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# A Kuratowski Theorem for Nonorientable Surfaces

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Let $\Sigma$ denote a surface. A graph $G$ is *irreducible* for $\Sigma$ provided that $G$ does not embed in $\Sigma$, but any proper subgraph does so embed. Let $I(\Sigma)$ denote the set of graphs without degree two vertices which are irreducible for $\Sigma$. Observe that a graph embeds in $\Sigma$ if and only if it does not contain a subgraph homeomorphic to a member of $I(\Sigma)$. For example, Kuratowski's theorem shows that $I(\Sigma)=\{K_{3,3}, K_5\}$ when $\Sigma$ is the sphere. In this paper we prove that the set $I(\Sigma)$ is finite for each nonorientable surface, setting in part a conjecture of Erdös from the 1930s. 

## 1. INTRODUCTION

Let $\Sigma_n$ denote the closed orientable surface of genus $n$, that is, the sphere with $n$ handles attached. Let $\Sigma_n^{\sim}$ denote the nonorientable surface of nonorientable genus $n$, that is, the sphere with $n$ crosscaps attached. As a special case we consider the sphere as the surface of both orientable and nonorientable genus zero.

For an orientable surface $\Sigma$, let $\gamma(\Sigma)$ denote its genus. Likewise if $\Sigma$ is nonorientable, let $\tilde{\gamma}(\Sigma)$ denote its nonorientable genus. For any surface $\Sigma$, let $\chi(\Sigma)$ denote its Euler characteristic and define the *Euler genus*, $\bar{\gamma}(\Sigma)$, as $2-\chi(\Sigma)$. Note that if $\Sigma$ is orientable, then $\bar{\gamma}(\Sigma)=2\gamma(\Sigma)$, and if $\Sigma$ is nonorientable, then $\bar{\gamma}(\Sigma)=\tilde{\gamma}(\Sigma)$.

In this paper all graphs are finite and are considered as topological spaces. An *embedding* of a graph $G$ into a surface $\Sigma$ is a one-to-one map $\phi$: $G\to\Sigma$. Define the *orientable genus* of $G$, as the least value of $\gamma(\Sigma)$ over all orientable surfaces $\Sigma$ in which $G$ embeds. Similarly define the *nonorientable*



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*genus* of $G$, $\tilde{\gamma}(G)$, and the *Euler genus*, $\bar{\gamma}(G)$. An embedding $\phi\colon G \to \Sigma$ will be called an *orientable genus embedding* provided that $\gamma(G)=\gamma(\Sigma)$. We similarly define a *nonorientable genus embedding* and an *Euler genus embedding*. It is well known that for any graph $G$, $\tilde{\gamma}(G) \leqslant 2\gamma(G)+1$. Also note that $\bar{\gamma}(G)=\min\{\tilde{\gamma}(G), 2\gamma(G)\}$.

Let $P$ be some property of a graph. We say that $G$ is *P-critical* provided that $G$ has property $P$, but no proper subgraph of $G$ has property $P$. For example, if $P$ is the property that $\gamma(G) \geqslant 1$, then the $P$-critical graphs, or $(\gamma \geqslant 1)$-critical graphs, are the homeomorphs of the two Kuratowski graphs $K_5$ and $K_{3,3}$. In general, if $P$ is the property that $\gamma(G) \geqslant n$, then a $(\gamma \geqslant n)$-critical graph does not embed in $\Sigma_{n-1}$, but every proper subgraph of $G$ does embed in $\Sigma_{n-1}$. Such a graph is called *irreducible* for the surface $\Sigma_{n-1}$. Similarly a $(\tilde{\gamma} \geqslant n)$-critical graph is irreducible for $\Sigma^{\sim}_{n-1}$. There is no commonly accepted name for a $(\bar{\gamma} \geqslant n)$-critical graph.

We are now ready to state the main result of this paper.

THEOREM 1.1. *There exists a function $f$ such that for any graph $G$, if $G$ is either $(\bar{\gamma} \geqslant n)$-critical or $(\tilde{\gamma} \geqslant n)$-critical, then $G$ contains at most $f(n)$ vertices which are not of degree* 2.

The proof of Theorem 1.1 appears in Section 3 of this paper.

For any surface $\Sigma$, let $I(\Sigma)$ denote the set of graphs which have no degree two vertices and which are irreducible for $\Sigma$. Restating part of the above theorem, we get the following.

THEOREM 1.2. *$I(\Sigma^{\sim}_n)$ is finite for each $n$.*

The basic idea of the proof is that there are only finitely many irreducible graphs for the surface $\Sigma^{\sim}_{n-1}$, and for any one of these, there are only a finite number of minimal ways to create a graph that does not embed in $\Sigma^{\sim}_n$. Specifically, in Section 4 we examine properties of a graph which is irreducible for $\Sigma^{\sim}_{n-1}$ and which is embedded in $\Sigma^{\sim}_n$ (in truth, it is here that we need the added complication of examining $(\bar{\gamma} \geqslant n)$-critical graphs). In Sections 5–7 we add selected subgraphs to this embedded $H$ in order to further restrict its possible embeddings in $\Sigma^{\sim}_n$. Sections 8–11 are concerned with bounding the number and size of the bridges of $H$ in $G$, and hence with bounding the number of vertices in $G$. These results are summarized in Section 12 and then used in Section 3 to prove Theorem 1.1.

The proofs of Theorems 1.1 and 1.2 do not extend to orientable surfaces. The reasons for this are discussed in Section 4 of this paper.

The study of irreducible graphs has a rich history, beginning in 1930 when Kuratowski [K] showed that the irreducible graphs for the sphere were $K_{3,3}$ and $K_5$. This result is commonly stated as an "excluded subgraph" characterization of planar graphs; $G$ is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or to $K_5$. In the 1930s Erdös conjectured that $I(\Sigma)$ was finite for each surface $\Sigma$, i.e., that there was a finite list of graphs whose exclusion characterized the graphs which embed in $\Sigma$. Little progress was made on this problem for the next 40 years, although the special case of finding the cubic irreducible graphs was recognized. Let $I_3(\Sigma)$ denote the set of cubic irreducible graphs for a surface $\Sigma$.

Being the simplest surface other than the sphere, attention focused on $\Sigma_1^{\sim}$, the real projective plane. The first breakthrough against Erdös' conjecture came from M. Milgram [M1] who proved that $I_3(\Sigma_1^{\sim})$ was finite. He latter improved this [M2], showing that there were exactly six graphs in $I_3(\Sigma_1^{\sim})$. This result was shown independently by Glover and Huneke [GH1]. The latter two authors then showed that $I_n(\Sigma_1^{\sim})$ was finite for all $n$ [GH2]; $I_n(\Sigma)$ denotes the set of irreducible graphs for $\Sigma$ which are of maximum degree at most $n$. Finally, the showed [GH3] that $I(\Sigma_1^{\sim})$ is finite, the first surface other than the sphere for which Erdös' conjecture was shown. Continuing the work in the projective plane, Glover, Huneke, and Wang [GHW] exhibited a list of 103 irreducible graphs. Archdeacon [A1] (see also [A2] for discussion) showed that their list was complete, and hence that $|I(\Sigma_1^{\sim})| = 103$.

Turning attention away from the projective plane, Archdeacon and Huneke [AH] have shown that $I_3(\Sigma_n^{\sim})$ is finite for each nonorientable surface; specifically, they showed that the cubic analogue of Theorem 1.1 holds. The techniques used are similar to those of this paper, although this paper is essentially self-contained. Both proofs are in the "spirit" of Kuratowski's original proof, embedding subgraphs and attempting to extend these embeddings.

Using an entirely different approach involving graph minors, Robertson and Seymour [RS1] have proven a special of Wagner's conjecture. This implies the main result of this paper, as well as the orientable analogue. Our results were obtained independently of, and concurrently with, their work. Their work is substantially longer than ours, although their result is much more general. We refer the interested reader to [RS2, RS3] for surveys.

Before outlining the structure of this paper, we need some definitions. These concepts will be used throughout this paper.

Let $G$ be a graph and let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. A *topological vertex* of $G$ is a vertex which is not of degree 2. A *topological edge* of $G$ is a path $P$ such that the two endpoints of $P$ are topological vertices of $G$ and each vertex interior to $P$ is of degree 2 in $G$. A *piece* of $G$ is either a topological vertex or the interior (excluding the endpoints) of a topological edge of $G$. Let $V_t(G)$, $E_t(G)$, and $P_t(G)$ denote the set of topological vertices, topological edges, and pieces of $G$, respectively. We say that two pseudographs (allowing loops and multiple edges) are *homeomorphic* provided that they are homeomorphic as topological spaces, that is, if they can be made isomorphic by the subdivision of edges. If every component of $G$ contains a vertex of degree not equal to 2, then $G$ is homeomorphic to some pseudograph $P$ where $|V_t(G)| = |V(P)|$ and $|E_t(G)| = |E(P)|$. We will call this $P$ the *underlying pseudograph* of $G$.

A *pair* $(G, H)$ is a graph $G$ together with a subgraph $H$. A pair $(G, H)$ is *2-connected* provided that both $G$ and $H$ are vertex 2-connected. If a graph is not 2-connected we shall say that it is *separable*. Let $\Sigma$ be a surface. A *$\Sigma$-pair* $(G, H)$ is a pair such that:

(1) $G$ does not embed in $\Sigma$,

(2) $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$, and

(3) the underlying pseudograph of $H$ has no loops, and has at most two edges joining any pair of vertices.

For example, the pair $(K_{3,3}, K_{3,3} - e)$ is a $\Sigma$-pair for the sphere. Observe that it is possible that $H$ may not embed in $\Sigma$. In this case, as we construct various $\Sigma$-pairs and examine their properties, the necessary conditions will be vacuously satisfied. The possibility of parallel edges in the underlying pseudograph of $H$ is a technical consideration needed in Section 6. Restricting to at most two such edges is needed only for the following lemma.

LEMMA 1.3. *Let $(G, H)$ be a $\Sigma$-pair. Then*

$$|E_t(H)| \leqslant |V_t(H)| \, (|V_t(H)| - 1).$$

Thus when we wish to give upper bounds on the size of a graph as a function of $|V_t(H)|$, it will suffice to express these bounds as function of both $|V_t(H)|$ and $|E_t(H)|$.

A pair $(G', H')$ is a *refinement* of a $\Sigma$-pair $(G, H)$ provided that:

(1) $(G', H')$ is a $\Sigma$-pair,

(2) $G'$ is a subgraph of $G$, and

(3) $H'$ contains a subgraph which is homeomorphic to $H$.

Informally, in making a refinement we may delete some edges from $G$ (provided we maintain nonembedability) and add some edges to $H$ (or to a homeomorphic copy thereof).

Let $(G, H)$ be a pair. A *$(G, H)$-bridge* $B$ is the closure (in $G$) of a topological component of $G - H$. The *vertices of attachment* of $B$, henceforth denoted vofa$(B)$, are those vertices of $G$ which form $B \cap H$.

Finally, let $C$ be a cycle in a surface $\Sigma$. We say that $C$ is *contractible* provided that $C$ is contained in some disk $D$ contained in $\Sigma$, i.e., if $C$ is homotopic to a point. A cycle which is not contractible will be called *noncontractible.*

The following theorem is essential to this paper. Its proof is given in Section 12.

THEOREM 12.2. *Let $\Sigma$ be a surface and let $(G, H)$ be a 2-connected $\Sigma$-pair. Then there exists a 2-connected $K \subset G$ such that $K$ does not embed in $\Sigma$, $K$ contains a subgraph homeomorphic to $H$, and $|V_t(K)|$ is bounded by a function of $|V_t(H)|$.*

We now proceed to outline the paper. In Section 2 we present several results on connectedness. In particular these results cover $\Sigma$-pairs $(G, H)$ where either $G$ or $H$ is separable, allowing us to concentrate on 2-connected $\Sigma$-pairs. In Section 3 we then use these results together with Theorem 12.2 to prove Theorem 1.1, the main result of this paper. The remainder of the paper is then concerned with proving Theorem 12.2.

Theorem 12.2 is proved by taking a $\Sigma$-pair $(G, H)$, studying properties of how $H$ embeds in $\Sigma$, and attempting to extend these embeddings to include certain $(G, H)$-bridges. In particular, in Section 4 we examine properties of Euler genus embeddings. In Section 5 we examine certain subgraphs $K$ of $G$ which must contain a noncontractible cycle for any embedding of $H \cup K$ into $\Sigma$. We combine these two sections in Section 6 to construct a refinement $(G', H')$ of $(G, H)$ which satisfies certain properties. In Section 7 we construct a further refinement $(G'', H'')$ of $(G', H')$ which satisfies a more restrictive set of properties. It is the $\Sigma$-pair $(G'', H'')$ which we work with in Sections 8 through 11. In Section 8 we examine the types of $(G'', H'')$-bridges, including a bound on the size of any $(G'', H'')$-bridge. In Section 9 we then prove on bound on the maximum degree of $G''$. In Section 10 we find paths contained in topological edges of $H''$ which contain the vertices of attachment for a "large" number of $(G'', H'')$-bridges. In Section 11 we then prove a bound on the number of $(G'', H'')$-bridges in these paths. In Section 12 we gather these results together and prove a bound on the number of $(G'', H'')$-bridges. Note that by bounding the size and number of $(G'', H'')$-bridges, we obtain a bound on $|V_t(G'')|$. Finally we give the (by then easy) proof of Theorem 12.2.

The reader is advised to first skim the paper, paying special attention to the first paragraph of each section. These paragraphs emphasize how the results of that section fit into the overall proof. The reader is also advised to periodically refer to Section 12 to review the proof of Theorem 12.2.

We now proceed to the proofs.

## 2. Some Results on Connectedness

In this section we prove several results about connectedness. These results allow us to concentrate on pairs $(G, H)$ in which both graphs are 2-connected. Recall that a graph $G$ is separable if it is either not connected or contains a cut point. We first examine how the genera of a graph relate to the genera of its maximal 2-connected components.

For any graph $G$, $\tilde{\gamma}(G) \leqslant 2\gamma(G)+1$. If either equality holds, or if $G$ is planar, then $G$ is *orientably simple.* Define an equivalence relation on the edges of $G$ which are not cut edges, $e_1 \sim e_2$ if and only if there exists a simple cycle $C$ of $G$ containing $e_1$ and $e_2$. A *block* of $G$ is the subgraph induced by an equivalence class under this relation.

Lemma 2.1. *Let $\{B_i\}_{i=1}^n$ be the blocks of a graph $G$. Then $G$ is orientably simple if and only if each $B_i$ is orientably simple.*

*Proof.* See [SB]. ∎

Proposition 2.2. *Let $\{B_i\}_{i=1}^n$ be the blocks of a graph $G$. Then*

(1) $\gamma(G)=\sum_{i=1}^n \gamma(B_i)$,

(2) $\bar{\gamma}(G)=\sum_{i=1}^n \bar{\gamma}(B_i)$,

(3) $\tilde{\gamma}(G)=\sum_{i=1}^n \tilde{\gamma}(B_i)$ *if $G$ is not orientably simple, or if $G$ is planar, and*

(4) $\tilde{\gamma}(G)=1+\sum_{i=1}^n \tilde{\gamma}(B_i)$ *if $G$ is orientably simple and if $G$ is not planar.*

*Proof.* Conclusion 1 is the main result in [BHKY]. Conclusions 2, 3, and 4 are rewordings of the main results in [SB]. ∎

Lemma 2.3. *Let the graph $G$ be either $(\gamma \geqslant n)$-critical, $(\tilde{\gamma} \geqslant n)$-critical, or $(\bar{\gamma} \geqslant n)$-critical. Then $G$ does not contain a cut edge.*

*Proof.* By way of contradiction, let $e$ be a cut edge of such a $G$. The blocks of $G-e$ are the same as the blocks of $G$, except for the planar block $B=\{e\}$. Thus by Proposition 2.2, $\gamma(G-e)=\gamma(G)$, $\tilde{\gamma}(G-e)=\tilde{\gamma}(G)$, and $\bar{\gamma}(G-e)=\bar{\gamma}(G)$. In each case, this contradicts that $G$ is critical. ∎

Lemmas 2.1 through 2.3 will be used in Section 3 to handle the case of a $(\bar{\gamma} \geqslant n)$-critical or $(\tilde{\gamma} \geqslant n)$-critical graph $G$ which is separable. We now turn our attention to the case where $G$ is 2-connected. Recall that a pair $(G, H)$ is 2-connected if both $G$ and $H$ are 2-connected. The following lemma will allow us to assume that $H$ is also 2-connected.

PROPOSITION 2.4. *Let $(G, H)$ be a $\Sigma$-pair where $G$ is 2-connected and $H$ does not contain a cut edge. Then there exists a refinement $(G, K)$ which is 2-connected and has $|V_t(K)| \leqslant 9\,|V_t(H)|$.*

*Proof.* We proceed in two steps. First we shall find a connected graph $H' \supset H$ such that $(G, H')$ satisfies the hypotheses of this proposition. We will then find a 2-connected $K \supset H'$ as desired.

*Step* 1. Note that the number of connected components of $H$ is at most $|V_t(H)|/2$. If $H$ is connected, let $H' = H$. Otherwise, let $u$ and $v$ be vertices in distinct components of $H$. Let $C$ be a simple cycle in $G$ which contains $u$, $v$, and at least one other point of $H$, such that the number of edges of $C$ which are not in $H$ is minimized. Such a cycle exists because $G$ is a 2-connected graph and each component of $H$ contains at least two topological vertices. Then $(G, C \cup H)$ satisfies the hypotheses of this lemma, and

$$|V_t(C \cup H)| - |V_t(H)| \leqslant 2(1 + k_H - k_{C \cup H}),$$

where $k_L$ denotes the number of connected components in a graph $L$. If $C \cup H$ is connected, then let $H' = C \cup H$. Otherwise, at least $C \cup H$ has fewer components than $H$, so that repeating this process inductively eventually leads (in say $i_0$ steps) to a connected graph $H' \supset H$ with $(G, H')$ satisfying the hypothesis of Proposition 2.4, and with

$$|V_t(H')| - |V_t(H)| \leqslant 2(i_0 + k_H) \leqslant 4k_H.$$

Since $k_H \leqslant |V_t(H)|/2$, we have that $|V_t(H')| \leqslant 3\,|V_t(H)|$.

*Step* 2. We now construct the desired 2-connected $K$. Let $b_L$ denote the number of blocks of a graph $L$. If $b_{H'} = 1$, then let $K = H'$; the pair $(G, K)$ satisfies the conclusion of this lemma. If $b_{H'} > 1$, then let $P$ be a shortest path in $G - H'$ with endpoints in $H'$ but not in the same block of $H'$; $P$ exists since $G$ is 2-connected. Then $(G, P \cup H')$ satisfies the hypothesis of this lemma and $|V_t(P \cup H')| - |V_t(H')| \leqslant 2$. If $P \cup H'$ is 2-connected, then let $K = P \cup H'$; if not, at least $P \cup H'$ has fewer blocks than $H'$, so repeating this process inductively leads to a 2-connected graph $K \supset H'$ in fewer than $b_{H'}$ steps. Hence $|V_t(K)| - |V_t(H')| \leqslant 2\,b_{H'}$. Since $b_{H'} < |V_t(H')|$, we have that $|V_t(K)| \leqslant 3\,|V_t(H')|$.

Combining the inequalities of Step 1 and Step 2, we get that $|V_t(K)| \leqslant 9\,|V_t(H)|$ as desired. ∎

## 3. Proof of the Main Result

In this section we will prove the main result, Theorem 1.1. The proof will use the material from Section 2 as well as Theorem 12.2. Recall that Sections 4 through 12 are devoted to the proof of Theorem 12.2 and are independent of this section. For the reader's convenience we restate the following theorem.

THEOREM 12.2. *Let $(G, H)$ be a 2-connected $\Sigma$-pair. Then there exists a 2-connected $K \subset G$ such that $K$ does not embed in $\Sigma$, $K$ contains a subgraph homeomorphic to $H$, and $|V_t(K)|$ is bounded by a function of $|V_t(H)|$.*

We now prove our main theorem. The proof is a simultaneous induction on $n$. The simultaneous induction is needed because of the restriction that $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$, which in turn is necessary in the proof of Theorem 12.2.

THEOREM 1.1. *There exists a function $f$ such that for any graph $G$, if $G$ is either $(\bar{\gamma} \geqslant n)$-critical or $(\tilde{\gamma} \geqslant n)$-critical, then $G$ contains at most $f(n)$ vertices which are not of degree 2.*

*Proof.* We will define $f$ inductively. To start the induction we note that Kuratowski's theorem [K] characterizes both $(\tilde{\gamma} \geqslant 1)$-critical graphs and $(\bar{\gamma} \geqslant 1)$-critical graphs. Thus we define $f(1) = 6$.

For the induction step we assume that $f$ has been defined for all natural numbers strictly less than $n$. The proof breaks into four cases; the first two covering the possibility that $G$ is not 2-connected.

*Case* 1. Assume that $G$ is not 2-connected, and that $G$ is either $(\bar{\gamma} \geqslant n)$-critical, or is both $(\tilde{\gamma} \geqslant n)$-critical and not orientably simple.

Let $\{C_i\}_{i=1}^{k}$ be the blocks of $G$. Observe that planar blocks cannot increase either $\bar{\gamma}(G)$ or $\tilde{\gamma}(G)$. Thus $1 \leqslant \bar{\gamma}(C_i) \leqslant n-1$ for all $i$ and $2 \leqslant k \leqslant n$. Also, $G$ does not contain a cut edge.

Pick $i \in \{1, ..., k\}$ and $e \in C_i$. By the criticalness of $G$, Proposition 2.2 implies that $\bar{\gamma}(C_i - e) < \bar{\gamma}(C_i)$. Thus each $C_i$ is $(\bar{\gamma} \geqslant j_i)$-critical for $j_i = \bar{\gamma}(C_i)$. Therefore, by the inductive hypothesis, $|V_t(C_i)| \leqslant f(j_i)$. We have our desired bound, since

$$|V_t(G)| \leqslant k - 1 + \sum_{i=1}^{k} |V_t(C_i)| \leqslant k - 1 + \sum_{i=1}^{k} f(j_i).$$

*Case* 2. Assume that $G$ is not 2-connected, $(\tilde{\gamma} \geqslant n)$-critical, and that it is orientably simple.

As in Case 1, let $\{C_i\}_{i=1}^{k}$ be the blocks of $G$. We will show that each $C_i$ is $(\tilde{\gamma} \geqslant j_i)$-critical, where $j_i = \tilde{\gamma}(C_i)$. If so, then by the inductive hypothesis we again have $|V_t(C_i)| \leqslant f(j_i)$ and $|V_t(G)| \leqslant k-1+\sum_{i=1}^{k} |V_t(C_i)| \leqslant k-1+\sum_{i=1}^{k} f(j_i)$.

By Lemma 2.1, each $C_i$ is orientably simple. Hence for each $C_i$, $\tilde{\gamma}(C_i) = 2\gamma(C_i)+1$ as well as $\tilde{\gamma}(C_i) > \bar{\gamma}(C_i) = 2\gamma(C_i)$, and so $\bar{\gamma}(C_i) = \tilde{\gamma}(C_i) - 1$.

Let $e$ be an edge of $C_i$. If $C_i - e$ is orientably simple, then Proposition 2.2 shows that $\bar{\gamma}(C_i - e) < \bar{\gamma}(C_i)$. This implies that $\tilde{\gamma}(C_i - e) < \tilde{\gamma}(C_i)$. If $C_i - e$ is not orientably simple, then $\bar{\gamma}(C_i - e) \leqslant \bar{\gamma}(C_i)$ implies that $\tilde{\gamma}(C_i - e) \leqslant \tilde{\gamma}(C_i) - 1$ and again $\tilde{\gamma}(C_i - e) < \tilde{\gamma}(C_i)$. We conclude that $C_i$ is $(\tilde{\gamma} \geqslant j_i)$-critical where $j_i = \tilde{\gamma}(C_i)$, and hence that $|V_t(G)|$ is bounded as desired.

*Case* 3. Assume that $G$ is $(\tilde{\gamma} \geqslant n)$-critical and 2-connected.

Since $\tilde{\gamma}(G) \geqslant n$ and $\bar{\gamma}(G) \geqslant \tilde{\gamma}(G) - 1$, we know that $G$ contains a $(\bar{\gamma} \geqslant n-1)$-critical subgraph $G_1$. By the induction hypothesis $|V_t(G_1)| \leqslant f(n-1)$, so it will suffice to bound $|V_t(G)|$ by a function of $|V_t(G_1)|$. If $\bar{\gamma}(G_1) = n$ then $G_1 = G$ and we are done; hence we may assume that $\bar{\gamma}(G_1) = n-1$. Setting $\Sigma = \Sigma^{\sim}_{n-1}$, we see that the $\Sigma$-pair $(G, G_1)$, satisfies the hypotheses of Proposition 2.4, and so there exists a 2-connected refinement $(G, G_2)$ with $|V_t(G_2)|$ bounded by a function of $|V_t(G_1)|$. Applying Theorem 12.2 to the pair $(G, G_2)$ yields a 2-connected $G_3 \subset G$ with $|V_t(G_3)|$ bounded by a function of $|V_t(G_2)|$ such that $G_3$ does not embed in $\Sigma^{\sim}_{n-1}$. Thus $\tilde{\gamma}(G_3) \geqslant n$. Since $G$ is $(\tilde{\gamma} \geqslant n)$-critical we see that $G_3 = G$, so $|V_t(G)|$ is bounded as desired.

*Case* 4. Assume that $G$ is $(\bar{\gamma} \geqslant n)$-critical and 2-connected.

Since $\bar{\gamma}(G) \geqslant n$ there exists a $G_1 \subset G$ such that $G_1$ is $(\tilde{\gamma} \geqslant n)$-critical. By Cases 1–3, $|V_t(G_1)|$ is appropriately bounded. By Proposition 2.4, there exists a 2-connected $G_2$ with $G \supset G_2 \supset G_1$ and with $|V_t(G_2)| \leqslant 9\,|V_t(G_1)|$.

If $\bar{\gamma}(G_2) \geqslant n$, then $G = G_2$ and we have the desired bound. If $\bar{\gamma}(G_2) < n$, then $G_2$ is orientably simple, and every embedding of $G_2$ in $\Sigma_{(n-1)/2}$ is an Euler genus embedding. Thus, by Theorem 12.2, there exists a 2-connected $G_3 \subset G$ such that $G_3$ does not embed in $\Sigma_{(n-1)/2}$ and $|V_t(G_3)|$ is bounded by a function of $|V_t(G_2)|$. Thus $\bar{\gamma}(G_3) \geqslant n$.

Since $G$ is critical, $G_3 = G$ and we again have $|V_t(G)|$ appropriately bounded.

These four cases cover all of the possibilities. In each case we showed that $|V_t(G)|$ is bounded as a function of $f(i)$ for $1 \leqslant i < n$. Defining $f(n)$ to be the maximum of these four bounds completes the proof of the inductive step and of Theorem 1.1. ∎

We note again that the proofs of the bounds for $(\bar{\gamma} \geqslant n)$-critical and $(\tilde{\gamma} \geqslant n)$-critical graphs are intertwined. In particular, we cannot prove one bound without proving the other.

## 4. Properties of Euler Genus Embeddings

Let $(G, H)$ be a $\Sigma$-pair. Much of the remainder of this paper will be concerned with studying embeddings $\phi: H \to \Sigma$ and attempting to extend these to embeddings of $G$ into $\Sigma$. Recall that in the definition of a $\Sigma$-pair we had $\bar{\gamma}(G) \geqslant \bar{\gamma}(\Sigma)$; i.e., $H$ embedded in no surface of higher Euler characteristic than that of $\Sigma$. The four propositions of this section describe some useful properties of these Euler genus embeddings. Before giving these propositions, we need some more terminology and a description of our figures.

Let $\phi: H \to \Sigma$ be an embedding. When considering a fixed embedding, reference to $\phi$ will frequently be replaced by considering $H$ as a subspace of $\Sigma$. A *region* of $\phi$ is a connected component of $\Sigma - H$. Let $D$ denote the closed unit disk and let $D^\circ$ denote the interior of $D$. We say that $\phi$ is an *open* 2-*cell* embedding provided that each region is homeomorphic to $D^\circ$. Similiarly $\phi$ is a *closed* 2-*cell* embedding if the closure $\bar{R}$ (in $\Sigma$) of each region $R$ is homeomorphic to $D$. In the literature an open 2-cell embedding is commonly called a 2-cell, or cellular embedding, while a closed 2-cell embedding has been called circular. We say that $H$ is $\Sigma$ *open* 2-*cell* or $\Sigma$-OTC, if every embedding of $H$ into $\Sigma$ is an open 2-cell embedding. We similarly define $H$ to be $\Sigma$-*closed* 2-*cell*, or $\Sigma$-CTC.

Let $\phi: H \to \Sigma$ be an open 2-cell embedding and let $R$ be a region of $\phi$. Let $\psi: D \to \bar{R}$ be a continuous surjection such that the restriction $\psi \mid D^\circ$ is a homeomorphism with $R$. Note that the boundary of $D$ maps onto the boundary (in $\Sigma$) of $R$. Hence the boundary in $R$ is a closed walk in $H$. Let $v$ be a vertex of $H$ (possibly of degree 2). We call $|\psi^{-1}(v)|$ the *number of occurrences* of $v$ in the boundary of $R$, and each element in $\psi^{-1}(v)$ will be called an *occurrence* of $v$. We will often depict a region $R$ by labeling some, possibly not all, of the occurrences of $v$ on the boundary of the closed disk. For example, Fig. 4.1 shows two alternate depictions for a region $R$ of $\phi$: $H \to \Sigma$, where $\Sigma$ is the torus and $H$ is $K_{3,3}$.

Proposition 4.1. *Let $H$ be a connected graph and let $\Sigma$ be a surface with $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$. Then $H$ is $\Sigma$-OTC.*

*Proof.* See [Y]. ∎

Let $C$ be a simple cycle in a surface $\Sigma$. We say that $C$ is *orientable* if there exists a neighborhood of $C$ which is homeomorphic to a cylinder. $C$ is *nonorientable* provided that every sufficiently small neighborhood is homeomorphic to a Möbius strip. Note that every cycle is either orientable or nonorientable, but not both.

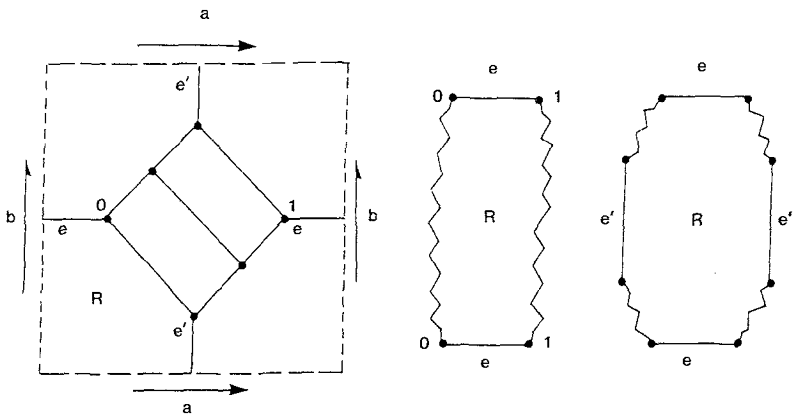


FIGURE 4.1

PROPOSITION 4.2. *Let $H$ be a graph and let $\Sigma$ be a surface with $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$. Then no embedding $\phi$: $H \to \Sigma$ has a region as depicted in Fig.* 4.2.

*Proof.* By way of contradiction, suppose that there does exist such a region $R$. Let $C$ be a cycle in $R \cup e$ which runs from one occurrence of the midpoint of $e$ to the other occurrence in the boundary walk of $R$; see Fig. 4.3. Note that $C$ is nonorientable. Delete the edge $e$ and re-embed it in $R$ connecting the occurrence of 1 and 2 as shown in Figure 4.3. The embedding of $H$ into $\Sigma$ thus constructed has the free crosscap $C$, contradicting that $H$ is $\Sigma$-OTC. ∎

Let $v$ be a vertex of a graph $H$, let $E_v$ be the set of edges incident with $v$, and let $\{E_1, E_2\}$ be a partition of $E_v$. We define a new graph, $S(E_1, E_2; H)$, or more simply $S(H)$, by

$$V(S(H)) = V(H) \cup \{v_1, v_2\} - \{v\}$$

and

$$E(S(H)) = E(H) \cup \{(v_i, u) \mid (v, u) \in E_i, i = 1, 2\} \cup \{(v_1, v_2)\} - E_v.$$

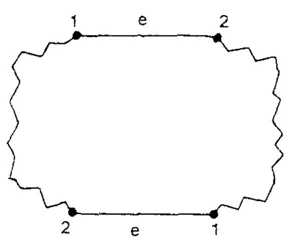


FIGURE 4.2

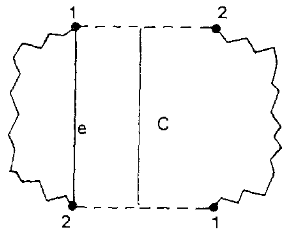


FIGURE 4.3

This process is called *splitting a vertex*. Let $e$ be an edge of a graph $G$. We define a new graph, $G/e$, by topologically contracting the edge $e$ to a point. This process is called *contracting an edge*. We note that these two processes are "inverses" of each other; in particular, splitting $v \in V(G)$ and then contracting the edge $(v_1, v_2)$ gives $G$ again.

LEMMA 4.3. *Let $e$ be an edge of a graph $G$ which is not a loop and let $\Sigma$ be a surface. If $G$ embeds in $\Sigma$, then $G/e$ also embeds in $\Sigma$.*

*Proof.* Considering $G$ as a subspace of $\Sigma$, we contract $e$ to a point in $\Sigma$. The resulting $G/e$ is embedded in the quotient space $\Sigma/e$, which is homeomorphic to $\Sigma$. ∎

Recall that a piece of a graph $H$ is either a topological vertex of $H$ or the interior (excluding the endpoints) of a topological edge of $H$.

PROPOSITION 4.4. *Let $H$ be a graph and let $\Sigma$ be a surface with $\bar{\gamma}(H) \geq \bar{\gamma}(\Sigma)$. Then no embedding $\phi$: $H \to \Sigma$ has a region $R$ with pieces $p_1$ and $p_2$ of $H$ in the boundary walk of $R$ as depicted in Fig.* 4.4.

*Proof.* By way of contradiction, let $R$ be such a region of an embedding $\phi$. If $p_1$ is a vertex of $H$, let $C$ be a path in $R$ connecting one occurrence of $p_1$ with the other occurrence; note that $C$ is a simple cycle in $\Sigma$. By considering a small neighborhood of $p_1$, we see that $C$ induces a natural bipartition on the edges incident with $p_1$. Let $H'$ be the graph formed by

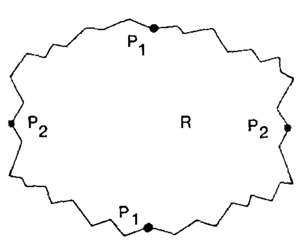


FIGURE 4.4

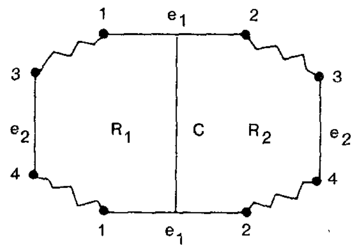


FIGURE 4.5

splitting the vertex $p_1$ using this bipartition. If $p_2$ is also a vertex we repeat this procedure, renaming $H'$ as the graph formed by both splittings. Reserving the construction of Lemma 4.3 gives an embedding $\phi'$: $H' \to \Sigma$. Thus we may assume that we have an embedding with a region as in Fig. 4.4, where both repeated pieces are edges of $H'$; call these $e_1$ and $e_2$.

As in the proof of Proposition 4.2, let $C$ be a simple cycle in $\Sigma$ which lies in $R \cup e_1$ connecting the two occurrences of $e_1$ in the boundary of $R$. By Proposition 4.2 this cycle is orientable. Moreover, the occurrences of $e_2$ in the boundary of $R$ imply that $C$ does not disconnect $H'$, and hence $C$ is not homologically null in $\Sigma$. Thus $R$ looks like the region of Fig. 4.5.

We now delete the edge $e_1$ and construct a new surface $\Sigma^-$ by deleting $C$ from $\Sigma$ and sewing in two closed 2-cells, i.e., capping of the handle represented by $C$. Since $C$ is orientable and is not homologically null, $\Sigma^-$ is connected and $\bar{\gamma}(\Sigma^-) = \bar{\gamma}(\Sigma) - 2$. We also have a natural embedding $\bar{\phi}$: $(H' - e_1) - \Sigma^-$ induced by $\phi$: $H' \to \Sigma$. Under $\bar{\phi}$, the edge $e_2$ bounds two regions, $R_1$ and $R_2$, as shown in Fig. 4.6. By sewing in a crosscap as shown in Fig. 4.7, we can extend $\bar{\phi}$ to an embedding of $H'$ into a new surface having Euler genus $\bar{\gamma}(\Sigma^-) + 1 = \bar{\gamma}(\Sigma) - 1$. Using Lemma 4.3 (if necessary) we get that $H$ also embeds in this new surface, contradicting that $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$. ∎

PROPOSITION 4.5. *Let $H$ be a graph and let $\Sigma$ be a surface with $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$. Then no embedding $\phi$: $H \to \Sigma$ has two regions as depicted in Fig.* 4.8.

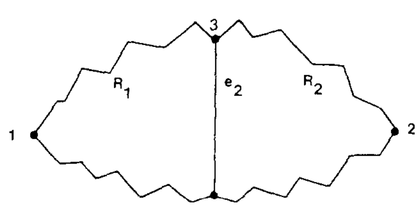


FIGURE 4.6

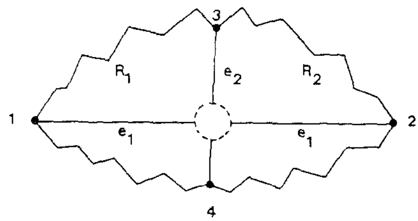


FIGURE 4.7

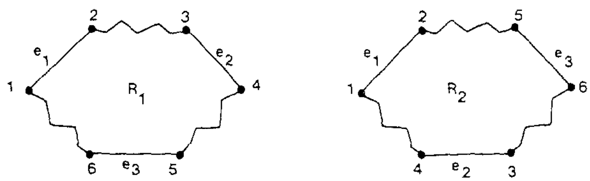


FIGURE 4.8

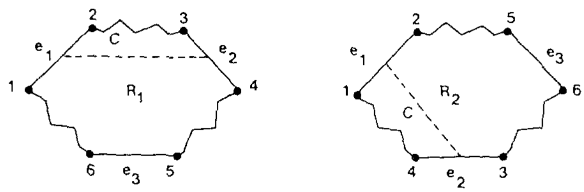


FIGURE 4.9

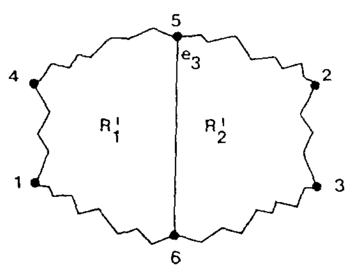


FIGURE 4.10

*Proof.* By way of contradiction, suppose that $\phi$: $H \to \Sigma$ is an embedding with regions $R_1$ and $R_2$ as in Fig. 4.8. We will construct an embedding $\phi'$: $H \to \Sigma'$ where $\bar{\gamma}(\Sigma') < \bar{\gamma}(\Sigma)$.

Let $C$ be a simple cycle in $\Sigma$ lying in $e_1 \cup R_1 \cup e_2 \cup R_2$ as shown in Fig. 4.9. Observe that $C$ is orientable. Since $e_3$ appears on either side of $C$, we have that $C$ does not disconnect $\Sigma$. Form a new surface, $\Sigma^-$, by deleting $C$ and sewing in two 2-cells (capping off the handle represented by $C$). There exists a naturally induced embedding $\bar{\phi}$: $(H - \{e_1, e_2\}) \to \Sigma^-$; moreover, $\bar{\gamma}(\Sigma^-) = \bar{\gamma}(\Sigma) - 2$. Under $\bar{\phi}$, the edge $e_3$ lies on the boundary of two regions, $R_1'$ and $R_2'$, as shown in Fig. 4.10. By sewing in a crosscap over the edge $e_3$ we construct a new surface $\Sigma'$, with $\bar{\gamma}(\Sigma') = \bar{\gamma}(\Sigma^-) + 1 = \bar{\gamma}(\Sigma) - 1$. Moreover, there is a modification of $\bar{\phi}$: $(H - \{e_1, e_2\}) \to \Sigma^-$ to an embedding $\phi'$: $H \to \Sigma'$ as shown in Fig. 4.11. This embedding contradicts that $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$, and hence the regions $R_1$ and $R_2$ of $\phi$ do not exist as hypothesized. ▌

We are done with our study of the properties of Euler genus embeddings. Recall that we will be taking a $\Sigma$-pair $(G, H)$ and attempting to extend embeddings of $H$ into $\Sigma$ to embeddings of $G$ into $\Sigma$. We would like to emphasize that the only uses of $\bar{\gamma}(H) \geqslant \bar{\gamma}(\Sigma)$ are in the four propositions of this section. These properties impose restrictions on the boundary walks of Euler genus embeddings. Each of these properties will be used in subsequent sections.

In the introduction we pointed out that we are unable to prove the orientable analogue of Theorem 1.1, that is, to prove the finiteness of the set of irreducible graphs for a given orientable surface. The reason for this lapse can now be made clearer. In [ABY], it is shown that there exist graphs of nonorientable genus one, but of arbitrarily high orientable genus. Thus if we start with a graph which is $(\gamma(H) \geqslant n)$-critical, we cannot deduce anything about $\bar{\gamma}(H)$. In particular, we cannot use the four propositions of this section.

If, on the other hand, one desires to forget about Euler genus embeddings altogether, and decides instead to study orientable genus embeddings

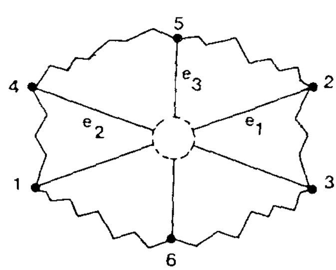


FIGURE 4.11

(those with $\gamma(H) \geqslant \gamma(\Sigma)$), then the conclusions of the propositions analogous to 4.4 and 4.5 are false. This may easily be seen by studying embeddings of $K_{3,3}$ into the torus.

Loosely speaking, because the step between $\Sigma_{n-1}$ and $\Sigma_n$ is twice as large—measured in terms of Euler characteristic—as the step between $\Sigma_{n-1}^{\sim}$ and $\Sigma_n^{\sim}$, region boundaries can be much more complicated. This allows more freedom in attaching $(G, H)$-bridges, and the analysis becomes prohibitive.

## 5. $\theta$-Graphs and $k$-Graphs

In Section 1 we defined a simple cycle in a surface $\Sigma$ to be contractible if it was homotopic to a point in $\Sigma$. The purpose of this section is to find certain subgraphs of a graph $G$, called $k$-graphs, such that for any embedding of $G$ into a surface there exists a nonctractible cycle contained in these subgraphs. We then use the existence of these $k$-graphs to bound $|V_t(G)|$ for certain types of $\Sigma$-pairs $(G, H)$. As a result of this section the size and types of $(G, H)$-bridges will be greatly restricted.

Let $K$ be an arbitrary subgraph of $G$. The *star* of $K$, st$(K)$, consists of $K$ together with all edges having at least one endpoint in $K$. Let $K$ be a subgraph of $G$ which is homeomorphic to the complete bipartite graph $K_{2,3}$ (respectively the complete graph $K_4$). We say that $K$ is a $K_{2,3}$ *$k$-graph* (respectively a $K_4$ *$k$-graph*) if there exists a subgraph $L$, $K \subset L \subset G$, with $L - \mathrm{st}(K)$ connected and the quotient $L/(L - \mathrm{st}(K))$ homeomorphic to $K_{3,3}$ (respectively to $K_5$).

The three minimal types of $k$-graphs (in terms of the number of topological edges in $L$) are illustrated in Fig. 5.1. The solid edges are in $K$ while the dashed edges are in $L - K$.

LEMMA 5.1. *Let $K$ be a $k$-graph of $G$ and suppose that $\phi$: $G \to \Sigma$ is an embedding. Then there exists a cycle $C$ of $K$ such that $\phi(C)$ is noncontractible.*

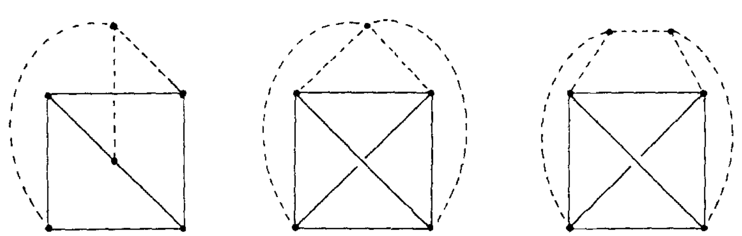

FIGURE 5.1

*Proof.* See Proposition 2.4 in [GH2]. ▮

A method of finding $k$-graphs is provided by the following:

LEMMA 5.2. *Let $e$ be an edge of a graph $G$, let $L$ be a 2-connected subgraph not containing $e$, and let $H$ be a connected component of $G-\mathrm{st}(L)$ not containing $e$. If $G$ does not embed in $\Sigma$ but $\phi$: $(G-e)\rightarrow\Sigma$ is an embedding with each cycle in $\phi(L)$ contractible in $\Sigma$, then there exists a $k$-graph of $G$ which is disjoint from $H$.*

*Proof.* See Lemma 4.5 in [GH2]. ▮

We say that a $\Sigma$-pair $(G, H)$ is *critical* provided that $G-e$ embeds in $\Sigma$ for every edge $e$ of $G-H$. No restriction is made on whether $G-e$ embeds for edges $e$ in $E(H)$. The following lemma is immediate.

LEMMA 5.3. *Let $(G, H)$ be a $\Sigma$-pair. Then there exists a refinement $(G', H)$ which is a critical $\Sigma$-pair.*

In the proofs which follow we will need to find such a critical refinement. Had we insisted that $(G-e)\rightarrow\Sigma$ for every edge of $G$ (not just those of $G-H$) this refinement would not necessarily exist.

Recall that a graph $H$ is $\Sigma$-CTC if for each embedding of $H$ into $\Sigma$ and each region $R$ of this embedding, $R$ together with the boundary of $R$ is homeomorphic to a closed 2-cell. This prevents the repetition of edges and vertices in the boundary walk of a region. A pair $(G, H)$ is *$\Sigma$-effectively closed 2-cell*, henceforth $\Sigma$-ECTC, provided that $H$ is $\Sigma$-OTC and that for any embedding $\phi$: $H\rightarrow\Sigma$ and for any region $R$, if $e$ is a topological edge of $H$ appearing twice on the boundary of $R$, then $e$ is a topological edge of $G$. Notice that for the purpose of augmenting embeddings of $H$ into $\Sigma$ by adding certain $(G, H)$-bridges, the condition that $(G, H)$ is $\Sigma$-ECTC allows us to "pretend" that no edge of $H$ occurs twice on the boundary of a region. No restriction is placed, however, on vertex repetitions.

A graph $T\subset G$ is a *$\theta$-graph* provided that it is homeomorphic to $K_{2,3}$ (i.e., to the greek letter theta). A pair $(G, H)$ is *$\theta$-less* provided that for each topological edge $e$ of $H$ and each $(G, H-e)$-bridge $B$, $B-\mathrm{vofa}(B)$ does not contain a $\theta$-graph. Note that there may be a $\theta$-graph in $(G-H)\cup\{v\}$ for $v\in V_t(H)$, but that our definition precludes there being a $\theta$-graph disjoint from $H$. Observe that if pair is $\theta$-less we have restrictions on both the complexity of individual $(G, H)$-bridges as well as restrictions on how several bridges may attach along an edge of $H$.

We now prove the main proposition of this section, which will essentially allow us to assume that our pair is $\theta$-less.

Proposition 5.4. *Let* $(G, H)$ *be a* $\Sigma$*-pair and suppose that* $(G, H')$ *is critical for all* $H'$ *homeomorphic to* $H$. *Suppose that either*:

(1) *There exists a* $\theta$*-graph of* $G$ *which is disjoint from* $H$, *or*

(2) $(G, H)$ *is* $\Sigma$*-ECTC and is not* $\theta$*-less,*

*then* $|V_t(G)| \leqslant |V_t(H)| + 8$.

*Proof.* Suppose that there exists a $\theta$-graph which is disjoint from $H$ (or disjoint from $H-e$ using the second hypothesis). It suffices to show that there is a $k$-graph of $G$ which is disjoint from $H$ (or respectively $H-e$). If so, then there exists a graph $K$, $H \subset K \subset G$, with a $k$-graph of $K$ disjoint from $H$ (or respectively from $H-e$). By Lemma 5.1 for any embedding of $K$ into $\Sigma$, this $k$-graph contains a noncontractible cycle. But since $H$ is $\Sigma$-OTC by Proposition 4.1 (and respectively $\Sigma$-ECTC by hypothesis) this $k$-graph is contained in a disk, a contradiction. We conclude that $K$ does not embed in $\Sigma$. Because $(G, H)$ is critical, $G = K$. Finally, since $(G, H')$ is critical for all $H'$ homeomorphic to $H$, this $k$-graph must be one of the minimal types shown in Fig. 5.1. Hence $|V_t(K)| \leqslant |V_t(H)| + 8$ as desired (for more details, see either Lemma 4.2 in [GH2] or Proposition 3.4 in [AH]).

To establish this $k$-graph, let $B$ denote the $(G, H)$-bridge (or respectively the $(G, H-e)$-bridge) containing the $\theta$-graph. Set $J = B - \text{vofa}(B)$, and note that the $\theta$-graph is contained in $J$. Either $J$ contains the desired $k$-graph, or $J$ contains $(L \cup e')$ where $L$ is a simple cycle and $e'$ is a topological edge of $J$ such that $e'$ and $H$ (or respectively and $H-e$) are in distinct $(G, L)$-bridges. We observe that there exists a homeomorph $H'$ of $H$ which is disjoint from $e'$ since $L$ is connected. Hence there exists an embedding $\phi$: $(G-e') \rightarrow \Sigma$. Because $(G, H)$ is $\Sigma$-OTC (and respectively $\Sigma$-ECTC), it follows that $\phi(L)$ is contractible. Thus by Lemma 5.2 there is a $k$-graph of $G$ which is disjoint from $H$ (or respectively $H-e$) as was desired. ∎

## 6. Construction of the First $\Sigma$-Pair

In this section we construct our first major refinement of a 2-connected $\Sigma$-pair. The construction proceeds in two steps. Pairs $(G, H)$ which have bridges whose vertices of attachment all lie in the interior of a single topological edge of $H$ are difficult to deal with. This possibility is eliminated by using Proposition 6.1. Proposition 6.2 then constructs a $\Sigma$-ECTC pair. This property restricts the way in which $(G, H)$-bridges may embed in the regions of an embedding of $H$ in $\Sigma$. In Section 7 we will construct a second refinement which satisfies a much more restrictive set of properties. We proceed with the propositions.

PROPOSITION 6.1. *Let $(G, H)$ be a 2-connected $\Sigma$-pair. Then there exists a critical refinement $(G', K)$ such that each $(G', K)$-bridge has vertices of attachment in at least two pieces of $K$ and with $|V_t(K)| \leqslant |V_t(H)| + 4|E_t(H)|$.*

*Proof.* Let $(G', \bar{H})$ be the critical $\Sigma$-pair with $G' \subset G$ and with $\bar{H}$ homeomorphic to $H$ that minimizes $|V_t(G')| + |E_t(G')|$. Note that we proved the existence of at least one such pair in Lemma 5.3. If $G'$ is not 2-connected, then, because it is critical and does not embed in $\Sigma$, Proposition 2.2 implies that $G'$ has exactly two blocks, $H$ and a bridge $B$ which is either $K_{3,3}$ or $K_5$. Hence $|V_t(G')| \leqslant |V_t(H)| + 7$. In this case the pair $(G', G')$ satisfies the conclusion of this proposition. Thus we may assume that $G'$ is 2-connected. This implies that for $K \subset G$, each $(G, K)$-bridge has at least two vertices of attachment. It is possible that the vertices of attachment all lie in a single topological edge of $K$.

Let $H' \subset G'$ be homeomorphic to $H$. By Lemma 5.3 there exists a critical pair $(G'', H')$ with $G'' \subset G'$. As $G'$ was chosen to minimize the number of topological edges and vertices over all such pairs, $G'' = G'$. Hence $(G', H')$ is critical for all $H' \subset G'$ homeomorphic to $H$. Now select $H'$ as that homeomorph of $H$ which minimizes the number of $(G', H')$-bridges. We will eventually form $K$ by augmenting $H'$ with selected paths in $G' - H'$.

If $G'$ contains a $\theta$-graph which is disjoint from $H'$, then by Proposition 5.4 $|V_t(G')| \leqslant |V_t(H')| + 8$. Defining the pair $(G', K)$ as $(G', G')$ satisfies the conclusion of the lemma. Hence we assume that there is no $\theta$-graph disjoint from $H'$.

For each $e \in E_t(H')$, let $\mathscr{B}_e$ denote the set of all $(G', H')$ bridges $B$ with $\text{vofa}(B) \subset e$. Let $a$ and $b$ be two vertices of $G$ contained in the arc $e$, and let $[a, b]$ denote that segment of $e$ with endpoints $a$ and $b$. We now define a special subset $\mathscr{B}'_e$ of $\mathscr{B}_e$.

Let $v_1$ and $v_2$ be the endpoints of the topological edge $e$ in $H'$. Let $b_1$ be the vertex in the interior of $e$ which is closest to $v_1$ that there exists a bridge $B \in \mathscr{B}_e$ with $\text{vofa}(B) \subset [v_1, b_1]$. Pick $a_1$ to be a vertex of $[v_1, b_1]$ such that there exists a bridge $B_1 \in \mathscr{B}_e$ with $a_1 \in \text{vofa}(B_1)$ and with $\text{vofa}(B_1) \subset [a_1, b_1]$. Next let $b_2$ be the vertex of $[b_1, v_2]$ closest to $b_1$ such that there exists a bridge $B$ with $\text{vofa}(B) \subset [b_1, b_2]$, and pick $a_2$ as a vertex in $[b_1, b_2]$ such that there exists a bridge $B_2$ with $a_2 \in \text{vofa}(B_2) \subset [a_2, b_2]$. Continuing in this way inductively, we obtain a sequence of bridges $B_1, ..., B_n$ and a sequence of vertices $a_1, b_1, ..., a_n, b_n$ (where possibly $b_i = a_{i+1}$) contained in $e$ in that order. Define $\mathscr{B}'_e$ as the set $\{B_1\}_{i=1}^n$. Note that by the way we selected the $[a_i, b_i]$, any bridge $B \in \mathscr{B}_e$ with $\text{vofa}(B) \subset [b_{i-1}, b_i]$ must have a vertex of attachment at $b_i$, or else that bridge would have been chosen in place of bridge $B_i$. Also note that since $G'$ is 2-connected, $a_i \neq b_i$ for any $i$.

Next, for each bridge $B \in \mathscr{B}'_e$, let $a$ and $b$ denote those vertices of attachment of $B$ such that vofa$(B) \subset [a, b]$. Let $P_B$ denote the shortest path from $a$ to $b$ in $B$ which is internally disjoint from $H'$. Now define $K$ to be the union of $H'$ and the arcs $P_B$ for each $e \in E_t(H')$ and each $B \in \mathscr{B}'_e$ (see Fig. 6.1).

Note that by construction the homeomorphic copy of $K$ with no degree 2 vertices will have at most two parallel edges joining a given pair of vertices and no loops, and hence $(G', K)$ is a $\Sigma$-pair. It is for this construction that we allowed two parallel edges rather than insisting that the subgraph the homeomorphically simple. Since $K \supset H'$ and $(G', H')$ is critical, so is $(G', K)$. To see that $|V_t(K)| \leqslant |V_t(H)| + 4\,|E_t(H')|$, it suffices to show that for each $e \in E_t(H')$, $|\mathscr{B}'_e| \leqslant 2$. We will prove this shortly. First we will show that each $(G', K)$-bridge has vertices of attachment in at least two pieces of $K$.

First observe that each $(G', K)$-bridge $B$ is contained in a $(G', H')$-bridge $B'$, since $K \supset H'$. Note that if $B'$ has vertices of attachment in at least two pieces of $H'$, then $B$ will also be a $(G', K)$-bridge with this same property. Moreover, if $B = B' \in \mathscr{B}_e - \mathscr{B}'_e$, then by our earlier observation, $B$ now has a vertex of attachment at some $b_i$ and hence vertices of attachment in at least two pieces of $K$. Thus if there is a bridge $B$ with vertices of attachment in a single piece of $K$, $B$ must be contained in a bridge $B' \in \mathscr{B}'_e$ for some $e$. Since each such bridge has a vertex of attachment in the path $P_{B'}$, the topological edge must be $P_{B'}$. Since $(G', H')$ is critical, $G' - H'$ has no parallel edges. Thus if $B$ consists of a single edge of $G'$ we contradict our choice of $P_{B'}$ as the shortest path. If $B$ consists of more than a single edge, then we must necessarily have a $\theta$-graph disjoint from $H'$, again a contradiction. Thus each $(G', K)$-bridge $B$ has vertices of attachment in at least two topological edges of $K$ as desired.

We need one more fact before proving our bound on $\mathscr{B}'_e$. For $e \in E_t(H')$ and $B \in \mathscr{B}'_e$, consider $a$, $b$, vofa$(B)$, $[a, b]$, and $P_B \subset B$ as before. We claim that $\{a, b\}$ forms a cut set of $G'$ which separates $B$ and $H'$. If not, then there must exist a vertex $c \in ([a, b] - \{a, b\})$ such that $c$ is a vertex of attachment for some $(G', H')$-bridge $B' \neq B$. Define $H'' = (H' - [a, b]) \cup P_B$. Observe that $H''$ is homeomorphic to $H$. Also since no $(G', K)$-bridge has all its vertices of attachment in $P_B$, $(B - P_B) \cup [a, b] \cup B'$ is a $(G', H'')$-bridge. Hence $(G', H'')$ has strictly fewer bridges than $(G', H')$.

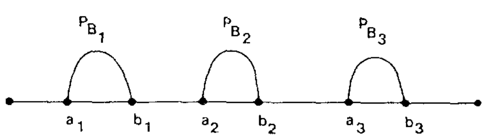


FIGURE 6.1

This contradicts our choice of $H'$. Thus each $P_B$ has as endpoints $\{a, b\}$ which form a cut set of $G'$.

It remains to show that $|\mathscr{B}'_e| \leqslant 2$ for each $e \in E_t(H')$. By way of contradiction, assume that $e \in E_t(H')$ has three such bridges, say $B_i \in \mathscr{B}'_e$ for $i = 1, 2, 3$. Further assume that the subscripts are chosen as shown in Fig. 6.1 (where possibly $b_i = a_{i+1}$).

Let $\bar{C}_i$ be that component of $G' - \{a_i, b_i\}$ which contains $H' - [a_i, b_i]$ and let $C_i = G' - \bar{C}_i$. Observe that $C_i$ is $[a_i, b_i]$ together with all bridges $B$ with $\text{vofa}(B) \subset [a_i, b_i]$. Also observe that $G' = C_i \bigcup_{\{a,b\}} \bar{C}_i$.

We note that each $C_i \cup A_i$ must be nonplanar, where $A_i$ is a path in $H' - [a_i, b_i]$ with end points $a_i$ and $b_i$. If not, then we can embed $G' - C_i$ in $\Sigma$ (by criticality) and can extend this embedding to include all of $G'$ by replacing the arc $[a_i, b_i] \subset C_i$ with the planar graph $C_i$ in a small $\varepsilon$-neighborhood of $[a_i, b_i]$, contradicting that $G'$ does not embed in $\Sigma$. Thus in any embedding of $H' \cup C_i$ in $\Sigma$, the subgraph $C_i \cup [a_i, b_i]$ must contain a noncontractible cycle $C$. Since $H'$ is $\Sigma$-OTC, the edge $e$ must be orientable by Proposition 4.2.

Now let $\phi_1$ be an embedding of $G' - B_1$ into $\Sigma$. By the preceding comments, $\phi_1(C_2)$ must lie in $R \cup [a_2, b_2]$, a cylinder. Hence $C_2$ must be planar. Let $\phi'_1$ embed $C_2$ into the sphere. Finally note that since $\phi_1(C_2)$ contains a noncontractible cycle, there does not exist a bridge $B \subset G' - C_2$ with vertices of attachment in both $[a_1, a_2]$ and $[b_1, b_2]$.

Next let $\phi_2$: $G' - C_2 \to \Sigma$. Again we have that $\phi_2(C_1)$ and $\phi_2(C_3)$ lie in a single region of $\phi_2 | H'$ as shown in Fig. 6.2.

We now cap off the handle represented by $C$ to obtain an embedding $\bar{C}_2 \to \Sigma'$, where $\tilde{\gamma}(\Sigma') = \tilde{\gamma}(\Sigma) - 2$. Recall that we also had an embedding of $C_2$ into the sphere. By joining $\Sigma'$ to the sphere by attaching two cylinders (one each to reconnect the vertices $\{a_2, b_2\} = C_1 \cap \bar{C}_2$) we construct an embedding $G' \to \Sigma$, a contradiction. Hence $|\mathscr{B}_e| < 3$ as desired, and the proposition is established. ∎

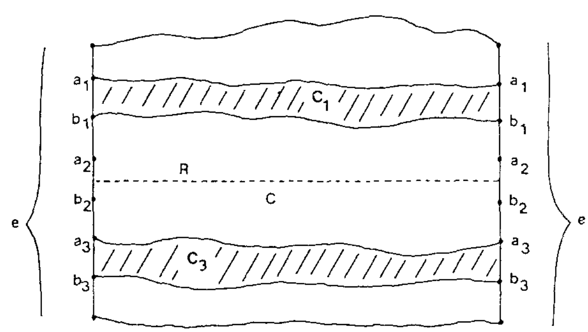


FIGURE 6.2

We now proceed with the main construction of this section.

PROPOSITION 6.2. *Let $(G, H)$ be a 2-connected $\Sigma$-pair. Then there exists a refinement $(G', K)$ which is a $\Sigma$-ECTC critical $\Sigma$-pair; morever, for some $n \leqslant |V_t(H)| + 4\,|E_t(H)|$,*

$$|V_t(K)| \leqslant n + 2n(n-1)(n^2-1)[2n(n-1)(n^2-1)+1].$$

*Proof.* Let $(G', H')$ be the pair constructed from $(G, H)$ by applying Proposition 6.1. Note that $(G', H')$ is critical and that for any $K \supset H'$, $(G', K)$ is also critical. Also if $n = |V_t(H')|$, then we have the bound on $n$ given by Proposition 6.1.

Let $\mathscr{A}$ denote the collection of all simple paths $A$ in $G$ with $A \cap H'$ the endpoints of $A$. We will construct $K$ by augmenting $H'$ with selected paths from $\mathscr{A}$. We will first give the construction of $K$, then show the bound on the size of the vertex set, and finally show that $K$ is $\Sigma$-ECTC.

Let $\{e_i\}_{i=1}^{|E_t(H')|}$ be an indexing of all the topological edges of $H'$, and let $\{P_i\}_{i=1}^{|V_t(H')| + E_t(H')|}$ be an indexing of all the topological pieces. Let $v_i^1$ and $v_i^2$ denote the endpoints of $e_i$. For each ordered pair $(e_i, P_j)$ with $e_i \neq P_j$ we will select two paths $A_{i,j}^1$ and $A_{i,j}^2$ each with one endpoints in $e_i$ and one in $P_j$. These paths will be chosen inductively, using the lexicographic order on the triple $(i, j, k)$ which indexes $A_{i,j}^k$. If there does not exist an $A \in \mathscr{A}$ with one endpoint in $e_i$ and the other endpoint in $P_j$, then we set $A_{i,j}^1 = A_{i,j}^2 = \varnothing$. Otherwise, let $u_{i,j}^k$ be the vertex nearest to $v_i^k$ in $e_i$ which has a path $A \in \mathscr{A}$ joining $u_{i,j}^k$ to $P_j$. Define $A_{i,j}^k$ as the path from $u_{i,j}^k$ which inductively adds the minimal number of topological vertices and edges. Observe that this minimality condition implies that $A_{i',j'}^{k'} \cap A_{i,j}^k$ is connected for each $(i', j', k') < (i, j, k)$. Furthermore, the addition of $A_{i,j}^k$ creates at most two new vertices in $A_{i',j'}^{k'}$ and introduces no new multiple topological edges. Hence when we are inductively attaching the paths $A_{i,j}^k$ to $H'$, the $m$th path increases the number of topological vertices by at most $2 + 2(m-1) = 2m$.

We now define

$$K = H' \cup \Bigl( \bigcup_{\text{all } (i,j,k)} A_{i,j}^k \Bigr).$$

Let $n = |V_t(H')|$. Note that $|E_t(H')| \leqslant n(n-1)$ and $|P_t(H')| \leqslant n^2$. Hence, by the above observations, we have

$$|V_t(K)| \leqslant |V_t(H')| + \sum_{m=1}^{2[n(n-1)][n^2-1]} 2m.$$

Calculating this sum yields the desired bound.

Since $(G', H')$ is a $\Sigma$-pair, so is $(G', K)$. Summarizing, we now have a

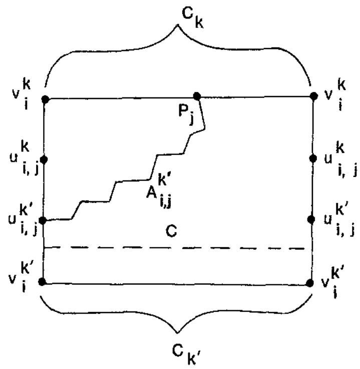


FIGURE 6.3

refinement of $(G, H')$ which is a critical $\Sigma$-pair and which satisfies the appropriate bound. It remains to show that $(G', K)$ is $\Sigma$-ECTC.

By way of contradiction, we assume that $(G', K)$ is not $\Sigma$-ECTC. Let $\phi$: $K \to \Sigma$ be an embedding, and let $\bar{e}_i$ be a topological edge of $K$ which appears twice on the boundary of the region $\bar{R}$. Since $(G', K)$ is not $\Sigma$-ECTC and $G$ is 2-connected, there exists a path $P$ in $G' - K$ with an endpoint in the interior of $\bar{e}_i$. By construction $H' \subset K$, so we have a restriction of $\phi$ which embeds $H'$ into $\Sigma$. Since $H'$ is $\Sigma$-OTC, $\bar{e}_i$ lies in some $e_i \in E_t(H')$. Let $B$ be the $(G', H')$-bridge containing $P$. Since $B$ has vertices of attachment in two pieces of $H'$, we may without loss of generality assume that the other endpoint of $P$ lies in a piece $p_j \neq e_i$.

Let $R$ be the region of $\phi \mid H'$ which contains $\bar{R}$. Let $C$ be a simple cycle in $R$ with $(C \cap H') \subset e_i$ such that $C$ runs from one occurrence of $e_i$ in the boundary cycle of $R$ to the other occurrence. By Proposition 4.2, $C$ is orientable. Thus the path $[u_{i,j}^1, u_{i,j}^2] \subset e_i$ separates the boundary walk of $R$ into two subwalks, say $C_1$ and $C_2$. By Proposition 4.4, we know that $p_j$ lies on exactly one of the $C_k$. Let $k'$ be the other index, i.e., $k' = 3 - k$. The edge $\bar{e}_i$ must lie in the arc $[u_{i,j}^1, u_{i,j}^2]$. The path $A_{i,j}^{k'}$ must embed under $\phi$ in $R$. This contradicts the assumption that $\bar{e}_i$ appears in the boundary of only one region $R$ (see Fig. 6.3). We conclude that $(G', K)$ is $\Sigma$-ECTC as desired. ▌

## 7. CONSTRUCTION OF THE FINAL $\Sigma$-PAIR

In this section we start with a 2-connected $\Sigma$-pair $(G, H)$ and construct a refinement $(G', H')$ which satisfies a highly technical set of properties, known collectively as Condition 7. The nature of these properties necessitates a rather convoluted construction. This section will complete our construction of refinements. The pair constructed herein will be examined in Section 8 through Section 11, where we will prove that $|V_t(G')|$ is bounded in terms of $|V_t(H')|$. Before proceeding to the main construction in Theorem 7.3, we need several propositions concerning $(G, H)$-bridges. These propositions will also be crucial in the proof of Theorem 12.2. Proposition 7.1 bounds the number of vertices of attachment for any bridge. In Section 8 this proposition will be used to bound the size of any $(G', H')$-bridge. Proposition 7.2 bounds the number of bridges which have vertices of attachment in at least three pieces of $H$. We proceed with the proofs.

PROPOSITION 7.1. *Let $(G, H)$ be a $\theta$-less $\Sigma$-pair and let $B$ be a $(G, H)$-bridge. Then* $|\text{vofa}(B)| \leqslant |V_t(H)| + 2\,|E_t(H)|$.

*Proof.* If $|\text{vofa}(B)|$ is strictly greater than desired, then there exists a topological edge $e$ of $H$ which contains three distinct vertices of attachment other than its endpoints. This contradicts the hypothesis that $(G, H)$ is $\theta$-less. ∎

Let $(G, H)$ be a $\Sigma$-pair and let $R$ be a region of an embedding $\phi$: $H \to \Sigma$. In Section 4 we defined an occurrence of a vertex $v \in V(H)$ on the boundary of $R$ by considering a map $\psi$ from a closed disk to $\bar{R}$ (the closure of $R$ in $\Sigma$) which was a homeomorphism onto $R$ when restricted to $D^\circ$ (the interior of $D$). Each element of $\psi^{-1}(v)$ was an occurrence of $v$. Let $B$ be a $(G, H)$-bridge and suppose that $\phi'$: $H \cup B \to \Sigma$ extends $\phi$. We say that $\phi'$ *attaches* $B$ at an occurrence $v'$ of $v$ in $R$ if $\phi'(B) \subset \bar{R}$ and $v' \in \overline{\psi^{-1}(\phi(B) - v)}$. We will analogously say that $\phi$ attaches several bridges at several different occurrences.

PROPOSITION 7.2. *Let $(G, H)$ be a critical $\Sigma$-pair and let $\mathscr{B}$ be the set of all $(G, H)$-bridges having vertices of attachment in at least three topological pieces of $H$. Then* $|\mathscr{B}| \leqslant \binom{V+E}{3}(2 - V + E)(2E)^3 + 1$, *where* $V = |V_t(H)|$ *and* $E = |E_t(H)|$.

*Proof.* By way of contradiction, suppose that $|\mathscr{B}|$ is strictly greater than the desired bound and let $e$ be an edge of an arbitrary $B \in \mathscr{B}$. Since $(G, H)$ is critical, there exists an embedding $\phi$: $(G - e) \to \Sigma$. $\mathscr{B}$ contains at least $\binom{V+E}{3}(2 - V + E)(2E)^3 + 1$ bridges which do not contain $e$. Since $V + E$ is the number of pieces of $H$, the pigeonhole principle implies that there exist three pieces $p_1$, $p_2$, and $p_3$, and $(2 - V + E)(2E)^3 + 1$ bridges in $\mathscr{B}$ which do not contain $e$ and which have a vertex of attachment in each $p_i$. We note that $2 - V + E$ is one plus the Betti number of $H$. Since the latter is an upper bound on the total number of regions in the embedding $\phi$, at least $(2E)^3 + 1$ of the bridges incident with $p_1$, $p_2$, and $p_3$ must be in the same region of the embedding $\phi$. Next, observe that the maximum length of the cycle bounding this region is $2E$, since any edge can occur at most twice. Thus, any piece occurs in the boundary cycle at most $2E$ times. Since we have $(2E)^3+1$ bridges incident with $p_1$, $p_2$, and $p_3$, at least two of these bridges must embed under $\phi$ at the same occurrence of $p_1$, $p_2$, and $p_3$. We observe that a tree joining these three occurrences in the boundary cycle separates the region into three components, none of which are incident with all three occurrences. Thus the two bridges cannot simultaneously embed in this region with vertices of attachment at the same occurrences of $p_1$, $p_2$, and $p_3$. With this contradiction, the proposition is demonstrated. ∎

We now state Condition 7, the collection of properties which we will use in Sections 8 through 12 to bound $|V_t(G)|$. Let $\deg_B(v)$ denote the degree of $v$ in a graph $B$. A $\Sigma$-pair $(G, H)$ satisfies *Condition* 7 provided that:

(1) $(G, H)$ is 2-connected,

(2) $(G, H')$ is critical, where $H'$ is any subgraph of $G$ which is homeomorphic to $H$,

(3) $(G, H)$ is $\theta$-less,

(4) $(G, H)$ is $\Sigma$-ECTC,

(5) for any pair of topological edges $e_1$ and $e_2$ of $H$ which are not topological edges of $G$ and for any homeomorph $H'$ of $H$ formed by replacing $e_1$ and $e_2$ by topological edges $e_1'$ and $e_2'$,

$$|V_t(G) \cap H| \leqslant |V_t(G) \cap H'|;$$

moreover, if equality holds in the previous equation, then

$$\sum_{v \in V_t(G) \cap H} \deg_G(v) \leqslant \sum_{v \in V_t(G) \cap H'} \deg_G(v)$$

and

(6) for any $(G, H)$-bridge $B$ and any $v \in \text{vofa}(B)$, $\deg_B(v) \leqslant 2$.

We observe that part 5 of Condition 7 ensures that we have picked the homeomorph of $H$ which, loosely speaking, contains as little of $G$ as possible. Thus we are trying to make the $(G, H)$-bridges as large as possible. This is similar in spirit to the usual proof of Kuratowski's theorem, in which bridges of a minimal cycle $C$ are examined. Indeed, if $C$ is a minimal cycle of a Kuratowski graph $G$, then $(G, C)$ satisfies all parts of Condition 7 where $\Sigma$ is the sphere (except that $C$ has no topological vertices and hence cannot be part of a $\Sigma$-pair). We would like to simplify part 5 by requiring that the minimality conditions hold over all $H'$ homeomorphic to $H$. The difficulty arises when trying to construct a pair satisfying all of these conditions simultaneously. Finally we note that Propositions 7.1 and 7.2 apply to graphs which satisfy Condition 7.

THEOREM 7.3. *Let* $(G, H)$ *be a 2-connected* $\Sigma$*-pair. Then there exists a refinement* $(G', H')$ *which satisfies Condition* 7. *Moreover,* $|V_t(H')|$ *is bounded by a function of* $|V_t(H)|$.

*Proof.* By Proposition 6.2, we know that there exists a $\Sigma$-ECTC critical $\Sigma$-pair $(G_1, H_1)$ with $|V_t(H_1)|$ bounded by a function of $|V_t(H)|$. Let $(G_2, H_2')$ be the $\Sigma$-ECTC critical $\Sigma$-pair with $H_2'$ homeomorphic to $H_1$ which minimizes $|V_t(G_2)| + |E_t(G_2)|$. Next, let $\mathscr{H}$ be the set of all subgraphs $H_2$ of $G$ such that $H_2$ is homeomorphic to $H_1$ and $(G_2, H_2)$ is a $\Sigma$-ECTC $\Sigma$-pair. From $\mathscr{H}$, we pick the subgraph $H_2$ which minimizes $|V(G_2) \cap H_2|$, where if it is possible to pick more than one $H_2$, then we choose the one which also minimizes $\sum_{v \in V_t(G) \cap H_2} \deg_G(v)$. Before constructing the graph $H'$ promised by the theorem, we investigate properties of $(G_2, H_2)$. In particular we will show that this pair satisfies parts 1 through 5 of Condition 7.

If $G_2$ is not 2-connected, then $G_2$ is the union along at most one point of $H_2$ and a Kuratowski graph, and hence $|V_t(G_2)| \leqslant |V_t(H_2) + 7$. In this case, we apply Proposition 2.4 to the $\Sigma$-pair $(G, G_2)$ to construct a 2-connected $\Sigma$-pair $(G, G_3)$ with $|V_t(G_3)|$ bounded by a function of $|V_t(H)|$. Defining $(G', H')$ as the pair $(G_3, G_3)$ completes the proof of this theorem. Hence we may assume that $G_2$ is 2-connected.

Next, let $H'$ be a homeomorph of $H_2$ in $G_2$. If $(G_2, H')$ is not critical, then there exists a $G_2' \subset G_2$ with $(G_2', H_2)$ critical. The pair $(G_2', H_2)$ is necessarily $\Sigma$-ETC since $(G_2, H_2)$ is $\Sigma$-ECTC. This contradicts our choice of $G_2$ as the smallest graph in a $\Sigma$-ECTC critical $\Sigma$-pair. Hence $(G_2, H')$ is critical for all $H'$ homeomorphic to $H_2$.

If $(G_2, H_2)$ is not $\theta$-less, then by Proposition 5.4, $|V_t(G_2)| \leqslant |V_t(H_2)| + 8$. In this case, the pair $(G_2, G_2)$ satisfies the conclusion of this theorem, and thus we may also assume that $(G_2, H_2)$ is $\theta$-less.

By our choice of $H_2$, the pair $(G_2, H_2)$ is $\Sigma$-ECTC.

Finally, let $e_1$ and $e_2$ be topological edges of $H_2$ which are not topological edges of $G_2$ and let $H_2'$ be formed by replacing $e_1$ and $e_2$ with topological edges $e_1'$ and $e_2'$, respectively. Since $e_1$ and $e_2$ are not topological edges of $G_2$, $H_2'$ is also $\Sigma$-ECTC. By our choice of $H_2 \in \mathscr{H}$, we have the appropriate minimality conditions.

We have shown that the pair $(G_2, H_2)$ satisfies parts 1 through 5 of Condition 7. Unfortunately, it need not satisfy part 6. We now describe the construction of $(G', H')$ which will also satisfy this final condition.

Let $\mathscr{B}$ be the set of $(G_2, H_2)$ bridges $B$ for which vofa$(B)$ lie in at least three pieces of $H_2$. For each $B \in \mathscr{B}$, let $T_B$ be a tree such that:

(1) $T_B \subset B$,

(2) $\text{vofa}(B) = \{v \in T_B \mid \deg_T(v) = 1\}$,

(3) If $T'_B$ is another tree satisfying 1 and 2, then

(a) $|V_t(G_2) \cap T_B| \leqslant |V_t(G_2) \cap T'_B|$; moreover, if equality holds, then

(b) $\sum_{v \in V_t(G_2) \cap T_B} \deg_{G_2}(v) \leqslant \sum_{v \in V_t(G_2) \cap T'_B} \deg_{G_2}(v)$.

Now, define the pair

$$(G', H') = \left(G_2, H_2 \cup \left(\bigcup_{B \in \mathscr{B}} T_B\right)\right).$$

We have the final pair which will satisfy the conclusions of this theorem. Before verifying that $(G', H')$ satisfies Condition 7 we will show that $|V_t(H')|$ is bounded by a function of $|V_t(H_2)|$.

Since $(G_2, H_2)$ is $\theta$-less Proposition 7.1 says that for each $B \in \mathscr{B}$, $|\{v \in T_B \mid \deg_T(v) = 1\}| \leqslant |V_t(H_2)| + 2\,|E_t(H_2)|$. But for any tree, if $n$ is the number of degree 1 vertices and $m$ is the number of vertices with degree greater than 2, then $m \leqslant n - 2$. Thus $|V_t(T_B)| \leqslant 2\,|V_t(H_2)| + 4\,|E_t(H_2)| - 2$. We note that $|V_t(H_2 \cup T_B)| \leqslant |V_t(H_2)| + |V_t(T_B)|$. If we let $N$ be the bound on $|\mathscr{B}|$ given by Proposition 7.2, we get that $|V_t(H')| \leqslant |V_t(H_2)| + [2\,|V_t(H_2)| + 4\,|E_t(H_2)| - 2]N$. Since $|E_t(H_2)| \leqslant |V_t(H_2)|\,(|V_t(H_2)| - 1)$, $|V_t(G_2)| = |V_t(H_1)|$, and $|V_t(H_1)|$ is bounded by a function of $|V_t(H)|$, we have that $|V_t(H')|$ is bounded by a function of $|V_t(H)|$.

Having shown the appropriate bound on the size of $H'$, we begin to show that the pair $(G', H')$ satisfies Condition 7. The first four parts are quickly handled; parts 5 and 6 are more difficult.

Since $(G_2, H_2)$ was a 2-connected $\Sigma$-pair, it follows from the construction of $H'$ that $(G', H')$ is also a 2-connected $\Sigma$-pair.

Next let $H''$ be any homeomorph of $H'$. Then since $H'$ contains $H_2$, $H''$ must contain a subgraph $H'_2$ which is homeomorphic to $H_2$. Thus $(G, H'_2)$ is critical, which implies that $(G', H'')$ is also critical.

Since $(G_2, H_2)$ is a $\theta$-less $\Sigma$-ECTC $\Sigma$-pair it follows that $(G', H')$ is as well. It remains to show that $(G', H')$ satisfies parts 5 and 6 of Condition 7. We proceed with part 5.

To avoid unnecessarily complicated notation, we define $\psi(H)$ as the ordered pair

$$\left(|V(G') \cap H|, \sum_{v \in V(G') \cap H} \deg_{G'}(v)\right),$$

for any subgraph $H \subset G'$. We say that $\psi(H) < \psi(H')$ if the inequality holds in the lexicographic order on ordered pairs of integers.

Let $\bar{H}$ be a homeomorphic copy of $H'$ formed by deleting a pair of topological edges $(e_1, e_2)$ from $H'$ which are not edges of $G'$ and replacing them with topological edges $(e_1', e_2')$. Then we need to show that $\psi(\bar{H}) \geqslant \psi(H')$.

Recall that by the construction of $(G', H')$, $H'$ contains the subgraph $H_2$. We consider three cases:

*Case* 1. Assume that $e_1$ and $e_2$ are both in $H_2$.

The let $\bar{H}_2 = (H_2 - \{e_1, e_2\}) \cup \{e_1', e_2'\}$. Since $e_1$ and $e_2$ are paths in $G'$, not edges, $\bar{H}_2$ is also $\Sigma$-ECTC. Since $H_2$ satisfied the minimality conditions, $\psi(H_2) \leqslant \psi(\bar{H}_2)$. But $H' - H_2 = \bar{H} - \bar{H}_2$, so $\psi(H') \leqslant \psi(H_2)$ as well.

*Case* 2. Assume that $e_1$ and $e_2$ both lie in $H' - H_2$.

Recall that $H'$ was constructed from $H_2$ by adding in trees $T_B$. Let $T_{B_i}$ be the tree containing $e_i$ ($T_{B_1}$ may possible equal $T_{B_2}$). The endpoints of $e_i$ are not both degree 1 vertices of $B_i$ since trees were chosen only for bridges with at least three vertices of attachment. Thus $e_i'$ has at least one endpoint in the interior of $B_i$. Hence $\bar{T}_{B_i} = (T_{B_i} - e_i) \cup e_i'$ is also a tree in $B_i$ (we do both modifications simultaneously if the paths lie in the same bridge). By minimality condition 3 on the choice of the $T_B$'s, $\psi(T_{B_i}) \leqslant \psi(\bar{T}_{B_i})$. Again, we note that $H' - (T_{B_1} \cup T_{B_2}) = \bar{H} - (\bar{T}_{B_1} \cup \bar{T}_{B_2})$; thus $\psi(H') \leqslant \psi(\bar{H})$.

*Case* 3. Assume that $e_1$ is in $H_2$ and that $e_2$ is in $H' - H_2$.

We will form $\bar{H}$ from $H$ by replacing the edges sequentially, first forming $H$ by replacing $e_2$ with $e_2'$, then replacing $e_1$ with $e_1'$ in $H$. We need to show that $e_2'$ does not intersect the interior of $e_1$. This follows because $e_2$ is an edge of a tree $T_B$. Again, the endpoints of $e_2$ are not both degree 1 vertices of $B$, since trees were chosen only for bridges with at least three vertices of attachment. Thus $e_2'$ has at least one endpoint in the interior of the $(G', H_2)$-bridge $B$. If $e_2'$ intersected $e_1$ in the interior of $e_1$, then there would exist a vertex of attachment for $B$ in the interior of $e_1$. This contradicts that $e_1$ is a topological edge of $H'$.

Next consider $H = (H' - e_2) \cup e_2'$. Since $e_2'$ is disjoint from $e_1$, $H$ is homeomorphic to $H'$. Because we only replaced an edge in $H' - H_2$, the argument used in Case 2 shows that $\psi(H') \leqslant \psi(H)$. Now form $\bar{H} = (H - e_1) \cup e_1'$. The argument of Case 1 shows that $\psi(H) \leqslant \psi(\bar{H})$. Thus $\psi(H') \leqslant \psi(\bar{H})$ as desired.

These three cases exhaust the possibilities; hence we have shown that $(G', H')$ satisfies part 5 of Condition 7.

It remains to show part 6 of Condition 7. By way of contradiction, let $B'$ be a $(G, H')$-bridge, $v \in \text{vofa}(B')$, and assume that $\deg_{B'}(v) \geqslant 3$. Let $B'' = (B' - \bigcup_{u \in \text{vofa}(B')} \text{st}(u)) \cup \text{st}(v)$, where $\text{st}(u)$ is the vertex $u$ together with the set of edges of $B'$ incident with $u$. Consider the following two cases.

*Case* 1. Assume that $B''$ contains a $k$-graph of $G$.

Let $e$ be any edge of $G'$ such that $(G'-e) \supset (H' \cup K)$, where $K$ is a minimal Kuratowski graph containing the hypothesized $k$-graph. If no such $e$ exists, then $G' = H' \cup K$, $|V_t(G')| \leqslant |V_t(H')| + 8$, and the pair $(G', G')$ satisfies the conclusion of the theorem.

Let $\phi$: $(G'-e) \to \Sigma$. Recall the graph $H_2$ used in the construction, and let $R$ be that region of $\phi$ restricted to $H_2$ which contains $\phi(B')$. Since $B''$ contains a $k$-graph, $\phi(B'')$ contains a noncontractible cycle. Because $B'' \cap H_2 = v$, this cycle embeds in $R \cup \phi(v)$ from one occurrence of $v$ to another. These two occurrences of $v$ in the boundary walk separate this walk into two subwalks, $C_1$ and $C_2$ (see Fig. 7.1). These walks are disjoint, except at $v$, by Proposition 4.4.

Let $B_2$ be the $(G, H_2)$-bridge containing $B'$ and let $T_{B_2}$ be the tree used in the construction of $H_2$. Since $H'$ is 2-connected, $T_{B_2}$ intersects at least one of the boundary walks, say $C_1$. Moreover, $T_{B_2}$ does not intersect $C_2$, or else the tree $T_{B_2}$ connecting $C_1$ to $C_2$ would separate the two occurrences of $v$, contradicting that the $(G', H')$-bridge $B'$ contains an essential cycle connecting them. It follows that the bridge $B'$ also does not intersect the cycle $C_2$, since $(\text{vofa}(B') \cap H_2) \subset (\text{vofa}(B_2) \cap H_2) = (T_{B_2} \cap H_2)$.

We modify the embedding $\phi$: $(G'-e) \to \Sigma$ by detaching the $(G', H')$-bridge $B'$ from the left-hand occurrence of $v$ in $R$, bending the bridge down along $C_2$ and reconnecting to the right-hand occurrence of $v$ in Fig. 7.1, calling this new embedding $\phi'$. This is possible since $(\text{vofa}(B') \cap H_2) \subset C_1$. But $\phi'(B'')$ which must contain a noncontractible cycle, a contradiction since the interior of $R$ together with a single occurrence of $v$ on the boundary is contained in a closed disk of $\Sigma$.

*Case* 2. Assume that $B''$ does not contain a $k$-graph.

We note that $B'' - \text{st}(v)$ is connected and $\deg_{B''}(v) = 3$; hence $B''$ contains a $\theta$-graph. If none of the three cycles in the $\theta$-graph disconnect the remaining arc from $H'$, then the $\theta$-graph is a $k_{2,3}$ $k$-graph, a contradiction. Hence let $L$ be such a disconnecting cycle, let $e$ be an edge in the remaining arc of the $\theta$-graph, and let $H$ denote the $(G', L)$-bridge containing $H'$.

Consider $\phi$: $(G'-e) \to \Sigma$. If $\phi(L)$ is contractible in $\Sigma$, then we define $\phi'$: $(G'-e) \to \Sigma$ to be $\phi$. If $\phi(L)$ is not contractible, then by the same

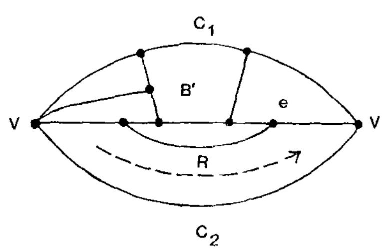


FIGURE 7.1

procedure as in Case 1 we know that $(\mathrm{vofa}(B) \cap H') \subset C_1$, and we again (by bending under) modify the embedding $\phi$ to a new $\phi'$: $(G' - e) \to \Sigma$ with $\phi'(L)$ contractible. In either case, we now apply Lemma 5.2 using $e$, $L$, $H$, as defined and using $\phi'$ with $\phi'(L)$ contractible. We thus conclude that $B''$ contains a $k$-graph, contradicting the hypothesis for Case 2.

Having shown that the pair $(G', H')$ satisfies parts 1 through 6 of Condition 7 and having demonstrated the bound on $|V_t(H)|$, the theorem is established. ∎

## 8. Types and Sizes of Bridges

We have established the $\Sigma$-pair $(G, H)$ which we will use for the next four sections. In this section we will study some properties of $(G, H)$-bridges. We begin (after two preliminary lemmas) in Proposition 8.3 by bounding the size of any bridge. As an aside, we note that in order to bound $|V_t(G)|$ by a function of $|V_t(H)|$, we need only then bound the number of bridges. Continuing in this section, we classify the bridges according to how many pieces (topological vertices or edges) of $H$ contain vertices of attachment for that bridge. Recall that in Proposition 6.1, we showed that there existed a $\Sigma$-pair $(G, H)$ such that any $(G, H)$-bridge had vertices of attachment in at least two pieces of $H$. In Proposition 8.4 we establish this property for our current $\Sigma$-pair. In Proposition 8.5 we show that if a bridge has vertices of attachment in exactly two pieces of $H$, then it is one of four specific types of bridges, and $H$-, $X$-, $Y$-, or $I$-bridge. In Proposition 7.2 we bounded the number of bridges which have vertices of attachment in three or more pieces of $H$. Thus, after this section, we will only need to bound the number of $H$-, $X$-, $Y$-, and $I$-bridges which hit exactly two pieces of $H$. It is this final bound that is the subject of Sections 9, 10, and 11.

LEMMA 8.1. *Let $B'$ be a graph with $n$ vertices of degree* 1, *no vertices of degree* 2, *no cycles with fewer than three vertices, no cubic vertex in a* 3-*cycle, and containing no $\theta$-graph. Then* $|V(B')| \leqslant 3n - 4$.

*Proof.* See Lemma 4.10 in [GH2]. ∎

LEMMA 8.2. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition* 7, *let $v$ be a vertex of $G$ with* $\deg_G(v) = 3$, *let $L$ be a cycle in $G$ containing $v$ and exactly two other topological vertices of $G$, and let $e$ be the topological edge of $G$ in $L$ not incident with $v$. Then $e \subset H$ and $v \in H$.*

*Proof.* By way of contradiction, assume that either $e$ is in $G - H$ or alternatively that $e \subset H$ and $v \in G - H$. Under the first possibility $G - e$

embeds in $\Sigma$ by the critical part of Condition 7. Under the second possibility, $v$ is not a vertex of $H$ so $(L-e) \subset (G-H)$. Hence $(H-e) \cup (L-e)$ is homeomorphic to $H$ and by part 2 of Condition 7, $G-e$ embeds in $\Sigma$. Thus in either case we have established a $\phi$: $(G-e) \to \Sigma$. Since $v$ is a cubic vertex in the topological 3-cycle $L$, $\phi$ extends to an embedding $\bar{\phi}: G \to \Sigma$ with $\bar{\phi}(e)$ embedded in a neighborhood of $\phi(L-e)$: This contradicts that $G$ does not embed in $\Sigma$. ▮

PROPOSITION 8.3. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Then for any $(G, H)$-bridge $B$, $|V_t(B)| \leqslant 6\,|V_t(H)| + 12\,|E_t(H)| - 4$.*

*Proof.* Form $B'$ by replacing each edge $e = (v_1, v_2)$ where $v_1 \in \text{vofa}(B)$ with a new edge $(v_e, v_2)$. These new vertices $v_e$ are assumed to be all distinct. Observe that by Proposition 7.1, $|\text{vofa}(B)| \leqslant |V_t(H)| + 2\,|E_t(H)|$. By part 6 of Condition 7, for each $v \in \text{vofa}(B)$, $\deg_B(v) \leqslant 2$. As there are no degree 1 vertices in $B - \text{vofa}(B)$, the number of degree 1 vertices in $B'$ is thus less than or equal to $2\,|V_t(H)| + 4\,|E_t(H)|$. Because we are only interested in bounding topological vertices we may assume that $B'$ has no vertices of degree 2. Since $(G, H)$ is critical any two topologically parallel edges must both be in $H$, so $B'$ has no such edges. Since $(G, H)$ is $\theta$-less, $B'$ does not contain a $\theta$-graph. Finally Lemma 8.2 shows that $B'$ cannot contain a cubic vertex in a 3-cycle. We have shown that $B'$ satisfies the conditions of Lemma 8.1. By applying Lemma 8.1 we get the proper bound on $|V_t(B')|$ and hence on $|V_t(B)|$. ▮

PROPOSITION 8.4. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Then any $(G, H)$-bridge contains vertices of attachment in at least two distinct pieces of $H$.*

*Proof.* Since $G$ is 2-connected, each bridge $B$ has $|\text{vofa}(B)| \geqslant 2$. If these vertices are contained in a single piece of $H$, then there must exist at least two vertices of attachment in the interior of a topological edge of $H$. By part 3 of Condition 7, $(G, H)$ is $\theta$-less. So $B$ must be a single edge. It is now easy to contradict part 5 of Condition 7 (one of the minimality conditions), since $(G, H)$ being critical implies that $G$ contains no topologically parallel edges unless they both lie in $H$. ▮

Let $(G, H)$ be a pair. A $(G, H)$-bridge $B$ is an $H$-bridge if it is homeomorphic to the letter $H$, or equivalently to $K_{3,3} - K_{2,2}$. Similarly, $B$ is an *X-bridge* if it is homeomorphic to an $X$, $(K_{1,4})$, $B$ is a *Y-bridge* if it is homeomorphic to a $Y$, $(K_{1,3})$, and $B$ is an *I-bridge* if it is homeomorphic to $K_2$. Examples of these bridges are given in Fig. 8.1.

PROPOSITION 8.5. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $B$*

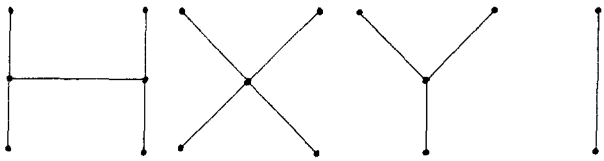

FIGURE 8.1

*be a $(G, H)$-bridge with vertices of attachment in exactly two pieces of $H$. Then $B$ is either an $H$-, $X$-, $Y$-, or $I$-bridge.*

*Proof.* We first observe that $|\text{vofa}(B)| \leqslant 4$. This follows since $|\text{vofa}(B)| \geqslant 5$ implies that there exists an $e \in E_t(H)$ containing at least three vertices of attachment. Since $B - \text{vofa}(B)$ is connected, this contradicts that $(G, H)$ is $\theta$-less.

If $|\text{vofa}(B)| = 4$, then since $G$ is 2-connected any cycle in $B$ would again contradict that $(G, H)$ is $\theta$-less. Hence $B$ is a tree with four vertices of degree 1, and thus is either an $H$-bridge or an $X$-bridge.

If $|\text{vofa}(B)| = 3$, then since $B$ does not contain a cubic vertex in a 3-cycle and since $G$ is 2-connected, a cycle in $B$ would lead to a contradiction of $(G, H)$ $\theta$-less. Thus $B$ is a tree and hence a $Y$-bridge.

If $|\text{vofa}(B)| = 2$, then let $K$ be a the quotient $B/\text{vofa}(B)$. If $K$ is non-planar, then there exists a $\theta$-graph disjoint from $H$, a contradiction. Thus $K$ is planar. Since $(G, H)$ is a critical $\Sigma$-pair it follows that $B$ must be a single edge and hence an $I$-bridge.

Finally, if $|\text{vofa}(B)| \leqslant 1$ we contradict Proposition 8.4. ∎

Summarizing, we have bounded the size and number of each type of $(G, H)$-bridge, with the exception of the number of $H$-, $X$-, $Y$-, or $I$-bridges which have vertices of attachment in exactly two pieces of $H$. We examine these in the next three sections.

## 9. A Bound on the Maximum Degree

Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. The purpose of this section is to prove that the maximum degree of $G$ is bounded by a function of $|V_t(H)|$. This bound together with Proposition 8.3, will allow us to assume that if $G$ is large in relation to $H$, then $G$ will contain many disjoint $(G, H)$-bridges. Once we obtain many disjoint bridges, the proof more closely follows that of the cubic case in [AH].

In Proposition 8.3 we bounded the number of topological vertices in an arbitrary $(G, H)$-bridge. Thus, to show that the maximum degree $\Delta$ of $G$ is bounded, it suffices to bound the number of $(G, H)$-bridges incident with a vertex of $H$. Proposition 7.2 bounds the number of bridges with vertices of attachment in three or more pieces of $H$. Proposition 8.4 shows that each bridge must have vertices of attachment in two or more pieces of $H$; hence we concentrate on bridges which have vertices of attachment in exactly two pieces of $H$. In Proposition 8.5 we show that such a bridge is either an $H$-, $X$-, $Y$-, or $I$-bridge. Lemma 9.1 bounds the number of $H$- or $X$-bridges. Lemma 9.2 bounds the number of $Y$-bridges, and Lemma 9.3 bounds the number of $I$-bridges. The results of this section are then summarized in Theorem 9.4.

LEMMA 9.1. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $v \in V(G)$ and let $\mathscr{B}_v$ be the set of H-bridges and X-bridges $B_i$ with $v \in \text{vofa}(B_i)$ and with* $\text{vofa}(B_i)$ *contained in exactly two pieces of H.* Then $|\mathscr{B}_v| \leqslant 2$.

*Proof.* By way of contradiction, suppose that $|\mathscr{B}_v| \geqslant 3$. Let $\{B_i\}_{i=1}^{3} \subset \mathscr{B}_v$. By part 3 of Condition 7 and because $|\text{vofa}(B_i)| = 4$, $v$ must be in the interior of a topological edge $e$ of $H$; moreover, each $B_i$ contains another vertex of attachment $v_i$ also in $e$. Because $v$ separates $e$ into two components, at least two of the $v_i$, say $v_1$ and $v_2$, lie in the same component of $e - v$. The edge $e$, a path in $B_1$ from $v$ to $v_1$, and a path in $B_2$ from $v$ to $v_2$ form a $\theta$-graph contained in $G - (H - e)$, contradicting that $(G, H)$ is $\theta$-less. ∎

LEMMA 9.2. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $v \in V(G)$, and let $\mathscr{B}_v$ be the set of Y-bridges B with $v \in \text{vofa}(B)$ and* $\text{vofa}(B)$ *contained in exactly two pieces of H. Then $|\mathscr{B}_v| \leqslant [36(2E)^2 + 1](V + E)E + 2$ where $V = |V_t(H)|$ and $E = |E_t(H)|$.*

*Proof.* By way of contradiction, suppose that $|\mathscr{B}_v|$ is greater than the desired bound. Our first goal is to show that there exist bridges $B_i$, $\bar{B}_i$, $i = 1, ..., 5$, an edge $e$, and an embedding $\phi'$: $(G - e) \to \Sigma$ as depicted in either Fig. 9.1 or Fig. 9.3 (Cases 1 and 2, respectively).

Each $B \in \mathscr{B}_v$ has $|\text{vofa}(B)| = 3$, with two of these vertices in a topological edge of $H$ and the third in a different piece of $H$. By an argument similar to that used in Lemma 9.1, no more than two of these bridges have a vertex of attachment other than $v$ which is in the same topological edge of $H$ as $v$. Thus at least $[36(2E)^2 + 1](V + E)E + 1$ of these bridges have as vertices of attachment $v$ and two other vertices which are in a topological edge of $H$ not containing $v$. By the pigeonhole principle, at least $[36(2E)^2 + 1](V + E) + 1$ of these bridges $B$ have $\text{vofa}(B) \subset (v \cup e)$, where $e$ is the interior of some fixed topological edge of $H$ not containing $v$.

We fix a distinguished endpoint of this edge $e$ and label those bridges just found with vertices of attachment in $v \cup e$, calling them $B_i$ for $i = 1, ..., n$. For each $B_i$, let $i_1$ denote the vertex of attachment of $B_i$ which is closest to the distinguished endpoint in $e$, let $i_2$ denote the other vertex of attachment in $e$, and let $i$ be the cubic vertex in the interior of $B_i$.

There exists a vertex of $G$, say $\bar{\imath}$, in the subset of $e$ joining $i_1$ and $i_2$. Otherwise, $i$ would be a vertex in a topological 3-cycle with $\deg_G(i)=3$, and $i \in (G-H)$ would contradict Lemma 8.2.

Let $\bar{B}_i$ be the $(G, H)$-bridge incident with $\bar{\imath}$ and let $\tilde{\imath}$ be in vofa$(\bar{B}_i)$ and not in $e$. By the pigeonhole principle, at least $36(2E)^2+2$ of these bridges $B_i$ have $\tilde{\imath}$ in the same piece, say $\tilde{p}$, of $H$. We observe that $(G, H)$ $\theta$-less implies that $\bar{B}_i \neq B_j$; moreover, $\bar{B}_i \neq B_j$ for any $i$ and $j$.

Let $e'$ be an edge in $B_1$. We consider $\phi'$: $(G-e') \to \Sigma$. Since $(G, H)$ is a $\Sigma$-pair, the vertex $v$ occurs at most $2E$ times on boundary paths and $e$ occurs at most twice on boundary paths of regions made by $\phi'$. Of the remaining $36(2E)^2+1$ bridges $B_i$, at least $36(2E)+1$ of them embed in a region $R_1$ under $\phi'$ at the same occurrence of $v$. Since $(G, H)$ is $\Sigma$-ECTC, $e$ only occurs once on the boundary of $R_1$, so all $36(2E)+1$ of these $B_i$ embed at that one occurrence of $e$ in $R_1$.

For each of the corresponding $36(2E)+1$ bridges $\{\bar{B}_i\}$, $\phi'$ attaches $\bar{B}_i$ to $\bar{\imath}$ at the other occurrence of $e$ in a region $R_2 \neq R_1$. Again, the piece $\tilde{p}$ occurs at most $2E$ times on a boundary walk, so there exist 37 bridges with $\tilde{\imath} \in$ vofa$(\bar{B}_i)$ at the same occurrence of the piece $\tilde{p}$ in the boundary of a region $R_2$. Renaming, call these $\bar{B}_i$, $i=1, ..., 37$.

It is the existence of these 37 bridges and the embedding $\phi'$: $(G-e') \to \Sigma$ which we use to reach a contradiction. We consider several cases.

*Case* 1. Assume that at least five of the $\tilde{\imath}$ are at the same vertex of $G$.

We call this vertex $u$ and refer the reader to Fig. 9.1. Note that $u$ may be $v$. Recall that the vertices of $B_i$ are named $i$, $i_1$, $i_2$, and $v$. We will show that $\{2_1, 4_2, u, v\}$ is a cut set of $G$ which separates $G$ into exactly two parts, or into three parts if one is the edge $(u, v)$. This follows because the walk $(u, \bar{2}$,

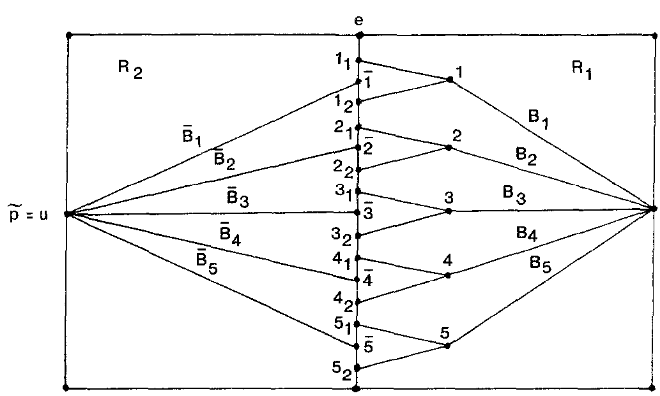


FIGURE 9.1

$2_1$, 2, $v$, 4, $4_2$, $\bar{4}$) bounds an open disk region $R'$ under $\phi$, and hence this walk separates $G$. Any bridge with a vertex of attachment in the open path $(u, \bar{2}, 2_1)$ (or respectively in $(u, \bar{4}, 4_2)$ creates a $\theta$-graph contained in $(G-H) \cup e$, contradicting part 3 of Condition 7 (see Fig. 9.1). Thus $\{2_1, 4_2, u, v\}$ is a cut set of $G$. To show this set separates $G$ into exactly two parts (or exactly three parts as hypothesized), we observe that any component of $G - \text{st}(\{2_1, 4_2, u, v\})$ which intersects three of these vertices contradicts the existence of the embedding $\phi'$. If there is a component intersecting exactly two of these vertices (which is not the edge $(u, v)$) and which does not contradict the embedding $\phi'$, it must one of the edges $(u, 2_1)$, $(u, 4_2)$, $(v, 2_1)$, or $(v, 4_2)$. Any of these four edges creates a 3-cycle of $G$ which contains a cubic vertex 2 or 4, contradicting Lemma 8.2.

Let $C_1$ denote that component of $G - \{2_1, 4_2, u, v\}$ containing the vertex 3, and let $C_2$ be the part of $G - \{2_1, 4_2, u, v\}$ containing $H - e$; we add $(u, v)$ to $C_2$ if $(u, v)$ is a topological edge of $G$. Thus $G = C_1 \bigcup_{\{2_1, 4_2, u, v\}} C_2$. Note that the edge $e'$ used in the construction of $\phi'$ lies in $C_2$.

Next we consider the embedding $\phi$: $((G - C_1) \cup B_2 \cup \bar{B}_2) \to \Sigma$. We observe that $\phi | H$ has two regions bounding the edge $e$ since $(G, H)$ is $\Sigma$-ECTC. Furthermore $\phi(B_2)$ and $\phi(\bar{B}_2)$ must be in different regions as shown in Fig. 9.2.

Let $N$ be a neighborhood in $\Sigma$ of $\phi((B_2 \cup \bar{B}_2 \cup [2_1, 4_2]) - \{u, v, 2_1, 4_2\})$ such that $N$ is homeomorphic to an open disk, $N \cap \phi(C_2) = \varnothing$, and the boundary of $N$ intersect $\phi(C_2)$ in exactly $\{u, v, 2_1, 4_2\}$. Such an $N$ is indicated in Fig. 9.2 by the dashed lines.

$R' \cup \{\phi'(u), \phi'(v), \phi'(2_1), \phi'(4_2)\}$ and $N \cup \{\phi(u), \phi(v), \phi(2_1), \phi(4_2)\}$ are both open disks with four (or three if $u = v$) points on the boundary. Let $\psi$ be a homeomorphism from the former to the latter. Then $(\psi\phi' | C_1) \cup (\phi | C_2)$ is a map embedding $C_1 \bigcup_{\{2_1, 4_2, u, v\}} C_2$ into $\Sigma$. But $G$ does not embed in $\Sigma$, a contradiction.

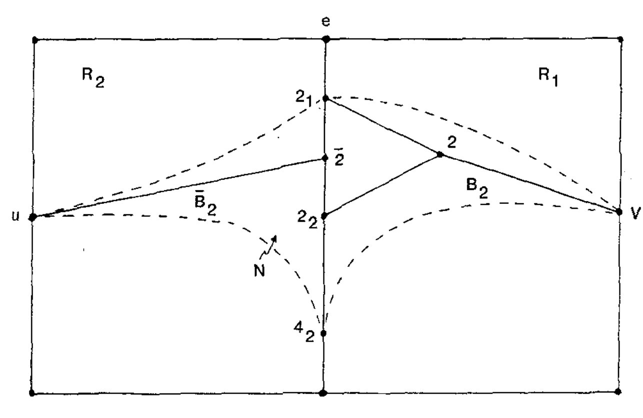


FIGURE 9.2

Technical procedures similar to this construction of the embedding of $C_1 \cup C_2$ from embeddings $\phi' | C_1$ and $\phi | C_2$ will be hereafter called *glueing* the embeddings $\phi' | C_1$ and $\phi | C_2$ together along $C_1 \cap C_2$. We will occasionally glue along a finite set of vertices as well as along a simple cycle.

*Case* 2. Assume that at most four of the $\tilde{i}$'s are at a common vertex.

Then these 37 $\tilde{i}$'s are at least 10 distinct vertices. Moreover, $\tilde{p}$ must be a topological edge of $H$ and there is a set of five $i$'s, say $i = 1, ..., 5$ such that the $\tilde{i}$'s are distinct and $v$ is not in $[1^{\sim}, 5^{\sim}]$. We refer the reader to Fig. 9.3.

We turn our attention to the embedding $\phi_3$: $(G - (3, v)) \to \Sigma$. Again the edge $e$ bounds two regions, $R_1$, $R_2$, of $\phi_3 | H$. Also $\phi_3$ maps the bridges $B_i$, $\bar{B}_i$ into different regions for $i \neq 3$; moreover, $\phi_3$ maps $\bar{B}_3$ and $B_3 - (3, v)$ into different regions. We now consider several subcases.

*Case* 2.1. Assume that $\phi_3(2)$ and $\phi_3(4)$ lie in different regions.

Without loss of generality, assume that $\phi_3(2)$ and $\phi_3(3)$ lie in the region $R_1$ and that $\phi_3(4)$ lies in $R_2$. Because $\tilde{p}$ is a topological edge of $H$ and $(G, H)$ is $\Sigma$-ECTC, $\tilde{p}$ occurs exactly once in the boundary cycle of $R_2$ and $\phi_3(\bar{3}, 3^{\sim})$ lies in $R_2$ as shown in Fig. 9.4.

Since $G$ does not embed in $\Sigma$, we cannot extend the embedding $\phi$ to include the edge $(3, v)$. Thus there exists a bridge with a vertex of attachment in the closed subpath $[2_2, 3_1]$ of $e$ and another vertex of attachment which is neither in this same path nor at $v$. By Fig. 9.3, another vertex of attachment must be in $\tilde{p}$ and $\text{vofa}(B) \subset [2_2, 3_1] \cup [2^{\sim}, 3^{\sim}]$. Recall that $v$ is not in $[2^{\sim}, 3^{\sim}]$.

The cycle $C = (2^{\sim}, \bar{2}, 2_2, 3_1, \bar{3}, 3^{\sim})$ is contractible; hence it separates $G$. Moreover, if $B$ is any $(G, H \cup \{(\bar{2}, 2^{\sim}), (\bar{3}, 3^{\sim})\})$-bridge containing vertices of attachment in both the open path $(\bar{2}, 2^{\sim}, 3^{\sim}, \bar{3})$ and the closed path $[\bar{2}, 2_2, 3_1, 3]$, then $\phi_3(B)$ is contained in the closed 2-cell bounded by $C$.

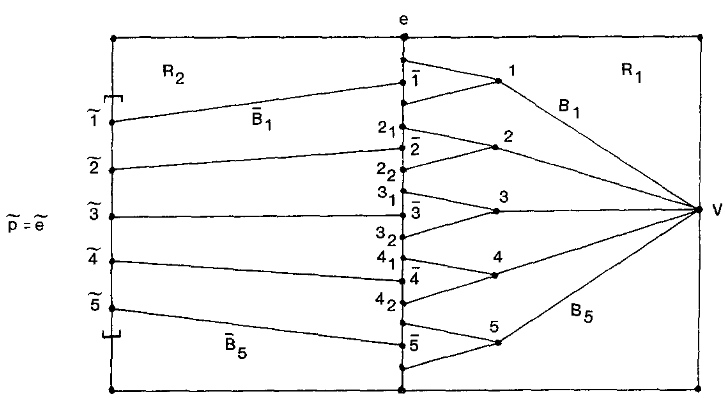


FIGURE 9.3

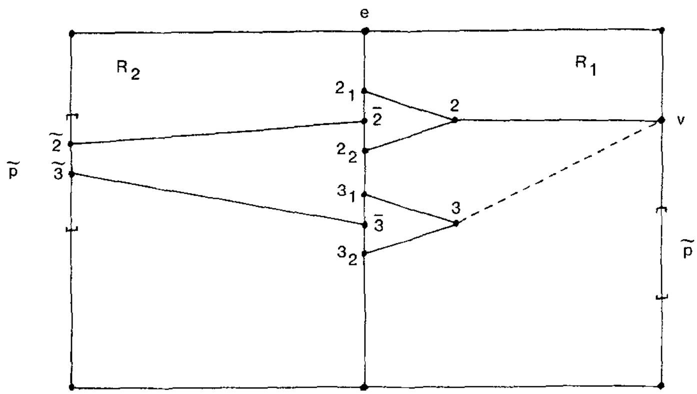


FIGURE 9.4

Let $K_1$ be the subgraph of $G$ consisting of the cycle $C$ together with all $(G, H \cup C)$-bridges $B$ with $\phi_3(B)$ contained in the region bounded by $C$. Let $K_2 = G - (K_1 - C)$.

We construct $\phi$: $(G - (3, v)) \to \Sigma$ by glueing $\phi' | K_1$ to $\phi_3 | K_2$ along $C$. There does not exist a bridge blocking the extension of $\phi$ to include the edge $(3, v)$, because any such bridge is in $K_1$. Thus $\phi$ extends to an embedding of all of $G$, a contradiction.

*Case* 2.2. Assume that $\phi_3(2)$ and $\phi_3(4)$ both lie in the same region.

Call this region $R_1$. We note that $\phi_3(\bar{B}_2) \cup \phi_3(\bar{B}_4)$ is contained in $R_2$ as shown in Fig. 9.5. Again we examine Fig. 9.3. Let $C$ be the cycle $(2^{\sim}, \bar{2}, 2_2, 2, v, 4, 4_1, \bar{4}, 4^{\sim}, 3^{\sim})$. Let $K_1 = C \cup \{B | B$ is a $(G, H \cup C)$-bridge and $\phi(B)$ is contained in the open disk region bounded by $C\}$. Let $K_2 = G - (K_1 - C)$. Thus $G = K_1 \bigcup_C K_2$.

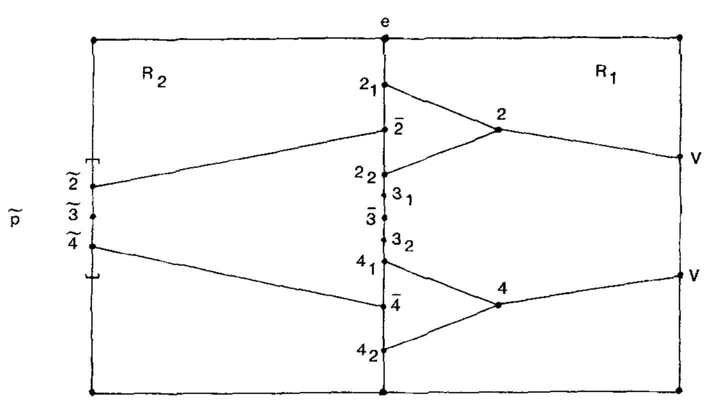


FIGURE 9.5

We consider $\phi_3 | K_2$. If there exists a bridge $B$ with a vertex of attachment in the closed path $[2_2, 3_1, \bar{3}, 3_2, 4]$, then $\phi'(B)$ lies in the region bounded by $C$; thus $B$ is not a subset of $K_2$. We may now modify $\phi_3 | K_2$ to an embedding $\phi_3'$: $K_2 \to \Sigma$ by deleting the edge $(4, v)$ from $\phi_3 | K_2$ and re-embedding it in a small neighborhood of the closed path $[4, 4_1, 3_2, \bar{3}, 3_1, 2_2, 2, v]$. We now construct $\phi$ by glueing $\phi_3' | K_2$ to $\phi' | K_1$ along $C$. The map $\phi$ embeds $G$ into $\Sigma$, a contradiction. ▮

LEMMA 9.3. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7, let $v \in V(G)$, and let $\mathscr{B}_v$ denote the set of all I-bridges which are incident with $v$. Then*

$$|\mathscr{B}_v| \leqslant (V+E)^2 [2(2E)(3N^2+2)+1],$$

*where $V = |V_t(H)|$, $E = |E_t(H)|$, and $N$ is the bound given in Lemma 9.2.*

*Proof.* The proof is similar in nature to the proof of Lemma 9.2. By way of contradiction, suppose that $|\mathscr{B}_v|$ is larger than the desired bound. We will first show that there exists $\phi'$: $(G-e') \to \Sigma$ with some appropriate properties, resembling either Fig. 9.6 or Fig. 9.9.

By the pigeonhole principle, at least $(V+E)[2(2E)(3N^2+2)+1]+1$ of the bridges in $\mathscr{B}_v$ have an endpoint other than $v$ in the same piece of $H$. Since $(G, H)$ is a critical $\Sigma$-pair, the endpoints of the $B_i$ other than $v$ are all distinct, so this piece must be an edge; call it $e$. Label these bridges $B_i$ with the subscript order induced by considering $e$ as a directed edge. Let $i$ be the vertex of $B_i$ in $e$. We note that there must exist a topological vertex of $G - (\bigcup B_i)$, say $\bar{i}$, in the half open interval $[i, i+1]$. If not, then the vertex $i$ is a cubic vertex in the 3-cycle $(i, i+1, v)$ and the edge opposite $i$ is not in $H$, which would contradict Lemma 8.2. Let $\bar{B}_i$ be a bridge with $\bar{i} \in \text{vofa}(\bar{B}_i)$ and let $\bar{\bar{i}} \in \text{vofa}(\bar{B}_i)$ be a vertex which is not in the topological edge $e$ and

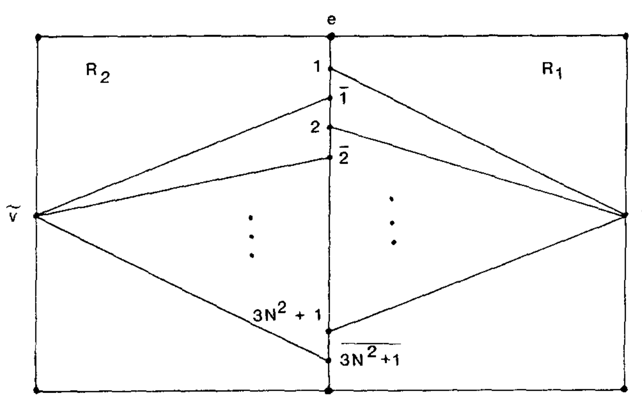


FIGURE 9.6

which is not $v$. Again, at least $2(2E)(3N^2+2)+2$ of these $\bar{B}_i$ have $\bar{\imath}$ in the same piece, say $\tilde{p}$, of $H$.

Let $e'$ be the edge $(1, v)$, and consider the embedding $\phi'$: $(G-e') \to \Sigma$. Let $R_1$ and $R_2$ denote the two regions of $\phi'|H$ which contain $e$ in their boundary; $R_1 \neq R_2$ since $e \in E(G)$ and $(G, H)$ is $\Sigma$-ECTC. Of the $2(2E)(3N^2+2)+1$ $B_i$'s which are not $e'$, at least $3N^2+3$ of them must map in the same region, say $R_1$, at the same occurrence of $v$ in the boundary cycle of $R_1$. With the exception of the two $\bar{B}_i$ with extreme subscripts, each of the remaining $3N^2+1\bar{B}_i$ must map into $R_2$ under $\phi$. Thus we have shown the existence of bridges $B_i$, $\bar{B}_i$ and the embedding $\phi'$ as depicted in either Fig. 9.6 or Fig. 9.9 (in Fig. 9.9 we shall need only five of the bridges). We relabel, preserving order, such that these bridges are $\{B_i\}_{i=1}^{3N^2+1}$ (or respectively $\{B_i\}_{i=1}^{5}$).

*Case* 1. Assume that the $\bar{\imath}$'s are all the same vertex of $G$.

We call this vertex $\tilde{v}$. Consider the cycles $(v, i, \bar{\imath}, i+1)$. Any $Y$-bridge mapped by $\phi'$ inside this cycle must have $v$ as a vertex of attachment, by Lemma 9.2. This occurs at most $N$ times. Thus there exists a string of at least $3N+1$ consecutive cycles which do not contain a $Y$-bridge. Repeating this for the cycles $(\bar{\imath}, i+1, \overline{i+1}, \tilde{v})$ we get a string of four consecutive cycles which do not contain a $Y$ bridge. Thus (relabeling) we have the situation depicted in Fig. 9.7, such that there does not exist a $(G, H)$-bridge $B$ with $\phi'(B)$ contained in $(\tilde{v}, \bar{1}, 1, v, 5, \bar{5})$ other than those shown. In particular the set $\{\bar{1}, 5, \tilde{v}, v\}$ separates $G$ into exactly two parts (or exactly three parts if one is the edge $(v, \tilde{v})$). Define $C_1$ as that part containing the vertex 3 and $C_2 = G - C_1$. In particular, note that the edge $e'$ used in constructing $\phi'$ is in $C_2$ and $G = C_1 \bigcup_{\{\bar{1}, 5, \tilde{v}, v\}} C_2$.

Consider the embedding $\phi_3$: $G-(v, 3) \to \Sigma$. Let $R_1$, $R_2$ denote the regions of $\phi_3|H$ incident with $e$, where $R_2$ is the region containing

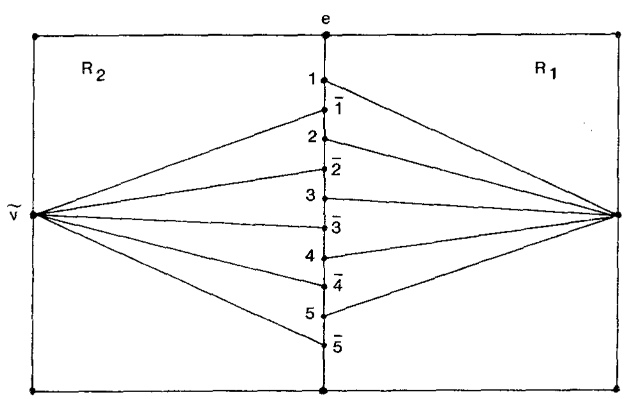


FIGURE 9.7

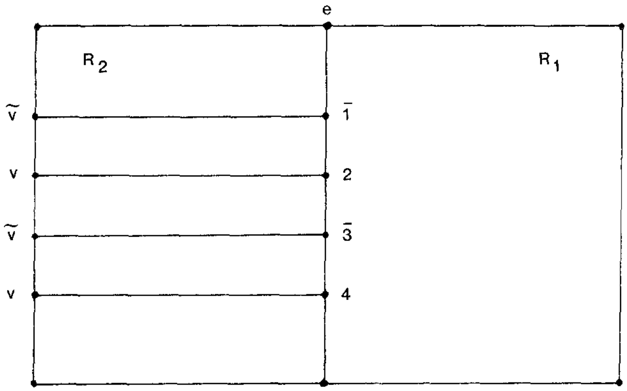


FIGURE 9.8

$\phi_3((\bar{1}, \tilde{v}))$. If $\phi_3$ maps on edge $(i, v)$ in one region and an edge $(\bar{j}, \tilde{v})$ in the other region, then by glueing $\phi' | C_1$ to $\phi_3 | C_2$ along $\{\bar{1}, 5, \tilde{v}, v\}$ we construct an embedding of $G$ into $\Sigma$, a contradiction. Thus $\phi((2, v)) \subset R_2$ and $\phi((4, v)) \subset R_2$, which forces $\phi((\bar{3}, \tilde{v})) \subset R_2$. Since the vertices 1, 2, $\bar{3}$, and 4 are all distinct, this implies that the vertices $v$ and $\tilde{v}$ occur in the boundary cycle of $R_2$ as shown in Fig. 9.8. This contradicts Proposition 4.4, since $\tilde{\gamma}(H) \geqslant \tilde{\gamma}(\Sigma)$.

*Case* 2. Assume that at least two of the $\tilde{i}$'s are distinct.

This implies that the piece $\tilde{p}$ of $H$ is an edge; call it $\tilde{e}$. Moreover, since $\tilde{e}$ is not an edge of $G$, $(G, H)$ $\Sigma$-ECTC implies that $\tilde{e}$ can only occur once in the cycle bounding any region $R$ of any embedding $\psi$: $H \to \Sigma$. We will only use five of the $3N^2 + 1$ bridges, labeled as depicted in Fig. 9.9, and select them such that $v$ is not in $[1^{\sim}, 5^{\sim}]$.

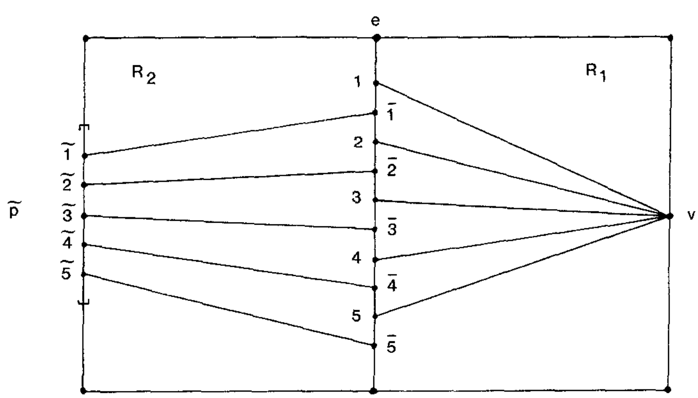


FIGURE 9.9

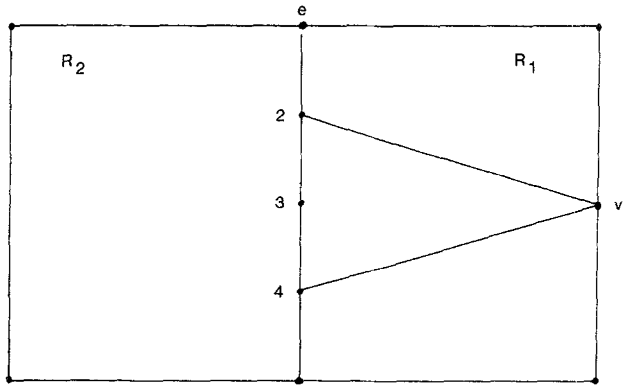


FIGURE 9.10

As before let $\phi_3$: $(G-(3, v)) \to \Sigma$ and let $R_1$ and $R_2$ be the regions of $\phi_3 | H$ which are incident with $e$. We break Case 2 into three subcases.

*Case* 2.1. Assume that $\phi_3$ maps $(2, v)$ and $(4, v)$ as shown in Fig. 9.10. Let $C$ be the cycle $(v, 2, \bar{2}, 3, \bar{3}, 4)$. Let $C_1$ be $C$ together with all edges of $G$ mapped by $\phi'$ to the open disk region in $R_1$ which is bounded by $C$ and let $C_2 = G-(C_1 - C)$. Thus $G = C_1 \bigcup_C C_2$. Note that $e'$ is an edge of $C_2$. By glueing $\phi' | C_1$ to $\phi_3 | C_2$ along $C$, we construct an embedding $G \to \Sigma$, a contradiction.

*Case* 2.2. Assume that $\phi_3$ maps $(2, v)$ and $(4, v)$ as shown in Fig. 9.11. We try to extend $\phi_3$ to include $(3, v)$ by adding in this edge alongside of the edge $(2, v)$. Because this embedding does not extend, the edge $(\bar{2}, 2^{\sim})$ must embed as shown. Since $\tilde{e}$ occurs only once in the cycle bounding $R_1$, $\phi_3$ must map the edges $(\bar{1}, 1^{\sim})$ and $(5, 5^{\sim})$ into region $R_2$. Let $C$ be the cycle $(1^{\sim}, \bar{1}, 2, \bar{2}, 3, \bar{3}, 4, \bar{4}, 5, \bar{5}, 5^{\sim}, 4^{\sim}, 3^{\sim}, 2^{\sim})$, let $C_1$ be

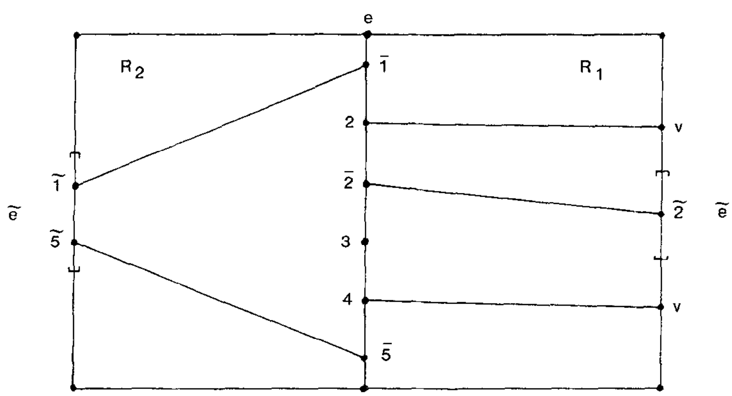


FIGURE 9.11

the subgraph $C$ together with all edges mapped by $\phi'$ into the open disk region in $R_2$ bounded by $C$, and let $C_2 = G-(C_1-C)-(3, v)$. Thus $G-(3, v) = C_1 \bigcup_C C_2$. Define $\phi'_3$ as $\phi_3 | C_2$ glued along $C$ to $\phi' | C_1$; then $\phi'_3$: $(G-(3, v)) \to \Sigma$. There does not exist the blocking edge $(\bar{2}, 2^{\sim})$ in $R_1$ for $\phi'_3$; thus we may extend the embedding $\phi'_3$ to an embedding of all of $G$, a contradiction.

*Case* 2.3. Assume that $\phi_3$ maps $(2, v)$ and $(4, v)$ as shown in Fig. 9.12. Again we attempt to extend $\phi_3$ to include the edge $(3, v)$. Because we cannot add in the edge $(3, v)$ alongside $(2, v)$, the edge $(\bar{2}, 2^{\sim})$ embeds in $R_1$ as shown. Since $(3, v)$ cannot be added in alongside $(4, v)$ in $R_2$, there exists either an edge $(\bar{3}, 3^{\sim})$ or $(\bar{4}, 4^{\sim})$ blocking $(4, v)$. Thus $\phi_3(\bar{5}, 5^{\sim}) \subset R_1$, as shown. Mimicking the procedure of the previous case wa may embed any bridge with vertices of attachment in both the closed path $[3, \bar{3}, 4]$ and $\tilde{e}$ in $R_1$ using $\phi'$. This modified embedding extends to an embedding of $G$ by adding $(3, v)$ in a neighborhood of $[3, 4] \cup [4, v]$.

These subcases exhaust Case 2, so the proof of Lemma 9.3 is complete. ▮

THEOREM 9.4. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Then for any $v \in V(G)$, $\deg_G(v)$ is bounded by a function of $|V_t(H)|$.*

*Proof.* It suffices to find bounds as a function of $|V_t(H)|$ and $|E_t(H)|$ (see Lemma 1.3). If $v \in (G-H)$, then it and all of its neighbors lie in some $(G, H)$-bridge $B$.

In Proposition 8.3 we bound $|V_t(B)|$, and thus $\deg_G(v)$ is also appropriately bounded. It follows that we may assume $v \in H$. The edges incident with $v$ are either in $H$ or in some $(G, H)$-bridge $B$. Those edges in $H$ are bounded by $2\,|E_t(H)|$. Since $\deg_B(v) \leqslant 2$ for each $(G, H)$-bridge $B$

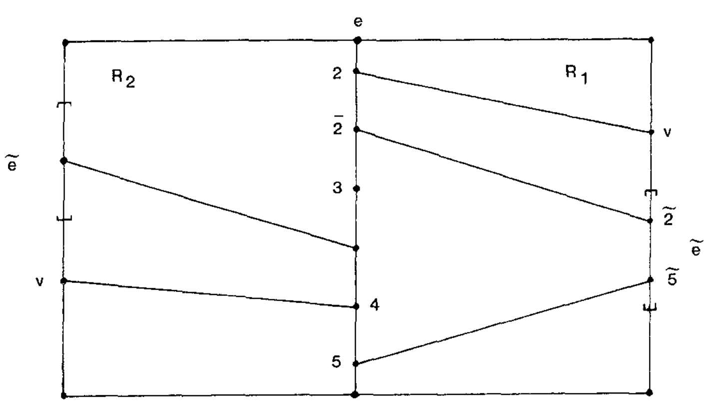


FIGURE 9.12

and each vertex of attachment $v$, it suffices to bound the number of bridges having $v$ as a vertex of attachment. There are no bridges with all vertices of attachment in a single piece by Proposition 8.4. Those with vertices of attachment in exactly two pieces of $H$ are either $H$-, $X$-, $Y$-, or $I$-bridges by Proposition 8.5. The number of these are bounded by Lemmas 9.1 through 9.3. Finally, Proposition 7.2 bounds the total number of bridges which attach at three or more pieces of $H$. Thus $\deg_G(v)$ is bounded as desired. ∎

Let $\Delta$ denote the maximum degree of a vertex in $G$. In the following section our bounds will be in terms of $|V_t(H)$, $|E_t(H)|$, and $\Delta$. Any such bound can now be rewritten as a function of $|V_t(H)|$ alone.

## 10. The Construction of Some Special Subarcs

We are now entering into the most technical portion of the argument. The types of bridges which are most difficult to bound are those whose vertices of attachment are contained in the interior of two topological edges of $H$. The purpose of this section is to find subarcs $\tilde{e}_1 \subset e_1$ and $\tilde{e}_2 \subset e_2$ and a set of bridges $\mathscr{B}$ with certain properties collectively known as Condition 10. Among these properties is that any bridge which has a vertex of attachment in $\tilde{e}_1$ or in $\tilde{e}_2$ must have all vertices of attachment in $e_1 \cup e_2$. We will also show that a "large" number of bridges with vertices of attachment contained in $e_1 \cup e_2$ are in $\mathscr{B}$. To bound the total number of bridges with vertices of attachment contained in $e_1 \cup e_2$, it will suffice to bound $|\mathscr{B}|$. This latter bound will be shown in Section 11.

Before stating Condition 10 we need a definition. We will call two $(G, H)$-bridges *disjoint* provided that they have no vertices in common. Note that if the graph $G$ is cubic and $H$ is 2-connected, then any pair of $(G, H)$-bridges are disjoint. This does not hold for noncubic graphs, as bridges may intersect at their vertices of attachment. However, by the bound on the maximum degree of $G$ given by Theorem 9.4, any sufficiently large set of bridges will contain a large subset of pairwise disjoint bridges.

We now state Condition 10. Let $(G, H)$ be a $\Sigma$-pair, let $e_i \in E_t(H)$ and let $\tilde{e}_i$ be a subarc of $e_i$ for $i = 1, 2$. Suppose that $\mathscr{B}$ is a family of $(G, H)$-bridges. We say that $\tilde{e}_1$, $\tilde{e}_2$, and $\mathscr{B}$ satisfy *Condition* 10 provided that:

(1) the bridges in $\mathscr{B}$ are pairwise disjoint,

(2) for each $B \in \mathscr{B}$, $\mathrm{vofa}(B) \subset (\tilde{e}_1 \cup \tilde{e}_2)$,

(3) for each $B \in \mathscr{B}$, $e \in E_t(B)$, and $\phi$: $(G - e) \to \Sigma$, $e_1$ and $e_2$ bound two common regions $R_1$ and $R_2$ of $\phi | H$ such that $e_1 \cup R_1 \cup e_2 \cup R_2$ is a cylinder, and

(4) any $(G, H)$-bridge $B$ with a vertex of attachment in $\tilde{e}_1$ or in $\tilde{e}_2$ has $\text{vofa}(B) \subset (e_1 \cup e_2)$.

We first show how to construct a set of disjoint bridges.

LEMMA 10.1. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7, let $e_1$ and $e_2$ be topological edges of $H$, and let $\Delta$ denote the maximum degree of $G$. Suppose that $\mathscr{B}$ is the set of bridges $B$ with* $\text{vofa}(B) \subset (e_1 \cup e_2)$. *If $|\mathscr{B}| \geqslant (N-1)(4\Delta+1)+1$, then there exists a set $\mathscr{B}' \subset \mathscr{B}$ of $N$ pairwise disjoint bridges.*

*Proof.* Given $B \in \mathscr{B}$, Proposition 8.5 implies that $|\text{vofa}(B)| \leqslant 4$. Thus there are at most $4\Delta$ bridges which are not disjoint from $B$. The proof now follows is easily by induction on $N$. ▌

The following notation will come in handy. Let $(G, H)$ be a $\Sigma$-pair, let $\phi$: $H \to \Sigma$ be an embedding with regions $\{R_i\}$, let $C_i$ denote the boundary cycle of $R_i$, and let $B_1$, $B_2$ be $(G, H)$-bridges. We say that $B_1$ is *$R_i$-admissible* if there exists an embedding $\phi'$: $(H \cup B_i) \to \Sigma$ with $\phi' \mid H = \phi$ and with $\phi'(B_1) \subset (R_i \cup C_i)$. Two $R_i$-admissible bridges are *$R_i$-parallel* if there exists a $\phi'$: $(H \cup B_1 \cup B_2) \to \Sigma$ with $\phi' \mid H = \phi$ and with $\phi'(B_1 \cup B_2) \subset (R_i \cup C_i)$. If $B_1$ and $B_2$ are each $R_i$-admissible but are not $R_i$-parallel we will call them *$R_i$-skew*. If the region and embedding are clear from context we will just say admissible, parallel, and skew, respectively. We now show how to find a family of bridges satisfying part 3 of Condition 10.

LEMMA 10.2. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7 and let $e_1$ and $e_2$ be topological edges of $H$. Let $\mathscr{B}$ be a set of disjoint $(G, H)$-bridges such that for each $B \in \mathscr{B}$,* $\text{vofa}(B) \subset (e_1 \cup e_2)$. *If $|\mathscr{B}| \geqslant 27 + N$, then there exist at least $N$ disjoint bridges $\{B_i\}_{i=1}^{N} \subset \mathscr{B}$ such that for any $e \in B_i$ and any embedding $\phi$: $(G - e) \to \Sigma$, $\phi \mid H$ has $e_1$ and $e_2$ bounding two common regions $R_1$ and $R_2$ with $e_1 \cup R_1 \cup e_2 \cup R_2$ a cylinder.*

*Proof.* By way of contradiction, suppose that there exist 28 bridges in $\mathscr{B}$, say $\{B_i\}_{i=1}^{28}$, such that for each $i$, $B_i$ contains an edge $e_i$ together with an embedding $\phi_i$: $(G - e_i) \to \Sigma$ which does not have the desired regions of $\phi_i \mid H$.

Note that for each $i$, $B_i \cap e_1$ and $B_i \cap e_2$ are nonempty by Proposition 8.4. Hence $\phi_i \mid H$ has a region which contains $B_j$, where $j \neq i$, and so $e_1$ and $e_2$ are in the boundary of this region.

Next we will show by contradiction that for at most four of the $\phi_i$, $\phi_i \mid H$ has only one region $R_i$ containing $e_1 \cup e_2$ in its boundary. Assume that five of the $\phi_i$ have this property, say $\phi_i$ for $i = 1, \ldots, 5$. Under the embedding $\phi_5$, the bridges $B_1$, $B_2$, $B_3$, and $B_4$ all embed in region $R_5$. Suppose that the order induced by the indices on these four disjoint bridges agrees with the order induced by the arc $e_1$ under $\phi_5$ (see, e.g., Fig. 10.1).

Next consider $\phi_2$: $(G-e_2) \to \Sigma$. Since $\phi_2$ does not extend to an embedding of $G$ and $B_2$ is $R_2$-admissible, there exists a bridge $\bar{B}_2$ such that $\bar{B}_2$ is $R_2$-skew to $B_2$. Note that $\bar{B}_2$ is not $B_1$, $B_3$, or $B_4$. Also note that $\text{vofa}(\bar{B}_2) \subset (e_1 \cup e_2)$ and that $\bar{B}_2$ has vertices of attachment in both $e_1$ and $e_2$. Finally consider $\phi_4$: $(G-e_4) \to \Sigma$. Bridges $B_2$ and $\bar{B}_2$ must both embed in $R_4$ under $\phi_4$. Because $(\text{vofa}(B_2) \cup \text{vofa}(\bar{B}_2)) \subset (e_1 \cup e_2)$ and $B_2$ is $R_2$-skew to $\bar{B}_2$, these two bridges are also $R_4$-skew, a contradiction.

We have shown that for at most four of the $\phi_i$, $\phi_i | H$ has only one region $R_i$ containing both $e_1$ and $e_2$. Hence we have 24 of the bridges $B_i$ in $\mathscr{B}$ with embeddings $\phi_i$ such that $\phi_i | H$ has two regions, say $R_{1,i}$ and $R_{2,i}$, with $e_1 \cup e_2$ in the boundary of each and with $e_1 \cup R_{1,i} \cup e_2 \cup R_{2,i}$ not a cylinder. For these 24 embeddings, this union must be a Möbius strip.

If $B_i$ and $B_j$ are any two of these 24 bridges, then for any collection of $(G, H)$-bridges with vertices of attachment in $e_1 \cup e_2$ which are pairwise $R_{1,i}$-parallel, the same bridges are pairwise $R_{2,i}$-skew. Likewise if they are $R_{2,i}$-parallel, then they are pairwise $R_{1,i}$-skew. Similar statements hold for the regions of $\phi_j$.

Now consider a particular $B_{i_0}$ in this collection of 24 bridges. Under this embedding at least 12 of the remaining 23 bridges must embed in the same region, $R_{1,i_0}$ or $R_{2,i_0}$. By re-indexing if necessary, assume that these 12 bridges are $B_i$ for $i=1, ..., 12$ and that the order induced by the indices coincides with the order induced by the arc $e_1$ (see, e.g., Figure 10.2). Also, for each $j \leqslant 12$, let $R_{1,j}$ be the region of $\phi_j | H$ containing at least ten of the $\phi_j(B_i)$ for $i \neq j$, $i=1, ..., 12$. Hence for each $j \leqslant 12$, the bridges $\{B_i | i \neq j, 1 \leqslant i \leqslant 12\}$ are pairwise $R_{1,j}$-parallel and $R_{2,j}$-skew. Note also that $B_i$ is $R_{1,j}$-admissible $(i \neq j)$ so it is also $R_{1,i}$-admissible. It follows that $\phi_{12}$ maps at least ten of the $\{B_i | 1 \leqslant i \leqslant 11\}$ into the region $R_{1,12}$. Again, renaming if

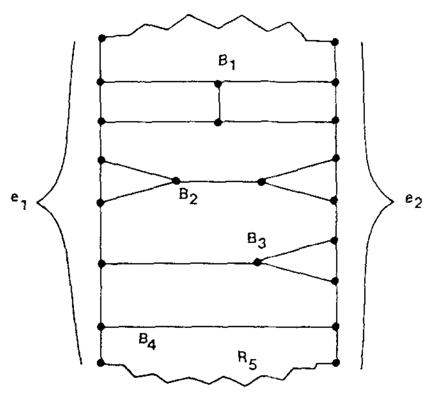


FIGURE 10.1

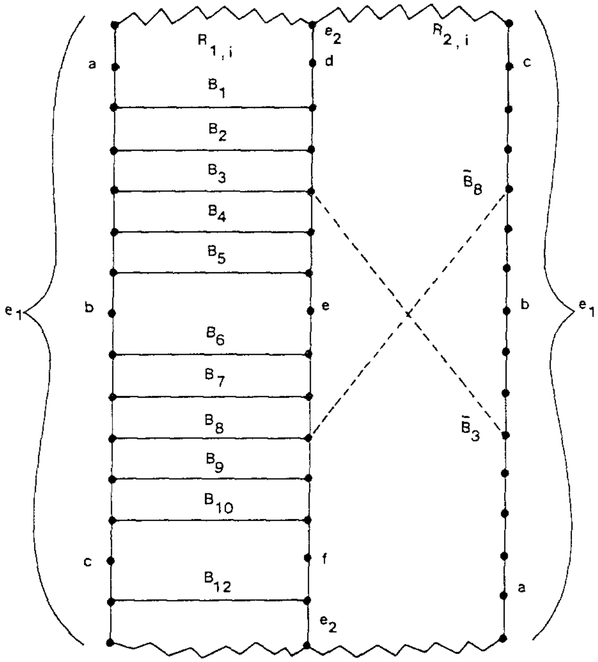


FIGURE 10.2

necessary say that these 10 bridges are $B_i$ for $i = 1, ..., 10$ and that the order induced by the indices agrees with the order induced by $e_1$. Finally, name points $a$, $b$, $c$, $d$, $e$, and $f$ in $e_1 \cup e_2$ which are not vertices of $G$ as shown in Fig. 10.2.

The embedding $\phi_3$: $(G - e_3) \to \Sigma$ does not extend to an embedding of $G$ in $\Sigma$. However, $B_3$ is $R_{1,3}$-admissible, so there exists a $(G, H)$-bridge $\bar{B}_3$ such that $B_3$ and $\bar{B}_3$ are $R_{1,3}$-skew. Observe that $\bar{B}_3 \neq B_i$ for $i = 1, ..., 12$. Also observe that since $B_1$ and $B_2$ are $R_{1,3}$-parallel, they are $R_{2,3}$-skew. Thus under $\phi_3$ at least one of the two must embed in $R_{1,3}$. Similarly, at least one of the pair $B_4$, $B_5$ embeds in $R_{1,3}$ under $\phi_3$. Hence $\bar{B}_3$ must be $R_{1,3}$-parallel to both $B_1$ and to $B_5$. We conclude that $\text{vofa}(B_3) \cup \text{vofa}(\bar{B}_3) \subset ((a, b) \cup (d, e))$, where $(a, b)$ denotes the interior of that connected portion of $e_1$ with endpoints $a$ and $b$.

In a similar manner we use embedding $\phi_8$ to find the bridge $\bar{B}_8$. $\bar{B}_8$ is $R_{1,8}$-skew to $B_8$, $\bar{B}_8$ is not $B_i$ for $i = 1, ..., 12$, and $\text{vofa}(B_8) \cup \text{vofa}(\bar{B}_8) \subset ((b, c) \cup (e, f))$.

For the desired contradiction we again examine the embedding $\phi_{12}$. Since $B_3$ and $\bar{B}_3$ are $R_{1,3}$-skew, $B_3$ and $\bar{B}_3$ are also $R_{1,12}$-skew, so at least one of $B_3$ or $\bar{B}_3$ must embed in $R_{2,12}$ under $\phi_{12}$. Likewise either $B_8$ or $\bar{B}_8$ embeds in $R_{2,12}$. However, these two bridges are $R_{1,12}$-skew as can be seen by examining the intervals containing their vertices of attachment. With this contradiction, we establish that there are at most 27 $(G, H)$-bridges which do not have the desired property. ▮

Let $H$ be a $\Sigma$-OTC graph. Two embeddings $\phi, \psi: H \to \Sigma$ are *equivalent* if there exists a homeomorphism $f$ of $\Sigma$ which carries $\phi$ to $\psi$. Let $\Phi_H^\Sigma$ denote the number of pairwise nonequivalent embeddings of $H$ into $\Sigma$. The following bound is needed.

LEMMA 10.3. *Let $H$ be a 2-connected $\Sigma$-OTC graph. Then* $\Phi_H^\Sigma \leqslant 2^E((2E)!)^V$, where $E = |E_t(H)|$ and $V = |V_t(H)|$.

*Proof.* An embedding can be characterized in terms of a rotation scheme on a signed graph [S]. There are at most $(2E)!$ cyclic permutations of the edges incident with a vertex. $V$ such permutations form a rotation scheme. Finally, there are $2^E$ signatures for a graph. ▮

The following proposition completes this section.

PROPOSITION 10.4. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $e_1$ and $e_2$ be topological edges of $H$ and suppose that $\mathscr{B}$ is a set of bridges with* $\text{vofa}(B) \subset (e_1 \cup e_2)$ *for each $B \in \mathscr{B}$. Let $V = |V_t(H)|$, $E = |E_t(H)|$, $\Delta$ be the maximum degree of $G$, and let $\Phi_H^\Sigma$ be as in Lemma* 10.3. *Set $M_1 = 2\Delta(V+E-2)+1$ and set $M_2 = NM_1 + \Delta(6V+12E-3)(M_1-1)$. Then there exist subarcs $\tilde{e}_1 \subset e_1$ and $\tilde{e}_2 \subset e_2$ and a family $\mathscr{B}' \subset \mathscr{B}$ which satisfy Condition* 10. *Moreover $|\mathscr{B}'| \geqslant N$ if*

$$|\mathscr{B}| \geqslant [(2M_2+11)\Phi_H^\Sigma + 27](4\Delta+1)+1.$$

*Proof.* Suppose that $|\mathscr{B}|$ satisfies the desired inequality. By Lemma 10.1 there exists at least $(2M_2+11)\Phi_H^\Sigma + 28$ pairwise disjoint bridges in $\mathscr{B}$. By Lemma 10.2 there are at least $(2M_2+11)\Phi_H^\Sigma + 1$ of these bridges with the property that for any $e \in B$ and any $\phi: (G-e) \to \Sigma$, $e_1$ and $e_2$ bound two common regions $R_1$ and $R_2$ of $\phi \mid H$ with $e_1 \cup R_1 \cup e_2 \cup R_2$ a cylinder. We will choose $\mathscr{B}'$ from these bridges, and hence parts 1 and 3 of Condition 10 will be satisfied.

At least $2M_2 + 12$ of these bridges have an edge $e$ and an embedding $\phi$: $(G-e) \to \Sigma$ with $\phi \mid H$ some fixed $\phi_0: H \to \Sigma$. Moreover, there is a region $R_1$ of $\phi_0$ such that under a fixed one of the embeddings $\phi$ at least $M_2 + 6 = NM_1 + \Delta(6V+12E-3)(M_1-1)+6$ of the remaining bridges embed in $R_1$. Note that these bridges are thus pairwise $R_1$-parallel. We partition these bridges into three collections of bridges as follows: $M_1$ sets with $N$ bridges in each, denoted $\{\mathscr{B}_j \mid 1 \leqslant j \leqslant M_1\}$ (these will be the candidates for the $\mathscr{B}'$ promised by this proposition), $M_1 - 1$ sets each with $\Delta(6V+12E-3)$ bridges in each, denoted $\{\mathscr{A}_j \mid 1 \leqslant j \leqslant M_1 - 1\}$ (these separate the $\mathscr{B}_j$), and two sets with three bridges in each, denoted $\mathscr{C}_1$ and $\mathscr{C}_2$ (the two extreme sets). Moreover, we suppose that these sets are arranged by $\phi$ in $R_1$ as depicted in Fig. 10.3. We finally label the points $u_i^j$ and $v_i^j$ for $1 \leqslant i \leqslant M_1$ and $j = 1, 2$ as also shown in Fig. 10.3, where again these points in $e_1 \cup e_2$ are not vertices of $G$.

For each $i$, $1 \leqslant i \leqslant M_1$, the subarcs $[u_i^1, v_i^1] \subset e_1$ and $[u_i^2, v_i^2] \subset e_2$ with the sets $\mathscr{B}_i$ satisfy parts 1, 2, and 3 of Condition 10. We will proceed by contradiction, so that part 4 of Condition 10 fails. Thus for each $i$ there exists a bridge $B_i$ incident with either $[u_i^1, v_i^1]$ or $[u_i^2, v_i^2]$ and also incident with some piece of $H$ distinct from $e_1$ and from $e_2$. This $B_i$ contains a path $P_i$ from the appropriate subarc to some piece other than $e_1$ and $e_2$. Of these $M_1 = 2\Delta(V+E-2)+1$ paths, at least $\Delta(V+E-2)+1$ have an endpoint in one particular $e_i$, say $e_2$. Note that the endpoints of these paths

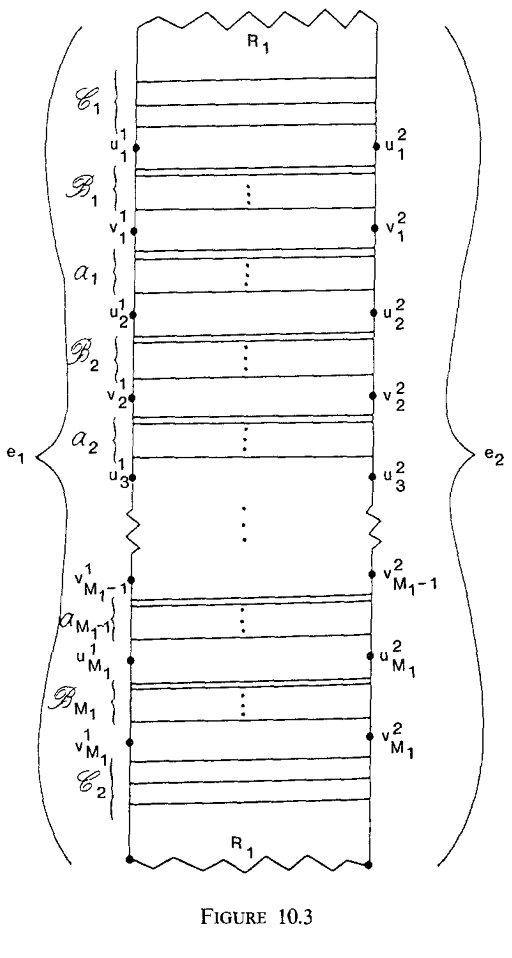


FIGURE 10.3

are separated in $e_2$ by at least $6V+12E-3$ vertices of $G$, since the sets $\mathscr{A}_j$ contained $\Delta(6V+12E-3)$ bridges for each $j$.

We will now show that any two of these paths must be disjoint (except possibly at the endpoint not in $e_2$). Suppose that two of these paths, say $P'$ and $P''$ intersect, and let $v'$ and $v''$ denote the endpoints of $P'$ and $P''$, respectively, which lie in $e_2$. These two paths must lie in the same $(G, H)$-bridge $B$. Thus there exists a path $P$ in $B$ from $v'$ to $v''$. Since $|V_t(B)| \leqslant 6V+12E-4$ by Lemma 8.3, the path $P$ contains at most this many vertices. Let $[v', v'']$ denote the path in $e_2$ with endpoints $v'$ and $v''$, and recall that $[v', v'']$ contains at least $6V+12E-3$ vertices of $G$. Thus the homeomorphic copy of $H$ created by replacing the subarc $[v', v'']$ with $P$ contains strictly fewer vertices of $G$ than does $H$, contradicting part 5 of Condition 7. We thus conclude that the $\Delta(V+E-2)+1$ paths constructed are internally pairwise disjoint.

Recall that each of these paths has an endpoint in a piece of $H$ which is distinct from $e_1$ and $e_2$. At most $\Delta V$ of these paths have endpoints which are topological vertices of $H$. Thus $\Delta(E-2)+1$ of these paths have endpoints which lie in topological edges of $H$. It follows that there exists a third edge $e_3$ such that the endpoints of at least $\Delta+1$ of the paths lie in $e_3$. Finally, we get two paths $P'$ and $P''$ which have distinct endpoints in the same edge $e_3$.

The sets $\{\mathscr{A}_i\}$ and $\{\mathscr{B}_i\}$ are used only to construct the paths $P'$ and $P''$ from $e_2$ to $e_3$. We now use these paths, $\mathscr{C}_1$ and $\mathscr{C}_2$, to reach our desired contradiction.

With respect to the embedding $\phi_0$ of $H$ previously fixed, let $R_2$ be the region of $\phi_0$ other than $R_1$ which contains $e_2$ in its boundary. Likewise label the points $a$ and $b$ as shown in Fig. 10.4 and let $\{B_i\}_{i=1}^{6}$ denote the bridges in $\mathscr{C}_1 \cup \mathscr{C}_2$. The arcs $e_1$ and $e_2$ partition the boundary walk of $R_1$ into two walks $D_1$ and $D_3$. Similarly $e_1$ and $e_2$ partition the boundary walk of $R_2$ into walks $D_2$ and $D_4$. Index these walks as shown in Fig. 10.4.

Recall that each $B_i$, for $1 \leqslant i \leqslant 6$, contains an edge $e_i'$ and an embedding $\phi_i$: $(G-e_i') \to \Sigma$ with $\phi_i | H = \phi_0$. Observe that $e_3$ lies in at least one of the $D_i$'s, since $P'$ must embed in $R_1 \cup e_2 \cup R_2 \cup e_3$. Without loss of generality, suppose that $e_3$ lies in the path $D_2$. Since $(G, H)$ is $\Sigma$-ECTC we have either that $e_3$ lies in no other $D_i$, or perhaps $e_3$ lies in $D_1$ and $D_2$ only, or finally perhaps $e_3$ lies in $D_3$ and $D_2$ only. We consider each of these three cases separately.

*Case* 1. Assume that $e_3$ lies in $D_2$ and no other $D_i$.

Consider $\phi_2$: $(G-e_2') \to \Sigma$. Under this embedding, the paths $P'$ and $P''$ must embed in $R_2$. Hence the bridges $B_1$ and $B_3$ both embed in the region $R_1$. Bridge $B_2$ is $R_1$-admissible, yet this embedding does not extend to an embedding of $G$. Thus there exists a bridge $\bar{B}_2$ such that $\bar{B}_2$ is $R_1$-skew to

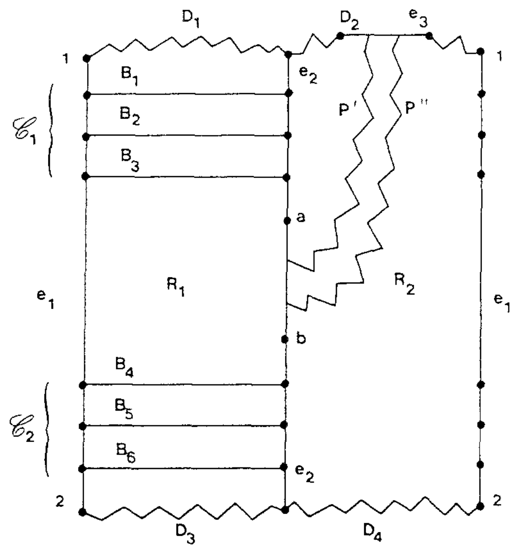


FIGURE 10.4

$B_2$, and $\bar{B}_2$ is $R_1$-parallel to both $B_1$ and to $B_3$. Since $(G, H)$ is $\theta$-less, Proposition 8.4 implies that vofa $(\bar{B}_2) \subset (e_1 \cup e_2)$ and that $\bar{B}_2$ has vertices of attachment in both pieces. Note that the vertices of attachment for $\bar{B}_2$ lie between those for $B_1$ and $B_3$. Next consider $\phi_5$: $(G - e_5') \to \Sigma$. The paths $P'$ and $P''$ again embed in $R_2$, which implies that $B_2$ and $\bar{B}_2$ both embed in $R_1$. This is a contradiction, since they are $R_1$-skew.

*Case* 2. Assume that $e_3$ lies in $D_1$ and $D_2$.

As in the argument of Case 1, $\phi_2$ embeds $B_1$ and $B_3$ in a common region $R$, and there is a $(G, H)$ bridge $\bar{B}_2$ such that $B_2$ and $\bar{B}_2$ are $R$-skew. Since the vertices of attachment of both $B_2$ and $\bar{B}_2$ lie in both $\bar{e}_1$ and in $\bar{e}_2$ and in no other edge, they are both $R_1$-skew and $R_2$-skew. However, $\phi_5$ embeds $B_2$ and $\bar{B}_2$ both in the region non containing $\phi_5(P')$, a contradiction.

*Case* 3. Assume that $e_3$ lies in $D_2$ and $D_3$.

By Proposition 4.5, $e_3$ is in $D_2$ such that $R_1 \cup e_2 \cup R_2 \cup e_3$ is a Möbius strip. Since $P'$ and $P''$ are $R_2$-parallel they must be $R_1$-skew. Thus under any embedding at least one of the two must embed in $R_2$. Hence an argument similar to that in Cases 1 and 2 also applies to establish a contradiction.

In summary, we have shown that under the assumption that there are not subarcs $\tilde{e}_1$, $\tilde{e}_2$ and bridges $\mathscr{B}$ satisfying Condition 10, there exist disjoint paths $P'$ and $P''$ joining $e_2$ to $e_3$. We then considered three exhaustive cases covering the possibilities for $e_3$ in the boundaries of regions $R_1$ and

$R_2$ and in each case reach a contradiction. We conclude that our assumption was wrong, which completes the proof of the proposition. ∎

## 11. A Bound on a Number of Bridges

Throughout this section we will dealing with a $\Sigma$-pair $(G, H)$ which satisfies Condition 7. We will also have subarcs $\tilde{e}_1$ and $\tilde{e}_2$ and a set of $(G, H)$-bridges $\mathscr{B}$ which satisfy Condition 10. The goal of this section is Proposition 11.4, which shows that $|\mathscr{B}| \leqslant 1979$. This final bound will be used in Section 12 to bound the total number of $(G, H)$-bridges and then to prove Theorem 12.2. A key result in this section is Lemma 11.1, in which we show that each bridge in $\mathscr{B}$ is an $I$-bridge. In Lemma 11.2 we forbid a certain configuration of $I$-bridges in $\mathscr{B}$. In Lemma 11.3 we use the minimality portions of Condition 7 (part 5) to forbid a second configuration of $I$-bridges in $\mathscr{B}$. Proposition 11.4 will then follow; its proof shows that an arbitrarily large set of bridges must contain one of these two forbidden configurations.

Lemma 11.1. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $\tilde{e}_1$, $\tilde{e}_2$, and $\mathscr{B}$ satisfy Condition 10. Then each $B \in \mathscr{B}$ is an $I$-bridge.*

*Proof.* By Proposition 8.5 we know that $B$ is either an $H$-bridge, an $X$-bridge, a $Y$-bridge, or an $I$-bridge. We proceed by way of contradiction.

*Case* 1. Suppose that $B$ is an $H$-bridge or an $X$-bridge.

Label the vertices of $B$ and the endpoints of $\tilde{e}_1$, $\tilde{e}_2$, $e_1$, $e_2$ as shown in Fig. 11.1. If $B$ is an $X$-bridge then the central vertex will be labeled both 9 and 10.

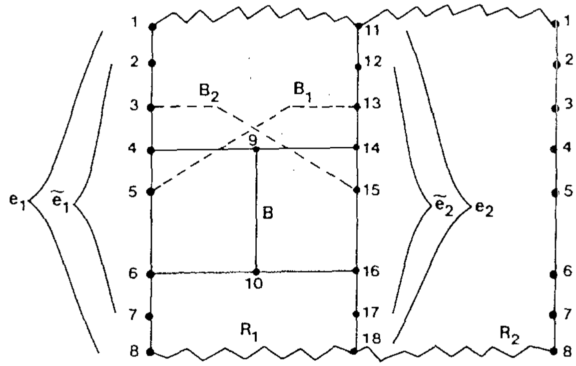


Figure 11.1

By Proposition 8.4 no bridge $B$ has vofa$(B) \subset e$ for any $e \in E_t(H)$. Also if $\phi$ is an embedding such that $\phi | H$ has regions $R_1$ and $R_2$ with $e_1 \cup R_1 \cup e_2 \cup R_2$ a cylinder, and if $B_1$, $B_2$ are two bridges with vofa$(B_1 \cup B_2) \subset (e_1 \cup e_2)$, then $B_1$ is $R_1$-skew to $B_2$ if and only if $B_1$ is $R_2$-skew to $B_2$. In such situations, we will refer to the bridges as being skew or parallel without designating the region involved.

Let $\phi$: $(G-(4,9)) \to \Sigma$, and let $R$ be the region of $\phi | H$ containing $\phi((9,10))$. $B$ is $R$-admissible and $\phi$ does not extend to an embedding including edge $(4,9)$. There exists a bridge $B_1$ which is skew to $B$ but parallel to $B-(4,9)$. The bridge $B_1$ must contain a vertex in the interval $(4,6)$. Since $(4,6) \subset e_1 = (2,7)$, $B_1$ must also contain a vertex, designated 13, in the interval $(11,14)$.

In a similar manner, by deleting edge $(9,14)$ we get a bridge $B_2$ with a vofa$(B_2)$ intersecting both intervals $(14,16)$ and $(1,4)$, at vertices designated 15 and 3, respectively. Observe that if $B_1 = B_2$ we violate that $(G, H)$ is $\theta$-less.

Consider $\phi$: $(G-(6,10)) \to \Sigma$. At least two of the bridges $\{B_1, B_2, B-(6,10)\}$ must embed in the same region. These three bridges are pairwise skew, a contradiction. Thus our assumption was wrong, and $B$ is not an $H$-bridge or an $X$-bridge.

*Case* 2. Suppose that $B$ is a $Y$-bridge.

Label the vertices of $B$ and the endpoints of $\tilde{e}_1$, $e_1$, $\tilde{e}_2$, $e_2$ as shown in Fig. 11.2. As before, for $B_1$ and $B_2$ bridges with vertices of attachment contained in $e_1 \cup e_2$ and embeddings with $e_1 \cup R_1 \cup e_2 \cup R_2$ a cylinder, we will refer to $B_1$ being skew (or parallel) to $B_2$ without mention of the region.

Consider the embedding $\phi$: $(G-(16,10)) \to \Sigma$. $B-(16,10)$ embeds in some region, but we cannot extend this embedding to admit all of $B$. Thus

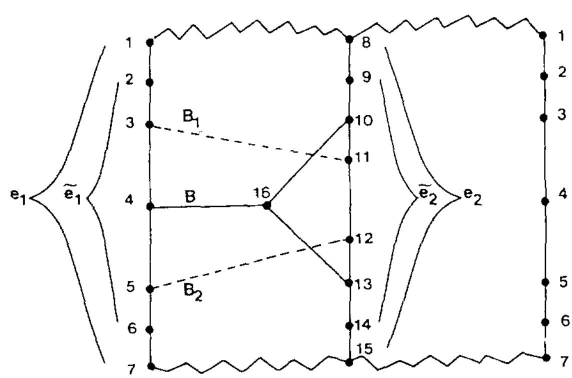


FIGURE 11.2

there exists a bridge $B_1$ which is skew to $B$ but parallel to $B-(16, 10)$. $B_1$ must have a vertex of attachment in the interval (10, 13). Since $(10, 13) \subset (9, 14) = \tilde{e}_2$, $\text{vofa}(B_1) \subset (e_1 \cup e_2)$. $B_1$ must also have a vertex of attachment in the interval (1, 4). Designate these two vertices of attachment by 11 and 3, respectively (see Fig. 11.2).

In a similar manner by deleting (16, 13) construct a bridge $B_2$ with $\{12, 5\} \subset \text{vofa}(B_2)$. We note that $B_1 \neq B_2$ as $B_1$ is parallel to $B-(10, 16)$ but skew to $B$, so $B_1$ is skew to $B-(13, 16)$ but $B_2$ is not.

Define $\bar{H} = (H-(10, 13)) \cup \{(10, 16), \quad (16, 13)\}$. $\bar{H}$ violates the minimality part of Condition 7. In particular if $11 \neq 12$ we violate the first inequality, and if $11 = 12$ we violate the second inequality.

By Proposition 8.5 $B$ must be either an $H$-bridge, an $X$-bridge, a $Y$-bridge, or an $I$-bridge. By eliminating the first three possibilities we conclude $B$ is an $I$-bridge. ∎

LEMMA 11.2. *Let* $(G, H)$ *be a* $\Sigma$*-pair satisfying Condition* 7. *Let* $\tilde{e}_1$, $\tilde{e}_2$, *and* $\mathscr{B}$ *satisfy Condition* 10. *Suppose that* $\{B_i\}_{i=1}^4 \subset \mathscr{B}$ *are as shown in Fig.* 11.3. *Then there does not exist a path P contained in* $G-H$ *as shown in that figure.*

*Proof.* By way of contradiction, suppose that there exists a configuration as in Fig. 11.3. Let $\phi$: $(G-B_2) \to \Sigma$, and let $R_1$ be the region of $\phi \mid H$ which contains $\phi(P)$. Since $e_1 \cup R_1 \cup e_2 \cup R_2$ is a cylinder and each $B_i$ is skew to $P$, $\phi(B_i) \subset R_2$ for $i=1$, 3, and 4. Label points $i$ as shown in Fig. 11.4, $i=1, ..., 6$, where $B_2$ is the edge (3, 4).

This embedding $\phi$ does not extend to an embedding of all $G$, and hence we cannot embed $B_2$ in $R_2$. This implies the existence of a bridge $\bar{B}_2$ with vertices of attachment, without loss of generality, $7 \in (3, 5]$ and $8 \in (4, 2]$—here $(a, b]$ denotes the path $[a, b]$ minus the vertex $a$.

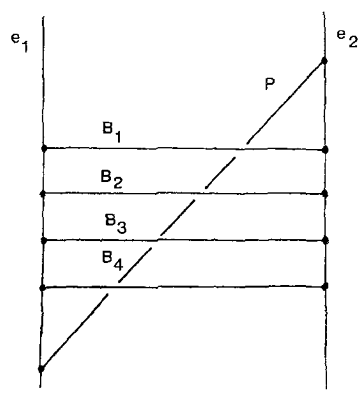


FIGURE 11.3

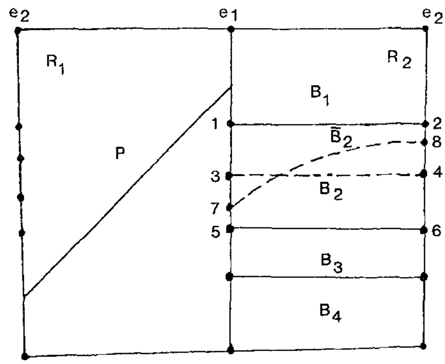


FIGURE 11.4

Next consider $\psi$: $(G-B_4)\to\Sigma$. Again, let $R_1$ be the region of $\psi|H$ containing $\psi(P)$. We have $\psi(B_2)\subset R_2$ and $\psi(\bar{B}_2)\subset R_2$, a contradiction. ∎

We now prove another "forbidden configuration" lemma.

LEMMA 11.3. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $\tilde{e}_1$, $\tilde{e}_2$, and $\mathcal{B}$ satisfy Condition 10. Then $\mathcal{B}$ does not contain a configuration as shown in Fig. 11.5.*

*Proof.* By way of contradiction, assume that $\mathcal{B}$ contains the configuration of Fig. 11.5. Label the points $t_1$, $t_2$, $b_1$, $b_2$ as in Fig. 11.5, partition $\mathcal{B}$ into sets $A_i$, $i=1, ..., 31$, as indicated, and label the vertices 1, ..., 8.

First observe that for each $A_i$, $i=1, ..., 30$, if $\bar{1}$, $\bar{2}$, $\bar{3}$, and $\bar{4}$ are the vertices in $A_i$ corresponding to the vertices labelled 1, 2, 3, and 4, respectively, in $A_1$, then $(\bar{1}, \bar{2})$ and $(\bar{3}, \bar{4})$ are topological edges of $G$. To see this, notice that the graph $H' = H\cup\{(\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (5, 8), (6, 7)\} - \{(\bar{1}, \bar{2}), (\bar{3}, \bar{4}), (5, 6), (7, 8)\}$ is homeomorphic to $H$. By part 5 of Condition 7, $H'$ contains at least as many topological vertices of $G$ as $H$ contains, and hence there is no topological vertex in $(\bar{1}, \bar{2})$. Since $\mathcal{B}$ is a set of topological edges of $G$, $\{(\bar{1}, \bar{2}), (\bar{3}, \bar{4}), (5, 6), (7, 8)\}$ is also a set of topological edges of $G$.

The technique of the previous paragraph will be referred to as *rerouting by paths* $(\bar{1}, \bar{4})$ *and* $(\bar{2}, \bar{3})$. The use of $(5, 8)$ and $(6, 7)$ is understood in a rerouting. Also for terminological convenience, if $a_i$ is a point in the open path $(t_i, b_i)\subset\tilde{e}_1$, $i=1, 2$, we say that set $\{a_1, a_2\}$ *separates* $\{\tilde{e}_1, \tilde{e}_2\}$ if for all $(G, H)$-bridges $B$ with $\text{vofa}(B)\subset(\tilde{e}_1\cup\tilde{e}_2)$ either $\text{vofa}(B)\subset([a_1, t_1]\cup[a_2, t_2])$ or $\text{vofa}(B)\subset([a_1, b_1]\cup[a_2, b_2])$.

Let $\mathcal{A}$ denote the set of $A_i$'s, $i=1, ..., 30$, such that the points, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$ in $A_i$ corresponding to those labelled 1, 2, 3, 4, respectively, in $A_1$ have both $\{\bar{1}, \bar{3}\}$ and $\{\bar{2}, \bar{4}\}$ separating $\{\tilde{e}_1, \tilde{e}_2\}$. We do not include $A_{31}$ in $\mathcal{A}$.

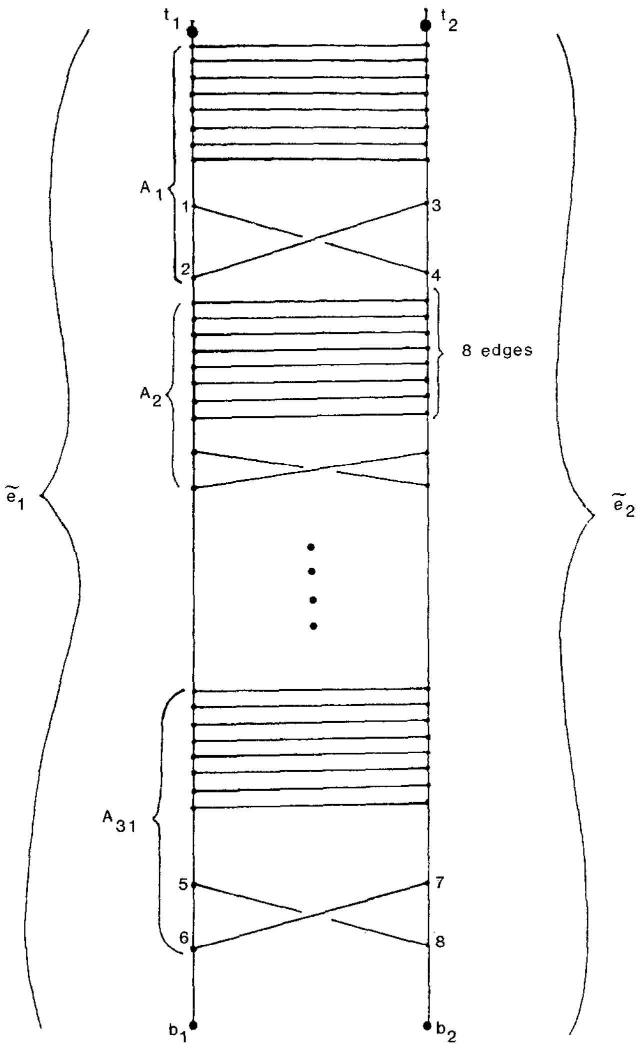


FIGURE 11.5

*Case* 1. Assume that $|\mathscr{A}| \leqslant 2$.

Then for at least 28 of the $A_i$'s each distinct from $A_{31}$, either $\{\bar{1}, \bar{3}\}$ or $\{\bar{2}, \bar{4}\}$ fails to separate $\{\bar{e}_1, \bar{e}_2\}$.

If an $A_i \in \mathscr{A}$, then there exists a bridge $\bar{B}_i$ which causes either $\{\bar{1}, \bar{3}\}$ or $\{\bar{2}, \bar{4}\}$ not to separate $\{\tilde{e}_1, \tilde{e}_2\}$. If this $\bar{B}_i$ is an $I$-bridge, then we contradict part 5 of Condition 7 (since (5, 6) and (7, 8) are established above to be in $E_t(G)$) by rerouting by $\bar{B}_i$ and either $(\bar{1}, \bar{4})$ or $(\bar{2}, \bar{3})$. Since $\bar{B}_i$ has a vertex of attachment in $\tilde{e}_1 \cup \tilde{e}_2$, part 4 of Condition 10 with Proposition 8.5 imply that $\bar{B}_i$ is either an $H$-bridge, an $X$-bridge, or a $Y$-bridge. We construct such a $\bar{B}_i$ for each $A_i$.

We note that $i > j$ implies that $\bar{B}_i$ is disjoint from $\bar{B}_j$. This follows using the set of eight parallel edges in $\bar{B}_j$ and Lemma 11.2. Thus we have a set of $H$-, $X$-, or $Y$-bridges $\{\bar{B}_i\}$ with $\mathrm{vofa}(\bar{B}_i) \subset \tilde{e}_1 \cup \tilde{e}_2$. By Lemma 10.2 at least one of these bridges, say $\bar{B}_1$, has the property that for any $e \in B_1$ and any $\phi$: $(G-e) \to \Sigma$, $e_1$ and $e_2$ bound $R_1$ and $R_2$ with their union, a cylinder. Since $\bar{B}_1$ is not an $I$-bridge, we contradict Lemma 11.1.

*Case* 2. Assume that $|\mathscr{A}| \geq 3$.

We have a set of bridges $\mathscr{B}' \subset \mathscr{B}$ as depicted in Fig. 11.6. Observe that the set of four vertices labelled $\{2, 9, 7, 14\}$ in this figure form a cut set of $G$. This follows from Condition 10 and the "separates" condition in the definition of $\mathscr{A}$.

Let $C_1$ be the maximal subgraph of $G$ which is separated from $H-(e_1 \cup e_2)$ by $\{2, 9, 7, 14\}$; equivalently, let $C_1$ be the topological closure of the component containing $e_4$ of the topological complement of $\{2, 9, 7, 14\}$ in $G$. Also let $C_2$ be the graph $G - C_1$. Observe that $C_1 \cap C_2 = \{2, 9, 7, 14\}$.

Consider $\phi_1$: $(G-e_1) \to \Sigma$. Let $C$ be a cylinder in $\Sigma$ such that $\phi_1(C_1) \subset C$ and $\phi_1(C_2 - e_1) \cap C = \{2, 9, 7, 14\}$. Such a cylinder exists since $\{2, 9, 7, 14\}$ is a cutset of $G$, $e_2$ is skew to $e_3$, and $e_6$ is skew to $e_7$. Next consider $\phi_4$: $(G-e_4) \to \Sigma$. Again there is a cylinder $C$ in $\Sigma$ such that $\phi_4(C_1 - e_4) \subset C$ and $\phi_4(C_2) \cap C = \{2, 9, 7, 14\}$ for the same reasons as above. We now glue $\phi_4 | C_2$ to $\phi_1 | C_1$ along $\{2, 9, 7, 14\}$ to construct an embedding of $G$ into $\Sigma$, a contradiction.

Thus in either Case 1 or Case 2 we reach a contradiction, and the lemma is shown. ▮

We are now able to prove the main proposition of this section.

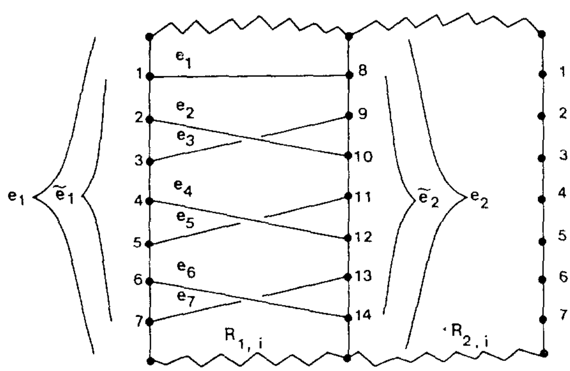


FIGURE 11.6

PROPOSITION 11.4. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition* 7. *Let $\tilde{e}_1$, $\tilde{e}_2$, and $\mathscr{B}$ satisfy Condition* 10. *Then* $|\mathscr{B}| \leqslant 1979$.

*Proof.* By way of contradiction, suppose that $|\mathscr{B}| > 1979$. By Lemma 11.1 each bridge is an $I$-bridge. We will show that there exists a configuration as shown in Fig. 11.5, thereby contradicting Lemma 11.3.

Let $B \in \mathscr{B}$ and $\phi$: $(G - B) \to \Sigma$. At least 990 of the remaining 1979 bridges in $\mathscr{B}$ embed in the same region of $\phi | H$, giving the situation depicted in Fig. 11.7. Group these bridges into a set $A_0$ of four bridges and sets $A_i$, $i = 1, 2, \ldots, 58$, $|A_i| = 17$, as shown in Fig. 11.7. Moreover, within each $A_i$

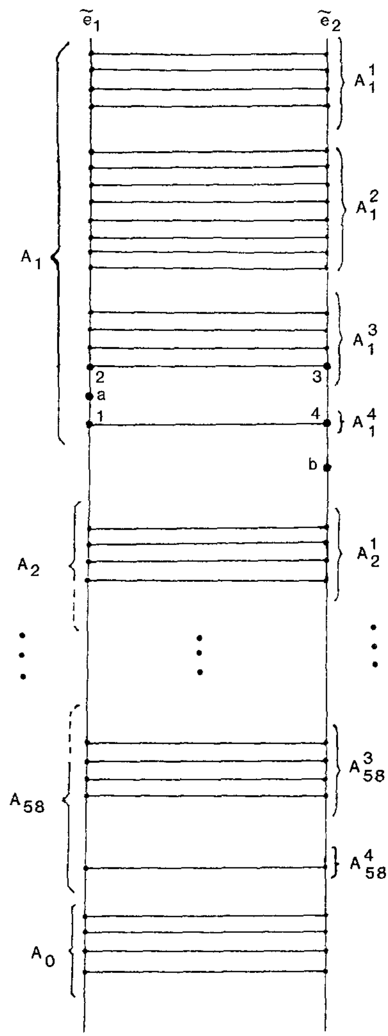


FIGURE 11.7

label subsets $A^j_i$, $j = 1, 2, 3, 4$, and finally label the points 1, 2, 3, and 4, all as in Fig. 11.7.

We examine the set $A_1$. Let $\phi$: $(G-(1,4)) \to \Sigma$. This embedding does not extend to an embedding of $G$, so we cannot embed the edge $(1, 4)$. Since $(1, 4)$ does not embed in a neighborhood of the path $(1, 2, 3, 4)$, there exists a bridge blocking the addition of this edge; call this bridge $\bar{A}^4_1$. Note $\bar{A}^4_1$ contains vertices of attachment in (without loss of generality) the half open path $(1, 2]$ and the open path $(4, 5)$. Moreover, by Lemma 11.2 the set $A^3_1$ guarantees $\bar{A}^4_1$ is disjoint from any bridge in $A^2_1$ and the set $A^1_2$ guarantees that $\bar{A}^4_1$ is disjoint from any bridge in $A^2_2$.

In a similar manner construct bridges $\bar{A}^4_i$, $i = 1, \ldots, 58$. By Lemma 10.2 at least 31 of these $\{\bar{A}^4_i\}$ have the property that for any $e \in \bar{A}^4_i$ and for any $\phi$: $(G-e) \to \Sigma$, $\phi \mid H$ has $e_1$, $e_2$ bounding $R_1$, $R_2$ with $e_1 \cup R_1 \cup e_2 \cup R_2$, a cylinder. Rename if necessary so that $\{\bar{A}^4_i\}^{31}_{i=1}$ all have this property. By Lemma 11.1, $\bar{A}^4_i$ must be an $I$-bridge. The arcs $\tilde{e}_1$ and $\tilde{e}_2$ together with the set of bridges $\{A^4_i\}^{31}_{i=1} \cup \{\bar{A}^4_i\}^{31}_{i=1} \cup \{A^2_i\}^{31}_{i=1}$ satisfy Condition 10. This contradicts Lemma 11.3. ∎

## 12. Proof of Two Bounding Theorems

The purpose of this section is the now easy proof of Theorem 12.2. This theorem in essence summarizes the results in Sections 4 through 10. Recall that this theorem was the principal ingredient in the proof of Theorem 1.1, the main result of this paper. We first prove the following theorem.

THEOREM 12.1. *Let $(G, H)$ be a $\Sigma$-pair satisfying Condition 7. Let $\mathscr{B}$ be the set of all $(G, H)$-bridges. Then $|\mathscr{B}|$ is bounded by a function of $|V_t(H)|$.*

*Proof.* By Proposition 8.4 each bridge has vertices of attachment in at least two pieces of $H$. In Proposition 7.2 we bound the number of bridges with vertices of attachment in three or more pieces of $H$. Hence we need only bound the number of bridges with vertices of attachment in exactly two pieces of $H$. If one of these pieces is a topological vertex of $H$ the bound is supplied by Theorem 9.4, which bounds the maximum degree of $G$. Thus each of the two pieces must be topological edges. By Lemma 1.3 it suffices to bound these bridges for a fixed pair of topological edges. Combining the bound of Proposition 11.4 with the inequality of Proposition 10.4 gives this bound in terms of $\Delta$ and $\Phi^{\Sigma}_H$, which are appropriately bounded by Theorem 9.4 and Lemma 10.3, respectively. ∎

THEOREM 12.2. *Let $(G, H)$ be a 2-connected $\Sigma$-pair. Then there exists a 2-connected $K \subset G$ such that $K$ does not embed in $\Sigma$, $K$ contains a subgraph homeomorphic to $H$, and $|V_t(K)|$ is bounded by a function of $|V_t(H)|$.*

*Proof.* By Theorem 7.3 there exists a refinement $(K, H')$ of $(G, H)$ which satisfies Condition 7 and which has $|V_t(H')|$ bounded by a function of $|V_t(H)|$. We note that $K$ does not embed in $\Sigma$, and by the definition of the refinement, $H'$ contains a subgraph homeomorphic to $H$. Also, by part 1 of Condition 7, $K$ is 2-connected. It suffices to show that $|V_t(K)|$ is bounded in term of $|V_t(H')|$, as it will then be bounded in terms of $|V_t(H)|$. But Proposition 8.3 bounds the size of any $(K, H')$-bridge and Theorem 12.1 bounds the number of such bridges. As both bounds are in terms of $|V_t(H')|$, the bound on $|V_t(K)|$ follows immediately. ∎

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