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Note

Determination of the star valency of a graph

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Abstract

The *star valency* of a graph G is the minimum, over all star decompositions π, of the maximum number of elements in π incident with a vertex. The *maximum average degree* of G, denoted by $d_{\text{max-ave}}(G)$, is the maximum average degree of all subgraphs of G. In this paper, we prove that the star valency of G is either $\lceil d_{\text{max-ave}}(G)/2 \rceil$ or $\lceil d_{\text{max-ave}}(G)/2 \rceil + 1$, and provide a polynomial time algorithm for determining the star valency of a graph.

Keywords: Graph; Star valency; Average degree; Orientation; Algorithm

1. Introduction

All graphs considered here are simple and connected. Until now, most of the results related to graph decomposition have concentrated on the minimum cardinality of the decomposition. However, people are somewhat interested in the minimum number of subgraphs, of course in the decomposition, that a vertex is incident with. Recent papers concerning this aspect are [2–4].

Let $\mathscr{H}$ be a family of graphs and G be a graph. *An $\mathscr{H}$-decomposition π of G* is a partition of $E(G)$ into disjoint sets $E(H_i)$ such that each of the subgraphs H_i induced by $E(H_i)$ is isomorphic to some $H \in \mathscr{H}$. We define *the $\mathscr{H}$ valency of G* as the minimum, over all $\mathscr{H}$-decompositions π of G, of the maximum number of elements

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in π containing a vertex. When $\mathscr{H}$ is the family of the stars forests, the $\mathscr{H}$ valency of a graph G is referred to as *the star valency of* G and is denoted by $\Lambda_s(G)$. In general, the determination of the $\mathscr{H}$ valency of a graph is hard. In this paper, we provide a polynomial time algorithm to determine the star valency of a graph.

2. The main results

2.1. Star decompositions and orientations

Let G be a graph and θ be an orientation of G. The indegree and the outdegree of a vertex $v \in V(G)$ under the orientation θ are denoted by $d^-(v;\theta)$ and $d^+(v;\theta)$, respectively. Let

$$\Delta^-(\theta) = \max\{d^-(v;\theta)\colon v \in V(G)\}$$

and let

$$\Delta^-_{\text{orient}}(G) = \min\{\Delta^-(\theta)\colon \text{ all orientations } \theta \text{ of } G\}.$$

Given a star decomposition π of G, we can easily connect π to an orientation of G by orienting the edges of G from the center of every star to its leaves, and it is sufficient to assume that every vertex of G is the center of at most one star of π. Thus, by the above discussion, we have the following fact:

Lemma 1.

$$\Delta^-_{\text{orient}}(G) \leqslant \Lambda_s(G) \leqslant \Delta^-_{\text{orient}}(G) + 1.$$

For convenience, $\Delta^-_{\text{orient}}(G)$ is called *the min–max indegree of* G. An orientation θ of G is said *to implement the min–max indegree of* G if $\Delta^-(\theta) = \Delta^-_{\text{orient}}(G)$. Similarly, there is the statement that a star decomposition implements the star valency of G.

2.2. Average degree and orientations

Somewhat different from usual practice, the length of a shortest directed path from u to v is denoted by $d(u,v)$.

For a real number a, the least integer that is greater than or equal to a is denoted by $\lceil a \rceil$. For a graph G and a vertex subset U of G, the subgraph of G induced by U, denoted by $G[U]$, is the subgraph of G consisting of all vertices of U and all edges of G with both endvertices in U. The edge set of the induced subgraph $G[U]$ is denoted by $E(G[U])$.

The following lemma provides a lower bound for $\Delta^-_{\text{orient}}(G)$.

Lemma 2. *For any orientation θ of a graph G,*

$$\Delta^-(\theta) \geqslant \max\left\{\left\lceil \frac{|E(G[U])|}{|U|} \right\rceil : \emptyset \neq U \subseteq V(G)\right\}.$$

Proof. It is obvious that maximum indegree of the induced subgraph $G[U]$ under the orientation θ is at least $\lceil |E(G[U])|/|U| \rceil$. □

In Theorem 4, we are to show the existence of an orientation θ that reaches the lower bound described in Lemma 2 (therefore, it is an orientation that implements the min–max indegree of G). The following lemma provides a structural description of such orientation.

Lemma 3. *Let G be a graph and θ be an orientation of G and $x \in V(G)$ with $d^-(x,\theta) = \Delta^-(\theta)$. Let*

$$O = \{v \in V(G){:}\ d^+(v;\theta) > 0,\ d^-(v;\theta) < \Delta^-(\theta) - 1\},$$

and

$$U = \{u \in V(G){:}\ d(u,x) < +\infty\}.$$

Then, there exists no directed path from some vertex $v \in O$ to x if and only if

$$O \cap U = \emptyset$$

and for every $u \in U$

$$d^-(u;\theta) = \text{either } \Delta^-(\theta) \text{ or } \Delta^-(\theta) - 1.$$

Furthermore, if there exists no directed path from some vertex $v \in O$ to x, then

$$\left\lceil \frac{|E(G[U])|}{|U|} \right\rceil = d^-(x;\theta) = \Delta^-(\theta).$$

Proof. The if and only if part of the lemma is obvious by the definitions of O and U.

Since no vertex $u \in U$ is dominated by any vertex of $G \setminus U$, for every $u \in U$, the indegree of u in the entire graph G is the same as the indegree of u in the induced subgraph $G[U]$. Thus,

$$\left\lceil \frac{|E(G[U])|}{|U|} \right\rceil = d^-(x;\theta) = \Delta^-(\theta). \qquad \square$$

Let G be a graph. The maximum average degree $d_{\text{max-ave}}(G)$ of the graph G is defined as follows:

$$d_{\text{max-ave}}(G) = \max\left\{ \frac{2|E(G[U])|}{|U|}{:}\ U \subseteq V(G),\ U \neq \emptyset \right\},$$

where, we notice that $2|E(G[U])|/|U|$ is the average degree of a subgraph $G[U]$.

The following results (together with Lemma 1) show the relations between the star valency $\Lambda_s(G)$, the maximum average degree $d_{\text{max-ave}}(G)$, and orientations that implement the min–max indegree of G.

Theorem 4. *For any graph G*,

$$\Delta^-_{\text{orient}}(G) = \left\lceil \frac{d_{\text{max-ave}}(G)}{2} \right\rceil.$$

Proof. Let G be a graph and θ be an orientation of G. For each positive integer i, let

$$S_i(\theta) = \{v \in V(G) \colon d^-(v;\theta) = i\}$$

and

$$O(\theta) = \{v \in V(G) \colon d^+(v;\theta) > 0,\ d^-(v;\theta) < \Delta^-(\theta) - 1\}.$$

Choose an orientation θ^* of G such that

(1) $\Delta^-(\theta^*) = \Delta^-_{\text{orient}}(G)$ and

(2) subject to (1), $S_{\Delta^-(\theta^*)}$ has the minimum cardinality.

Denote $S_{\Delta^-(\theta^*)}$ by S^*. Obviously, there is no directed path in G under the orientation θ^* from a vertex $v \in O(\theta^*)$ to a vertex $x \in S^*$. Otherwise, reverse the direction of the path to obtain a new orientation of G that contradicts the choice of θ^*. Let $x \in S^*$ and let

$$U^* = \{v \in V(G) \colon d(v,x) < +\infty\}.$$

By Lemma 3,

$$\begin{aligned}\Delta^-(\theta^*) &= d^-(x,\theta^*)\\ &= \left\lceil \frac{|E(G[U^*])|}{|U^*|} \right\rceil\\ &\leqslant \max\left\{ \left\lceil \frac{|E(G[U])|}{|U|} \right\rceil : U \subseteq V(G),\ U \neq \emptyset \right\}.\end{aligned}$$

By Lemma 2,

$$\Delta^-(\theta^*) = \Delta^-_{\text{orient}}(G).$$

That is, the orientation θ^* implements the min–max indegree of G. □

Corollary 5. *For any graph G, an orientation θ satisfies the condition described in Lemma* 3 *if and only if θ is an orientation implementing the min–max indegree of G*.

2.3. Star valency

From Lemma 1 and Theorem 4, we have the following immediate corollaries.

Corollary 6. *For any graph G*,

$$\left\lceil \frac{d_{\text{max-ave}}(G)}{2} \right\rceil \leqslant \Lambda_s(G) \leqslant \left\lceil \frac{d_{\text{max-ave}}(G)}{2} \right\rceil + 1.$$

Corollary 7. *For any planar graph* G, $\Lambda_s(G) \leqslant 4$.

Proof. Let G be a planar graph. Because every induced subgraph of a planar graph is also planar, we have that, for every $U \subseteq V(G)$ with $|U| \geqslant 3$,

$$|E(G[U])| \leqslant 3|U| - 6$$

(see [1, Corollary 9.5.2]). Thus, for every $U \subseteq V(G)$,

$$\frac{|E(G[U])|}{|U|} \leqslant 3,$$

and therefore,

$$\left\lceil \frac{d_{\text{max-ave}}(G)}{2} \right\rceil \leqslant 3.$$

By Corollary 6, $\Lambda_s(G) \leqslant 4$. □

2.4. Algorithm

The inductive and constructive proofs of Lemma 3 and Theorem 4 also imply a polynomial algorithm to determine a star decomposition that implements the star valency of G, the maximum of average degree of G.

Now we design an algorithm that determines the star valency of a graph.

ALGORITHM SV
INPUT: A graph G and an orientation θ of G,
OUTPUT: An integer i and an orientation θ of G.
METHOD:
1. Compute $\Delta^-(\theta) = \max\{d^-(v;\theta) \colon v \in V(G)\}$
2. $i \leftarrow \Delta^-(\theta)$, $S_i \leftarrow \emptyset$, $O_i \leftarrow \emptyset$
3. **for** all $v \in V(G)$ **do**
  if $d^-(v;\theta) = i$ **then** $S_i \leftarrow S_i \cup \{v\}$
  else if $d^+(v;\theta) > 0$ **and** $d^-(v;\theta) < i - 1$ **then** $O_i \leftarrow O_i \cup \{v\}$
4. **if** $S_i \neq \emptyset$, **then go to (5)**, **else**, **go to (7)**.
5. Choose an $x \in S_i$
6. **if** there exists a directed path from some $v \in O_i$ to x **then**
   begin
   reverse the direction of the path (modify θ),
   $S_i \leftarrow S_i \setminus \{x\}$,
   if $d^-(v;\theta) = i - 1$ **then** $O_i \leftarrow O_i \setminus \{v\}$
   go to 4
   end
  else go to (8)
7. $i \leftarrow i - 1$ **and go to (2)**.
8. **OUTPUTS:** the integer i (here $i = \lceil \Delta_{\text{max-ave}}/2 \rceil$), and the orientation θ with the minimum $\Delta^-(\theta)$, and a star decomposition π with the minimum star valency

$$\pi = \{S_v\colon v \in V(G) \text{ and } S_v \text{ is the star induced by } v \text{ and all arcs dominated by } v \text{ in } G\}.$$

The correctness of the Algorithm SV is guaranteed by Corollary 5 (followed by Lemmas 2 and 3 and Theorem 4): the modification of an existing orientation (reversing the direction of a path in the 6th step of the algorithm) will either reduce the size of S_i or reduce $i = \Delta^-(\theta)$. The algorithm will be processed until no such path exists (at that time, the resulting sets S_i and O_i are the sets U and O described in Theorem 4). By Theorem 4, the resulting orientation of G is an orientation that implements the min–max indegree of G.

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