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$\mathcal{D}$-forced spaces: A new approach to resolvability ☆

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Abstract

We introduce a ZFC method that enables us to build spaces (in fact special dense subspaces of certain Cantor cubes) in which we have "full control" over all dense subsets.

Using this method we are able to construct, in ZFC, for each uncountable regular cardinal λ a 0-dimensional T_2, hence Tychonov, space which is μ-resolvable for all $\mu < \lambda$ but not λ-resolvable. This yields the final (negative) solution of a celebrated problem of Ceder and Pearson raised in 1967: Are ω-resolvable spaces maximally resolvable? This method enables us to solve several other open problems concerning resolvability as well.

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1. Introduction

Resolvability questions about topological spaces were first studied by Hewitt [15], in 1943. Given a cardinal $\kappa > 1$, a topological space $X = \langle X, \tau_X \rangle$ is called *κ-resolvable* iff it contains κ disjoint dense subsets. X is *resolvable* iff it is 2-resolvable and *irresolvable* otherwise.

If X is κ-resolvable and $G \subset X$ is any non-empty open set in X then clearly $\kappa \leqslant |G|$. Hence if X is κ-resolvable then we have $\kappa \leqslant \Delta(X)$ where

$$\Delta(X) = \min\{|G|:\ G \in \tau_X \setminus \{\emptyset\}\}.$$

This observation explains the following terminology of Ceder [5]: a space X is called *maximally resolvable* iff it is $\Delta(X)$-resolvable.

Ceder and Pearson in [6], raised the question whether an ω-resolvable space is necessarily maximally resolvable? El'kin [12], Malykhin [18], Eckertson [11], and Hu [16], gave several counterexamples, but either these spaces were not even T_2 or their construction was not carried out in ZFC. Our Theorems 4.5 and 4.8 give a large number of 0-dimensional T_2 (and so Tychonov) counterexamples in ZFC. The question if this can be done has been asked much more recently again in [7,9].

Our results are obtained with the help of a new method that is presented in Section 2. Here we first introduce the new and simple concept of *$\mathcal{D}$-forced spaces*. Given a family $\mathcal{D}$ of dense subsets of the space X we say that X (or its topology) is *$\mathcal{D}$-forced* if any subset of X can only be dense in X if $\mathcal{D}$ forces this to happen. The exact formulation of this reads as follows: If S is dense in X then S includes a set of the form

$$M = \bigcup\{V \cap D_V:\ V \in \mathcal{V}\}$$

where $\mathcal{V}$ is a maximal disjoint collection of open sets in X and $D_V \in \mathcal{D}$ for all $V \in \mathcal{V}$. Such a set M, that is clearly dense in X, is called a $\mathcal{D}$-mosaic. Then, in Lemmas 2.6–2.8, and 2.10, we establish the basic properties of $\mathcal{D}$-forced spaces.

In the next section we prove our main result, Theorem 3.3, that will allow us to construct $\mathcal{D}$-forced subspaces of certain Cantor cubes with a wide range of resolvability, respectively irresolvability properties. Thus, in Sections 4 and 5, we shall be able to answer not only the problem of Ceder and Pearson mentioned above but several other open problems as well, like [1, Question 4.4], [2, Problem 8.6], [11, Questions 3.4, 3.6, 4.5], or a problem of Comfort and Hu mentioned in [9, Discussion 1.4].

In the remaining part of this introduction we summarize our further notation and terminology, most of it is standard.

A space X is called *open hereditarily irresolvable* (*OHI*) iff every non-empty open subspace of X is irresolvable. It is well known that every irresolvable space has a non-empty open subspace that is OHI. Clearly, X is OHI iff every dense subset of X contains a dense open subset, i.e., if $S \subset X$ dense in X implies that $\mathrm{Int}(S)$ is dense, as well.

Next, a space X is called *hereditarily irresolvable* (*HI*) iff all subspaces of X are irresolvable. Since a space having an isolated point is trivially irresolvable, any space is HI iff all its *crowded* subspaces are irresolvable. (Following van Douwen, we call a space crowded if it has no isolated points.) Having this in mind, if P is any resolvability or irresolvability property of topological spaces then the space X is called *hereditarily P* iff all *crowded* subspaces of X have property P.

Following the terminology of [10], a topological space X is called NODEC if all nowhere dense subsets of X are closed, and hence closed discrete. All spaces obtained by our main Theorem 3.3 will be NODEC.

A space is called *submaximal* (see [15]) iff all of its dense subsets are open. The following observation is easy to prove and will be used repeatedly later: a space is submaximal iff it is both OHI and NODEC.

A set $D \subset X$ is said to be *κ-dense* in X iff $|D \cap U| \geqslant \kappa$ for each non-empty open set $U \subset X$. Thus D is dense iff it is 1-dense. Also, it is obvious that the existence of a κ-dense set in X implies $\Delta(X) \geqslant \kappa$.

We shall denote by $\mathcal{N}(X)$ the family of all nowhere dense subsets of a space X. Clearly, $\mathcal{N}(X)$ is an ideal of subsets of X and the notation $=^*$ or $\subset^*$ will always be used to denote equality, respectively inclusion modulo this ideal.

Following the notation introduced in [8], we shall write

$$\mathrm{nwd}(X) = \min\bigl\{|Y|\colon\ Y \in \mathcal{P}(X) \setminus \mathcal{N}(X)\bigr\} = \text{non-}\bigl(\mathcal{N}(X)\bigr),$$

i.e. $\mathrm{nwd}(X)$ is the minimum cardinality of a somewhere dense subset of X.

Malykhin was the first to suggest studying families of dense sets of a space X that are *almost disjoint* with respect to the ideal $\mathcal{N}(X)$ rather than disjoint, see [19]. He calls a space X *extraresolvable* if there are $\Delta(X)^+$ many dense sets in X such that any two of them have nowhere dense intersection. Here we generalize this concept by defining a space X to be *κ-extraresolvable* if there are κ many dense sets in X such that any two of them have nowhere dense intersection. (Perhaps κ-almost resolvable would be a better name for this.) Note that, although κ-extraresolvability of X is mainly of interest if $\kappa > \Delta(X)$, it does make sense for $\kappa \leqslant \Delta(X)$ as well. Clearly, κ-resolvable implies κ-extraresolvable, moreover the converse holds if $\kappa = \omega$, however we could not decide if these two concepts coincide if

$$\omega < \kappa \leqslant \Delta(X).$$

In particular, we would like to know the answer to the following question.

Problem 1.1. Let X be an extraresolvable (T_2, T_3, or Tychonov) space with $\Delta(X) \geqslant \omega_1$. Is X then ω_1-resolvable?

Note that a counterexample to Problem 1.1 is also a counterexample to the Ceder–Pearson problem.

Finally we mention a variation of extraresolvability. The space X is called *strongly κ-extraresolvable* iff there are κ many dense subsets $\{D_\alpha\colon\ \alpha < \kappa\}$ of X such that $|D_\alpha \cap D_\beta| < \mathrm{nwd}(X)$ whenever $\{\alpha, \beta\} \in [\kappa]^2$. We say that X is *strongly extraresolvable* iff it is strongly $\Delta(X)^+$-extraresolvable. Clearly, strongly (κ)-extraresolvable implies (κ)-extraresolvable.

2. $\mathcal{D}$-forced spaces

Definition 2.1. Let $\mathcal{D}$ be a family of dense subsets of a space X. A subset $M \subset X$ is called a $(\mathcal{D}, X)$-*mosaic* iff there is a maximal disjoint family $\mathcal{V}$ of open subsets of X and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ such that

$$M = \bigcup \{V \cap D_V\colon\ V \in \mathcal{V}\}.$$

A set M of the above form with $\mathcal{V}$ disjoint, but not necessarily maximal disjoint, is called a *partial* $(\mathcal{D}, X)$-mosaic.

A set P of the form $P = D \cap U$, where $D \in \mathcal{D}$ and U is a non-empty open subset of X, is called a $(\mathcal{D}, X)$-*piece*. So, naturally, any (partial) $(\mathcal{D}, X)$-mosaic is composed of $(\mathcal{D}, X)$-pieces. Let

$$\mathfrak{M}(\mathcal{D}, X) = \big\{M\colon\ M \text{ is a } (\mathcal{D}, X)\text{-mosaic}\big\}$$

and

$$\mathcal{P}(\mathcal{D}, X) = \big\{P\colon\ P \text{ is a } (\mathcal{D}, X)\text{-piece}\big\}.$$

When the space X is clear from the context we will omit it from the notation: we will write $\mathcal{D}$-*mosaic* instead of $(\mathcal{D}, X)$-mosaic, and $\mathcal{D}$-*piece* instead of $(\mathcal{D}, X)$-piece, etc. The following statement is now obvious.

Fact 2.2. *Every $(\mathcal{D}, X)$-mosaic is dense in X and every $(\mathcal{D}, X)$-piece is somewhere dense in X.*

Thus we arrive at the following very simple but, as it turns out, very useful concept.

Main Definition 2.3. Let $\mathcal{D}$ be a family of dense subsets of a topological space X. We say that the space X (or its topology) is $\mathcal{D}$-*forced* iff every dense subset S of X includes a $\mathcal{D}$-mosaic M, i.e. there is $M \in \mathfrak{M}(\mathcal{D}, X)$ with $M \subset S$.

It is easy to check that one can give the following alternative characterization of being $\mathcal{D}$-forced.

Fact 2.4. *The space X is $\mathcal{D}$-forced iff every somewhere dense subset of X includes a $(\mathcal{D}, X)$-piece.*

Since X is always dense in X, the simplest choice for $\mathcal{D}$ is $\{X\}$.

Fact 2.5. *A subset $P \subset X$ is an $\{X\}$-piece iff it is non-empty open*; *M is an $\{X\}$-mosaic iff it is dense open in X. Consequently, X is $\{X\}$-forced iff it is OHI.*

Let us now consider a few further, somewhat less obvious, properties of $\mathcal{D}$-forced spaces. The first result yields a useful characterization of nowhere dense subsets in such spaces. Note that a subset Y of any space X is nowhere dense iff $S \backslash Y$ is dense in X for all dense subsets S of X. Not surprisingly, in a $\mathcal{D}$-forced space it suffices to check this for members of $\mathcal{D}$.

Lemma 2.6. *Assume that X is $\mathcal{D}$-forced. Then*

$$\mathcal{N}(X) = \{Y \subset X\colon\ D \setminus Y \text{ is dense in } X \text{ for each } D \in \mathcal{D}\}.$$

Proof. Assume that $Y \notin \mathcal{N}(X)$, i.e. Y is somewhere dense. Then, by Fact 2.4, Y contains some $\mathcal{D}$-piece $U \cap D$, where $D \in \mathcal{D}$ and U is a non-empty open subset of X. Then $(D \setminus Y) \cap U = \emptyset$, i.e. $D \setminus Y$ is not dense. This proves that the right-hand side of the equality includes the left one. The converse inclusion is obvious. □

The following result will be used to produce irreducible (even OHI) spaces. Of course, the superscript * in its formulation designates equality and inclusion modulo the ideal $\mathcal{N}(X)$ of nowhere dense sets.

Lemma 2.7. *Let X be $\mathcal{D}$-forced and $S \subset X$ be dense such that*

$$\text{for each } D \in \mathcal{D} \text{ we have } S \cap D =^* \emptyset \text{ or } S \subset^* D. \tag{†}$$

Then S, as a subspace of X, is OHI.

Proof. Let $T \subset S$ be dense in S, then T is also dense in X, hence it must contain a $\mathcal{D}$-mosaic, say $M = \bigcup\{V \cap D_V\colon\ V \in \mathcal{V}\}$. But then we have $S \subset^* D_V$ for each $V \in \mathcal{V}$ by (†). Consequently,

$$T \cap V \subset S \cap V \subset^* V \cap D_V \subset T \cap V$$

and so $T \cap V =^* S \cap V$ holds for all $V \in \mathcal{V}$. This clearly implies that $T =^* S$. In other words, we have shown that every dense subset T of S has nowhere dense complement in S, i.e. the subspace S of X is OHI. □

The following lemma will enable us to conclude that certain $\mathcal{D}$-forced spaces are not κ-(extra)resolvable for appropriate cardinals κ.

Lemma 2.8. *Assume that X is a topological space and $\mathcal{D}$ is a family of dense subsets of X. Assume, moreover, that $\mu \geqslant \hat{c}(X)$* (*i.e., X does not contain μ many pairwise disjoint open subsets*) *and*

$$\begin{aligned}&\text{for each } \mathcal{E} \in [\mathcal{D}]^{\mu} \text{ there is } \mathcal{F} \in [\mathcal{E}]^{\hat{c}(X)} \text{ such that}\\ &\quad D_0 \cap D_1 \text{ is dense in } X \text{ whenever } \{D_0, D_1\} \in [\mathcal{F}]^2.\end{aligned} \tag{$*$}$$

Then for any family of $\mathcal{D}$-pieces $\{P_i\colon\ i < \mu\} \subset \mathcal{P}(\mathcal{D})$ there is $\{i, j\} \in [\mu]^2$ such that $P_i \cap P_j$ is somewhere dense in X.

In particular, if X is $\mathcal{D}$-forced and $|\mathcal{D}|^+ \geqslant \hat{c}(X)$ then X is not $|\mathcal{D}|^+$-extraresolvable (*hence not $|\mathcal{D}|^+$-resolvable, either*).

Proof. Assume that $P_i = U_i \cap D_i$, where $D_i \in \mathcal{D}$ and U_i is a non-empty open subset of X for all $i \in \mu$. By $(*)$ there is $I \in [\mu]^{\hat{c}(X)}$ such that $D_i \cap D_j$ is dense for each $\{i, j\} \in [I]^2$. By the definition of $\hat{c}(X)$, there is $\{i, j\} \in [I]^2$ such that $U = U_i \cap U_j$ is non-empty. But then $U \cap D_i \cap D_j \subset P_i \cap P_j$, hence $P_i \cap P_j$ is dense in the non-empty open set U.

The last statement now follows because $\mathcal{D}$ trivially satisfies condition $(*)$ with $\mu = |\mathcal{D}|^+$ and, as X is $\mathcal{D}$-forced, every dense subset of X includes a $\mathcal{D}$-piece (even a $\mathcal{D}$-mosaic). □

The following fact is obvious.

Fact 2.9. *Let $\mathcal{D}$ be a family of dense sets in X and*

$$M = \bigcup\{V \cap D_V\colon\ V \in \mathcal{V}\}$$

be a partial $\mathcal{D}$-mosaic. If all the dense sets D_V are μ-(extra)resolvable for $V \in \mathcal{V}$ then so is M.

We finish this section with a result that, together with Fact 2.9, will be used to establish hereditary (extra)resolvability properties of several examples constructed later.

Lemma 2.10. *Let X be a $\mathcal{D}$-forced space in which every crowded subspace is somewhere dense. (This holds e.g. if X is NODEC.) Then for every crowded $S \subset X$ there is a partial $\mathcal{D}$-mosaic $M \subset S$ that is dense in S. So if, in addition, all $D \in \mathcal{D}$ are μ-resolvable (respectively μ-extraresolvable) then X is hereditarily μ-resolvable (respectively μ-extraresolvable).*

Proof. Let $\mathcal{V}$ be a maximal disjoint family of open sets V such that there is an element $D_V \in \mathcal{D}$ with $V \cap D_V \subset S$ and consider the partial $\mathcal{D}$-mosaic

$$M = \bigcup\{V \cap D_V\colon\ V \in \mathcal{V}\}.$$

Then $M \subset S$ is dense in S, since otherwise, in view of the maximality of $\mathcal{V}$, the set $S \setminus \overline{M} \neq \emptyset$ would be crowded and could not include any $\mathcal{D}$-piece. The last sentence now immediately follows using Fact 2.9. □

3. The Main Theorem

We have introduced the concept of $\mathcal{D}$-forced spaces but one question that immediately will be raised is if there are any beyond the obvious choice of $\mathcal{D} = \{X\}$? The aim of this section is to prove Theorem 3.3 that provides us with a large supply of such spaces. All these spaces will be dense subspaces of Cantor cubes, i.e., powers of the discrete two-point space $D(2)$. As is well known, there is a natural one-to-one correspondence between dense subspaces of size κ of the Cantor cube $D(2)^\lambda$ and independent families of 2-partitions of κ indexed by λ. (A partition of a set S is called a *μ-partition* if it partitions S into μ many pieces.) For technical reasons, we shall produce our spaces by using partitions rather than Cantor cubes.

We start with fixing some notation and terminology.

Let $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \mu \rangle$ be a sequence of cardinals. We set

$$\mathbb{FIN}(\vec{\lambda}) = \{\varepsilon : \varepsilon \text{ is a finite function with } \operatorname{dom}\varepsilon \in [\mu]^{<\omega} \text{ and } \varepsilon(\zeta) \in \lambda_\zeta \text{ for all } \zeta \in \operatorname{dom}\varepsilon\}.$$

Note that if $\lambda_\zeta = \lambda$ for all $\zeta < \mu$ then

$$\mathbb{FIN}(\vec{\lambda}) = Fn(\mu, \lambda).$$

Let S be a set, $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \mu \rangle$ be a sequence of cardinals, and $\mathbb{B} = \{\langle B^i_\zeta : i < \lambda_\zeta \rangle : \zeta < \mu\}$ be a family of partitions of S. Given a cardinal κ we say that $\mathbb{B}$ is *κ-independent* iff

$$\mathbb{B}[\varepsilon] \stackrel{\text{def}}{=} \bigcap \{B^{\varepsilon(\zeta)}_\zeta : \zeta \in \operatorname{dom}\varepsilon\}$$

has cardinality at least κ for each $\varepsilon \in \mathbb{FIN}(\vec{\lambda})$. $\mathbb{B}$ is *independent* iff it is 1-independent, i.e. $\mathbb{B}[\varepsilon] \neq \emptyset$ for each $\varepsilon \in \mathbb{FIN}(\vec{\lambda})$. $\mathbb{B}$ is *separating* iff for each $\{\alpha, \beta\} \in [S]^2$ there are $\zeta < \mu$ and $\{\rho, \nu\} \in [\lambda]^2_\zeta$ such that $\alpha \in B^\rho_\zeta$ and $\beta \in B^\nu_\zeta$.

We shall denote by $\tau_{\mathbb{B}}$ the (obviously zero-dimensional) topology on S generated by the subbase $\{B^i_\zeta : \zeta < \mu,\ i < \lambda_\zeta\}$, moreover we set $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}} \rangle$. Clearly, the family $\{\mathbb{B}[\varepsilon] : \varepsilon \in \mathbb{FIN}(\vec{\lambda})\}$ is a base for the space $X_{\mathbb{B}}$. Note that $X_{\mathbb{B}}$ is Hausdorff iff $\mathbb{B}$ is separating.

The following statement is very easy to prove and is well known. It can certainly be viewed as part of the folklore.

Observation 3.1. *Let κ and λ be infinite cardinals. Then, up to homeomorphisms, there is a natural one-to-one correspondence between dense subspaces X of $D(2)^\lambda$ of size κ and spaces of the form $X_{\mathbb{B}} = \langle \kappa, \tau_{\mathbb{B}} \rangle$, where $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi \rangle : \xi < \lambda\}$ is a separating and independent family of 2-partitions of κ. Moreover, X is μ-dense in $D(2)^\lambda$ iff $\mathbb{B}$ is μ-independent.*

The spaces obtained from our Main Theorem 3.3 will all be of the above form, with $\lambda = 2^\kappa$. The following fact will be instrumental in finding appropriate families of dense sets $\mathcal{D}$ to be used to produce $\mathcal{D}$-forced spaces.

Fact 3.2. *For each infinite cardinal κ, there is a family*

$$\mathbb{B} = \{\langle B^i_\xi : i < \kappa \rangle : \xi < 2^\kappa\}$$

of 2^κ many κ-partitions of κ that is κ-independent.

Indeed, this fact is just a reformulation of the statement that the space $\mathrm{D}(\kappa)^{2^\kappa}$, the 2^κth power of the discrete space on κ, contains a κ-dense subset of size κ. This, in turn, follows immediately from the Hewitt–Marczewski–Pondiczery theorem, see, e.g. [13, Theorem 2.3.15].

Main Theorem 3.3. *Assume that κ is an infinite cardinal and we are given $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi\rangle\colon \xi < 2^\kappa\}$, a κ-independent family of 2-partitions of κ, moreover a non-empty family $\mathcal{D}$ of κ-dense subsets of the space $X_\mathbb{B}$. Then there is another, always separating, κ-independent family $\mathbb{C} = \{\langle C^0_\xi, C^1_\xi\rangle\colon \xi < 2^\kappa\}$ of 2-partitions of κ that satisfies the following five conditions*:

(1) *every $D \in \mathcal{D}$ is also κ-dense in $X_\mathbb{C}$ (and so $\Delta(X_\mathbb{C}) = \kappa$),*
(2) *$X_\mathbb{C}$ is $\mathcal{D}$-forced,*
(3) $\mathrm{nwd}(X_\mathbb{C}) = \kappa$, *i.e.* $[\kappa]^{<\kappa} \subset \mathcal{N}(X_\mathbb{C})$,
(4) *$X_\mathbb{C}$ is NODEC.*

Moreover, if $J \subset 2^\kappa$ is given with $|2^\kappa \setminus J| = 2^\kappa$ then we can assume that

(5) $\mathbb{C} \restriction J = \mathbb{B} \restriction J$.

Proof. Assume that J is given and let $I = 2^\kappa \setminus J$. We partition I into two disjoint pieces, $I = I_0 \cup I'$, such that $|I_0| = \kappa^{<\kappa}$ and $|I'| = 2^\kappa$. Next we partition I_0 into pairwise disjoint countable sets $J_{A,\alpha} \in [I_0]^\omega$ for all $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$. If $\xi \in J_{A,\alpha}$ (for some $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$) then we let

$$C^0_\xi = \big(B^0_\xi \cup A\big) \setminus \{\alpha\},$$

and

$$C^1_\xi = \big(B^1_\xi \setminus A\big) \cup \{\alpha\}.$$

Next, let us fix any enumeration $\{F_\nu\colon \nu < 2^\kappa\}$ of $[\kappa]^\kappa$ and then by transfinite recursion on $\nu < 2^\kappa$ define

- sets $K_\nu \subset I'$ with $K_\nu = \emptyset$ or $|K_\nu| = \kappa$,
- partitions $\langle C^0_\sigma, C^1_\sigma\rangle$ of κ for all $\sigma \in K_\nu$,
- finite functions $\eta_\nu \in \mathrm{Fn}(2^\kappa, 2)$,

such that the inductive hypothesis

$$\forall \varepsilon \in \mathrm{Fn}\big(2^\kappa, 2\big)\ \forall D \in \mathcal{D}\ \big|D \cap \mathbb{B}_\nu[\varepsilon]\big| = \kappa \qquad (\phi_\nu)$$

holds, where

$$\mathbb{B}_\nu = \big\{\big\langle C^0_\sigma, C^1_\sigma\big\rangle\colon \sigma \in I_\nu\big\} \cup \big\{\big\langle B^0_\sigma, B^1_\sigma\big\rangle\colon \sigma \in 2^\kappa \setminus I_\nu\big\}$$

with

$$I_\nu = I_0 \cup \bigcup_{\zeta < \nu} K_\zeta.$$

Note that (ϕ_ν) simply says that every set $D \in \mathcal{D}$ is κ-dense in the space $X_{\mathbb{B}_\nu}$. We shall then conclude that $\mathbb{C} = \mathbb{B}_{2^\kappa}$ is as required.

Let us observe first that (ϕ_0) holds because, by assumption, we have $|\mathbb{B}[\varepsilon] \cap D| = \kappa$ for all $D \in \mathcal{D}$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$, moreover

$$\big|\mathbb{B}[\varepsilon] \triangle \mathbb{B}_0[\varepsilon]\big| < \kappa.$$

Clearly, if ν is a limit ordinal and (ϕ_ζ) holds for each $\zeta < \nu$ then (ϕ_ν) also holds. So the induction hypothesis is preserved in limit steps.

Now consider the successor steps. Assume that (ϕ_ν) holds. We distinguish two cases:

Case 1. F_ν contains a $(\mathcal{D}, X_{\mathbb{B}_\nu})$-piece, i.e. $F_\nu \supset D \cap \mathbb{B}_\nu[\eta_\nu]$ for some $\eta_\nu \in \mathrm{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$.

This defines η_ν and then we set $K_\nu = \emptyset$. The construction from here on will not change the partitions whose indices occur in $\mathrm{dom}(\eta_\nu)$, thus we shall have $\mathbb{B}_\nu[\eta_\nu] = \mathbb{B}_{2^\kappa}[\eta_\nu]$ and so at the end F_ν will include the $(\mathcal{D}, X_{\mathbb{B}_{2^\kappa}})$-piece $D \cap \mathbb{B}_{2^\kappa}[\eta_\nu]$. Also, in this case we have $\mathbb{B}_\nu = \mathbb{B}_{\nu+1}$, hence $(\phi_{\nu+1})$ trivially remains valid.

Case 2. F_ν does not include a $(\mathcal{D}, X_{\mathbb{B}_\nu})$-piece, i.e. $(D \cap \mathbb{B}_\nu[\varepsilon]) \setminus F_\nu \neq \emptyset$ for all $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$.

In this case we choose and fix any set

$$K_\nu \subset I' \setminus \Bigl(\bigcup\{\mathrm{dom}\,\eta_\zeta\colon \zeta < \nu\} \cup \bigcup\{K_\zeta\colon \zeta < \nu\}\Bigr)$$

of size κ and let $K_\nu = \{\gamma_{\nu,i}\colon i < \kappa\}$ be a 1–1 enumeration of K_ν. We also set $\eta_\nu = \emptyset$. We want to modify the partitions with indices in K_ν so as to make the set F_ν closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and hence in $X_{\mathbb{B}_{2^\kappa}}$ as well. To do this, we set for all $i < \kappa$

$$C^0_{\gamma_{\nu,i}} = \bigl(B^0_{\gamma_{\nu,i}} \setminus F_\nu\bigr) \cup \{i\},$$

and

$$C^1_{\gamma_{\nu,i}} = \bigl(B^1_{\gamma_{\nu,i}} \cup F_\nu\bigr) \setminus \{i\}.$$

Then for each $i \in \kappa$ we have $i \in C^0_{\gamma_{\nu,i}}$ and

$$F_\nu \cap C^0_{\gamma_{\nu,i}} \subset \{i\},$$

consequently F_ν is closed discrete in $X_{\mathbb{B}_{\nu+1}}$, hence F_ν will be closed discrete in $X_{\mathbb{B}_{2^\kappa}}$.

We still have to show that $(\phi_{\nu+1})$ holds in this case, too. Assume, indirectly, that for some $D \in \mathcal{D}$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ we have

$$\bigl|D \cap \mathbb{B}_{\nu+1}[\varepsilon]\bigr| < \kappa.$$

Then we can clearly find $\xi \in I_0 \setminus \mathrm{dom}\,\varepsilon$ with

$$\bigl(D \cap \mathbb{B}_{\nu+1}[\varepsilon]\bigr) \cup \mathrm{dom}(\varepsilon) \subset C^0_\xi,$$

and so for $\varepsilon^* = \varepsilon \cup \{\langle \xi, 1\rangle\}$ we even have

$$D \cap \mathbb{B}_{\nu+1}[\varepsilon^*] = \emptyset.$$

On the other hand, our choices clearly imply that

$$\mathbb{B}_{\nu+1}[\varepsilon^*] \supset \mathbb{B}_\nu[\varepsilon^*] \setminus F_\nu,$$

consequently

$$D \cap \mathbb{B}_{\nu+1}[\varepsilon^*] \supset \big(D \cap \mathbb{B}_\nu[\varepsilon^*]\big) \setminus F_\nu \neq \emptyset,$$

a contradiction. This shows that $(\phi_{\nu+1})$ is indeed valid, and the transfinite construction of $\mathbb{C} = \mathbb{B}_{2^\kappa}$ is thus completed. We show next that $\mathbb{C}$ satisfies all the requirements of our main theorem.

$\mathbb{C}$ is separating because, e.g. for any $\xi \in J_{\{\alpha\},\beta}$ the partition $\langle C_\xi^0, C_\xi^1\rangle$ separates α and β.

That $\mathbb{C}$ is κ-independent and that (1) holds (i.e. each $D \in \mathcal{D}$ is κ-dense in $X_\mathbb{C}$) both follow from (ϕ_{2^κ}).

If $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$, then for any $\xi \in J_{A,\alpha}$ we have $A \subset C_\xi^0$ and $\alpha \in C_\xi^1$, hence $\alpha \notin \overline{A}^{X_\mathbb{C}}$. Thus every member of $[\kappa]^{<\kappa}$ is closed and hence closed discrete in $X_\mathbb{C}$, and so (3) is satisfied.

Assume next that $F \in \mathcal{N}(X_\mathbb{C})$, we want to show that F is closed discrete. By (3) we may assume that $|F| = \kappa$ and so can find $\nu < 2^\kappa$ with $F = F_\nu$. Suppose that at step ν of the recursion we were in case 1; then we had $F \supset D \cap \mathbb{B}_\nu[\eta_\nu]$ for some $D \in \mathcal{D}$. But $\mathbb{B}_\nu[\eta_\nu] = \mathbb{B}_{2^\kappa}[\eta_\nu] = \mathbb{C}[\eta_\nu]$, so F would be dense in $\mathbb{C}[\eta_\nu]$. This contradiction shows that, at step ν, we must have been in case 2. However, in this case we know that $F = F_\nu$ was made to be closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and consequently in $X_\mathbb{C}$ as well. So $X_\mathbb{C}$ is NODEC, i.e. (4) holds.

It remains to check that $X_\mathbb{C}$ is $\mathcal{D}$-forced, i.e. that (2) holds. By 2.4 it suffices to show that any somewhere dense subset E of $X_\mathbb{C}$ includes a $(\mathcal{D}, X_\mathbb{C})$-piece. By (3) we must have $|E| = \kappa$ and hence we can pick $\nu < 2^\kappa$ such that $F_\nu = E$. Then at step ν of the recursion we could not be in case 2, since otherwise $F_\nu = E$ would have been made closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and so in $X_\mathbb{C}$ as well. Hence at step ν of the recursion we were in case 1, consequently $\eta_\nu \in \mathrm{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$ could be found such that $E = F_\nu \supset D \cap \mathbb{B}_\nu[\eta_\nu]$. However, by the construction, we have $\mathbb{C}[\eta_\nu] = \mathbb{B}_\nu[\eta_\nu]$, and therefore E actually includes the $(\mathcal{D}, X_\mathbb{C})$-piece $D \cap \mathbb{C}[\eta_\nu]$.

Finally, (5) trivially holds by the construction. □

4. Applications to resolvability

In this and the following section we shall present a large number of consequences of our Main Theorem 3.3. The key to most of these will be given by a judicious choice of a family $\mathcal{D}$ of κ-dense sets in a space $X_\mathbb{B}$, where $\mathbb{B} = \{\langle B_\xi^0, B_\xi^1\rangle: \xi < 2^\kappa\}$ is a κ-independent family of 2-partitions of some cardinal κ. In our first application, however, this choice is trivial, that is we have $\mathcal{D} = \{\kappa\}$.

In [1], the following results were proven:

(1) $\mathrm{D}(2)^\mathfrak{c}$ does not have a dense countable maximal subspace,
(2) $\mathrm{D}(2)^\mathfrak{c}$ has a dense countable irresolvable subspace,
(3) it is consistent that $\mathrm{D}(2)^\mathfrak{c}$ has a dense countable submaximal subspace,

and then the following natural problem was raised [1, Question 4.4]: *Is it provable in* ZFC *that the Cantor cube* $\mathrm{D}(2)^{\mathfrak{c}}$ *or the Tychonov cube* $[0,1]^{\mathfrak{c}}$ *has a dense countable submaximal subspace*? Our next result gives an affirmative answer to this problem.

Theorem 4.1. *For each infinite cardinal* κ *the Cantor cube* $\mathrm{D}(2)^{2^\kappa}$ *contains a dense submaximal subspace* X *with* $|X| = \Delta(X) = \kappa$.

Proof. Let us start by fixing any κ-independent family of 2-partitions $\mathbb{B} = \{\langle B_\xi^0, B_\xi^1\rangle\colon \xi < 2^\kappa\}$ of κ, and let $\mathcal{D} = \{\kappa\}$. Applying Theorem 3.3 with $\mathbb{B}$ and $\mathcal{D}$ we obtain a family of 2-partitions $\mathbb{C}$ of κ that satisfies 3.3(1)–(4). The space $X_{\mathbb{C}}$ is as required. Indeed, $\Delta(X_{\mathbb{C}}) = \kappa$ because of 3.3(1), $X_{\mathbb{C}}$ is NODEC by 3.3(4), while it is OHI by Lemma 2.7. But then it is submaximal. Finally, by Observation 3.1, $X_{\mathbb{C}}$ embeds into $\mathrm{D}(2)^{2^\kappa}$ as a dense subspace. □

That Theorem 4.1 fully answers [1, Question 4.4] follows from the following Fact 4.2 that may be already known, although we have not found it in the literature.

Fact 4.2. *Any countable dense subspace of* $\mathrm{D}(2)^{\mathfrak{c}}$ *is homeomorphic to a* dense *subspace of* $[0,1]^{\mathfrak{c}}$.

This fact, in turn, immediately follows from the next proposition. In it, as usual, we denote by $\mathbb{P}$ the space of the irrationals.

Proposition 4.3. *Assume that* κ *is an infinite cardinal,* $S \subset \mathrm{D}(2)^\kappa$ *is dense, moreover there is a partition* $\{I_\nu\colon \nu < \kappa\}$ *of* κ *into countably infinite sets such that for each* $\nu < \kappa$ *the set* $2^{I_\nu} \setminus (S \restriction I_\nu)$ *is dense* (*in other words*: $S \restriction I_\nu$ *is co-dense*) *in* 2^{I_ν}. (*The last condition is trivially satisfied if the cardinality of* S *is less than continuum.*) *Then* S *is homeomorphic to a dense subspace of the irrational cube* $\mathbb{P}^\kappa$ *and hence of the Tychonov cube* $[0,1]^\kappa$.

Proof. For each $\nu < \kappa$ we may select a countable dense subset of $D_\nu \subset 2^{I_\nu} \setminus (S \restriction I_\nu)$. The space $2^{I_\nu} \setminus D_\nu$ is known to be homeomorphic to $\mathbb{P}$ for all $\nu < \kappa$. Also, for each $\nu < \kappa$ we have $S \restriction I_\nu \subset 2^{I_\nu} \setminus D_\nu$ and therefore S is naturally homeomorphic to a dense subspace of the product space

$$\prod\{2^{I_\nu} \setminus D_\nu\colon \nu < \kappa\}.$$

This product, however, is homeomorphic to the cube $\mathbb{P}^\kappa$. □

Let us remark that, as far as we know, the first ZFC example of a countable regular, hence 0-dimensional, submaximal space was constructed by van Douwen in [10], by using an approach that is very different from and much more involved than ours. Also, it is not clear if his example embeds *densely* into the Cantor or Tychonov cube of weight $\mathfrak{c}$.

After proving in [2, Corollary 8.5] that every separable submaximal topological group is countable, Arhangel'skii and Collins raised the following question [2, Problem 8.6]: *Is there a crowded uncountable separable Hausdorff* (*or even Tychonov*) *submaximal space*? As it turns out, starting from any zero-dimensional countable submaximal space (e.g. the one obtained from the previous theorem or van Douwen's example from [10]) an affirmative answer can be given to this question, at least in the T_2 case. The regular or Tychonov cases of the problem, however, remain open.

Theorem 4.4. *There is a crowded, separable, submaximal T_2 space Y of cardinality $\mathfrak{c}$.*

Proof. Let τ be any fixed crowded, submaximal, 0-dimensional, and T_2 topology on ω. Since τ is not compact we can easily find $\{U_\sigma\colon \sigma \in 2^{<\omega}\}$, an infinite partition of ω into nonempty τ-clopen sets indexed by all finite 0–1 sequences σ.

The underlying set of Y will be $\omega \cup {}^{\omega}2$ and we let $X = \langle \omega, \tau \rangle$ be an open subspace of Y. Next, a basic neighbourhood of a point $f \in {}^{\omega}2$ will be of the form

$$\{f\} \cup \bigcup \{D_{f\restriction n}\colon n \geqslant m\},$$

where $m \in \omega$ and $D_{f\restriction n}$ is a dense (hence, as X is submaximal, open) subset of $U_{f\restriction n}$ for $m \leqslant n < \omega$. It is easy to see that Y is T_2, and Y is separable because ω is dense in it.

Now, assume that $D \subset Y$ is dense. Then $D \cap X$ is also dense hence open in X, and similarly $D \cap U_\sigma$ is dense open in U_σ for each $\sigma \in 2^{<\omega}$. So for each $f \in D$ the set $\{f\} \cup \bigcup \{D \cap U_{f\restriction n}\colon n \geqslant 0\} \subset D$ is a basic neighbourhood of f, showing that D is open in Y. □

In 1967 Ceder and Pearson [6], raised the question whether an ω-resolvable space is necessarily maximally resolvable? El'kin [12], constructed a T_1 counterexample to this question, and then Malykhin [18], produced a crowded hereditarily resolvable T_1 space (that is clearly ω-resolvable) which is not maximally resolvable. Eckertson [11], and later Hu [16], gave Tychonov counterexamples but not in ZFC: Eckertson's construction used a measurable cardinal, while Hu applied the assumption $2^\omega = 2^{\omega_1}$. Whether one could find a Tychonov counterexample to the Ceder–Pearson problem in ZFC was repeatedly asked as recently as in [7,9].

Our next theorem gives a whole class of 0-dimensional T_2 (hence Tychonov) counterexamples to the Ceder–Pearson problem in ZFC. Quite naturally, they involve applications of our Main Theorem 3.3 where the family of dense sets $\mathcal{D}$ forms a partition of the underlying set.

Recall that any application of Theorem 3.3 yields a dense NODEC subspace X of some Cantor cube $\mathrm{D}(2)^{2^\kappa}$ with the extra properties

$$|X| = \mathrm{nwd}(X) = \Delta(X) = \kappa.$$

From now on, we shall call any space having all these properties a $\mathcal{C}(\kappa)$-space. Of course, any $\mathcal{C}(\kappa)$-space is zero-dimensional T_2 and therefore Tychonov. Finally, with the intention to use Lemma 2.8, we recall that any $\mathcal{C}(\kappa)$-space X, being dense in a Cantor cube, is CCC, i.e. satisfies $\hat{\mathrm{c}}(X) = \omega_1$.

Theorem 4.5. *For any two infinite cardinals $\mu < \kappa$ there is a $\mathcal{C}(\kappa)$-space X that is the disjoint union of μ dense submaximal subspaces but is not μ^+-extraresolvable.* (*Of course, X is then μ-resolvable but not μ^+-resolvable, hence not maximally resolvable.*)

Proof. Using Fact 3.2 we can easily find a μ-partition $\langle D_i\colon\ i < \mu\rangle$ and a family of 2-partitions $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi\rangle\colon\ \xi < 2^\kappa\}$ of κ such that for each $i < \mu$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ we have

$$\big|\, D_i \cap \mathbb{B}[\varepsilon]\big| = \kappa.$$

We may then apply Theorem 3.3 to this $\mathbb{B}$ and the family $\mathcal{D} = \{D_i\colon\ i < \mu\}$ to get a collection $\mathbb{C}$ of 2-partitions of κ satisfying 3.3(1)–(4). We claim that the space $X_{\mathbb{C}}$ is as required.

Firstly, as the members of $\mathcal{D}$ partition κ and $X_{\mathbb{C}}$ is NODEC, Lemma 2.7 implies that each $D_i \in \mathcal{D}$ is a submaximal dense subspace of $X_{\mathbb{C}}$.

Secondly, since $X_{\mathbb{C}}$ is CCC and $|\mathcal{D}| = \mu \geqslant \omega$, Lemma 2.8 implies that $X_{\mathbb{C}}$ is not μ^+-extraresolvable. □

Theorem 4.5 talks about infinite cardinals, and with good reason; it has been long known that for any finite n there are say countable zero-dimensional spaces that are n-resolvable but not $(n+1)$-resolvable. In connection with this, Eckertson asked in [11, Question 4.5] the following question: *Does there exist for each infinite cardinal κ and for each natural number $n \geqslant 1$ a Tychonov space X with $|X| = \Delta(X) = \kappa$ such that X is n-resolvable but X contains no $(n+1)$-resolvable subspaces*? Li Feng [14], gave a positive answer to this question and the following corollary of Theorem 4.5 improves his result. Our example is a $\mathcal{C}(\kappa)$-space that is the disjoint union of n dense submaximal subspaces.

Corollary 4.6. *For each cardinal $\kappa \geqslant \omega$ and each natural number $n \geqslant 1$ there is a $\mathcal{C}(\kappa)$-space Y which is the disjoint union of n dense submaximal subspaces. Then Y, automatically, does not contain any $(n+1)$-resolvable subspaces.*

Proof. Consider the $\mathcal{C}(\kappa)$-space X given by Theorem 4.5 for any fixed pair of cardinals $\mu < \kappa$ and then set $Y = \bigcup\{D_i\colon\ i < n\}$. Here each subspace D_i of Y is submaximal and therefore HI. Consequently, every subspace of Y can be written as the union of at most n HI subspaces. By [17, Lemma 2], no such space can be $(n+1)$-resolvable, hence Y contains no $(n+1)$-resolvable subspaces. □

Another question that can be raised concerning Theorem 4.5 is whether it could be extended to apply to all infinite cardinals instead of just the successors μ^+. It is actually known that the answer to this question is negative.

Indeed, Illanes, and later Bashkara Rao proved the following two "compactness"-type results on λ-resolvability, for cardinals λ of countable cofinality.

Theorem. (Illanes [17]) *If a topological space X is k-resolvable for each $k < \omega$ then X is ω-resolvable.*

Theorem. (Bhaskara Rao [4]) *If λ is a singular cardinal with* $\mathrm{cf}(\lambda) = \omega$ *and X is any topological space that is μ-resolvable for each $\mu < \lambda$ then X is λ-resolvable.*

In contrast to these, our next result, Theorem 4.8, implies that no such compactness-phenomenon is valid for uncountable regular limit (that is inaccessible) cardinals. However, the following intriguing problem remains open.

Problem 4.7. Assume that λ is a singular cardinal with $\mathrm{cf}(\lambda) > \omega$ and X is a topological space that is μ-resolvable for all $\mu < \lambda$. Is it true then that X is also λ-resolvable?

Theorem 4.8 may be viewed as an extension of 4.5 from successors to all uncountable regular cardinals, providing counterexamples to the Ceder–Pearson problem in further cases. However, the spaces obtained here are quite different from the ones constructed in 4.5 because they are *hereditarily resolvable*.

Theorem 4.8. *For any two cardinals κ and λ with $\omega < \mathrm{cf}(\lambda) = \lambda \leqslant \kappa$ there is a $\mathcal{C}(\kappa)$-space that is not λ-extraresolvable* (*and hence not λ-resolvable*) *and still it is hereditarily μ-resolvable for all $\mu < \lambda$.*

Proof. Let us fix the sequence $\vec{\lambda} = \langle \lambda_\zeta \colon \zeta < \lambda \rangle$ by setting $\lambda_\zeta = \rho$ for each $\zeta < \lambda$ if $\lambda = \rho^+$ is a successor and by putting $\lambda_\zeta = \omega_\zeta$ for $\zeta < \lambda$ if λ is a limit cardinal (note that $\lambda = \omega_\lambda$ in the latter case).

Using Fact 3.2 we can find two families of partitions

$$\mathbb{D} = \left\{ \left\langle D^i_\zeta \colon i < \lambda_\zeta \right\rangle \colon \zeta < \lambda \right\} \quad \text{and} \quad \mathbb{B} = \left\{ \left\langle B^0_\xi, B^1_\xi \right\rangle \colon \xi < 2^\kappa \right\}$$

of κ such that $\mathbb{D} \cup \mathbb{B}$ is κ-independent, i.e. $|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]| = \kappa$ whenever $\eta \in \mathbb{FIN}(\vec{\lambda})$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$. Then

$$\mathcal{D} = \left\{ \mathbb{D}[\eta] \colon \eta \in \mathbb{FIN}(\vec{\lambda}) \right\}$$

is a family of κ-dense sets in the space $X_{\mathbb{B}}$, hence we can apply Theorem 3.3 with $\mathbb{B}$ and $\mathcal{D}$ to get a family $\mathbb{C}$ of 2-partitions of κ satisfying 3.3(1)–(4). We shall show that the $\mathcal{C}(\kappa)$-space $X_{\mathbb{C}}$ is as required.

Claim 4.8.1. *For every family $\mathcal{E} \in [\mathcal{D}]^\lambda$ there is $\mathcal{F} \in [\mathcal{E}]^\lambda$ such that $D \cap D' \in \mathcal{D}$* (*and hence is dense in $X_{\mathbb{C}}$*) *whenever $\{D, D'\} \in [\mathcal{F}]^2$.*

Proof. We can write $\mathcal{E} = \{\mathbb{D}[\eta_\gamma] \colon \gamma < \lambda\}$. Since $\lambda = \mathrm{cf}(\lambda) > \omega$ we can find $K \in [\lambda]^\lambda$ such that $\{\mathrm{dom}(\eta_\gamma) \colon \gamma \in K\}$ forms a Δ-system with kernel K^*. Then $\prod_{i \in K^*} \lambda_i < \lambda$, hence, as λ is regular, there are a set $I \in [K]^\lambda$ and a fixed finite function $\eta \in \prod_{i \in K^*} \lambda_i \subset \mathbb{FIN}(\vec{\lambda})$ such that $\eta_\gamma \restriction K^* = \eta$ for each $\gamma \in I$.

But then $\mathcal{F} = \{\mathbb{D}[\eta_\gamma] \colon \gamma \in I\}$ is as required: for any $\{\gamma, \delta\} \in [I]^2$ we have $\eta_\gamma \cup \eta_\delta \in \mathbb{FIN}(\vec{\lambda})$, consequently

$$\mathbb{D}[\eta_\gamma] \cap \mathbb{D}[\eta_\delta] = \mathbb{D}[\eta_\gamma \cup \eta_\delta] \in \mathcal{D}. \qquad \square$$

Now, since $\hat{c}(X_{\mathbb{C}}) = \omega_1$ and the above claim holds we can apply Lemma 2.8 to conclude that $X_{\mathbb{C}}$ is not λ-extraresolvable.

Let us now fix $\mu < \lambda$. We first show that every $\mathbb{D}[\eta] \in \mathcal{D}$ is μ-resolvable. Indeed, choose $\zeta \in \lambda \setminus \mathrm{dom}\,\eta$ with $\lambda_\zeta \geqslant \mu$. Clearly, then the family $\{\mathbb{D}[\eta \cup \{\langle \zeta, \gamma \rangle\}] \colon \gamma < \lambda_\zeta\}$ forms a partition of $\mathbb{D}[\eta]$ into $\lambda_\zeta \geqslant \mu$ many dense subsets.

Since $X_{\mathbb{C}}$ is NODEC and $\mathcal{D}$-forced, any crowded subspace S of $X_{\mathbb{C}}$ is somewhere dense. Consequently, Lemma 2.10 implies that $X_{\mathbb{C}}$ is hereditarily μ-resolvable. $\square$

Remark. It is well known that any dense subspace of the Cantor cube $D(2)^\lambda$ has weight (even π-weight) equal to λ. Consequently, any $\mathcal{C}(\kappa)$-space (that is, by definition, of cardinality κ) has maximum possible weight, that is 2^κ. Now, ZFC counterexamples to the Ceder–Pearson problem are naturally expected to have this property. Indeed, for instance the forcing axiom BACH (see, e.g. [20]) implies that every topological space X with $|X| = \Delta(X) = \omega_1$ and $\pi w(X) < 2^{\omega_1}$ is ω_1-resolvable. Consequently, under BACH, any ω-resolvable space X satisfying $|X| = \omega_1$ and $\pi w(X) < 2^{\omega_1}$ is maximally resolvable.

By [17, Lemma 4], any topological space that is not ω-resolvable contains a HI somewhere dense subspace. Theorem 4.8 shows that this badly fails if ω is replaced by an uncountable cardinal.

Again by [17, Lemma 4], if a space X can be partitioned into finitely many dense HI subspaces, then the number of pieces is uniquely determined. It follows from our next result, Theorem 4.9 below, that this is not the case for infinite partitions. In fact, for every infinite cardinal κ there is a $\mathcal{C}(\kappa)$-space that can be simultaneously partitioned into λ many dense submaximal (and so HI) subspaces for all infinite $\lambda \leqslant \kappa$.

Theorem 4.9 also gives an affirmative answer to the following question of Eckertson, raised in [11, 3.4 and 3.6]: *Does there exist, for each cardinal μ, a μ^+-resolvable space that can be partitioned into μ-many dense HI subspaces*?

The proof of Theorem 4.9 will require an even more delicate choice of the family of dense sets $\mathcal{D}$ than the one we used in the proof of 4.8.

Theorem 4.9. *For each infinite cardinal κ there is a $\mathcal{C}(\kappa)$-space that can be simultaneously partitioned into countably many dense hereditarily κ-resolvable subspaces and also into μ many dense submaximal* (*and therefore HI*) *subspaces for all infinite $\mu \leqslant \kappa$.*

Proof. Let us start by setting $\lambda_0 = \omega$, $\lambda_1 = \kappa$, and $\vec{\lambda} = \langle \lambda_i \colon i < 2\rangle$, moreover $\vec{\kappa} = \langle \kappa_n \colon n < \omega\rangle$, where $\kappa_0 = \omega$ and $\kappa_n = \kappa$ for $1 \leqslant n < \omega$.

By Fact 3.2 there are three families of partitions of κ, say

$$\mathbb{B} = \{\langle B^i_\zeta \colon i < 2\rangle \colon \zeta < 2^\kappa\},$$

$$\mathbb{E} = \{\langle E^j_n \colon j < \kappa_n\rangle \colon n < \omega\},$$

and

$$\mathbb{F} = \{\langle F^k_\ell \colon k < \lambda_\ell\rangle \colon \ell < 2\},$$

such that $\mathbb{B} \cup \mathbb{E} \cup \mathbb{F}$ is κ-independent, i.e. for each $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$, $\eta \in \mathbb{FIN}(\vec{\kappa})$, and $\rho \in \mathbb{FIN}(\vec{\lambda})$ we have

$$\bigl|\mathbb{B}[\varepsilon] \cap \mathbb{E}[\eta] \cap \mathbb{F}[\rho]\bigr| = \kappa. \tag{†}$$

Of course, (†) implies that all sets of the form $\mathbb{E}[\eta] \cap \mathbb{F}[\rho]$ are κ-dense in $X_{\mathbb{B}}$, however the family $\mathcal{D}$ of κ-dense sets that we need will be defined in a more complicated way.

To start with, let us write $\mathcal{F}_\ell = \{F^k_\ell \colon k < \lambda_\ell\}$ for $\ell < 2$ and then set

$$\mathcal{D}_{\mathbb{E}} = \{\mathbb{E}[\eta] \colon \eta \in \mathbb{FIN}(\vec{\kappa})\}$$

and

$$\mathcal{D}_{\mathbb{F}} = \mathcal{F}_0 \cup \mathcal{F}_1 = \{F_\ell^k\colon\ \ell < 2,\ k < \lambda_\ell\}.$$

Next let

$$\mathcal{D}_{\mathbb{E},\mathbb{F}} = \Big\{E \setminus \bigcup \mathcal{F}\colon\ E \in \mathcal{D}_{\mathbb{E}},\ \mathcal{F} \in [\mathcal{D}_{\mathbb{F}}]^{<\omega}\Big\}$$

and

$$\mathcal{D}_{\mathbb{F},\mathbb{E}} = \Big\{F_\ell^k \setminus \Big(\Big(\bigcup \mathcal{E}\Big) \cup \Big(\bigcup \mathcal{F}\Big)\Big)\colon\ F_\ell^k \in \mathcal{D}_{\mathbb{F}},\ \mathcal{E} \in [\mathcal{D}_{\mathbb{E}}]^{<\omega},\ \mathcal{F} \in [\mathcal{F}_{1-\ell}]^{<\omega}\Big\}.$$

Finally, we set

$$\mathcal{D} = \mathcal{D}_{\mathbb{E},\mathbb{F}} \cup \mathcal{D}_{\mathbb{F},\mathbb{E}}.$$

Every element of $\mathcal{D}$ contains some (in fact, infinitely many) sets of the form $\mathbb{E}[\eta] \cap \mathbb{F}[\rho]$ and so is κ-dense in $X_{\mathbb{B}}$ by (†).

Now we may apply Theorem 3.3 with $\mathbb{B}$ and $\mathcal{D}$ to obtain a family of partitions $\mathbb{C}$ of κ that satisfies 3.3(1)–(4). We shall show that $X_{\mathbb{C}}$ is as required.

Claim 4.9.1. *$E \cap F$ is nowhere dense in $X_{\mathbb{C}}$ whenever $E \in \mathcal{D}_{\mathbb{E}}$ and $F \in \mathcal{D}_{\mathbb{F}}$.*

Proof. According to 2.6 it suffices to show that $D \setminus (E \cap F)$ includes an element of $\mathcal{D}$ whenever $D \in \mathcal{D}$.

Now, if $D = E' \setminus \bigcup \mathcal{F} \in \mathcal{D}_{\mathbb{E},\mathbb{F}}$ then

$$D \setminus (E \cap F) \supset E' \setminus \Big(\bigcup \big(\mathcal{F} \cup \{F\}\big)\Big) \in \mathcal{D}_{\mathbb{E},\mathbb{F}}.$$

If, on the other hand, $D = F_\ell^k \setminus ((\bigcup \mathcal{E}) \cup (\bigcup \mathcal{F})) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}$ then

$$D \setminus (E \cap F) \supset F_\ell^k \setminus \Big(\Big(\bigcup \big(\mathcal{E} \cup \{E\}\big)\Big) \cup \Big(\bigcup \mathcal{F}\Big)\Big) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}. \qquad \square$$

Claim 4.9.2. *$F \cap F'$ is nowhere dense in $X_{\mathbb{C}}$ for all $\{F, F'\} \in [\mathcal{D}_{\mathbb{F}}]^2$.*

Proof. Again, by 2.6, it is enough to show that $D \setminus (F \cap F')$ includes an element of $\mathcal{D}$ for each $D \in \mathcal{D}$.

If $D = E \setminus \bigcup \mathcal{F} \in \mathcal{D}_{\mathbb{E},\mathbb{F}}$ then

$$D \setminus (F \cap F') \supset E \setminus \Big(\bigcup \big(\mathcal{F} \cup \{F\}\big)\Big) \in \mathcal{D}_{\mathbb{E},\mathbb{F}}.$$

If $D = F_\ell^k \setminus ((\bigcup \mathcal{E}) \cup (\bigcup \mathcal{F})) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}$ and $F \cap F' \neq \emptyset$ then we can assume that $F \in \mathcal{F}_\ell$ and $F' \in \mathcal{F}_{1-\ell}$. But then we have

$$D \setminus (F \cap F') \supset F_\ell^k \setminus \Big(\Big(\bigcup \mathcal{E}\Big) \cup \Big(\bigcup \big(\mathcal{F} \cup \{F'\}\big)\Big)\Big) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}. \qquad \square$$

Claim 4.9.3. *Every $D \in \mathcal{D}_{\mathbb{E},\mathbb{F}}$ is κ-resolvable.*

Proof. Let $D = E \setminus \bigcup \mathcal{F}$. Without loss of generality we can assume that $E = \mathbb{E}[\eta]$ with $\operatorname{dom} \eta = n \in \omega \setminus \{0\}$. But then D is the disjoint union of the $\kappa_n = \kappa$ many dense sets

$$\Bigl\{\mathbb{E}\bigl[\eta \cup \bigl\{\langle n, \zeta\rangle\bigr\}\bigr] \setminus \bigcup \mathcal{F}\colon\ \zeta < \kappa\Bigr\}. \qquad \square$$

Claim 4.9.4. *E_0^i is hereditarily κ-resolvable for each $i < \omega = \kappa_0$.*

Proof. Let us note first of all that for any

$$D = F \setminus \Bigl(\Bigl(\bigcup \mathcal{E}\Bigr) \cup \Bigl(\bigcup \mathcal{F}\Bigr)\Bigr) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}$$

we have $E_0^i \cap D \subset E_0^i \cap F \in \mathcal{N}(X_{\mathbb{C}})$ by Claim 4.9.1.

Now, let S be any crowded subspace of E_0^i. Since $X_{\mathbb{C}}$ is NODEC and $\mathcal{D}$-forced, by Lemma 2.10 there is a partial $(\mathcal{D}, X_{\mathbb{C}})$-mosaic

$$M = \bigcup \{V \cap D_V\colon\ V \in \mathcal{V}\} \subset S$$

that is dense in S. By our above remark, we must have $D_V \in \mathcal{D}_{\mathbb{E},\mathbb{F}}$ whenever $V \in \mathcal{V}$, consequently M and hence S is κ-resolvable by Claim 4.9.3 and Fact 2.9. $\square$

We have thus concluded that $\{E_0^i\colon\ i < \omega\}$ partitions $X_{\mathbb{C}}$ into countably many hereditarily κ-resolvable dense subspaces.

Claim 4.9.5. *$F_\ell^k \subset X_{\mathbb{C}}$ is submaximal for all $\ell < 2$ and $k < \lambda_\ell$.*

Proof. Since $X_{\mathbb{C}}$ is NODEC, so is its dense subspace F_ℓ^k, hence it suffices to show that F_ℓ^k is OHI. By Lemma 2.7, this will follow if we can show that for each $D \in \mathcal{D}$ either $F_\ell^k \cap D$ or $F_\ell^k \setminus D$ is nowhere dense in $X_{\mathbb{C}}$.

Case 1. $D = E \setminus \bigcup \mathcal{F} \in \mathcal{D}_{\mathbb{E},\mathbb{F}}$.

Then $D \cap F_\ell^k \subset E \cap F_\ell^k \in \mathcal{N}(X_{\mathbb{C}})$ by Claim 4.9.1.

Case 2. $D = F' \setminus ((\bigcup \mathcal{E}) \cup (\bigcup \mathcal{F})) \in \mathcal{D}_{\mathbb{F},\mathbb{E}}$.

If $F' \neq F_\ell^k$ then $F_\ell^k \cap D \subset F_\ell^k \cap F' \in \mathcal{N}(X_{\mathbb{C}})$ by Claim 4.9.2. Thus we may assume that $F' = F_\ell^k$ and hence $F_\ell^k \notin \mathcal{F}$ because $\mathcal{F} \subset \mathcal{F}_{1-\ell}$. But then

$$\begin{aligned} F_\ell^k \setminus D &= F_\ell^k \setminus \Bigl(F_\ell^k \setminus \Bigl(\Bigl(\bigcup \mathcal{E}\Bigr) \cup \Bigl(\bigcup \mathcal{F}\Bigr)\Bigr)\Bigr) \\ &= F_\ell^k \cap \Bigl(\Bigl(\bigcup \mathcal{E}\Bigr) \cup \Bigl(\bigcup \mathcal{F}\Bigr)\Bigr) = \bigcup_{E \in \mathcal{E}} (F_\ell^k \cap E) \cup \bigcup_{F \in \mathcal{F}} (F \cap F_\ell^k), \end{aligned}$$

where each $F_\ell^k \cap E$ is nowhere dense by Claim 4.9.1 and each $F \cap F_\ell^k$ is nowhere dense by Claim 4.9.2, i.e. $F_\ell^k \setminus D \in \mathcal{N}(X_{\mathbb{C}})$. $\square$

Claim 4.9.5 implies that $X_{\mathbb{C}}$ can be partitioned into μ many dense submaximal subspaces for both $\mu = \omega$ and $\mu = \kappa$. Since $\mathcal{C}(\kappa)$-spaces are CCC, it follows from Theorem 4.10 below that this is also valid for all μ with $\omega < \mu < \kappa$. □

The following result is somewhat different from the others in that it has no relevance to $\mathcal{D}$-forced spaces. Still we decided to include it here not only because it makes the proof of Theorem 4.9 simpler but also because it seems to have independent interest.

Theorem 4.10. *Let $\omega \leqslant \lambda < \mu < \kappa$ be cardinals and X be a topological space with $\mathrm{c}(X) \leqslant \mu$. If X can be partitioned into both λ many and κ many dense OHI subspaces then X can also be partitioned into μ many dense OHI subspaces.*

Proof. Let $\langle Y_\sigma\colon \sigma < \lambda\rangle$ and $\langle Z_\zeta\colon \zeta < \kappa\rangle$ be two partitions of X into OHI subspaces. For each $\sigma < \lambda$ let

$$\mathcal{U}_\sigma = \Big\{U \subset X\colon U \text{ is open and there is } I_{\sigma,U} \in [\kappa]^\mu \text{ such that } Y_\sigma \cap \bigcup\{Z_\zeta\colon \zeta \in I_{\sigma,U}\} \text{ is dense in } U\Big\}.$$

Since $\mathrm{c}(X) \leqslant \mu$ there is $\mathcal{U}^*_\sigma \in [\mathcal{U}_\sigma]^{\leqslant\mu}$ such that $U_\sigma = \bigcup \mathcal{U}^*_\sigma$ is dense in $\bigcup \mathcal{U}_\sigma$. Clearly, we also have $U_\sigma \in \mathcal{U}_\sigma$. Next we set $V_\sigma = X \setminus \overline{U_\sigma}$ and $Q_\sigma = X \setminus (U_\sigma \cup V_\sigma) = \mathrm{Fr}(U_\sigma)$.

Since $\lambda < \mu$ we can pick $I \in [\kappa]^\mu$ with

$$\bigcup\{I_{\sigma,U_\sigma}\colon \sigma < \lambda\} \subset I$$

and then can choose $J \in [\kappa \setminus I]^\lambda$. Let $Z = \bigcup\{Z_\zeta\colon \zeta \in I \cup J\}$.

For $\sigma \in \lambda$ let $R_\sigma = Y_\sigma \cap V_\sigma \cap Z$. Since $|I \cup J| = \mu$, it follows from the definition of $\mathcal{U}_\sigma$ and $V_\sigma = X \setminus \overline{\bigcup \mathcal{U}_\sigma}$ that

(∗) R_σ is nowhere dense in X for each $\sigma < \lambda$.

Let $P_\sigma = (Y_\sigma \cap U_\sigma) \setminus \bigcup\{Z_\zeta\colon \zeta \in I_{\sigma,U_\sigma}\}$ for $\sigma < \lambda$. Then P_σ is also nowhere dense because $\bigcup\{Z_\zeta\colon \zeta \in I_{\sigma,U_\sigma}\} \cap U_\sigma \cap Y_\sigma$ is dense in U_σ and Y_σ is OHI.

Now let $\{\sigma_\zeta\colon \zeta \in J\}$ be an enumeration of λ without repetition and for each $\zeta \in J$ set

$$T_\zeta = (Z_\zeta \cap U_{\sigma_\zeta}) \cup \big((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z\big).$$

Claim 4.10.1. *Each T_ζ is a dense OHI subspace of X.*

Proof. Z_ζ is dense in U_{σ_ζ} and

$$(Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z = (Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus R_{\sigma_\zeta}$$

is dense in V_{σ_ζ} because Y_{σ_ζ} is dense and $R_{\sigma_\zeta} = Y_{\sigma_\zeta} \cap V_{\sigma_\zeta} \cap Z$ is nowhere dense by (∗). Hence T_ζ is dense. T_ζ is OHI because both Z_ζ and Y_{σ_ζ} are. □

Claim 4.10.2. *The family $\{Z_\xi\colon \xi \in I\} \cup \{T_\zeta\colon \zeta \in J\}$ is disjoint.*

Proof. Assume first that $\xi \in I$ and $\zeta \in J$. Then $\xi \neq \zeta$ and hence

$$T_\zeta \cap Z_\xi = \big((Z_\zeta \cap U_{\sigma_\zeta}) \cup \big((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z\big)\big) \cap Z_\xi$$
$$\subset (Z_\zeta \cap Z_\xi) \cup (Z_\xi \setminus Z) = \emptyset.$$

Next if $\{\zeta, \xi\} \in [J]^2$, then

$$T_\zeta \cap T_\xi = \big((Z_\zeta \cap U_{\sigma_\zeta}) \cup \big((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z\big)\big) \cap \big((Z_\xi \cap U_{\sigma_\xi}) \cup \big((Y_{\sigma_\xi} \cap V_{\sigma_\xi}) \setminus Z\big)\big)$$
$$\subset (Z_\zeta \cap Z_\xi) \cup (Z_\zeta \setminus Z) \cup (Z_\xi \setminus Z) \cup (Y_{\sigma_\zeta} \cap Y_{\sigma_\xi}) = \emptyset. \qquad \square$$

Thus we would be finished if we could prove that

$$\{Z_\xi\colon \xi \in I\} \cup \{T_\zeta\colon \zeta \in J\}$$

covers X. However, we can only prove the following weaker statement.

Claim 4.10.3.

$$X = \bigcup\{Z_\xi\colon \xi \in I\} \cup \bigcup\{T_\zeta\colon \zeta \in J\} \cup \bigcup\{P_\sigma \cup Q_\sigma \cup R_\sigma\colon \sigma < \lambda\}.$$

Proof. Let $x \in X$ be any point then there is a unique $\sigma < \lambda$ with $x \in Y_\sigma$. If $x \notin U_\sigma \cup V_\sigma$ then, by definition, $x \in Q_\sigma$.

So assume now that $x \in U_\sigma$. If $x \notin \cup\{Z_\zeta\colon \zeta \in I_{\sigma,U_\sigma}\}$ then $x \in P_\sigma$. Otherwise $x \in Z_\zeta$ for some $\zeta \in I_{\sigma,U_\sigma} \subset I$.

Finally, assume that $x \in V_\sigma$ and let $\zeta \in J$ with $\sigma_\zeta = \sigma$. Now, if $x \notin Z$ then $x \in T_\zeta$ and if $x \in Z$ then $x \in R_\sigma$. $\square$

The pairwise disjoint dense OHI subspaces $\{Z_\xi\colon \xi \in I\} \cup \{T_\zeta\colon \zeta \in J\}$ thus cover X apart from the nowhere dense sets $P_\sigma \cup Q_\sigma \cup R_\sigma$ for $\sigma < \lambda$. But then, using the obvious fact that the union of a dense OHI subspace with any nowhere dense set is OHI, the latter can be simply "absorbed" by the former, and thus a partition of X into μ many dense OHI subspaces can be produced. $\square$

5. Applications to extraresolvability

In [9] Comfort and Hu investigated the following question: *Are maximally resolvable spaces* (*strongly*) *extraresolvable*? They presented several counterexamples, but the following question was left open (see [9, Discussion 1.4]): *Is there a maximally resolvable Tychonov space X with $|X| = \mathrm{nwd}(X)$ such that X is not extraresolvable*? Using our main Theorem 3.3 we can give an affirmative answer to this question in ZFC. Recall that if X is a $\mathcal{C}(\kappa)$-space then $|X| = \mathrm{nwd}(X) = \kappa$.

Theorem 5.1. *For every infinite cardinal κ there is a $\mathcal{C}(\kappa)$-space that is hereditarily κ-resolvable* (*and hence maximally resolvable*) *but not extraresolvable.*

Proof. Let $\vec{\kappa} = \langle \kappa, \kappa, \ldots \rangle$ be the constant κ sequence of length ω. By Fact 3.2 there are a countable family $\mathbb{D} = \{\langle D^i_m : i < \kappa \rangle : m < \omega\}$ of κ-partitions of κ and a family $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi \rangle : \xi < 2^\kappa\}$ of 2-partitions of κ such that $\mathbb{B} \cup \mathbb{D}$ is κ-independent, that is for each $\eta \in \mathbb{FIN}(\vec{\kappa}) = \mathrm{Fn}(\omega, \kappa)$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ we have

$$\left|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]\right| = \kappa.$$

Now let

$$\mathcal{D} = \left\{\mathbb{D}[\eta] : \eta \in \mathbb{FIN}(\vec{\kappa})\right\}$$

and apply Theorem 3.3 to $\mathbb{B}$ and $\mathcal{D}$ to get a family $\mathbb{C}$ of 2-partitions of κ satisfying 3.3(1)–(4).

Since $|\mathcal{D}| = \kappa$ and $\hat{c}(X_{\mathbb{C}}) = \omega_1$, it follows from Lemma 2.8 that $X_{\mathbb{C}}$ is not κ^+-extraresolvable (= extraresolvable).

Next, if $\mathbb{D}[\eta] \in \mathcal{D}$ then $\{\mathbb{D}[\eta^\frown\langle\zeta\rangle] : \zeta < \kappa\}$ partitions $\mathbb{D}[\eta]$ into κ many dense sets, i.e. $\mathbb{D}[\eta]$ is κ-resolvable. Hence, by Lemma 2.10, $X_{\mathbb{C}}$ is hereditarily κ-resolvable. □

Our next two results are natural analogues of Theorems 4.5 and 4.8 with μ-resolvability replaced by μ-extraresolvability. Before formulating them, however, we need a new piece of notation.

Definition 5.2. Given a family $\mathbb{D} = \{\langle D^0_\xi, D^1_\xi \rangle : \xi \in \rho\}$ of 2-partitions of a cardinal κ we set

$$\mathcal{I}(\mathbb{D}) = \left\{ D^0_\zeta \setminus \bigcup_{\xi \in \Xi} D^0_\xi : \zeta \in \rho \wedge \Xi \in \left[\rho \backslash \{\zeta\}\right]^{<\omega} \right\}.$$

Theorem 5.3. *For any infinite cardinals $\kappa \leqslant \lambda \leqslant 2^\kappa$ there is a λ-extraresolvable $\mathcal{C}(\kappa)$:*

- *space X that is not λ^+-extraresolvable. Moreover, every crowded subspace of X has a dense submaximal subspace.*

Proof. By Fact 3.2 there are families of 2-partitions of κ, say $\mathbb{D} = \{\langle D^0_\zeta, D^1_\zeta \rangle : \zeta < \lambda\}$ and $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi \rangle : \xi < 2^\kappa\}$, such that $\mathbb{B} \cup \mathbb{D}$ is κ-independent, i.e. $|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]| = \kappa$ for all $\eta \in \mathrm{Fn}(\lambda, 2)$ and $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$.

Then $\mathcal{D} = \mathcal{I}(\mathbb{D})$ is a family of κ-dense subsets of $X_{\mathbb{B}}$, hence we can apply the main Theorem 3.3 to $\mathbb{B}$ and $\mathcal{D}$ to obtain a family of partitions $\mathbb{C}$ satisfying 3.3(1)–(4). We shall show that $X_{\mathbb{C}}$ is as required.

Claim 5.3.1. $D^0_\zeta \cap D^0_\xi \in \mathcal{N}(X_{\mathbb{C}})$ *for each pair* $\{\zeta, \xi\} \in [\lambda]^2$.

Proof. Write $Y = D^0_\zeta \cap D^0_\xi$ and $D = D^0_\nu \setminus \bigcup_{\eta \in \Xi} D^0_\eta$ be an arbitrary member of $\mathcal{D}$. We can assume that $\xi \neq \nu$ and so

$$D \setminus Y = \left(D^0_\nu \setminus \bigcup_{\eta \in \Xi} D^0_\eta\right) \setminus \left(D^0_\zeta \cap D^0_\xi\right) \supset D^0_\nu \setminus \bigcup_{\eta \in \Xi \cup \{\xi\}} D^0_\eta \in \mathcal{D},$$

showing that $D \setminus Y$ is dense in $X_{\mathbb{C}}$. Hence, by Lemma 2.6, Y is nowhere dense in $X_{\mathbb{C}}$. □

Thus the family $\{D^0_\xi\colon \xi \in \lambda\}$ witnesses that $X_{\mathbb{C}}$ is λ-extraresolvable. On the other hand, since $|\mathcal{D}| = \lambda$ and $\mathrm{c}(X_{\mathbb{C}}) = \omega$, Lemma 2.8 implies that $X_{\mathbb{C}}$ is not λ^+-extraresolvable.

Claim 5.3.2. *Every $S \in \mathcal{D}$ is a submaximal subspace of $X_{\mathbb{C}}$.*

Proof. Let $S = D^0_\nu \setminus \bigcup_{\eta\in\Xi} D_\eta$, moreover $D = D^0_\mu \setminus \bigcup_{\eta\in\Psi} D^0_\eta$ be an arbitrary member of $\mathcal{D}$. If $\nu = \mu$ then, by Claim 5.3.1,

$$S \setminus D = \left(D^0_\nu \setminus \bigcup_{\eta\in\Xi} D^0_\eta\right) \setminus \left(D^0_\nu \setminus \bigcup_{\eta\in\Psi} D^0_\eta\right) \subset \bigcup_{\eta\in\Psi} D^0_\nu \cap D^0_\eta \in \mathcal{N}(X_{\mathbb{C}})$$

and so $S \subset^* D$. If, on the other hand, $\nu \neq \mu$ then we have

$$S \cap D = \left(D^0_\nu \setminus \bigcup_{\eta\in\Xi} D^0_\eta\right) \cap \left(D^0_\nu \setminus \bigcup_{\eta\in\Psi} D^0_\eta\right) \subset D^0_\nu \cap D^0_\mu \in \mathcal{N}(X_{\mathbb{C}})$$

by Claim 5.3.1 again, consequently $S \cap D =^* \emptyset$. Thus S is OHI by Lemma 2.7, and since $X_{\mathbb{C}}$ is NODEC, S is even submaximal. □

Claim 5.3.2 clearly implies that all $\mathcal{D}$-pieces and hence all partial $\mathcal{D}$-mosaics are submaximal subspaces of $X_{\mathbb{C}}$. But $X_{\mathbb{C}}$ is $\mathcal{D}$-forced and NODEC, and therefore, by Lemma 1.10, every crowded subspace of $X_{\mathbb{C}}$ includes a partial $\mathcal{D}$-mosaic as a dense subspace. □

Let us remark that Theorem 5.3 makes sense, and remains valid, for $\lambda < \kappa$ as well. However, in this case Theorem 4.5 yields a stronger result. This is the reason why we only formulated it for $\lambda \geqslant \kappa$. This remark also applies to our following result that implies an analogue of Theorem 4.8 for μ-extraresolvability instead of μ-resolvability.

Theorem 5.4. *Let $\kappa < \lambda = \mathrm{cf}(\lambda) \leqslant (2^\kappa)^+$ be infinite cardinals. Then there is a $\mathcal{C}(\kappa)$-space that is*

(1) *hereditarily κ-resolvable,*
(2) *hereditarily μ-extraresolvable for all $\mu < \lambda$,*
(3) *not λ-extraresolvable.*

Proof. Similarly as in the proof of 4.8, let the sequence $\vec{\lambda} = \langle \lambda_\zeta\colon \zeta < \lambda\rangle$ be given by $\lambda_\zeta = \omega_\zeta$ if λ is a limit (hence inaccessible) cardinal, and let $\lambda_\zeta = \rho$ for each $\zeta < \lambda$ if $\lambda = \rho^+$ is a successor.

Using Fact 3.2 again, we can find the following two types of families of 2-partitions of κ:

$$\mathbb{B} = \left\{\langle B^0_\xi, B^1_\xi\rangle\colon \xi < 2^\kappa\right\}$$

and

$$\mathbb{D}_\zeta = \left\{\langle D^0_{\zeta,\nu}, D^1_{\zeta,\nu}\rangle\colon \nu < \lambda_\zeta\right\}$$

for all $\zeta < \lambda$, moreover a countable family

$$\mathbb{G} = \{\langle G_n^i\colon i < \kappa\rangle\colon n < \omega\}$$

of κ-partitions of κ such that $\mathbb{B} \cup \bigcup_{\zeta<\lambda} \mathbb{D}_\zeta \cup \mathbb{G}$ is κ-independent.

Now let $\mathcal{D}$ be the family of all sets of the form $\bigcap_{i<n} E_i \cap \mathbb{G}[\eta]$ where $n < \omega$ and $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ with all the ζ_i distinct, moreover $\eta \in \mathrm{Fn}(\omega, \omega)$. It is easy to see that $\mathcal{D}$ is a family of κ-dense sets in $X_\mathbb{B}$, so we may apply Theorem 3.3 with $\mathbb{B}$ and $\mathcal{D}$ to get a family of partitions $\mathbb{C}$ satisfying 3.3(1)–(4). We claim that $X_\mathbb{C}$ is as required.

Indeed, as we have already seen many times, the $\mathbb{G}[\eta]$ components of the elements of $\mathcal{D}$ can be used to show that every $D \in \mathcal{D}$ is κ-resolvable. But then, as $X_\mathbb{C}$ is both $\mathcal{D}$-forced and NODEC, every crowded subspace of $X_\mathbb{C}$ is κ-resolvable by Lemma 2.10, hence (1) is proven.

To prove (2), we need the following statement.

Claim 5.4.1. *Assume that $\zeta < \lambda$ and $\{\nu, \nu'\} \in [\lambda_\zeta]^2$. Then*

$$Y = D^0_{\zeta,\nu} \cap D^0_{\zeta,\nu'} \in \mathcal{N}(X_\mathbb{C}).$$

Proof. Let $D = \bigcap_{i<n} E_i \cap G$ be an arbitrary element of $\mathcal{D}$, where $n \in \omega$, $\{\zeta_i\colon i < n\} \in [\lambda]^n$ with $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ for all $i < n$, and $G = \mathbb{G}[\eta]$ for some $\eta \in \mathrm{Fn}(\omega, \omega)$. Our aim is to check that $D \setminus Y$ is dense, hence, by shrinking D if necessary, we may assume that $\zeta_0 = \zeta$ and $E_0 = D^0_{\zeta,\varphi} \setminus \bigcup_{\xi\in\Psi} D^0_{\zeta,\xi}$. Since $\nu \neq \nu'$ we can assume that $\varphi \neq \nu$. Then

$$\begin{aligned} D \setminus Y &\supset \left(\bigcap_{i<n} E_i \cap G\right) \setminus D^0_{\zeta,\nu} \\ &= \left(D^0_{\zeta,\varphi} \setminus \bigcup_{\xi\in\Psi\cup\{\nu\}} D^0_{\zeta,\xi}\right) \cap \bigcap_{i=1}^{n-1} E_i \cap G \in \mathcal{D}. \end{aligned}$$

Hence, $D \setminus Y$ is indeed dense and so, by Lemma 2.6, Y is nowhere dense in $X_\mathbb{C}$. □

Assume now that $D = \bigcap_{i<n} E_i \cap G$ is again an arbitrary element of $\mathcal{D}$ with $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ for all $i < n$. By Claim 5.4.1, for every ζ that is distinct from all the ζ_i the collection

$$\{D \cap D^0_{\zeta,\nu}\colon \nu < \lambda_\zeta\}$$

consists of members of $\mathcal{D}$ that have pairwise nowhere dense intersections, hence D is λ_ζ-extraresolvable. Clearly, this implies that D is μ-extraresolvable for all $\mu < \lambda$. By Lemma 2.10, since $X_\mathbb{C}$ is $\mathcal{D}$-forced and NODEC it follows that $X_\mathbb{C}$ is hereditarily μ-extraresolvable for all $\mu < \lambda$ and thus (2) has been established.

Finally, a standard Δ-system and counting argument proves that for each $\mathcal{E} \in [\mathcal{D}]^\lambda$ there is $\mathcal{F} \in [\mathcal{E}]^\lambda$ such that $F \cap F' \in \mathcal{D}$ whenever $\{F, F'\} \in [\mathcal{F}]^2$. Hence, by Lemma 2.8, the space $X_\mathbb{C}$ is not λ-extraresolvable, proving (3). □

Having seen these parallels between resolvability and extraresolvability, it is interesting to note that we do not know if the analogue of Bashkara Rao's "compactness" theorem holds for extraresolvability.

Problem 5.5. Assume that λ is a singular cardinal with $\mathrm{cf}(\lambda) = \omega$ and the space X is μ-extraresolvable for all $\mu < \lambda$. Is it true then that X is also λ-extraresolvable?

Both Theorems 5.3 and 5.4 imply, in ZFC, that for every infinite cardinal κ there is a 2^κ-extraresolvable $\mathcal{C}(\kappa)$-space. However, Theorem 5.12 below implies that this fails for strong 2^κ-extraresolvability. To prove 5.12, however, we need some preparatory work.

Definition 5.6. Let κ be any cardinal. A topological space X is called *κ-fragmented* iff there is a κ-sequence $\langle A_\alpha\colon \alpha < \kappa\rangle$ of pairwise disjoint elements of $[X]^{<\kappa}$ such that $|A_\alpha| \leqslant |\alpha|$ for all $\alpha < \kappa$ and $\bigcup\{A_\alpha\colon \alpha \in I\}$ is κ-dense in X whenever $I \in [\kappa]^\kappa$. If, in addition,

$$\bigcup\{A_\alpha\colon \alpha \in K\} \in \mathcal{N}(X)$$

for each $K \in [\kappa]^{<\kappa}$ then X is called *κ-hyperresolvable.* Finally, we say that X is *fragmented (hyperresolvable)* iff it is $\Delta(X)$-fragmented ($\Delta(X)$-hyperresolvable).

We call a subfamily $\mathcal{F}$ of $[\kappa]^\kappa$ *boundedly* almost disjoint (BAD) if the intersection of any two members of $\mathcal{F}$ is bounded in κ. Of course, if κ is regular then any almost disjoint subfamily of $[\kappa]^\kappa$ is BAD. Moreover it is standard to show that for every infinite κ there is a BAD subfamily of $[\kappa]^\kappa$ of size κ^+. Thus from the above definitions we get the following fact, explaining the term hyperresolvable.

Fact 5.7. *Any hyperresolvable space X is extraresolvable and if, in addition, $\Delta(X) = \mathrm{nwd}(X)$ then X is strongly extraresolvable.*

Definition 5.8. Let X be a topological space and κ be an infinite cardinal. A point $p \in X$ is said to be a *κ-limit* iff there is a one-to-one κ-sequence of points converging to p in X.

Lemma 5.9. *If a topological space X contains a dense set D of size $\leqslant \kappa$ consisting of κ-limit points then X is κ-fragmented.*

Proof. Enumerate D as $\{d_\zeta\colon \zeta < \kappa\}$ with possible repetitions. For each $d \in D$ fix a one-to-one sequence $\{x_d(\xi)\colon \xi < \kappa\} \subset X \setminus \{d\}$ converging to d. By transfinite recursion on $\alpha < \kappa$ we may easily construct a sequence $\langle A_\alpha\colon \alpha < \kappa\rangle$ such that

(1) $A_\alpha \subset X \setminus \bigcup\{A_\delta\colon \delta < \alpha\}$,
(2) $|A_\alpha| \leqslant |\alpha|$,
(3) $A_\alpha \cap \{x_{d_\zeta}(\beta)\colon \beta \geqslant \alpha\} \neq \emptyset$ for all $\zeta \leqslant \alpha$.

It remains to show that $A_I = \bigcup\{A_\alpha\colon \alpha \in I\}$ is κ-dense in X whenever $I \in [\kappa]^\kappa$. So let $G \neq \emptyset$ be open and fix $d \in D \cap G$. There is $\zeta < \kappa$ with $d_\zeta = d$. Then for each $\alpha \in I \setminus \zeta$ we have $A_\alpha \cap \{x_d(\beta)\colon \beta \geqslant \alpha\} \neq \emptyset$. But the sequence $\{x_d(\xi)\colon \xi < \kappa\}$ is eventually in G and the A_α's are pairwise disjoint, consequently we have $|G \cap A_I| \geqslant \kappa$. □

The Cantor cube $\mathrm{D}(2)^{2^\kappa}$ has a dense subset of cardinality κ, moreover every point of $\mathrm{D}(2)^{2^\kappa}$ is a κ-limit point. Thus from Lemma 5.9 we obtain the following fact.

Fact 5.10. *For each cardinal κ, the Cantor cube* $\mathrm{D}(2)^{2^\kappa}$ *has a κ-fragmented, dense subspace X with $|X| = \Delta(X) = \kappa$.*

Using our main Theorem 3.3 we can improve this as follows.

Theorem 5.11. *For each cardinal κ there is a hyperresolvable* (*and hence strongly extraresolvable*) *$\mathcal{C}(\kappa)$-space.*

Proof. By 3.1 and Fact 5.10 we can find a κ-independent family

$$\mathbb{B} = \{\langle B^0_\xi, B^1_\xi\rangle \colon \xi < 2^\kappa\}$$

of 2-partitions of κ such that $X_\mathbb{B}$ is κ-fragmented by the sequence $\mathcal{A} = \langle A_\nu \colon \nu < \kappa\rangle$.

As above, for any $I \subset \kappa$ we write $A_I = \bigcup\{A_\nu \colon \nu \in I\}$. Then

$$\mathcal{D} = \{A_I \colon I \in [\kappa]^\kappa\}$$

is a family of κ-dense sets in $X_\mathbb{B}$. So we can apply Theorem 3.3 to $\mathbb{B}$ and $\mathcal{D}$ and get a family $\mathbb{C}$ of 2-partitions of κ satisfying 3.3(1)–(4).

We claim that the sequence $\mathcal{A}$ witnesses that $X_\mathbb{C}$ is $(\Delta(X_\mathbb{C}) =)\kappa$-hyperresolvable. Indeed, every A_I remains κ-dense in $X_\mathbb{C}$ for $I \in [\kappa]^\kappa$ because $A_I \in \mathcal{D}$. Moreover, if $J \in [\kappa]^{<\kappa}$ then for each $I \in [\kappa]^\kappa$, we have $A_I \setminus A_J = A_{I\setminus J} \in \mathcal{D}$, consequently Lemma 2.6 may be applied to conclude that A_J is nowhere dense in $X_\mathcal{C}$. □

Remark. The spaces obtained from Theorem 5.11 do not contain non-trivial convergent sequences of any length because they are NODEC. This shows that the converse of Lemma 5.9 fails.

After this preparation we are now ready to formulate and prove Theorem 5.12.

Theorem 5.12. *Let $\kappa = \mathrm{cf}(\kappa) < \lambda$ be two infinite cardinals. Then the following three statements are equivalent*:

(i) *There is a strongly λ-extraresolvable but not λ^+-extraresolvable $\mathcal{C}(\kappa)$-space.*

(ii) *There is a strongly λ-extraresolvable space X with*

$$|X| = \mathrm{nwd}(X) = \kappa.$$

(iii) *There is an almost disjoint family $\mathcal{T} \subset [\kappa]^\kappa$ of size λ.*

Proof. Clearly (i) implies (ii) implies (iii). To prove that (iii) implies (i), we again use Fact 5.10 and Observation 3.1 to find an independent, separating family $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi\rangle \colon \xi < 2^\kappa\}$ of 2-partitions of κ such that $X_\mathbb{B}$ is κ-fragmented by the sequence $\mathcal{A} = \langle A_\nu \colon \nu < \kappa\rangle$.

Since $A_I = \bigcup\{A_\nu \colon \nu \in I\}$ is κ-dense in $X_\mathbb{C}$ for each $I \in [\kappa]^\kappa$, we may apply Theorem 3.3 to $\mathbb{B}$ and the family of κ-dense sets

$$\mathcal{D} = \{A_T \colon T \in \mathcal{T}\}$$

to get a family $\mathbb{C}$ of 2-partitions of κ that satisfies 3.3(1)–(4).

Since κ is regular, the family of dense sets $\{A_T\colon T \in \mathcal{T}\} \subset [\kappa]^\kappa$ is almost disjoint because $\mathcal{T}$ is. This, together with $\mathrm{nwd}(X_\mathbb{C}) = \kappa$ clearly implies that $X_\mathbb{C}$ is strongly λ-extraresolvable.

Moreover, as $|\mathcal{D}| = \lambda$ and $\mathrm{c}(X_\mathbb{C}) = \omega$, Lemma 2.8 implies that $X_\mathbb{C}$ is not λ^+-extraresolvable. □

It is known (see, e.g. [3]) that if one adds at least ω_3 Cohen reals to a model of GCH then in the resulting generic extension there is no almost disjoint subfamily of $[\omega_1]^{\omega_1}$ of size ω_3. Consequently, by Theorem 5.12, in such a ZFC model, although 2^{ω_1} is as big as you wish, there is no strongly ω_3-extraresolvable space X with $|X| = \mathrm{nwd}(X) = \omega_1$.

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