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A Completeness Theorem for a Nonlinear Multiparameter Eigenvalue Problem

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In this paper we study the linked nonlinear multiparameter system

$$y_r''(x_r) + M_r y_r + \sum_{s=1}^{k} \lambda_s (a_{rs}(x_r) + P_{rs})\, y_r(x_r) = 0, \qquad r = 1,\dots, k,$$

where $x_r \in [a_r, b_r]$, y_r is subject to Sturm–Liouville boundary conditions, and the continuous functions a_{rs} satisfy $|A|(x) = \det\{a_{rs}(x_r)\} > 0$. Conditions on the polynomial operators M_r, P_{rs} are produced which guarantee a sequence of eigenfunctions for this problem $y^n(x) = \prod_{r=1}^{k} y_r^n(x_r)$, $n \geq 1$, which form a basis in $L^2([a, b], |A|)$. Here $[a, b] = [a_1, b_1] \times \cdots \times [a_k, b_k]$.

1. INTRODUCTION

In a recent paper [3], K. J. Brown proved interesting completeness results for a Sturm–Liouville eigenvalue problem perturbed by certain nonlinear factors. His main theorem is a direct application of a bifurcation result of Crandall and Rabinowitz [8]. The Crandall–Rabinowitz result follows readily from an implicit function theorem in Banach space. It is our purpose here to use the implicit function theorem to produce a result similar to that of Crandall and Rabinowitz and then, following the ideas of Brown, to obtain a completeness theorem for a linked system of nonlinear second-order ordinary differential equations.

During the past few years, increasing interest has been paid to the linked system of linear differential equations

$$\frac{d^2 y_r(x_r)}{dx_r^2} + q_r(x_r)\, y_r(x_r) + \sum_{s=1}^{k} \lambda_s a_{rs}(x_r)\, y_r(x_r) = 0, \qquad r = 1, 2,\dots, k, \tag{1}$$

where $x_r \in [a_r, b_r]$, λ_s are complex parameters, and $q_r(x_r)$, $a_{rs}(x_r)$ are continuous real-valued functions defined on $[a_r, b_r]$, $r, s = 1, 2,\dots, k$.

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Sturm–Liouville boundary conditions are imposed; viz

$$y_r(a_r)\cos\alpha_r + y_r'(a_r)\sin\alpha_r = 0, \qquad 0 \leqslant \alpha_r < \pi,$$
$$y_r(b_r)\cos\beta_r + y_r'(b_r)\sin\beta_r = 0, \qquad 0 < \beta_r \leqslant \pi, \quad r = 1, 2,..., k.$$

The functions $a_{rs}(x_r)$ are assumed to satisfy the definiteness condition

$$|A|(x) = \det\{a_{rs}(x_r)\} > 0$$

for $x = (x_1 ,..., x_n) \in [a, b] = [a_1, b_1] \times \cdots \times [a_k, b_k]$. A k-tuple of (necessarily real) numbers $\lambda = (\lambda_1 ,..., \lambda_k)$ and a function $f(x) = f_1(x_1) \cdots f_k(x_k)$ are then called an eigenvalue and eigenfunction, respectively, if λ and $f_1 ,..., f_k$ satisfy the above equations and boundary conditions. In this manner a so called multiparameter eigenvalue problem is formulated.

It is known that $L^2([a, b], |A|)$ the space of all real-valued measureable functions f defined on $[a, b]$ for which

$$\|f\|^2 = \int_{[a,b]} |f(x)|^2 |A|(x)\, dx < \infty$$

has a complete orthonormal basis of eigenfunctions of the above multiparameter problem. The inner product in this space is given by

$$(f, g) = \int_{[a,b]} f(x)\, g(x) |A|(x)\, dx.$$

For details of this result and an overview of recent developments in multiparameter spectral theory, the reader may consult Atkinson [1, 2], Browne [4–7], Faierman [10], and Sleeman [12–16].

We intend perturbing the above linear equations by nonlinear operators acting on the functions $y_r(x_r)$. Our main result will be that the new linked problem has a sequence of eigenvalues λ^n and corresponding eigenfunctions $f^n(x)$ forming a basis for $L^2([a, b], |A|)$. In this context the word "basis" means that $f \in L^2([a, b], |A|)$ will be expressible uniquely as $f = \sum_{j=1}^{\infty} c_j f^j$, $c_1, c_2, ...$ real numbers.

To conclude our introductory remarks we state two propositions which will be needed in the sequel.

PROPOSITION 1. *Let $\{u_n\}$ be a complete orthonormal system for a Hilbert space H. If $\{v_n\}$ is a sequence of vectors in H such that $\sum_{j=1}^{\infty} \| u_j - v_j \|^2 < 1$, then $\{v_n\}$ is a basis for H.*

Proof. See [11, Sec. 5, Theorem 2.20 et seq., p. 265].

PROPOSITION 2 (The Implicit Function Theorem). *Let E, F, G be three Banach spaces, f a continuous mapping of an open subset A of $E \times F$ into G. Let the* *map $y \to f(x, y)$ of $A_x = \{y \in F \mid (x, y) \in A\}$ into G be differentiable in A_x for each $x \in E$ such that $A_x \neq \varnothing$, and assume the derivative of this map, denoted by f_y, is continuous on A. Let $(x_0, y_0) \in A$ be such tuat $f(x_0, y_0) = 0$ and $f_y(x_0, y_0)$ is a linear homeomorphism of F onto G. Then there are neighborhoods U of x_0 in E and V of y_0 in F such that*

(i) $U \times V \subset A$,

(ii) *there is exactly one function u: $U \to V$ satisfying $f(x, u(x)) = 0$ for $x \in U$,*

(iii) *the mapping u of* (ii) *is continuous.*

If, moreover, the mapping f is k times continuously differentiable on A, then (iii) *above may be replaced by*

(iv) *u is k times continuously differentiable.*

Proof. This theorem is stated by Crandall and Rabinowitz [8] who quote [9, Theorem 10.2.1, p. 270] as a suitable reference.

2. The Bifurcation Problem

We adopt the following notation. The subspace D_r of $L^2([a_r, b_r])$ is defined by

$$D_r = \{f_r(x_r) \in L^2([a_r, b_r]) \mid f_r'(x_r) \quad \text{is absolutely continuous,}$$

$$f_r''(x_r) + q_r(x_r) f_r(x_r) \in L^2([a_r, b_r])$$

and

$$f_r(a_r) \cos \alpha_r + f_r'(a_r) \sin \alpha_r = 0,$$

$$f_r(b_r) \cos \beta_r + f_r'(b_r) \sin \beta_r = 0\}.$$

$V_{rs}: L^2([a_r, b_r]) \to L^2([a_r, b_r])$ denotes the continuous linear map $(V_{rs} f_r)(x_r) = a_{rs}(x_r) f_r(x_r)$.

It will be assumed that the linear problem (1) is in a "normalized" form in the sense that

$$\sum_{s=1}^{k} a_{rs}(x_r) = 1, \qquad \forall x_r \in [a_r, b_r], \qquad r = 1, 2, \ldots, k.$$

This is no real restriction as is discussed in [16, Sect. 3, pp. 203–206]. Lemma 2 of this reference states that we may select real numbers $\mu_1, \ldots, \mu_k$ such that $\sum_{s=1}^{k} \mu_s a_{rs}(x_r) > 0 \; \forall x_r \in [\alpha_r, b_r]$, $r = 1, 2, \ldots, k$. As stated the lemma allows the possibility of the intervals $[a_r, b_r]$ being half-lines. The result is not correct in this case. It is essential that these intervals be compact—the proof given tacitly makes this assumption. We also see that we may select each $\mu_s \neq 0$. This is an immediate consequence of the continuity of the functions $a_{rs}(x_r)$.

T_r will denote the operator with domain D_r defined by $T_r f_r = f'' + q_r f_r$. T_r is self-adjoint in $L^2([a_r, b_r])$. X_r will denote the space D_r equipped with the graph norm $|||f_r||| = \|f_r\| + \|T_r f_r\|$. Under this norm, X_r is a Banach space.

We shall denote by M_r, P_{rs}, nonlinear maps defined on X_r and taking values in $L^2([a_r, b_r])$. These maps are assumed to be continuously Fréchet differentiable and satisfy $M_r(0) = P_{rs}(0) = 0$, $M_r'(0) = P_{rs}'(0) = 0$, $r, s = 1, 2,..., k$.

Now let $\lambda = (\lambda_1, ..., \lambda_k)$ be an eigenvalue of the linear multiparameter problem (1) with associated $|A|$-normalized eigenfunction $u(x) = u_1(x_1) \cdots u_k(x_k)$. Then 0 is a simple eigenvalue of each Sturm–Liouville operator

$$T_r + \sum_{s=1}^{k} \lambda_s V_{rs} : D_r \to L^2([a_r, b_r]), \quad r = 1, 2,..., k.$$

Let Z_r be any complement of $\text{span}\{u_r\}$ in X_r. We define a map $F_r: \mathbb{R} \times (\mathbb{R} \times Z_r) \to L^2([a_r, b_r])$ as follows. If $\alpha_r \in \mathbb{R}$, $(t_r, z_r) \in \mathbb{R} \times Z_r$, we put

$$\begin{aligned} F_r(\alpha_r, t_r, z_r) &= (1/\alpha_r)\{T_r(\alpha_r u_r + \alpha_r z_r) + M_r(\alpha_r u_r + \alpha_r z_r) \\ &\quad + \sum_{s=1}^{k} (\lambda_s + t_r)(V_{rs} + P_{rs})(\alpha_r u_r + \alpha_r z_r)\} \qquad \text{if} \quad \alpha_r \neq 0, \\ &= T_r(u_r + z_r) + \sum_{s=1}^{k} (\lambda_s + t_r)\, V_{rs}(u_r + z_r) \qquad \text{if} \quad \alpha_r = 0, \end{aligned}$$

$r = 1, 2,..., k$. F_r is continuously differentiable in (t_r, z_r) for each fixed α_r and satisfies $F_r(0, 0, 0) = 0$. $F_r(0, t_r, z_r)$ has as its (t_r, z_r) derivative at the point $(t_r, z_r) = (0, 0)$ the linear map defined on $\mathbb{R} \times Z_r$ with values in $L^2([a_r, b_r])$ given by

$$(t_r, z_r) \to T_r z_r + \sum_{s=1}^{k} \lambda_s V_{rs} z_r + t_r u_r. \tag{2}$$

We have made use here of the normalization $\sum_{s=1}^{k} a_{rs}(x_r) = 1$. Notice that $T_r z_r + \sum_{s=1}^{k} \lambda_s V_{rs} z_r$ is orthogonal to u_r and indeed the simplicity of the eigenvalue 0 of $T_r + \sum_{s=1}^{k} \lambda_s V_{rs}$ implies that the linear map (2) is a linear homeomorphism of $\mathbb{R} \times Z_r$ onto $L^2([a_r, b_r])$. We now apply the implicit function theorem and claim the existence of continuously differentiable functions $t_r(\alpha_r): \mathbb{R} \to \mathbb{R}$ and $z_r(\alpha_r): \mathbb{R} \to Z_r$ satisfying $t_r(0) = 0$, $z_r(0) = 0$ and

$$T_r(\alpha_r u_r + \alpha_r z_r) + M_r(\alpha_r u_r + \alpha_r z_r) + \sum_{s=1}^{k} (\lambda_s + t_r)(V_{rs} + P_{rs})(\alpha_r u_r + \alpha_r z_r) = 0.$$

These remarks hold for each choice of $r = 1, 2,..., k$.

The implicit function theorem guarantees that the functions $t_r(\alpha_r)$, $z_r(\alpha_r)$ are once continuously differentiable. Higher degrees of differentiability of these functions will depend on the nature of the maps M_r, P_{rs}. To produce bifurcation from the eigenvalue λ it will be necessary to impose conditions on the maps M_r, P_{rs} which ensure the possibility of selecting $\alpha_1, \ldots, \alpha_k$ so that $t_1(\alpha_1) = \cdots = t_k(\alpha_k)$ thus preserving the linking in our eigenvalue problem. In the remainder of this section we display some simple situations in which this bifurcation occurs.

Our first example is covered by

HYPOTHESIS 1. Assume $P_{rs} = 0$, $r, s = 1, 2, \ldots, k$ and that each M_r is a polynomial operator with constant coefficients

$$M_r f_r = \sum_{i=0}^{p} A_r^i f_r^{i+2},$$

where $A_r^{2i+1} \geqslant 0$, $i = 0, 1, \ldots, [p/2 - \frac{1}{2}]$, $r = 1, 2, \ldots, k$, with at least one of these inequalities being strict for each r.

Here $[\cdot]$ represents the integer part function. We are interested in the first nonvanishing derivative of the functions $t_r(\alpha_r)$ at $\alpha_r = 0$. If this derivative has odd order then $t_r(\alpha_r)$ takes both positive and negative values for α_r ranging through some neighborhood of 0. Suppose then that $0 = t_r(0) = t_r'(0) = \cdots = t_r^{(n-1)}(0)$, $t^{(n)}(0) \neq 0$ where n is even. The value of n will, in general, be different for each r. Then by the implicit function theorem and [8, Eq. (1.20)] we have

$$(\partial^n/\partial\alpha_r^n) F_r(0, 0, 0) + (\partial F_r/\partial t_r)(0, 0, 0)\, t_r^{(n)}(0) + (\partial F_r/\partial z_r)(0, 0, 0)\, z_r^{(n)}(0) = 0.$$

Evaluating these derivatives we obtain

$$n!\, A_r^{n-1} u_r^{n+1} + t_r^{(n)}(0)\, u_r + \left(T_r + \sum_{s=1}^{k} \lambda_s V_{rs}\right) z_r^{(n)}(0) = 0.$$

Multiply this equation throughout by u_r and integrate over the interval $[a_r, b_r]$. Recall that $T_r + \sum_{s=1}^{k} \lambda_s V_{rs}$ is self-adjoint and that u_r is in its kernel. These calculations yield

$$\begin{aligned} t_r^{(n)}(0) &= \left[n!\, A_r^{n-1} \int_{a_r}^{b_r} (u_r(x_r))^{n+2}\, dx_r \Big/ \int_{a_r}^{b_r} (u_r(x_r))^2\, dx_r\right] \\ &> 0 \end{aligned}$$

in view of Hypothesis 1. Thus assuming the first nonvanishing derivative of t_r at $\alpha_r = 0$ has even order leads to the conclusion that this derivative must in fact be positive. Notice that Hypothesis 1 and the above calculations show that not all derivatives of t_r at $\alpha_r = 0$ can vanish. We conclude in this case that there is a neighborhood of $\alpha_r = 0$ in which t_r has positive values.

We now appeal to the intermediate value theorem and claim the existence of

$\epsilon_0 > 0$ such that for $0 \leqslant t < \epsilon_0$, there exist $\alpha_1, \ldots, \alpha_k$ with $t_1(\alpha_1) = \cdots = t_k(\alpha_k) = t$. Accordingly we can find solutions $t, f_1, \ldots, f_k$ of the linked problem

$$T_r f_r + M_r f_r + \sum_{s=1}^{k} (\lambda_s + t) V_{rs} f_r = 0, \qquad r = 1, 2, \ldots, k$$

with f_r of the form $\alpha_r u_r + \alpha_r z_r$. Regarding t and z_r as functions of $\alpha = (\alpha_1, \ldots, \alpha_k)$ we have $t \to 0$, $z_r \to 0$ as $\alpha \to 0$.

It is clear that corresponding results can be obtained by methods similar to the above if Hypothesis 1 is replaced by any of the following.

HYPOTHESIS 2. Assume $P_{rs} = 0$, $r, s = 1, 2, \ldots, k$, and that each M_r is a polynomial operator

$$(M_r f_r)(x_r) = \sum_{i=0}^{p} A_r^i(x_r) f_r^{i+2}(x_r), \qquad A_r^i \in C[a_r, b_r],$$

where $A_r^{2i+1}(x_r) \geqslant 0$, $i = 0, 1, \ldots, [p/2 - \frac{1}{2}]$, $r = 1, 2, \ldots, k$, with at least one of the inequalities being strict for each r.

HYPOTHESIS 3. Assume $P_{rs} = 0$, $r, s = 1, 2, \ldots, k$, and that each M_r is a polynomial operator

$$(M_r f_r)(x_r) = \sum_{i=0}^{p} A_r^i(x_r) f_r^{i+2}(x_r), \qquad A_r^i \in C[a_r, b_r],$$

where $A_r^{2i+1}(x_r) \leqslant 0$, $i = 0, 1, \ldots, [p/2 - \frac{1}{2}]$, $r = 1, 2, \ldots, k$, with at least one of the inequalities being strict for each r.

HYPOTHESES 4, 5, 6. Assume $M_r = 0$, $r = 1, 2, \ldots, k$, and that each P_{rs} is a polynomial operator with no constant or linear terms and such that the operators $\sum_{s=1}^{k} \lambda_s P_{rs}$ satisfy the conditions of Hypotheses 1, 2, 3, respectively, $r = 1, 2, \ldots, k$.

These hypotheses provide a range of simple situations in which we have a bifurcation phenomenon at the simple eigenvalue λ.

Notice that in case $k = 1$ we may return to the hypotheses of Brown [3], for then the problem of preserving the linking in the eigenvalue problem no longer arises.

3. THE COMPLETENESS THEOREM

THEOREM. *Assume that one of Hypotheses* 1, 2, 3 *holds or that one of Hypotheses* 4, 5, 6 *holds at each eigenvalue of the linear problem* (1). *Then the problem*

$$T_r f_r + M_r f_r + \sum_{s=1}^{k} \mu_s (V_{rs} + P_{rs}) f_r = 0, \qquad r = 1, 2, \ldots, k$$

has a sequence of eigenvalues $\mu^n = (\mu_1{}^n, \ldots, \mu_k{}^n)$ *and corresponding eigenfunctions* $f_r{}^n$ *such that* $(\prod_{r=1}^k f_r{}^n)_{n=1}^\infty$ *forms a basis for* $L^2([a, b], | A |)$.

Proof. Let $\lambda^n = (\lambda_1{}^n, \ldots, \lambda_k{}^n)$ denote the eigenvalues of the linear problem (1) with $u^n(x) = u_1{}^n(x_1) \cdots u_k{}^n(x_k)$, $n = 1, 2 \ldots$ the corresponding orthonormal basis of eigenfunctions for $L^2([a, b], | A |)$. We shall denote by $\| \, \|_A$, the norm in this space. Note that this norm is equivalent to the usual unweighted Lebesque norm in $L^2([a, b])$.

Applying the theory of paragraph 2, at each λ^n, we select $\alpha_r{}^n \neq 0$, $r = 1, 2, \ldots, k$ so that $t_1(\alpha_1{}^n) = \cdots = t_k(\alpha_k{}^n) = t_n$, say. We shall further require that if $v_r{}^n = u_r{}^n + z_r(\alpha_r{}^n)$ and $v^n = v_1{}^n \cdots v_k{}^n$ then $\| u^n - v^n \|_A < 1/2^{(n+1)}$. That this is possible follows from the continuity of the functions $z_r(\alpha_r)$ discussed in paragraph 2. We now have $\sum_{n=1}^\infty \| u^n - v^n \|^2 < 1$, so that by Proposition 1, $\{v^n\}$ forms a basis for $L^2([a, b], | A |)$. The claim of the theorem now follows by setting $f_r{}^n = \alpha_r{}^n v_r{}^n$, $r = 1, 2, \ldots, k$, $n = 1, 2, \ldots$, and noting that $f_r{}^n$, $r = 1, 2, \ldots, k$ are solutions of the nonlinear system for $\mu_s = \lambda_s{}^n + t_n$.

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