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A numerical study for off-centered stagnation flow towards a rotating disc

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Abstract In this investigation, a semi-numerical method based on Bernstein polynomials for solving off-centered stagnation flow towards a rotating disc is introduced. This method expands the desired solutions in terms of a set of Bernstein polynomials over a closed interval and then makes use of the tau method to determine the expansion coefficients to construct approximate solutions. This method can satisfy boundary conditions at infinity. The properties of Bernstein polynomials are presented and are utilized to reduce the solution of governing nonlinear equations and their associated boundary conditions to the solution of algebraic equations. Graphical results are presented to investigate the influence of the rotation ratio α on the radial velocity, azimuthal velocity and the induced velocities. A comparative study with the previous results of viscous fluid flow in the literature is made.

1. Introduction

Many of the mathematical modeling, which appears in many areas of scientific fields such as fluid dynamics, plasma physics and solid state physics, can be modeled by nonlinear ordinary or partial differential equations. Such problems often require advanced numerical methods or powerful analytical methods to solve the governing equations. These known methods are for example, Runge-Kutta method [1], spectral methods [2,3], the δ-expansion method [4], the Adomian decomposition method [5], the variational iteration method [6,7], the homotopy perturbation method [8,9] and the homotopy analysis method [10,11].

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The flow due to rotating disks is one of the classical problems of fluid mechanics which has many practical applications in manufacturing processes in industry, such as rotating machinery, crystal growth processes, cooling of silicon wafers and chemical vapor deposition processes etc. The pioneering study of fluid flow due to an infinite rotating disk has been carried out by von Karman [12]. He first formulated the problem and then reduced the governing partial differential equations to the ordinary differential equations by defining appropriate transformations. The stagnation flow problem was studied by Homann [13]. Hannah [14] introduced and discussed the combination of axisymmetric stagnation flow on a rotating disc. Tifford et al. [15] extended Hannah's work and combined it with torque. In the von Karman's rotating disc problem and Homann's stagnation flow problem an axisymmetric stagnation flow aligned with the axis of the rotating disc is considered. Since the flow is axisymmetric, the governing equations are greatly simplified.

Here, we have considered the off-centered stagnation flow toward a rotating disc. This problem was first studied by Wang [16] in 2008 where they implemented a similarity solution. Dinarvand [17] used the homotopy analysis method to solve off-centered stagnation flow toward a rotating disc. Nourbakhsh et al. [18] obtained an approximate analytical solution via homotopy analysis method with two auxiliary parameters. Rashidi et al. [19] introduced a combination of the differential transformation method and the Padé approximants for solving off-centered stagnation flow toward a rotating disc.

In this study, we are going to introduce and implement a new algorithm based on Bernstein polynomials [20] to find the approximate solution of the off-centered stagnation flow toward a rotating disc. Bernstein polynomials have many useful properties, such as, the positivity, the continuity, and unity partition of the basis set over the interval $[a, b]$. The Bernstein polynomials vanish except the first polynomial at $x=a$, which is equal to 1 and the last polynomial at $x=b$, which is also equal to 1 over the interval $[a, b]$. This provides greater flexibility in imposing boundary conditions at the end points of the interval.

The Bernstein polynomials are widely used for numerical solutions of differential, integral, and integro-differential equations which we point to some of them studied in recent years briefly. In Ref. [21], an algorithm for solving KdV equation using modified Bernstein polynomials is presented. These polynomials are applied to numerical solution of some classes of integral equations In Ref. [22]. Bhatti et al. [23] used the Bernstein polynomial basis to solve differential equations. Bhattacharya and Mandal [24] obtained numerical solutions of Volterra integral equations using the Bernstein polynomials. Chakrabarti and Martha [25] proposed an effective approach using the Bernstein polynomials to obtain approximate solutions of Fredholm integral equations of the second kind. Singh et al. [26] and Yousefi and Behroozifar [27] have proposed some operational matrices in different ways for solving differential equations. Authors of [28] applied the operational matrices of Bernstein polynomials for solving the parabolic equation subject to specification of the mass. Doha et al. [29,30] employed two attractive algorithms based on the derivatives and Integrals of Bernstein polynomials for solving high even-order differential equations. In Ref. [31] a numerical solution of the nonlinear age-structured population models by using the operational matrices of Bernstein polynomials is presented. Maleknejad et al. [32] proposed a Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations. Rostamy and Karimi [33] solved fractional heat- and wave-like equations using Bernstein polynomials. Some numerical integration methods based on Bernstein polynomials are introduced in Ref. [34]. In Ref. [35], a combination of homotopy analysis and tau Bernestein polynomial method is applied to solve singularly perturbed boundary value problems.

This paper is divided to the following sections. In Section 2, the flow analysis and mathematical formulation are presented. Section 3 describes the basic formulations of Bernstein polynomials required for our subsequent development. In Section 4, the approximate solution of the governing equations using Bernstein polynomials is presented. Section 5 contains the results and discussion. Finally, conclusions are made in Section 6.

2. Flow analysis and mathematical formulation

The description of the physical problem closely follows that of Wang [16]. Let u, v and w be the velocity components along x, y and z directions, respectively. A stagnation flow along the z axis impinging on a rotating disc whose axis is distance b from that axis is depicted in Figure 1.

The appropriate conditions at infinity on the potential stagnation flow are

$$u = ax, \quad v = ay, \quad w = -2az, \tag{1}$$

where a is the strength of the stagnation flow. The boundary conditions on the disc are

$$u = -\Omega y, \quad v = \Omega(x-b), \quad w = 0, \tag{2}$$

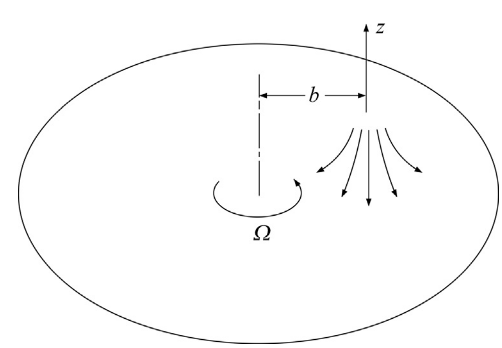

Figure 1 Configuration of the off-centered stagnation flow on a rotating disc [16,19].

where Ω is the angular velocity. Wang in Ref. [16] introduced the following similarity transforms

$$u = axf'(\eta) - \Omega yg(\eta) + b\Omega k(\eta), \tag{3}$$

$$v = ayf'(\eta) + \Omega xg(\eta) + b\Omega h(\eta), \tag{4}$$

$$w = -2\sqrt{a\nu}f(\eta), \tag{5}$$

where ν is the kinematic viscosity and $\eta = \sqrt{a/\nu z}$. Consequently by using Eqs. (3)–(5), the constant property Navier-Stokes equations reduced to the following system of ordinary differential equations:

$$f''' - (f')^2 + \alpha^2 g^2 + 2ff'' + 1 = 0, \tag{6}$$

$$g'' - 2gf' + 2fg' = 0, \tag{7}$$

$$k'' - kf' + \alpha hg + 2fk' = 0, \tag{8}$$

$$h'' - \alpha kg - hf' + 2fh' = 0, \tag{9}$$

where $\alpha = \Omega/a$ is rotation ratio parameter. The boundary conditions are

$$f(0) = f'(0) = 0, \;\; f'(\infty) = 1, \tag{10}$$

$$g(0) = 1, \;\; g(\infty) = 0, \tag{11}$$

$$k(0) = 0, \;\; k(\infty) = 0, \tag{12}$$

$$h(0) = 1, \;\; h(\infty) = 0. \tag{13}$$

The pressure p can be expressed by

$$p = p_0 - \frac{\rho a^2 \left(x^2 + y^2\right)}{2} - \rho\left(\frac{w^2}{2} - \nu w_z\right), \tag{14}$$

where p_0 is the pressure at the origin and ρ is the fluid density. The shear stress on the disc is given by

$$\tau_x = \rho\nu\frac{\partial u}{\partial z}|_{z=0} = \rho a\sqrt{\nu a}[xf''(0) - \alpha yg'(0) + b\alpha k'(0)], \tag{15}$$

$$\tau_y = \rho\nu\frac{\partial v}{\partial z}|_{z=0} = \rho a\sqrt{\nu a}[yf''(0) + \alpha xg'(0) + b\alpha h'(0)]. \tag{16}$$

The shear center, where shear stress is zero, can be obtained by setting Eqs. (15) and (16) to zero and solving for x and y. Furthermore, the torque experienced by the disc of radius $\tilde{R}$ is given by

$$M = \frac{\pi}{2}\alpha g'(0)\tilde{R}^4\rho a\sqrt{\nu a}. \tag{17}$$

3. Bernstein polynomials and their properties

3.1. The definition of the Bernstein polynomials

The Bernstein polynomials of degree n are defined on the interval $[0, \;\; R]$ as Ref. [23]:

$$B_{i,n}(x) = \binom{n}{i}\frac{x^i(R-x)^{n-i}}{R^n}, \quad 0 \le i \le n, \tag{18}$$

where the binomial coefficients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}. \tag{19}$$

These Bernstein polynomials form a basis on $[0, \;\; R]$. There are $n+1$, nth-degree polynomials. For convenience, we set $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$. Moreover, the recursive definition for the Bernstein polynomials over the interval $[0, \;\; R]$ is as follows:

$$B_{i,n}(x) = \frac{R-x}{R}B_{i,n-1}(x) + \frac{x}{R}B_{i-1,n-1}(x). \tag{20}$$

The binomial expansion of the right-hand side of the equality $R^n = (x + (R-x))^n$ shows that the sum of all Bernstein polynomials of degree n is the constant 1, that is, $\sum_{i=0}^{n} B_{i,n}(x) = 1$. Also, for all $i = 0, 1, \cdots, n$ and all x in $[0, \;\; R]$, we have $B_{i,n}(x) \ge 0$. The derivatives of the nth degree Bernstein polynomials are polynomials of degree $n-1$, which can be formulated as Ref. [23]:

$$\frac{d}{dx}B_{i,n}(x) = \frac{n}{R}\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right]. \tag{21}$$

In Ref. [29], an explicit expression for the derivatives of Bernstein polynomials on $[0, \;\; 1]$ of any degree and for any order in terms of the Bernstein polynomials is given as follows:

$$\begin{aligned}\frac{d^k}{dx^k}B_{i,n}(x) &= \frac{n!}{(n-k)!}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}B_{i-j,n-k}(x)\\ &= \frac{n!}{(n-k)!}\sum_{j=\max\{0,i+k-n\}}^{\min\{i,k\}}(-1)^{j+k}\binom{k}{j}B_{i-j,n-k}(x).\end{aligned} \tag{22}$$

It can easily be shown tshat for Bernstein polynomials on $[0, \;\; R]$.

$$\begin{aligned}\frac{d^k}{dx^k}B_{i,n}(x) &= \frac{1}{R^k}\frac{n!}{(n-k)!}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}B_{i-j,n-k}(x)\\ &= \frac{1}{R^k}\frac{n!}{(n-k)!}\sum_{j=\max\{0,i+k-n\}}^{\min\{i,k\}}(-1)^{j+k}\binom{k}{j}B_{i-j,n-k}(x).\end{aligned} \tag{23}$$

Corollary 1. For $k=0,1,\cdots,n$, we have

$$\frac{d^k}{dx^k}B_{i,n}(0)=\frac{(-1)^{i+k}}{R^k}\frac{n!}{(n-k)!}\binom{k}{i}\gamma_{i,k}, \tag{24}$$

where

$$\gamma_{i,k}=\begin{cases}1, & i\le k,\\ 0, & i>k.\end{cases} \tag{25}$$

Proof. For a fixed value of k, we consider the following cases:

1) If $i=0,1,\cdots,k$, then $\min\{i,k\}=i$. Furthermore $B_{0,n}(0)=1$ and $B_{i,n}(0)=0$, $i=1,2,\cdots,n$. So, from Eq. (23), we can get $j\le i$ and $\frac{d^k}{dx^k}B_{i,n}(0)=\frac{(-1)^{i+k}}{R^k}\frac{n!}{(n-k)!}\binom{k}{i}$.
2) If $i=k+1,k+2,\cdots,n$, then $\min\{i,k\}=k$. So, from Eq. (23), we can get $j<i$ and $\frac{d^k}{dx^k}B_{i,n}(0)=0$.

Corollary 2. For $k=0,1,\cdots,n$, we have

$$\frac{d^k}{dx^k}B_{i,n}(R)=\frac{(-1)^{n-i}}{R^k}\frac{n!}{(n-k)!}\binom{k}{n-i}\gamma_{n-i,k}. \tag{26}$$

Proof. The proof is similar to that of Corollary 1.

The integrals of the products of the Bernstein basis functions can be found by using

$$\int_0^1 x^i(1-x)^r dx=\frac{1}{(r+i+1)\binom{r+i}{i}},\quad i,r\in\mathbb{N}\cup\{0\}, \tag{27}$$

as follows:

$$I_{k,i}^{n,j}=\int_0^R B_{k,n}(x)B_{i,j}(x)dx=\frac{\binom{n}{k}\binom{j}{i}}{(j+n+1)\binom{j+n}{k+i}}R, \tag{28}$$

and

$$I_{k,i,p}^{n,j,q}=\int_0^R B_{k,n}(x)B_{i,j}(x)B_{p,q}(x)dx=\frac{\binom{n}{k}\binom{j}{i}\binom{q}{p}}{(j+n+q+1)\binom{j+n+q}{k+i+p}}R. \tag{29}$$

All Bernstein polynomials of the same order have the same definite integral over the interval $[0,\ R]$, namely

$$\int_0^R B_{i,n}(x)dx=\frac{R}{n+1},\quad i=0,1,\cdots,n. \tag{30}$$

3.2. Function approximation

Suppose that $H=L^2[0,R]$ where $R\in\mathbb{R}$, let $\{B_{0,n}(x), B_{1,n}(x),\cdots,B_{n,n}(x)\}\subset H$ be the set of Bernstein polynomials of nth degree, and suppose that

$$Y=\text{span}\{B_{0,n}(x),B_{1,n}(x),\cdots,B_{n,n}(x)\}. \tag{31}$$

Theorem 1. For every given x in a Hilbert space H and every given closed subspace Z of H there is a unique best approximation to x from Z.

Proof. See Ref. [36].

Since $H=L^2[0,R]$ is Hilbert space and Y is finite-dimensional subspace, so Y is a closed subspace of H, therefore Y is a complete subspace of H. So, if f be an arbitrary element in H, by Theorem 3.3, f has unique best approximation from Y such as f^*, that is

$$\exists f^*\in Y;\quad \forall g\in Y\|f-f^*\|_2\le\|f-g\|_2, \tag{32}$$

Where $\|f\|_2=\sqrt{\langle f,f\rangle}$. Since $f^*\in Y$, there exist unique coefficients $f_0,f_1,\cdots,f_n$ such that

$$f(x)\simeq f^*(x)=\sum_{i=0}^{n}f_iB_{i,n}(x), \tag{33}$$

where the coefficients $f_0,f_1,\cdots,f_n$ can be obtained by solving the following linear system

$$\sum_{i=0}^{n}f_i\langle B_{i,n}(x),B_{j,n}(x)\rangle=\langle f(x),B_{j,n}(x)\rangle,\quad j=0,1,\cdots,n. \tag{34}$$

Let f, g be two arbitrary elements in Y, that is

$$f(x)=\sum_{i_1=0}^{n}f_{i_1}B_{i_1,n}(x), \tag{35}$$

$$g(x)=\sum_{i_2=0}^{n}g_{i_2}B_{i_2,n}(x). \tag{36}$$

For any $k,l\in\mathbb{N}\cup\{0\}$ and $i_1,i_2=0,1,\cdots,n$, we define

$$\psi_{i_1,i_2}^{(k,l)}(f,g)(x)=\left(\beta_{k,l}\sum_{j_1=0}^{k}\sum_{j_2=0}^{l}(-1)^{j_1+j_2}\binom{k}{j_1}\binom{l}{j_2}B_{i_1-j_1,n-k}(x)B_{i_2-j_2,n-l}(x)\right)f_{i_1}g_{i_2} \tag{37}$$

where

$$\beta_{k,l}=\left(\frac{-1}{R}\right)^{k+l}\frac{(n!)^2}{(n-k)!(n-l)!},\quad k,l\in\mathbb{N}\cup\{0\}. \tag{38}$$

Lemma 1. Let $f,g\in Y$ and $k,l\in\mathbb{N}\cup\{0\}$, then

$$\textbf{(I)}\ f^{(k)}(x)g^{(l)}(x)=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\psi_{i_1,i_2}^{(k,l)}(f,g)(x). \tag{39}$$

(II) $f^{(k)}(x)=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\psi_{i_1,i_2}^{(k,0)}(f,1)(x).$ (40)

(III) $$\tilde{\psi}_{i_1,i_2}^{k,l,s}(f,g)=\int_0^R\psi_{i_1,i_2}^{(k,l)}(f,g)(x)B_{s,n}(x)dx$$
$$=\left(\beta_{k,l}\sum_{j_1=0}^{k}\sum_{j_2=0}^{l}(-1)^{j_1+j_2}\binom{k}{j_1}\times\binom{l}{j_2}I_{i_1-j_1,i_2-j_2,s}^{n-k,n-l,n}\right)f_{i_1}g_{i_2},\qquad 0\le s\le n. \tag{41}$$

Proof (I):. For each $k,l\in\mathbb{N}\cup\{0\}$ we have

$$f^{(k)}(x)=\sum_{i_1=0}^{n}f_{i_1}\frac{d^k}{dx^k}B_{i_1,n}(x), \tag{42}$$

$$g^{(l)}(x)=\sum_{i_2=0}^{n}g_{i_2}\frac{d^l}{dx^l}B_{i_2,n}(x). \tag{43}$$

By using Eq. (23) the following equations will be obtained:

$$f^{(k)}(x)=\frac{1}{R^k}\frac{n!}{(n-k)!}\times\sum_{i_1=0}^{n}\sum_{j_1=0}^{k}(-1)^{j_1+k}\binom{k}{j_1}B_{i_1-j_1,n-k}(x)f_{i_1}, \tag{44}$$

$$g^{(l)}(x)=\frac{1}{R^l}\frac{n!}{(n-l)!}\times\sum_{i_2=0}^{n}\sum_{j_2=0}^{l}(-1)^{j_2+l}\binom{l}{j_2}B_{i_2-j_2,n-l}(x)g_{i_2}, \tag{45}$$

According to the equations Eqs. (44), (45) and (37), the equation achieved as below

$$f^{(k)}(x)g^{(l)}(x)=\frac{1}{R^{k+l}}\frac{(n!)^2}{(n-k)!(n-l)!}\times\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\sum_{j_1=0}^{k}\sum_{j_2=0}^{l}(-1)^{j_1+j_2+k+l}\times\binom{k}{j_1}\binom{l}{j_2}B_{i_1-j_1,n-k}(x)B_{i_2-j_2,n-l}(x)f_{i_1}g_{i_2}$$
$$=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\left(\left(\beta_{k,l}\sum_{j_1=0}^{k}\sum_{j_2=0}^{l}\left((-1)^{j_1+j_2}\binom{k}{j_1}\times\binom{l}{j_2}B_{i_1-j_1,n-k}(x)B_{i_2-j_2,n-l}(x)\right)f_{i_1}g_{i_2}\right.\right.$$
$$=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\psi_{i_1,i_2}^{(k,l)}(f,g)(x).$$

Proof (II). By substituting $g(x)=1=\sum_{i_2=0}^{n}B_{i_2,n}(x)$ and $l=0$ in I we can conclude II.

Proof (III). By using the Eqs. (29) and (37), the result can be shown easily.

4. Solution of the problem

The tau approach is a modification of the Galerkin method that is applicable to problems with no-periodic boundary conditions [37,38]. In this section we apply Bernstein-tau method (BTM) for the computation of off-centered stagnation flow towards a rotating disc based on the Bernstein polynomials. The physical region $[0,\infty)$ is transformed into the region $[0,R]$ using the domain truncation technique. In practice, R should be chosen sufficiently large so that the numerical solution closely approximates the terminal boundary conditions Eqs. (10)–(13). If R is not large enough, the numerical solution will not only depend on the physical parameter α, but also on R. For a given α, we first solve for the functions f and g from Eqs. (6), (7), (10) and (11), then solve for the functions k and h from Eqs. (8), (9), (12) and (13).

For an arbitrary natural number n, we suppose that the approximate solutions $f(\eta)$ and $g(\eta)$ of Eqs. (6) and (7) are as follows:

$$f(\eta)\simeq\sum_{i=0}^{n}f_iB_{i,n}(\eta), \tag{46}$$

$$g(\eta)\simeq\sum_{i=0}^{n}g_iB_{i,n}(\eta), \tag{47}$$

and the residual functions associated to the differential Eqs. (6) and (7) are

$$\text{RES}\,f(\eta)=f'''(\eta)-(f'(\eta))^2+\alpha^2g^2(\eta)+2f(\eta)f''(\eta)+1, \tag{48}$$

$$\text{RES}\,g(\eta)=g''(\eta)-2g(\eta)f'(\eta)+2f(\eta)g'(\eta). \tag{49}$$

By substituting Eqs. (46) and (47) in the above residual functions and using Lemma 1-I and II, we obtain

$$\text{RES}\,f(\eta)=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}[\psi_{i_1,i_2}^{(3,0)}(f,1)(\eta)-\psi_{i_1,i_2}^{(1,1)}(f,f)(\eta)+\alpha^2\psi_{i_1,i_2}^{(0,0)}(g,g)(\eta)+2\psi_{i_1,i_2}^{(0,2)}(f,f)(\eta)]+1, \tag{50}$$

Table 1 Illustrating the variation of $f''(0)$, $g'(0)$, $k'(0)$ and $h'(0)$ for various α.

α	0	0.5	1	2	3	5	7	10
$f''(0)$								
Hannah [14]	1.312	1.379	1.575	2.295	———	———	———	———
Wang [16]	1.31194	1.3787	1.5739	2.2951	3.3657	6.2602	9.9165	16.5229
DTM-Padé [19]	1.310358	1.380676	1.573657	2.284367	3.362927	6.263483	10.09378	16.527112
BTM	1.311958	1.378749	1.573930	2.295639	3.365647	6.259869	9.916513	16.522816
NM	1.311958	1.378749	1.573930	2.295639	3.365647	6.259869	9.916513	16.522816
$g'(0)$								
Hannah [14]	−1.075	−1.084	−1.110	−1.197	———	———	———	———
Wang [16]	−1.07467	−1.0839	−1.1100	−1.1968	−1.3055	−1.5320	−1.7451	−2.0330
DTM-Padé [19]	−1.073704	−1.076563	−1.109411	−1.211464	−1.308748	−1.536611	−1.758276	−2.036709
BTM	−1.074697	−1.083934	−1.110020	−1.196841	−1.305526	−1.531983	−1.745106	−2.033028
NM	−1.074697	−1.083934	−1.110020	−1.196841	−1.305526	−1.531983	−1.745106	−2.033028
$k'(0)$								
Wang [16]	0	0.1380	0.2700	0.5040	0.6972	0.9992	1.2353	1.5227
DTM-Padé [19]	0	0.139298	0.269021	0.499471	0.693201	0.996602	1.244999	1.525583
BTM	0	0.137977	0.270021	0.503886	0.697144	0.999261	1.235242	1.522695
NM	0	0.137977	0.270021	0.503886	0.697144	0.999261	1.235242	1.522695
$h'(0)$								
Wang [16]	−0.93872	−0.9495	−0.9787	−1.0787	−1.2003	−1.4466	−1.6728	−1.9735
DTM-Padé [19]	−0.937147	−0.949198	−0.968431	−1.082066	−1.210548	−1.441987	−1.661361	−1.968989
BTM	−0.938803	−0.949555	−0.979767	−1.078848	−1.200249	−1.446531	−1.672846	−1.973452
NM	−0.938803	−0.949555	−0.979767	−1.078848	−1.200249	−1.446531	−1.672846	−1.973452

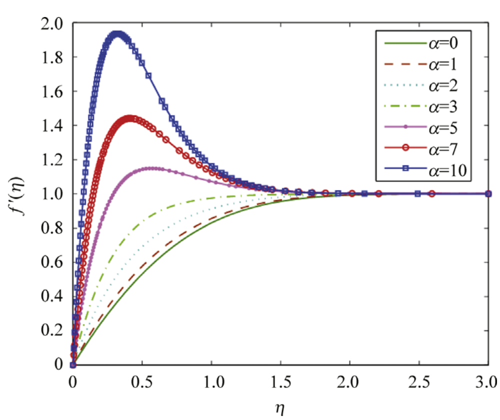

Figure 2 Influence of the rotation ratio α on the radial velocity $f'(\eta)$.

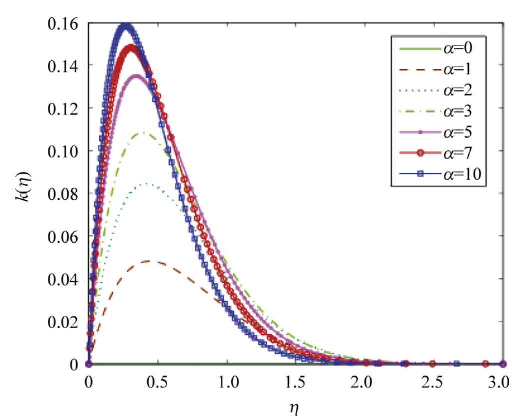

Figure 4 Influence of the rotation ratio α on the induced velocity $k(\eta)$.

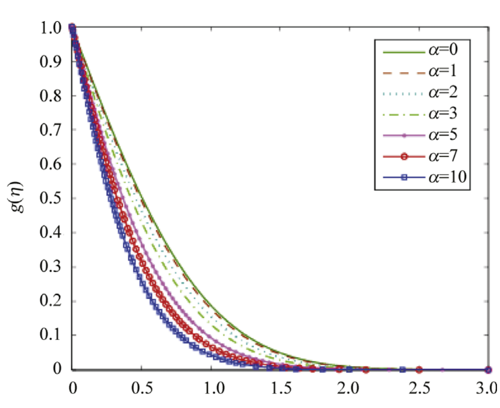

Figure 3 Influence of the rotation ratio α on the azimuthal velocity $g(\eta)$.

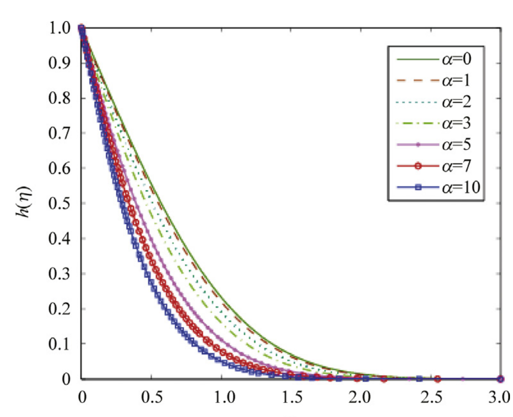

Figure 5 Influence of the rotation ratio α on the induced velocity $h(\eta)$.

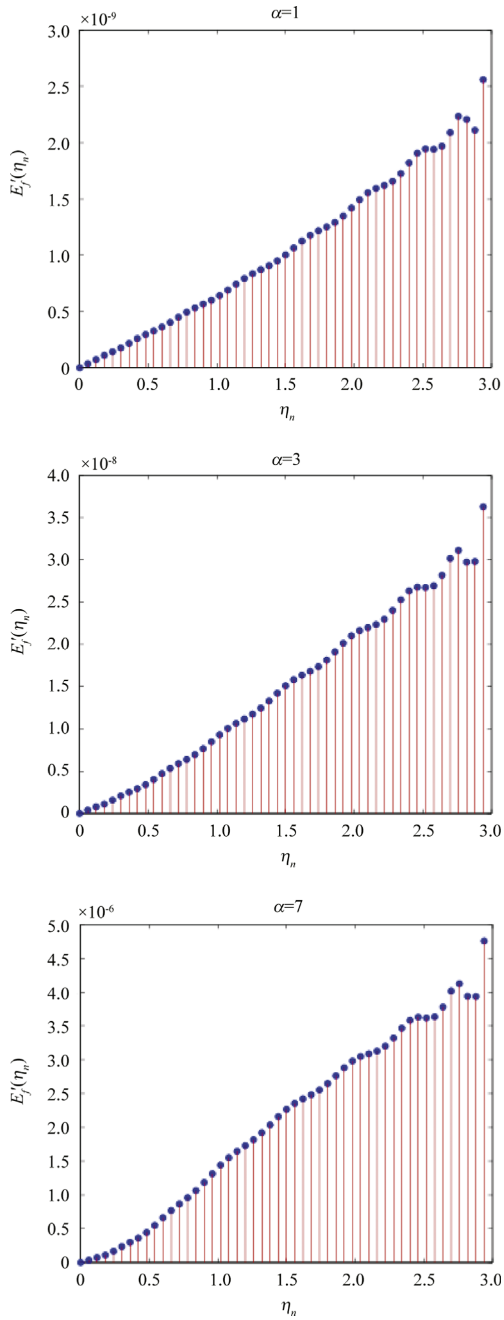

Figure 6 Plots of error values $E_f^{'}(\eta_n)$ with $M=50$.

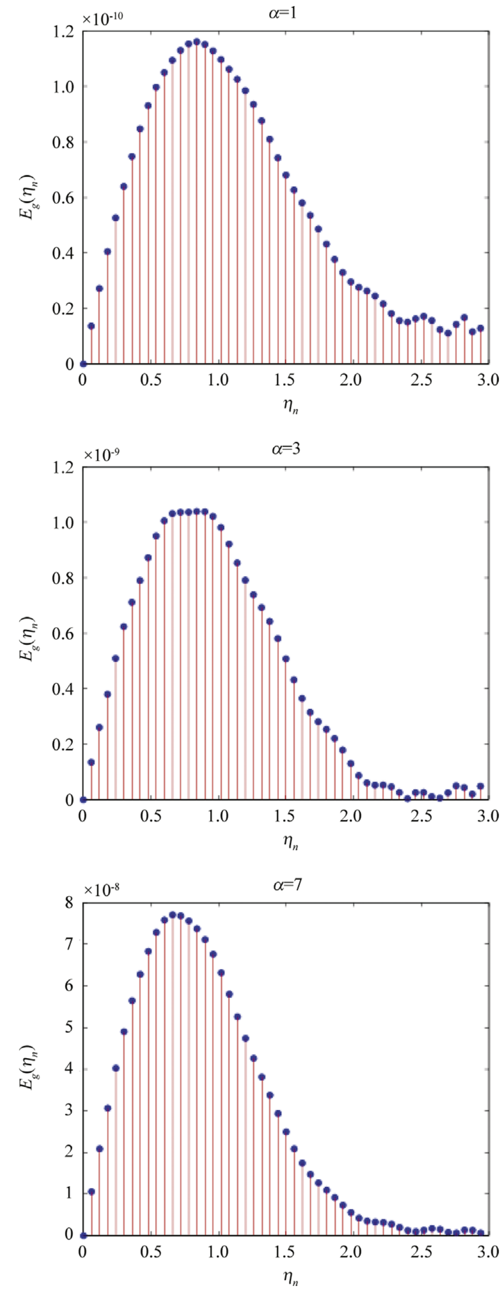

Figure 7 Plots of error values $E_g(\eta_n)$ with $M=50$.

$$\text{RES } g(\eta)=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}[\psi_{i_1,i_2}^{(2,0)}(g,1)(\eta)-2\psi_{i_1,i_2}^{(0,1)}(g,f)(\eta)+2\psi_{i_1,i_2}^{(0,1)}(f,g)(\eta)]. \quad (51)$$

Via the tau method, we get the inner product of the above equations with $B_{s,n}(\eta)$:

$$\langle \text{RES } f(\eta),B_{s,n}(\eta)\rangle=0,\quad s=0,1,\cdots,n-3, \quad (52)$$

$$\langle \text{RES } g(\eta),B_{s,n}(\eta)\rangle=0,\quad s=0,1,\cdots,n-2, \quad (53)$$

where $\langle f,g\rangle=\int_0^R f(\eta)g(\eta)d\eta$. Now by using Lemma 1-III and Eq. (30), we rewrite Eqs. (52) and (53) as follows:

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\left[\tilde{\psi}_{i_1,i_2}^{(3,0,s)}(f,1)-\tilde{\psi}_{i_1,i_2}^{(1,1,s)}(f,f)+\alpha^2\tilde{\psi}_{i_1,i_2}^{(0,0,s)}(g,g)+2\tilde{\psi}_{i_1,i_2}^{(0,2,s)}(f,f)\right]+\frac{R}{n+1}=0,\ s=0,1,\cdots,n-3, \quad (54)$$

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}[\tilde{\psi}_{i_1,i_2}^{(2,0,s)}(g,1)-2\tilde{\psi}_{i_1,i_2}^{(0,1,s)}(g,f)+2\tilde{\psi}_{i_1,i_2}^{(0,1,s)}(f,g)]=0,\ s=0,1,\cdots,n-2. \quad (55)$$

Also by imposing the boundary conditions Eqs. (10) and (11), we have

$$\sum_{i=0}^{n} f_i B_{i,n}(0) = 0, \quad \sum_{i=0}^{n} f_i B'_{i,n}(0) = 0, \quad \sum_{i=0}^{n} f_i B'_{i,n}(R) = 1, \tag{56}$$

$$\sum_{i=0}^{n} g_i B_{i,n}(0) = 1, \quad \sum_{i=0}^{n} g_i B_{i,n}(R) = 0. \tag{57}$$

By using Corollaries 1 and 2, we rewrite the above equations as follows:

$$f_0 = 0, \; -\frac{n}{R} f_0 + \frac{n}{R} f_1 = 0, \; -\frac{n}{R} f_{n-1} + \frac{n}{R} f_n = 1, \tag{58}$$

$$g_0 = 1, \quad g_n = 0. \tag{59}$$

From Eqs. (54), (55), (58) and (59), a nonlinear system of $2n+2$ equations and $2n+2$ unknown coefficients is resulted. Solving this system, we can obtain unknown coefficients f_i and $g_i, i = 0, 1, \cdots, n$ and therefore $f(\eta)$ and $g(\eta)$ are identified. Now, we suppose that the approximate solutions $k(\eta)$ and $h(\eta)$ of Eqs. (8) and (9) are as follows

$$k(\eta) \simeq \sum_{i=0}^{n} k_i B_{i,n}(\eta), \tag{60}$$

$$h(\eta) \simeq \sum_{i=0}^{n} h_i B_{i,n}(\eta). \tag{61}$$

By a similar manner, we can obtain the linear system for finding unknown coefficients k_i and $h_i, i = 0, 1, \cdots, n$ as follows:

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} [\tilde{\psi}^{(2,0,s)}_{i_1,i_2}(k,1) - \tilde{\psi}^{(0,1,s)}_{i_1,i_2}(k,f) + \alpha \tilde{\psi}^{(0,0,s)}_{i_1,i_2}(h,g) + 2\tilde{\psi}^{(0,1,s)}_{i_1,i_2}(f,k)] = 0, \; s = 0, 1, \ldots, n-2, \tag{62}$$

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} [\tilde{\psi}^{(2,0,s)}_{i_1,i_2}(h,1) - \alpha \tilde{\psi}^{(0,0,s)}_{i_1,i_2}(k,g) - \tilde{\psi}^{(0,1,s)}_{i_1,i_2}(h,f) + 2\tilde{\psi}^{(0,1,s)}_{i_1,i_2}(f,h)] = 0, \; s = 0, 1, \ldots, n-2, \tag{63}$$

$$k_0 = 0, \quad k_n = 0, \tag{64}$$

$$h_0 = 1, \quad h_n = 0. \tag{65}$$

Solving this system, we can obtain unknown coefficients k_i and $h_i, i = 0, 1, \cdots, n$ and therefore $k(\eta)$ and $h(\eta)$ are identified.

5. Results and discussion

In this section, we present the approximate solutions of the governing Eqs. (6)–(13) using the Bernstein-tau method (BTM). We remark that, all the BTM results presented in this work are obtained by using $n = 20$ and $R = 3$ to simulate $\eta = \infty$ for this problem. Table 1 displays the numerical results for $f''(0)$, $g'(0)$, $k'(0)$ and $h'(0)$ for several values of the rotation ratio α and gives a comparison between the presented method, numerical method (NM) (fourth-order Runge-Kutta method and shooting method), DTM-Padé technique [19] and the results reported by Hannah [14] and Wang [16].

To study the behavior of the radial velocity $f'(\eta)$, azimuthal velocity $g(\eta)$ and the induced velocities $k(\eta)$ and $h(\eta)$ due to the non-alignment of the stagnation flow and the rotating disc, curves are drawn for various values of the rotation ratio α. Figure 2 depicts the variations of the radial component of velocity $f'(\eta)$ for different values of

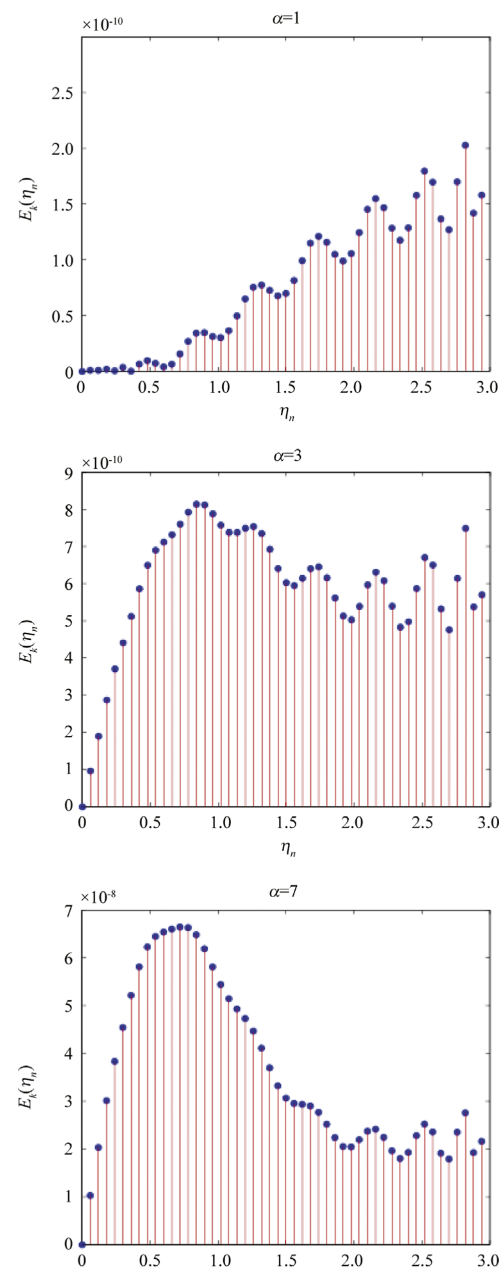

Figure 8 Plots of error values $E_k(\eta_n)$ with $M = 50$.

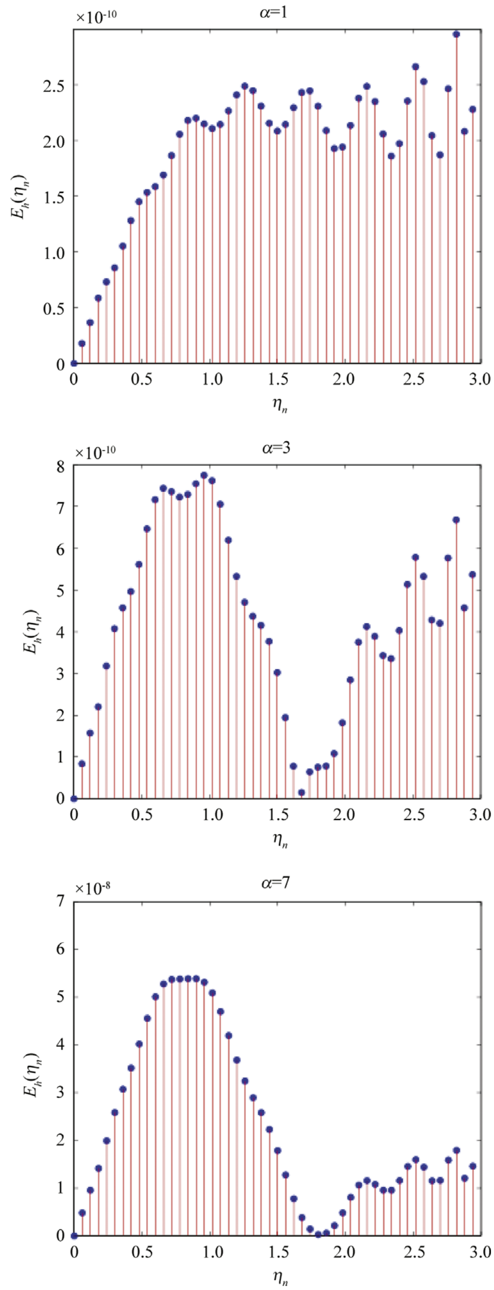

Figure 9 Plots of error values $E_h(\eta_n)$ with $M = 50$.

the rotation ratio α. It is observed that with increased α, the velocity for any fixed value of η increases. For Homann's stagnation flow ($\alpha = 0$), the radial velocity increases monotonically to the finite velocity caused by the pressure gradient at infinity. In Figure 3 we plot the azimuthal component of velocity $g(\eta)$. It is seen that the azimuthal velocity decreases with increasing rotation ratio α. Figure 4 shows the effect of the rotation ratio α on the induced velocity function $k(\eta)$ obtained using presented method. It is observed from Figure 4 that the induced velocity $k(\eta)$ increases with increasing α. Figure 5 shows the induced velocity function $h(\eta)$ for several values of the rotation ratio α and we see that the induced velocity $h(\eta)$ decreases with increasing α.

We define the following error values to evaluate the accuracy of the presented method as:

$$E_f(\eta_n) = |f(\eta_n) - f_{RK4}(\eta_n)|, \quad n = 0, 1, \ldots, M-1,$$

and

$$E_f'(\eta_n) = |f'(\eta_n) - f'_{RK4}(\eta_n)|, \quad n = 0, 1, \ldots, M-1,$$

where $\eta_n = (nR/M)$, $n = 0, 1, \ldots, M-1$ and $M \in \mathbb{N}$. Figures 6–9 give a comparison between the present BTM results and the numerical method (NM) for $\alpha = 1, 3, 7$, respectively, with $M = 50$. We can clearly observe from Figures 6–9 that the solutions obtained by the proposed method are in good agreement with the RK4-based solutions. Hence, this leads the confidence of the presented results.

6. Conclusions

An efficient and accurate semi-numerical scheme based on the Bernstein-tau method is developed for solving the system of nonlinear ordinary differential equations derived from similarity transformation for off-centered stagnation flow towards a rotating disc. This algorithm reduces the solution of a system of ordinary differential equations to the solution of a system of algebraic equations. Excellent agreement is seen between the results obtained by Hannah [14], Wang [16], Dinarvand [17], Rashidi et al. [19] and approximate solutions computed semi-numerically by choosing a few terms for the truncated series. The present success of the proposed method for off-centered stagnation flow towards a rotating disc verifies that the method is a useful tool for the solution of nonlinear problems in fluid mechanics.

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