Radical formula and prime submodules

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Abstract

Let $B$ be a submodule of an $R$-module $M$. The intersection of all prime submodules of $M$ containing $B$ is denoted by $\text{rad}(B)$. For every positive integer $n$, a generalization of $E(B)$ denoted by $E_n(B)$ of $M$ will be introduced. Moreover, $\langle E(B) \rangle \subseteq \langle E_n(B) \rangle \subseteq \text{rad}(B)$. In this paper we will study the equality $\langle E_n(B) \rangle = \text{rad}(B)$. It is proved that if $R$ is an arithmetical ring of finite Krull dimension $n$, then $\langle E_n(B) \rangle = \text{rad}(B)$.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider $R$ to be a ring, $M$ a unitary $R$-module and $\mathbb{N}$ the set of positive numbers.

Let $N$ be a proper submodule of $M$. It is said that $N$ is a prime submodule of $M$, if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M) = \{t \in R \mid tM \subseteq N\}$, we say that $N$ is a $P$-prime submodule of $M$, and it is easy to see that $P$ is a prime ideal of $R$. Prime submodules have been studied in several papers such as [1–6,9–14,17].

Recall that for an ideal $I$ of a ring $R$, the radical of $I$ denoted by $\sqrt{I}$ is defined to be $\sqrt{I} = \{r \in R \mid r^n \in I, \text{ for some } n \in \mathbb{N}\}$.

Also for any subset $B$ of $M$, the envelope of $B$, $E(B)$ is defined to be:

$$E(B) = \{x \mid x = ra, \ r^n a \in B, \text{ for some } r \in R, \ a \in M, \ n \in \mathbb{N}\}.$$
\[ \langle E(B) \rangle \] is a module version of the radical of ideals and evidently \( B \subseteq \langle E(B) \rangle \).

We know that for an ideal \( I \) of \( R \) we have, \( \sqrt{\sqrt{I}} = \sqrt{I} \). So in studying the radical of an ideal the number of radicals are not important. But for its generalization to modules, i.e., \( \langle E(B) \rangle \), it is not correct. For an example see the following.

**Example 1.** Consider \( R = \mathbb{Z}[X] \) and let the \( R \)-module \( M \) to be \( M = R \oplus R \) and \( N = \{(r, s) \in M \mid 4r - sX \in RX^2\} \). Then according to [17, p. 110], for the submodule \( N \) we have, \( \langle E(N) \rangle = R(0, 4) + XM \neq R(0, 2) + XM \). Hence \( (0, 2) \notin \langle E(N) \rangle \), however \( 2^2(0, 1) = (0, 4) \in \langle E(N) \rangle \) and consequently \( (0, 2) = 2(0, 1) \in E(\langle E(N) \rangle) \). Thus \( \langle E(N) \rangle \neq \langle E(\langle E(N) \rangle) \rangle \).

On the other hand, if \( \langle E(B) \rangle \) is a module version of the radical of ideals, so are \( \langle E(\langle E(B) \rangle) \rangle \), \( \langle E(\langle E(\langle E(B) \rangle)) \rangle \rangle \) and so on.

This discussion leads us to consider a generalization of \( E(B) \) in the following definition.

**Definition.** For a submodule \( B \) of \( M \), we will define \( E_0(B) = B \), \( E_1(B) = E(B) \), \( E_2(B) = E(\langle E(B) \rangle) \) and for any positive number \( n \), it will be defined \( E_{n+1}(B) = E(\langle E_n(B) \rangle) \) inductively. We will call \( E_n(B) \) the \( n \)-th-envelope of \( E(B) \).

Recall that for an ideal \( I \) of \( R \) we have, \( \sqrt{I} = \bigcap_{P \text{ prime ideal}, I \subseteq P} P \) (see [15, p. 3]).

Let \( B \) be a proper submodule of \( M \). The intersection of all prime submodules of \( M \) containing \( B \) is denoted by \( \text{rad}(B) \). If there does not exist any prime submodule of \( M \) containing \( B \), then we say \( \text{rad}(B) = M \). It is said that \( M \) satisfies the radical formula, if for every submodule \( B \) of \( M \), \( \langle E(B) \rangle = \text{rad}(B) \). It is said that a ring \( R \) satisfies the radical formula, if every \( R \)-module satisfies the radical formula (see, for example, [4,6,11,13,14,16,17]).

In [13], the authors characterize all commutative Noetherian rings which satisfy the radical formula. In particular it is proved that a commutative Noetherian domain satisfies the radical formula if and only if it is a Dedekind domain. Thus it is a natural question whether Prüfer domains satisfy the radical formula.

Recall that a ring \( R \) is said to be an arithmetical ring, if for all ideals \( I, J \) and \( K \) of \( R \) we have, \( I + (J \cap K) = (I + J) \cap (I + K) \) (see [7,8]). Obviously Prüfer domains and Dedekind domains are arithmetical.

In this paper we will prove that every arithmetical ring \( R \) with \( \dim R \leq 1 \) satisfies the radical formula.

**Definition.** Let \( n \) be a non-negative number. If \( \langle E_n(B) \rangle = \text{rad}(B) \), for every submodule \( B \) of \( M \), we will say that \( M \) satisfies the radical formula of degree \( n \). It will be said that the ring \( R \) satisfies the radical formula of degree \( n \), if every \( R \)-module satisfies the radical formula of degree \( n \).

We will show that every arithmetical ring of finite Krull dimension \( n \) satisfies the radical formula of degree \( n \).

**2. Radical formula**

**Lemma 2.1.** Let \( B \) be a submodule of an \( R \)-module \( M \), \( S \) a multiplicatively closed subset of \( R \) and \( n \) a non-negative number.

(i) If \( x \in E_n(B) \), then \( Rx \subseteq E_n(B) \).
(ii) $(\langle E_n(B) \rangle)_S = \langle E_n(B_S) \rangle$.
(iii) $B \subseteq \langle E(B) \rangle \subseteq \langle E_2(B) \rangle \subseteq \langle E_3(B) \rangle \subseteq \cdots \subseteq \text{rad}(B)$.
(iv) $(\bigcup_{m \in \mathbb{N}} \langle E_m(B) \rangle)_S = \bigcup_{m \in \mathbb{N}} \langle E_m(B_S) \rangle$.
(v) $(\text{rad}(B))_S \subseteq \text{rad}(B_S)$.
(vi) If for every maximal ideal $p$ of $R$ and every $R_p$-module, rad$(0) = \langle E_n(0) \rangle$, then the ring $R$
 satisfies the radical formula of degree $n$.

**Proof.** The proof is easy and is left to the reader. □

**Example 2.** In Example 1, we have

$$N \subseteq \langle E(N) \rangle = R(0, 4) + XM \subseteq R(0, 2) + XM = \langle E_2(N) \rangle = \text{rad}(N).$$

**Proof.** By [17, p. 110], we have, $\langle E(N) \rangle = N + XM = R(0, 4) + XM$ and $\text{rad}(N) = R(0, 2) + XM$. In Example 1, we showed that $(0, 2) \subseteq E_2(N)$. So $R(0, 2) \subseteq \langle E_2(N) \rangle$. By Lemma 2.1(iii),

$$\text{we have, } N \subseteq \langle E_n(N) \rangle \subseteq \text{rad}(N), \text{ for every } n \in \mathbb{N}. \text{ Hence } N \subseteq \langle E(N) \rangle = R(0, 4) + XM \subseteq R(0, 2) + XM \subseteq \langle E_2(N) \rangle \subseteq \text{rad}(N) = R(0, 2) + XM. \text{ Therefore } N \subseteq \langle E(N) \rangle = R(0, 4) + XM \subseteq R(0, 2) + XM = \langle E_2(N) \rangle = \text{rad}(N).$$

**Lemma 2.2.** A ring $R$ is arithmetical if and only if for each prime ideal $p$ of $R$, every two ideals of the ring $R_p$ are comparable.

**Proof.** See [7, Theorem 1]. □

**Lemma 2.3.** Let $R$ be a local arithmetical ring, $r$ a non-unit element of $R$, and $I = \bigcap_{n \in \mathbb{N}} R \cdot r^n$. Then

(i) If for some $k \in \mathbb{N}$, $r^k \in I$, then $r^k = 0$.
(ii) $I$ is a prime ideal of $R$, or $r$ is a nilpotent element of $R$.

**Proof.** (i) Note that $r^k \in I \subseteq \bigcap_{n \in \mathbb{N}} R \cdot r^{k+1}$. So for some $c \in R$ we have, $r^k = cr^{k+1}$, that is, $r^k(1 - cr) = 0$. Since $1 - cr$ is a unit, $r^k = 0$.

(ii) Let $ab \in I = \bigcap_{n \in \mathbb{N}} R \cdot r^n$, where $a, b \in R$, $a \notin I$ and $b \notin I$. Then $a \notin R \cdot r^n$ and $b \notin R \cdot r^m$ for some $n, m \in \mathbb{N}$. By Lemma 2.2, every two ideals of $R$ are comparable, so $R \cdot r^n \subseteq Ra$ and $R \cdot r^m \subseteq Rb$. Thus $r^{n+m} \in R \cdot r^{n+m} \subseteq Ra \cdot r^m \subseteq I$. Consequently by part (i), $r^{n+m} = 0$. □

**Theorem 2.4.**

(i) Every arithmetical ring of finite Krull dimension $k$ satisfies the radical formula of degree $k$.
(ii) Every arithmetical ring $R$ with $\dim R \leq 1$ satisfies the radical formula.

**Proof.** (i) Consider $E^0(0) = 0$, $E^1(0) = E(0)$, $E^2(0) = E(E(0))$ and for every $n \in \mathbb{N}$, define $E^{n+1}(0) = E(E^n(0))$ inductively. It is easy to see that for every $\delta \in \mathbb{N}$ we have

$$E^\delta(0) = \{ x | \text{ for } i = 1, 2, \ldots, \delta, \exists r_i \in R, \exists x_i \in M, \exists n_i \in \mathbb{N}, \exists x = r_1 x_1, \ldots, r_\delta x_\delta = 0 \}. \quad (*)$$
Let \( R \) be an arithmetical ring and \( \dim R = k \). By Lemma 2.1(vi), it is enough to show that for every maximal ideal \( p \) of \( R \), for every \( R_p \)-module we have, \( \text{rad}(0) = (E_k(0)) \). Indeed we will show that \( \text{rad}(0) \subseteq E^k(0) \), and it is easy to see that \( E^0(0) \subseteq E_k(0) \subseteq (E_k(0)) \). Also by Lemma 2.1(iii), \( (E_k(0)) \subseteq \text{rad}(0) \). Therefore we will have \( E^k(0) = E_k(0) = (E_k(0)) = \text{rad}(0) \).

By localization and Lemma 2.2, we may assume that \( R \) is a local ring with a maximal ideal \( m \) such that every two ideals of \( R \) are comparable. Let \( M \) be an \( R \)-module. Consider \( x \in \text{rad}(0) \). We will prove that \( x \in E^k(0) \).

Since \( m \) is a maximal ideal of \( R \), \( mM \) is a prime submodule of \( M \) or \( mM = M \). Thus \( x \in mM \). Then \( x = \sum_{i=1}^l r_i a_i \) such that for each \( i \), \( 1 \leq i \leq l \), \( r_i \in m \) and \( a_i \in M \). Every two ideals of \( R \) are comparable, then \( \{Rr_i, \ i = 1, 2, 3, \ldots, l\} \) is a chain of ideals of \( R \). Without loss of generality we may suppose that \( Rr_1 \) is the maximal element of this chain. So \( x = r_1 x_1 \), for some \( x_1 \in M \). Let \( S_1 = \{r_1^n x_1 \mid n \in \mathbb{N}\} \). Now define the set \( T_1 \) as follows:

\[
T_1 = \left\{ K \mid K \text{ is a submodule of } M, \ K \cap S_1 = \emptyset, \bigcap_{n \in \mathbb{N}} Rr_1^n \subseteq (K : M) \right\}.
\]

First we will show that \( T_1 = \emptyset \). If \( T_1 \neq \emptyset \), then by Zorn’s Lemma, \( T_1 \) has a maximal element. Let \( N \) be a maximal element of \( T_1 \). We show that \( N \) is a prime submodule of \( M \). Suppose \( ay \in N \), where \( a \in R \) and \( y \in M \). We have one of the following.

(i) \( Ra \subseteq Rr_1^n \), for every \( n \in \mathbb{N} \).

(ii) \( Ra \not\subseteq Rr_1^d \), for some \( d \in \mathbb{N} \).

If (i) holds, then \( a \in Ra \subseteq \bigcap_{n \in \mathbb{N}} Rr_1^n \subseteq (N : M) \), so we have the proof.

If (ii) is satisfied, since every two ideals of \( R \) are comparable, we have, \( Rr_1^d \subseteq Ra \). Let \( r_1^d = ba \), where \( b \in R \). If \( y \notin N \), then \( \bigcap_{n \in \mathbb{N}} Rr_1^n \subseteq (N : M) \subseteq (N + Ry : M) \) and \( N \) is a maximal element of \( T_1 \), then \( (N + Ry) \cap S_1 \neq \emptyset \). Consider \( r_1^t x_1 \in (N + Ry) \cap S_1 \), where \( t \in \mathbb{N} \). Then \( r_1^t x_1 = n' + cy \), where \( n' \in N \) and \( c \in R \). Now \( r_1^{d+t} x_1 = r_1^d (ba x_1) = ban' + cbay \in N \), that is \( N \cap S_1 \neq \emptyset \), which is a contradiction. So \( y \in N \). Therefore \( N \) is a prime submodule of \( M \) and since \( N \subseteq T_1 \) we have, \( N \cap S_1 = \emptyset \), which is a contradiction with the fact that \( r_1 x_1 = x \in S_1 \cap \text{rad}(0) \subseteq S_1 \cap N \). Consequently \( T_1 = \emptyset \).

Put \( \bigcap_{n \in \mathbb{N}} Rr_1^n = I_1 \). By Lemma 2.3(ii), \( r_1 \) is a nilpotent element or \( I_1 \) is a prime ideal of \( R \). If \( r_1 \) is nilpotent and \( r_1^a \neq 0 \), for some \( a \in \mathbb{N} \), then \( r_1^a x_1 = 0 \). So \( x = r_1 x_1 \in E(0) \subseteq E^k(0) \). Now assume that \( I_1 \) is a prime ideal of \( R \). Note that \( T_1 = \emptyset \), then \( I_1 M \neq T_1 \) and since \( I_1 \subseteq (I_1 M : M) \) we have, \( I_1 M \cap S_1 \neq \emptyset \). Then let \( r_1^n x_1 = \sum_{j=1}^h i_j m_j \), where \( n_1 \in \mathbb{N} \), \( i_j \in I_1 \), \( m_j \in M \), for each \( 1 \leq j \leq h \). Note that \( \{Ri_j, \ j = 1, 2, 3, \ldots, h\} \) is a chain of ideals of \( R \), then we may assume that \( r_1^n x_1 = r_2 x_2 \), where \( r_2 \in I_1 \) and \( x_2 \in M \).

Now consider \( S_2 = \{r_2^n x_2 \mid n \in \mathbb{N}\} \), and define the set \( T_2 \) as follows:

\[
T_2 = \left\{ K \mid K \text{ is a submodule of } M, \ K \cap S_2 = \emptyset, \bigcap_{n \in \mathbb{N}} Rr_2^n \subseteq (K : M) \right\}.
\]

A similar proof to that of above will show that \( T_2 \) is an empty set and \( r_2 \) is a nilpotent element or \( I_2 = \bigcap_{n \in \mathbb{N}} Rr_2^n \) is a prime ideal of \( R \). If \( r_2 \) is nilpotent, then for some positive number \( n_2 \) we have, \( r_2^{n_2} = 0 \). Therefore we have, \( x = r_1 x_1, \ r_1^n x_1 = r_2 x_2, \ r_2^{n_2} x_2 = 0 \). So by (*) we have, \( x \in E^2(0) \subseteq E^k(0) \). Now suppose that \( r_2 \) is not a nilpotent element. Then \( I_2 \) is a prime ideal.
of $R$. Since $r_2 \in I_1$, $I_2 \subseteq I_1$. If $I_2 = I_1$, then $r_2 \in I_1 = I_2 = \bigcap_{n \in \mathbb{N}} Rr_2^n$. Now by Lemma 2.3(i), $r_2 = 0$. So $r_2$ is a nilpotent element, which is a contradiction. Hence $I_2 \subsetneq I_1$. Also note that $I_1 \subseteq m$, otherwise $r_1 \in m = I_1$. Thus by Lemma 2.3(i), $r_1 = 0$. Then $r_1$ is a nilpotent element, which is a contradiction.

By continuing this argument we will get elements $r_1, r_2, \ldots \in R$, $x_1, x_2, \ldots \in M$ and $n_1, n_2, \ldots \in \mathbb{N}$ such that $r_j^{n_j}x_j = r_{j+1}x_{j+1}$ for each $j \in \mathbb{N}$, and if $I_j = \bigcap_{n \in \mathbb{N}} Rr_j^n$, then $\cdots \subset I_3 \subset I_2 \subset I_1 \subset m$ is a chain of prime ideals of $R$. Moreover, $r_{j+1} \in I_j$, for each $j$. Since $\dim R = k$, we have, $I_{k+1} = I_k$. Consequently the following is a descending chain of cyclic submodules of $M$:

\[\cdots \subseteq I_k \subseteq I_{k-1} \subseteq \cdots \subseteq I_2 \subseteq I_1 \subset m,\]

which is a contradiction.

By continuing this process, for each $1 < n$, we will get elements $r_{n+1}x_{n+1}$ such that

\[r_{n+1}x_{n+1} = r_{n+2}x_{n+2} = r_{n+3}x_{n+3} = \cdots = r_{n+k}x_{n+k} = 0.\]

Consequently by (\*) we have, $x = r_1x_1 \in E^k(0)$ and the proof is completed.

(ii) If $\dim R \leq 1$, then by part (i), for every submodule $B$ of an $R$-module $M$ we have, $\langle E(B) \rangle = \langle E_1(B) \rangle = \text{rad}(B)$.

\textbf{Corollary 2.5.} Let $R$ be an arithmetical ring with DCC on prime ideals. Then for every submodule $B$ of an $R$-module $M$, $\lim_n \langle E_n(B) \rangle = \text{rad}(B)$.

\textbf{Proof.} Follow the proof of Theorem 2.4. \qed

\textbf{Lemma 2.6.} Let $M$ be an $R$-module with DCC on cyclic submodules, $B$ a submodule of $M$, and $S$ a multiplicatively closed subset of $R$.

(i) $M/B$ as an $R$-module has DCC on cyclic submodules.
(ii) $M_S$ as an $R_S$-module has DCC on cyclic submodules.

\textbf{Proof.} (i) Let $\cdots \subseteq R(x_3 + B) \subseteq R(x_2 + B) \subseteq R(x_1 + B)$ be a descending chain of cyclic submodules of $M/B$, where $x_1, x_2, x_3, \ldots \in M$. Since $R(x_2 + B) \subseteq R(x_1 + B)$, there exist $r_2 \in R$ and $b_2 \in B$ such that $x_2 = r_1x_1 + b_2$. Now since $R(x_3 + B) \subseteq R(x_2 + B) = R(x_2 - b_2 + B)$, there exist $r_3 \in R$ and $b_3 \in B$ such that $x_3 = r_2x_2 + b_3$. By continuing this process, for each $1 < n \in \mathbb{N}$, we will get $b_{n+1} \in B$ and $r_{n+1} \in R$ such that $x_{n+1} = r_{n+1}(x_n - b_n) + b_{n+1}$. Consequently the following is a descending chain of cyclic submodules of $M$:

\[\cdots \subseteq R(x_3 - b_3) \subseteq R(x_2 - b_2) \subseteq R(x_1).\]

Hence by our assumption there exists $m \in \mathbb{N}$ such that for each $k > m$ we have, $R(x_k - b_k) = R(x_m - b_m)$, which implies that $R(x_k + B) = R(x_m + B)$.

(ii) Obviously every cyclic submodule of $M_S$ is of the form $R_S^{\frac{x}{1}}$ where $x \in M$. Then let $\cdots \subseteq R_S^{\frac{x_3}{1}} \subseteq R_S^{\frac{x_2}{1}} \subseteq R_S^{\frac{x_1}{1}}$ be a descending chain of cyclic submodules of $M_S$, where $x_1, x_2, x_3, \ldots \in M$.

Since $R_S^{\frac{x_2}{1}} \subseteq R_S^{\frac{x_1}{1}}$, we have, $\frac{x_2}{1} = \frac{r_1x_1}{s_1}$, for some $r_1 \in R$ and $s_1 \in S$. Then there exists $s_2 \in S$ such that $s_2s_1x_2 = s_2r_1x_1$. Put $t_2 = s_2x_1$. Then $t_2 \in S$ and we have, $Rt_2x_2 \subseteq Rx_1$. Now since $R_S^{\frac{x_1}{1}} \subseteq R_S^{\frac{s_1x_1}{s_1}} = R_S^{\frac{t_2x_1}{s_1}}$, similarly there exists $t_3 \in S$ such that $Rt_3x_3 \subseteq Rt_2x_2$.

By continuing this process, for each $1 < n \in \mathbb{N}$, we will get $t_{n+1} \in S$ such that $Rt_{n+1}x_{n+1} \subseteq Rt_nx_n$. Consequently the following is a descending chain of cyclic submodules of $M$, $\cdots \subseteq
$R_{t_3}x_3 \subseteq R_{t_2}x_2 \subseteq Rx_1$. Hence by our assumption there exists $m \in \mathbb{N}$ such that for each $k > m$ we have, $R_{t_k}x_k = R_{t_m}x_m$, which implies that $R_{s_{\frac{k}{m}}} = R_{s_{\frac{m}{m}}}$.

**Corollary 2.7.** Let $R$ be an arithmetical ring, and $M$ an $R$-module with DCC on cyclic submodules. Then for every submodule $B$ of $M$, $\lim_{n \in \mathbb{N}}(E_n(B)) = \text{rad}(B)$.

**Proof.** By Lemma 2.6(i), $M/B$ as an $R$-module has DCC on cyclic submodules. Now by Lemma 2.6(ii), every localization of $M/B$ has DCC on cyclic submodules. So by localization we may assume that $R$ is a local arithmetical ring such that every two ideals of $R$ are comparable, and it is enough to show that $\bigcup_{n \in \mathbb{N}}(E_n(0)) = \text{rad}(0)$.

Let $x \in \text{rad}(0)$. By following the proof of Theorem 2.4, we will get a sequence $\{r_{j}x_j\}_{j \in \mathbb{N}}$ of elements of $M$ such that for each $j$, $r_{j}^{n_{j}}x_{j} = r_{j+1}x_{j+1}$. Since the chain $\cdots \subseteq R_{t_3}x_3 \subseteq R_{t_2}x_2 \subseteq R_1x_1$ stops, for some $k \in \mathbb{N}$ we have, $R_{t_k}x_k = R_{t_{k+1}}x_{k+1} = R_{t_{k}}^{n_k}\ x_k$. Therefore there exists $v \in R$ such that $r_k(1 - r_k^{n_k-1})v = 0$. Note that $r_k \in m$, then $1 - r_k^{n_k-1}v$ is a unit. So $r_k^{-1}x_{k-1} \neq 0$. Thus by $(\ast)$ in Theorem 2.4, we have, $x = r_1x_1 \in E_k^{-1}(0) \subseteq E_{k-1}(0) \subseteq \bigcup_{n \in \mathbb{N}}(E_n(0))$. Therefore $\bigcup_{n \in \mathbb{N}}(E_n(0)) = \text{rad}(0)$.

**Remark.** Let $B$ be a submodule of an $R$-module $M$.

(i) If $M$ satisfies the radical formula of any degree we have, $\bigcup_{n \in \mathbb{N}}(E_n(B)) = \text{rad}(B)$, by Lemma 2.1(iii). Now suppose that $R$ is an arithmetical ring. According to Theorem 2.4, Corollaries 2.5 and 2.7, $\bigcup_{n \in \mathbb{N}}(E_n(B)) = \text{rad}(B)$, if one of the following holds:

(a) $\dim R < \infty$.

(b) $R$ has DCC on prime ideals.

(c) $M$ has DCC on cyclic submodules.

These facts inspire us to define the radical envelope of a submodule $B$ to be $\bigcup_{n \in \mathbb{N}}(E_n(B))$ and not just $(E(B))$.

(ii) In general $E_k(B)$ is not necessarily a submodule of $M$. Now suppose that $R$ is a local arithmetical ring of finite Krull dimension $k$. According to the proof of Theorem 2.4, we have, $E_k^k(B) = E_k(B) = \text{rad}(B)$. Consequently $E_k^k(B) = E_k(B)$ and it is a submodule of $M$. Moreover, $(E_k^n(B)) = (E_k(B)) = \bigcup_{n \in \mathbb{N}}(E_n(B)) = \text{rad}(B)$.

(iii) Let $R$ be an arithmetical ring of finite Krull dimension $k$. In Lemma 2.1, we may replace $E_n(B)$ with $E^n(B)$. Now by the new version of this lemma and part (ii) of this remark, we have $((E_k(B)))_P = (E_k(BP)) = (E_k(B))_{(P)}$, for each prime ideal $P$ of $R$. Consequently $(E_k^n(B)) = (E_k^n(B)) = \text{rad}(B)$.

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**References**


