C) \to A$: this accounts for the unpleasant asymmetry in the main result in [19].

It can easily be seen, following the proof of Prop. 3.9 that the functionals $A \xrightarrow{\varphi} C$ which are continuous for the locally m-convex topology $\{p\}$, can be lifted all the way up, or equivalently, they are morphisms at all levels, and therefore every such functional determines a point of $X = \boldsymbol{B}(B,C)$. This means that X contains the classical spectrum of $(A, \{p\})$, but might, a priori, be larger.

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