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Regularity, Decay, and Best Constants for Dispersive Equations[1]

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We show that the Strichartz $L^2_w(L^2)$-estimates for solutions to the (pseudo-) differential equations

are equivalent. A necessary and sufficient condition for decay and regularity for solutions to the equation

$$\varphi(\sqrt{-\Delta_x}\,)\,u = i\partial_t u$$

is given.

Key Words: oscillatory integrals; dispersive equations; weighted and mixed norm inequalities; global smoothing and decay.

1. PURPOSE

1.1. In this paper we generalise previous work (cf., e.g., Ben-Artzi and Klainerman [3], Ben-Artzi and Nemirovsky [4], Kato and Yajima [8], [19, 20]) on decay and regularity for oscillatory integrals.

Let u_1 and u_2 be functions on $\mathbf{R}^{n+1}$. The tempered distribution f belongs to the Sobolev space $H^s(\mathbf{R}^n)$ if and only if the function $\xi \mapsto (1+|\xi|^2)^s |\hat{f}(\xi)|^2$ is integrable on $\mathbf{R}^n$. Here $\hat{f}$ is the Fourier transform of f. Consider the following two statements:

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Statement 1. Let $-\Delta_x u_1 = i\partial_t u_1$, $u_1(x,0) = f(x)$, $x \in \mathbf{R}^n$, $n \geqslant 3$. Then there exists a number C independent of f such that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u_1(x,t)|^2 \frac{dt\,dx}{1+|x|^2} \leqslant C\,\|f\|^2_{H^{-1/2}(\mathbf{R}^n)}.$$

Statement 2. Let $\sqrt{-\Delta_x + 1}\, u_2 = i\partial_t u_2$, $u_2(x,0) = f(x)$, $x \in \mathbf{R}^n$, $n \geqslant 3$. Then there exists a number C independent of f such that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u_2(x,t)|^2 \frac{dt\,dx}{1+|x|^2} \leqslant C\,\|f\|^2_{L^2(\mathbf{R}^n)}.$$

The estimates in these statements are examples of Strichartz estimates with weights. Strichartz estimates have been treated in many papers during recent years. See, e.g., Ben-Artzi and Devinatz [1, 2], Ben-Artzi and Klainerman [3], Bourgain [5], Georgiev *et al.* [6], Ginibre and Velo [7], Kato and Yajima [8], Keel and Tao [9], Klainerman and Machedon [11], and Montgomery-Smith [12] and the references cited in these papers. Suitable introductions to the subject may be found in Stein [15, Chapter VIII, Sects. 5.16, 5.18, 5.19] and Strauss [17].

THEOREM A [3, Corollary 2, p. 28; 8, (1.5), p. 482; 20, Theorem 2.2(a), p. 385]. *Statement* 1 *is true.*

1.2. According to Ben-Artzi and Nemirovsky [4, Theorem 3A, p. 35] Statement 2 also is true. The main purpose of this paper is to show that Statements 1 and 2 are equivalent. See Example 4.1 with $a = 2$ and Example 4.2 with $a_1 a_2 = 1$ and $a_1 = 2$.

THEOREM 1. *Statements* 1 *and* 2 *are equivalent.*

1.3. Let B^n denote the open unit ball of $\mathbf{R}^n$. Statement 1 is sharp with respect to both decay and regularity. This is the content of Theorems B and C below.

THEOREM B [20, Theorem 2.2(b), p. 385]. *Assume that* $n \geqslant 3$ *and that there is a number* C *independent of* f *such that*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u_1(x,t)|^2 \frac{dt\,dx}{(1+|x|)^b} \leqslant C\,\|\hat{f}\|^2_{L^2(B^n)}, \qquad \operatorname{supp} \hat{f} \subseteq B^n.$$

Then $b \geqslant 2$.

THEOREM C (Sjögren and Sjölin [14, Theorem 4, p. 5]). *Assume that there is a number C independent of f such that*

$$\|u_1\|_{L^2(B^{n+1})} \leqslant C \, \|f\|_{H^s(\mathbf{R}^n)}.$$

Then $s \geqslant -1/2$.

From the point of view of gain of regularity in the L^2-sense the (pseudo-) differential equation

$$\sqrt{-\Delta_x + 1}\; u = i\partial_t u$$

in Statement 2 is equivalent to the classical wave equation simply because $\sqrt{|\xi|^2+1}$ behaves like $|\xi|$ as ξ goes to infinity. It is well known that there is no gain of regularity in the L^2-sense for solutions to the wave equation. See also [21]. Hence Statement 2 is sharp with respect to regularity:

THEOREM D. *Assume that there is a number C independent of f such that*

$$\|u_2\|_{L^2(B^{n+1})} \leqslant C \, \|f\|_{H^s(\mathbf{R}^n)}.$$

Then $s \geqslant 0$.

Another purpose of this paper is to show that Statement 2 is sharp with respect to decay. See Example 4.2 with $a_1 a_2 = 1$ and $a_1 = 2$.

THEOREM 2. *Assume that* $n \geqslant 3$ *and that there is a number C independent of f such that*

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}} |u_2(x,t)|^2 \frac{dt\,dx}{(1+|x|)^b} \leqslant C \, \|\hat{f}\|^2_{L^2(B^n)}, \qquad \operatorname{supp} \hat{f} \subseteq B^n.$$

Then $b \geqslant 2$.

2. NOTATION

2.1. *Oscillatory integrals.* For x and ξ in $\mathbf{R}^n$ we let $x\xi = x_1\xi_1 + \cdots + x_n\xi_n$. If f is in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ and if m is any essentially bounded and measurable function on $\mathbf{R}^n \times \mathbf{R}_+$ we define

$$(S^{\varphi}_m f)[x](t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} m(x, |\xi|)\, e^{i(x\xi - t\varphi(|\xi|))} \hat{f}(\xi)\, d\xi. \tag{1}$$

Here $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(\xi)=\int_{\mathbf{R}^n} e^{-ix\xi} f(x)\,dx. \tag{2}$$

If $m=1$ we write S^{φ} instead of S_1^{φ}. We will also need the modified operator $\widetilde{S_m^{\varphi,\eta}}$ defined by

$$(\widetilde{S_m^{\varphi,\eta}} f)[x](t)=g(|x|)^{1/2}\int_{\mathbf{R}^n} m(x,|\xi|)\,e^{i(x\xi-t\varphi(|\xi|))}\eta(|\xi|)\,f(\xi)\,d\xi, \tag{3}$$

where we assume g and η to be positive measurable functions. Again, if $m=1$ we write $\widetilde{S^{\varphi,\eta}}$ instead of $\widetilde{S_1^{\varphi,\eta}}$.

The conditions on φ will be made precise in Theorem 4.1.

2.2. *Bessel functions.* For real numbers $\lambda>-1/2$ we define the Bessel function of order λ by

$$J_\lambda(\rho)=\frac{\rho^\lambda}{2^\lambda\Gamma(\lambda+1/2)\,\Gamma(1/2)}\int_{-1}^{1} e^{ir\rho}(1-r^2)^{\lambda-1/2}\,dr. \tag{4}$$

Here Γ is the gamma function.

Bessel functions of order $n/2+k-1$ are important when describing the symmetry properties of the Fourier transform. See Theorem 5.2. We set

$$\nu(k)=\frac{n}{2}+k-1.$$

2.3. *Auxiliary notation.* By B^n we denote the open unit ball in $\mathbf{R}^n$. (B^1 will be denoted by B.) We will use auxiliary functions χ and ψ such that $\chi\in\mathscr{C}_0^\infty(\mathbf{R})$ is even,

$$\chi(\mathbf{R}\setminus 2B)=0,\qquad \chi(\mathbf{R})\subseteq[0,1]\qquad\text{and}\qquad \chi(B)=1$$

and $\psi=1-\chi$.

Unless otherwise explicitly stated all functions f are supposed to belong to $\mathscr{S}(\mathbf{R}^n)$.

3. SOME EXAMPLES AND PREVIOUS RESULTS

3.1. The expression

$$e^{i(x\xi-t\varphi(|\xi|))}\hat{f}(\xi)$$

solves for each fixed ξ the equation

$$\varphi(|\xi|)\, u = i\partial_t u.$$

Hence the expression

$$u(x, t) = (S^{\varphi} f)[x](t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i(x\xi - t\varphi(|\xi|))} \hat{f}(\xi)\, d\xi \tag{5}$$

solves the pseudo-differential equation

$$\varphi(\sqrt{-\Delta_x}\,)\, u = i\partial_t u$$

with initial data $u(x, 0) = f(x)$. If $\varphi(\rho) = \rho^2$ then u given by (5) will be a solution to the free time-dependent Schrödinger equation as in Statement 1. If $\varphi(\rho) = \sqrt{\rho^2 + 1}$ then u given by (5) will instead be a solution to the free time-dependent relativistic Schrödinger equation as in Statement 2.

If we use the modified operator $\widetilde{S_m^{\varphi, \eta}}$ then the estimates in Statements 1 and 2 can both be expressed as

$$\|\widetilde{S^{\varphi, \eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant C\, \|f\|_{L^2(\mathbf{R}^n)}$$

with the same function $g(x) = (1 + |x|^2)^{-1}$ but with different functions φ and η and with $n \geqslant 3$. Here C is a number independent of f. In the case of $\varphi(\rho) = \rho^2$ we have $\eta(\rho) = (1 + \rho^2)^{1/4}$ (or equivalently $\eta(\rho) = (1 + \rho)^{1/2}$) whereas in the case of $\varphi(\rho) = \sqrt{\rho^2 + 1}$ we have $\eta(\rho) = 1$. What we among other things aim to prove is that the estimates of both Statements 1 and 2 are equivalent to the estimate

$$\sup \left\{ (1 + \rho^2)^{1/2} \int_0^\infty J_{\nu(k)}(r\rho)^2 \frac{r\, dr}{1 + r^2} : \rho > 0, k \in \mathbf{N} \right\} < \infty \tag{6}$$

which we consider for $n \geqslant 3$. ($\mathbf{N}$ denotes the set of nonnegative integers.) Estimate (6) can be verified (see [20, pp. 390–392]) in a straightforward manner.

The estimates in Statement 1 and Theorem A both concern the case $n \geqslant 3$. For the case $n = 2$ the condition $b \geqslant 2$ has to be replaced by the condition $b > 2$. This is due to the *local* asymptotics of the Bessel function J_0 and may be expressed as follows: *the function*

$$\rho \mapsto \int_0^\infty J_0(r\rho) \frac{r\, dr}{1 + r^b}, \qquad 0 \leqslant \rho \leqslant 1 \tag{7}$$

is bounded if and only if $b > 2$. See [20, Sect. 4.6, p. 392].

3.2. Let us now consider the case $\varphi(|\xi|) = |\xi|^a$, $a \neq 2$, $a > 1$, $\eta(\rho) = (1+\rho^2)^{-s/2}$, $g(r) = (1+r^b)^{-1}$, and $n \geqslant 2$. As in the case $a = 2$ we can classify the decay and regularity. We have the following theorems.

THEOREM A′ [19, Theorem 14.7(a) and 14.8(b)]. *Assume that $n \geqslant a$, $s \geqslant (1-a)/2$, and either $b > a = n$ or $b \geqslant a \neq n$. Then there is a number C independent of f such that*

$$\|\widetilde{S_m^{\varphi,\eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant C\, \|f\|_{L^2(\mathbf{R}^n)}.$$

We aim to prove that the estimate in Theorem A′ is equivalent to

$$\sup\left\{\rho^{2-a}(1+\rho^2)^{-s}\int_0^\infty J_{\nu(k)}(r\rho)^2\,\frac{r\,dr}{1+r^b} : \rho > 0, k \in \mathbf{N}\right\} < \infty \tag{8}$$

under the stated assumptions. The estimates (6) and (7) are implied by (8), an estimate which likewise can be verified (see [19, pp. 225–228]) in a straightforward manner.

THEOREM B′ [19, Theorem 14.7(b) and 14.8(b)]. *Assume that there is a number C independent of f such that*

$$\|\widetilde{S^{\varphi,\eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant C\, \|\hat{f}\|_{L^2(B^n)}, \qquad \operatorname{supp} \hat{f} \subseteq B^n.$$

Then either $b > a = n$ or $b \geqslant a \neq n$.

THEOREM C′. *Assume that there is a number C independent of f such that*

$$\|\widetilde{S^{\varphi,\eta}} f\|_{L^2(B^{n+1})} \leqslant C\, \|f\|_{L^2(\mathbf{R}^n)}.$$

Then $s \geqslant (1-a)/2$.

A proof of Theorem C′ together with related material [21] will appear elsewhere.

4. MAIN RESULT AND EXAMPLES

THEOREM 4.1. *Let $n \geqslant 2$. Assume that φ is injective on $\mathbf{R}_+$ with range $\Omega \subseteq \mathbf{R}$ and that φ' is well defined on $\mathbf{R}_+$.*

(a) *Assume that*

$$\alpha = \sup\left\{\rho\eta(\rho)^2\,\varphi'(\rho)^{-1}\int_0^\infty J_{\nu(k)}(r\rho)^2\,g(r)\,r\,dr : \rho > 0, k \in \mathbf{N}\right\} < \infty. \tag{9}$$

Then

$$\|\widetilde{S_m^{\varphi,\eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant (2\pi)^{(n+1)/2}\, \alpha^{1/2}\, \|m\|_{L^\infty(\mathbf{R}^n,\,\mathbf{R}_+)}\, \|f\|_{L^2(\mathbf{R}^n)};$$

i.e., the linear mapping $\widetilde{S_m^{\varphi,\eta}}$ *can be extended to a bounded linear mapping* $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^{n+1})$ *with norm at most*

$$(2\pi)^{(n+1)/2}\, \alpha^{1/2}\, \|m\|_{L^\infty(\mathbf{R}^n,\,\mathbf{R}_+)}.$$

(b) *Conversely, if the mapping* $\widetilde{S^{\varphi,\eta}}$ *can be extended to a bounded linear mapping* $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^{n+1})$ *then the norm of the extension is* $(2\pi)^{(n+1)/2}\, \alpha^{1/2}$.

EXAMPLE 4.1. For $a>1$ set $\varphi(\rho)=\rho^a$, $\eta(\rho)=(1+\rho^2)^{-s/2}$, and $g(r)=(1+r^b)^{-1}$. It is straightforward to show that (9) is equivalent to (8). That (8) holds under the assumptions $n\geqslant a$, $s\geqslant(1-a)/2$ and either $b>a=n$ or $b\geqslant a\neq n$ was shown in [19, pp. 225–228]. Conversely, if (9) holds then also

$$\sup\left\{\rho\eta(\rho)^2\,\varphi'(\rho)^{-1}\int_0^\infty J_{\nu(k)}(r\rho)^2\,g(r)\,r\,dr : 0\leqslant\rho\leqslant 1,\, k\in\mathbf{N}\right\}<\infty$$

which in our case reads

$$\sup\left\{\rho^{2-a}\int_0^\infty J_{\nu(k)}(r\rho)^2\,\frac{r\,dr}{1+r^b} : 0\leqslant\rho\leqslant 1,\, k\in\mathbf{N}\right\}<\infty \tag{10}$$

from which it follows that $n\geqslant a$ and either $b>a=n$ or $b\geqslant a\neq n$.

This example is an illustration of Theorems A′ and B′ in Section 3.2 and also of the following theorem:

THEOREM. *Let* $\varphi(\rho)=\rho^a$ *and* $\eta=g=\chi$. *Assume that there is a number* C *independent of* f *such that*

$$\|\widetilde{S^{\varphi,\eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant C\, \|f\|_{L^2(\mathbf{R}^n)}.$$

Then $n\geqslant a$.

This theorem should be compared with Vega [18, Theorem 3′, p. 878] from which it follows that *given* $\varphi(\rho)=\rho^a$ *for any* $a>1$ *and* $\eta=g=\chi$ *there is a number* C *independent of* f *such that*

$$\|\widetilde{S^{\varphi,\eta}} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant C\, \|f\|_{L^2(\mathbf{R}^n)}.$$

Cf. also the interesting example in Wang [23].

EXAMPLE 4.2. For $a_1 > 1$ set $\varphi(\rho) = (\rho^{a_1}+1)^{a_2}$, $\eta(\rho) = (1+\rho^2)^{-s/2}$, and $g(r) = (1+r^b)^{-1}$. Again, it is straightforward to show that (9) is equivalent to (8) with a replaced by a_1 and s replaced by $a_1(a_2-1)/2+s$. Hence (9) holds under the assumptions $n \geqslant a_1$, $s \geqslant (1-a_1a_2)/2$, and either $b > a_1 = n$ or $b \geqslant a_1 \neq n$ (cf. Example 4.1). In particular, (9) holds if $a_1a_2 = 1$ and $a_1 = 2 \neq n$. Thus we have proved Theorem 1 in Section 1.2.

Conversely, if (9) holds then as in Example 4.1 (10) holds with a replaced by a_1 from which it follows that $n \geqslant a_1$ and either $b > a_1 = n$ or $b \geqslant a_1 \neq n$. We may now conclude that Theorem 2 in Section 1.3 holds by choosing $a_1 = 2$ and $a_2 \neq 0$.

EXAMPLE 4.3. For $a \neq 0$ set $\varphi(\rho) = \rho^a$, $\eta(\rho) = \rho^{-s}$, and $g(r) = r^{-b}$. Now (9) is equivalent to

$$\sup\left\{\rho^{2-a-2s}\int_0^\infty J_{\nu(k)}(r\rho)^2\, r^{1-b}\, dr : 0 \leqslant \rho < \infty, k \in \mathbf{N}\right\} < \infty$$

which in turn is equivalent to $s = (b-a)/2$ and $1 < b < n$.

This example is an illustration of [22, Theorem 2.6, p. 157] (choose $q_1 = 2$ and $\gamma = 0$) and for $a = b = 2$ of Simon [13, (3), p. 66].

EXAMPLE 4.4. Set $\varphi(\rho) = \rho^2$, $\eta(\rho) = \rho^{1/2}$, and $g(r) = (1+r^2)^{-1}$. Now (9) is equivalent to

$$\sup\left\{\rho\int_0^\infty J_{\nu(k)}(r\rho)^2\,\frac{r\,dr}{1+r^2} : 0 \leqslant \rho < \infty, k \in \mathbf{N}\right\} < \infty \tag{11}$$

and for $n \geqslant 3$ (11) clearly follows from (6). This example is an illustration of Simon [13, (2), p. 66].

EXAMPLE 4.5. For $a > 1$ set $\varphi(\rho) = \rho^a$, $\eta(\rho) = \psi(\rho)\,\rho^{(a-1)/2}$, and $g(r) = (1+r^b)^{-1}$. Now (9) is equivalent to

$$\sup\left\{\rho\int_0^\infty J_{\nu(k)}(r\rho)^2\,\frac{r\,dr}{1+r^b} : 1 \leqslant \rho < \infty, k \in \mathbf{N}\right\} < \infty$$

which in turn is equivalent to $b > 1$.

This example is an illustration of [22, Theorem 2.14, p. 159].

EXAMPLE 4.6. For $a > 1$ set $\varphi(\rho) = \rho^a$, $\eta(\rho) = \rho^{(a-1)/2}$, and $g = \chi$. Now (9) is equivalent to

$$\sup\left\{\rho\int_0^\infty J_{\nu(k)}(r\rho)^2\,\chi(r)\, r\, dr : 0 \leqslant \rho < \infty, k \in \mathbf{N}\right\} < \infty. \tag{12}$$

Equation (12) clearly follows from (11).

This example is an illustration of Kenig *et al.* [10, Theorem 4.1, p. 54].

5. PREPARATION

5.1. Readers familiar with the classical preparatory results in [20, Sect. 3, pp. 386–387] may skip reading this section and go directly to Section 6.

5.2. *Notation.* P will always denote a *solid spherical harmonic* (cf. Stein and Weiss [16, pp. 140–141]) of nonnegative degree k such that $\|P\|_{L^2(\Sigma^{n-1})} = 1$.

Let $\mathfrak{H}_k(\mathbf{R}^n)$ be the linear space of all finite linear combinations of functions of the form

$$x \mapsto P(x)\, f_0(|x|)\, |x|^{-n/2-k+1/2},$$

where $f_0 \in L^2(\mathbf{R}_+)$ (cf. [16, p. 138]). $\mathfrak{H}_k(\mathbf{R}^n)$ is an infinite dimensional Hilbert subspace of $L^2(\mathbf{R}^n)$ (with the inherited inner product). It is generated by a *finite* number of spherical harmonics with radial functions as coefficients.

THEOREM 5.1 [16, Lemma 2.18, p. 151]. *The complete orthogonal decomposition*

$$L^2(\mathbf{R}^n) = \bigoplus \mathfrak{H}_k(\mathbf{R}^n)$$

holds in the sense that

(a) *each subspace $\mathfrak{H}_k(\mathbf{R}^n)$ is closed;*

(b) *$\mathfrak{H}_{k_1}(\mathbf{R}^n)$ is orthogonal to $\mathfrak{H}_{k_2}(\mathbf{R}^n)$ if $k_1 \neq k_2$;*

(c) *every f can be written as a sum*

$$f = \sum f_k, \qquad f_k \in \mathfrak{H}_k(\mathbf{R}^n)$$

with convergence in $L^2(\mathbf{R}^n)$.

THEOREM 5.2 (cf. [16, Theorem 3.10, p. 158]). *Let $f(\xi) = P(\xi)\, f_0(|\xi|)$. Then*

$$\hat{f}(x) = (2\pi)^{n/2}\, i^{-k}\, |x|^{-\nu(k)}\, P(x) \int_0^\infty f_0(\rho)\, J_{\nu(k)}(\rho\, |x|)\, \rho^{n/2+k}\, d\rho.$$

COROLLARY 5.1. *The tempered distribution μ_P defined by*

$$\mu_P(f) = \int_{\Sigma^{n-1}} f(x')\, P(x')\, d\sigma(x')$$

has Fourier transform $\widehat{\mu_P}$ given by

$$\widehat{\mu_P}\,(\xi) = (2\pi)^{n/2}\, i^{-k}\, |\xi|^{-\nu(k)}\, P(\xi)\, J_{\nu(k)}(|\xi|).$$

6. PROOF OF THE MAIN RESULT

6.1. *Proof of part* (a) *in Theorem* 4.1. The proof follows closely the proof of [20, Theorem 2.1(a) and 2.2(a), p. 385].

For $\rho \in \Omega$ we define

$$\widetilde{f^{\varphi,\eta}}[x](\rho) = \widetilde{f^{\varphi,\eta}}(\rho)[x] = g(|x|)^{1/2}\, \eta(\varphi^{-1}(\rho))\, \varphi^{-1}(\rho)^{n-1}\, (D\varphi^{-1})(\rho)$$
$$\times \int_{\Sigma^{n-1}} e^{i\varphi^{-1}(\rho)\, x\xi'} f(\varphi^{-1}(\rho)\, \xi')\, d\sigma(\xi')$$

and by $\tilde{f}(\rho) = 0$ for $\rho \notin \Omega$. Let us write $m(x, \rho) = m[x](\rho)$. For fixed x function $m[x]$ is extended to $\mathbf{R}$ by $m(\rho) = 0$ for $\rho \leqslant 0$. The formula

$$(\widetilde{S_m^{\varphi,\eta} f})[x](t) = \int_{\mathbf{R}} e^{-it\rho} m(x, \varphi^{-1}(\rho))\, \widetilde{f^{\varphi,\eta}}[x](\rho)\, d\rho \tag{13}$$

follows by polar coordinates and change of variables in (3). It tells us that $(\widetilde{S_m^{\varphi,\eta} f})[x]$ is obtained from $\widetilde{f^{\varphi,\eta}}[x]$ via a Fourier multiplier transformation on $L^2(\mathbf{R})$ with norm $(2\pi)^{1/2}\, \|m[x]\|_{L^\infty(\mathbf{R})}$. We now have the estimate

$$\|S_m^{\varphi,\eta} f\|_{L^2(\mathbf{R}^{n+1})} \leqslant (2\pi)^{1/2}\, \|m\|_{L^\infty(\mathbf{R}^n \times \mathbf{R}_+)}\, \|\widetilde{f^{\varphi,\eta}}\|_{L^2(\mathbf{R}^{n+1})}. \tag{14}$$

Hence, to prove our statement it is sufficient to prove that

$$\|\widetilde{f^{\varphi,\eta}}\|_{L^2(\mathbf{R}^{n+1})} \leqslant (2\pi)^{n/2}\, \alpha^{1/2}\, \|f\|_{L^2(\mathbf{R}^n)}. \tag{15}$$

It is a consequence of orthogonality (cf. [20, Sect. 4.2.3, p. 389] and Theorem 5.1) that it is sufficient to prove the estimate (15) for $f \in (\mathfrak{H}_k \cap \mathscr{S})(\mathbf{R}^n)$. In turn, it is a consequence of orthogonality (cf. [20, Sect. 4.2.4, pp. 389–390]) that it is sufficient to prove the estimate (15) for

$$f\colon \xi \mapsto P(\xi)\, f_0(|\xi|)\, |\xi|^{-n/2-k+1/2}, \qquad f_0 \in \mathscr{C}_0^\infty(\mathbf{R}_+).$$

Straightforward computations using change of variables, Corollary 5.1, and polar coordinates show that for such f

$$\|\widetilde{f^{\varphi,\eta}}\|^2_{L^2(\mathbf{R}^{n+1})} = (2\pi)^n \int_0^\infty \left(\rho\eta(\rho)^2\, \varphi'(\rho)^{-1} \int_0^\infty J_{\nu(k)}(\rho r)^2\, g(r)\, r\, dr \right) |f_0(\rho)|^2\, d\rho$$
$$\leqslant (2\pi)^n\, \alpha\, \|f_0\|^2_{L^2(\mathbf{R}_+)} = (2\pi)^n\, \alpha\, \|f\|^2_{L^2(\mathbf{R}^n)}. \tag{16}$$

6.2. *Proof of Part* (b) *in Theorem* 4.1. If $m = 1$ then equality will occur in (14) with $\|m\|_{L^\infty(\mathbf{R}^n, \mathbf{R}_+)} = 1$. Moreover, from (16) it is clear that the norm of the bounded linear mapping $f \mapsto \widetilde{f^{\varphi,\eta}}$ is $(2\pi)^{n/2}\, \alpha^{1/2}$.

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