On the order of Schur multiplier of non-abelian $p$-groups

Peyman Niroomand

School of Mathematics and Computer Science, Damghan University of Basic Sciences, Damghan, Iran

ARTICLE INFO

Article history:
Received 23 February 2009
Available online 7 October 2009
Communicated by E.I. Khukhro

Keywords:
Schur multiplier
Non-abelian $p$-groups

ABSTRACT

Let $G$ be a finite $p$-group of order $p^n$. Green proved that $\mathcal{M}(G)$, its Schur multiplier is of order at most $p^\frac{1}{2}(n-1)$. Later Berkovich showed that the equality holds if and only if $G$ is elementary abelian of order $p^n$. In the present paper, we prove that if $G$ is a non-abelian $p$-group of order $p^n$ with derived subgroup of order $p^k$, then $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-k-2)(n-k-1)+1}$. In particular, $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}$, and the equality holds in this last bound if only if $G = H \times Z$, where $H$ is extra special of order $p^3$ and exponent $p$, and $Z$ is an elementary abelian $p$-group.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction


Let $G$ be a $p$-group of order $p^n$, Green [3] gave an upper bound $p^{\frac{1}{2}n(n-1)}$ for its Schur multiplier. Also, Berkovich [1] proved that the equality holds if and only if $G$ is an elementary abelian $p$-group of order $p^n$.

The Schur multiplier of abelian groups may be calculated easily by a result [9] which was obtained by Schur. So in this paper, we focus on non-abelian $p$-groups.

This paper is devoted to the derivation of certain upper bound for the Schur multiplier of non-abelian $p$-groups of order $p^n$ with derived subgroup of order $p^k$. We prove that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-k-2)(n-k-1)+1}$. In particular, if $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1}$, we characterize the structure of $G$.

If $G$ is a $p$-group of order $p^n$, Jones [4] proved that $|\mathcal{M}(G)||G'| \leq p^{\frac{1}{2}n(n-1)}$ which shows that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)-1}$ when $G$ is a non-abelian $p$-group of order $p^n$. So, the general bound given above is better than Jones’s bound unless $|G| = p^2$, in which case the two bounds are the same.

E-mail address: p_niroomand@yahoo.com.

0021-8693/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
Main Theorem. Let $G$ be a non-abelian finite $p$-group of order $p^n$. If $|G'| = p^k$, then we have

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$  

In particular,

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-2)+1},$$

and the equality holds in this last bound if and only if $G = H \times Z$, where $H$ is an extra special $p$-group of order $p^3$ and exponent $p$, and $Z$ is an elementary abelian $p$-group.

At first we list the following theorems, which are used in our proofs.

**Theorem 1.1.** (See [5, Theorem 3.1 and Theorem 4.1].) Let $G$ be a finite $p$-group and let $N$ be a central subgroup of $G$. Then

$$|\mathcal{M}(G/N)| \leq |\mathcal{M}(G)| \frac{|G'| \cap N}{|G/\mathcal{M}(G)| \otimes |G/N \otimes N|}.$$  

**Theorem 1.2.** (See [7, Theorem 3.3.6].) Let $G$ be an extra special $p$-group of order $p^{2m+1}$. Then:

(i) If $m \geq 2$, then $|\mathcal{M}(G)| = p^{2m^2 - m - 1}$.

(ii) If $m = 1$, then $|\mathcal{M}(G)| \leq p^2$, and the equality holds if and only if $G$ is of exponent $p$.

**Theorem 1.3.** (See [7, Theorem 2.2.10].) For every finite groups $H$ and $K$, we have

$$\mathcal{M}(H \times K) \cong \mathcal{M}(H) \times \mathcal{M}(K) \times \frac{H \otimes K}{H' \otimes K'}.$$  

**Corollary 1.4.** If $G \cong C_{m_1} \times \cdots \times C_{m_k}$, where $m_{i+1}$ divides $m_i$ for all $i$, $1 \leq i \leq k$, then

$$\mathcal{M}(G) \cong C_{m_2} \times C_{m_3}^{(2)} \times \cdots \times C_{m_k}^{(k-1)}.$$  

2. Proof of the Main Theorem

In this section we want to prove our result. The following technical lemmas shorten the proof of our Main Theorem.

**Lemma 2.1.** Let $G$ be a finite $p$-group of order $p^n$ such that $G/G'$ is elementary of order $p^{n-1}$, then $G$ is a central product of an extra special $p$-group $H$ and $Z(G)$, such that $H \cap Z(G) = G'$.

**Proof.** Let $H/G'$ be the complement of $Z(G)/G'$ in $G/G'$. Then $G = HZ(G)$, so $G' = H'$ and $Z(H) = Z(G) \cap H$. On the other hand, $1 \neq Z(G) \cap H \leq G'$, and the result follows. \qed

**Lemma 2.2.** Let $G$ be an abelian $p$-group of order $p^n$ which is not elementary abelian. Then $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)}$.

**Proof.** The result is obtained obviously if $G$ is cyclic. So, let $G \cong C_{p^{m_1}} \times C_{p^{m_2}} \times \cdots \times C_{p^{m_k}}$ such that $\sum_{i=1}^{k} m_i = n$ and $m_1 \geq m_2 \geq \cdots \geq m_k$. We know that $m_1 \geq 2$, and then, by using Corollary 1.4,
Lemma 2.3. Let $G$ be a non-abelian $p$-group of order $p^n$ with derived subgroup of order $p$ such that $G/G'$ is not elementary abelian, then $|\mathcal{M}(G)| < p^{\frac{1}{2}(n-1)(n-2)+1}$.

Proof. By using Theorem 1.1 and Lemma 2.2,

$$|\mathcal{M}(G)| \leq p^{-1} |\mathcal{M}(\frac{G}{G'})| \left| \frac{G}{G'} \otimes G' \right|$$

$$\leq p^{-1} p^{\frac{1}{2}(n-2)(n-3)} p^{(n-1)}$$

$$< p^{\frac{1}{2}(n-1)(n-2)+1},$$

which completes the proof. □

Lemma 2.4. Let $G$ be a non-abelian $p$-group of order $p^n$, such that $G/G'$ is elementary abelian of order $p^{n-1}$, then $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}$ and the equality holds if and only if $G = H \times Z$, where $H$ is an extra special $p$-group of order $p^3$ and exponent $p$, and $Z$ is an elementary abelian $p$-group.

Proof. By Lemma 2.1, $G$ is a central product of $H$ and $Z(G)$, and by Theorem 1.2, we may assume that $|Z(G)| \geq p^2$. Let $|H| = p^{2m+1}$, so $|Z(G)| = p^{n-2m}$.

Suppose first that $m \geq 2$. If $Z(G)$ is elementary abelian, let $T$ be a group such that $Z(G) \cong G' \times T$. By using Theorems 1.2 and 1.3, we have

$$|\mathcal{M}(G)| = |\mathcal{M}(H \times T)|$$

$$= |\mathcal{M}(H)||\mathcal{M}(T)| \left| \frac{H}{H'} \otimes T \right|$$

$$= p^{2m^2-m-1} p^{\frac{(n-2m-1)(n-2m-2)}{2}} p^{2m(n-2m-1)}$$

$$= p^{\frac{1}{2}(n^2-3n)}$$

$$< p^{\frac{1}{2}(n-1)(n-2)+1}. $$

Now assume that $Z(G)$ is not elementary abelian. Then, Theorems 1.1 and 1.3 imply that

$$|\mathcal{M}(G)| \leq p |\mathcal{M}(H \times Z(G))| = p |\mathcal{M}(H)||\mathcal{M}(Z(G))| \left| \frac{H}{H'} \otimes Z(G) \right|.$$ 

Hence by using Theorem 1.2 and Lemma 2.2, we have

$$|\mathcal{M}(G)| \leq pp^{2m^2-m-1} p^{\frac{1}{2}(n-2m-1)(n-2m-2)} p^{2m(n-2m-1)}$$

$$< p^{\frac{1}{2}(n-1)(n-2)+1}. $$

If $H$ is extra special of order $p^3$ and $Z(G)$ is not elementary abelian, then Theorem 1.1 implies that
\[ |M(G)| \leq p^{-1} |M\left( \frac{G}{Z(G)} \right)| |M(Z(G))| \left| \frac{G}{Z(G)} \otimes Z(G) \right| \]
\[ \leq p^{\frac{1}{2}(n-3)+1} \]
\[ < p^{\frac{1}{2}(n-2)+1}. \]

By Theorem 1.2, it is easy to see that if \( Z(G) \) is elementary abelian, then \( |M(G)| = p^{\frac{1}{2}(n-1)(n-2)+1} \) if \( H \) is extra special of order \( p^3 \) and exponent \( p \); and in other cases \( |M(G)| < p^{\frac{1}{2}(n-1)(n-2)+1} \). □

**Proof of the Main Theorem.** We prove the theorem by induction on \( k \). If \( k = 1 \) the result is obtained by Lemmas 2.3 and 2.4. Let \( G \) be a non-abelian \( p \)-group of order \( p^n \) with derived subgroup of order \( p^k \) \( (k \geq 2) \). Choose \( K \) in \( G' \cap Z(G) \) of order \( p \). By using induction hypothesis, we have

\[ |M\left( \frac{G}{K} \right)| \leq p^{\frac{1}{2}(n+k-4)(n-k-1)+1}. \]

On the other hand, Theorem 1.1, implies that

\[ |M(G)| \leq p^{-1} |M\left( \frac{G}{K} \right)| |M(K)| \left| \left( \frac{G}{G'} \otimes K \right) \right| \]
\[ \leq p^{\frac{1}{2}(n+k-4)(n-k-1)} p^{n-k} \]
\[ = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}. \]

Now let \( G \) be a \( p \)-group of order \( p^n \) such that \( |M(G)| = p^{\frac{1}{2}(n-1)(n-2)+1} \). If \( |G'| \geq p^2 \), then \( |M(G)| \leq p^{\frac{1}{2}(n-1)(n-2)} \), which is a contradiction.

Since \( |G'| = p \), Lemma 2.3 implies that \( G/G' \) is elementary abelian. Hence Lemma 2.4 shows that \( G = H \times Z \), where \( H \) is an extra special \( p \)-group of order \( p^3 \) and exponent \( p \), and \( Z \) is an elementary abelian \( p \)-group, so the result follows. □

**References**


