A uniform asymptotic expansion for the incomplete gamma function

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Abstract

We describe a new uniform asymptotic expansion for the incomplete gamma function $\Gamma(a,z)$ valid for large values of $z$. This expansion contains a complementary error function of an argument measuring transition across the point $z = a$ (which is different from that in the well-known uniform expansion for large $a$ of Temme), with easily computable coefficients that do not involve a removable singularity at $z = a$. Our expansion is, however, valid in a smaller domain of the parameters than that of Temme. Numerical examples are given to illustrate the accuracy of the expansion.

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1. Introduction

The incomplete gamma functions are defined by the integrals

$$
\gamma(a,z) = \int_0^z t^{a-1} e^{-t} \, dt, \quad \Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t} \, dt,
$$

(1.1)

where $a$ and $z$ are complex variables and the integration paths do not cross the negative real axis. The definition of $\gamma(a,z)$ requires the condition $\text{Re}(a) > 0$ while in that for $\Gamma(a,z)$ it is assumed that $|\arg z| < \pi$. These functions are among the most fundamental and important special functions of mathematical analysis. This viewpoint has been reinforced during the past decade by the use of $\Gamma(a,z)$ as a terminator in the development of the theory of exponentially improved asymptotic expansions and the associated new interpretation of the Stokes phenomenon [10]. The uniform asymptotics
of the incomplete gamma function also play an important role in recent methods of computation of the Riemann zeta function high up on the critical line [11,12].

The asymptotic expansion of $\Gamma(a,z)$ when either $a$ or $z$ is large is well known and quite straightforward; see, for example, [17, Section 11.2.1]. The treatment when both $a$ and $z$ assume large (complex) values is more difficult since the resulting expansions have to take into account the presence of the transition point at $z = a$, about which the asymptotic structure of $\Gamma(a,z)$ changes. For example, in the case of large positive variables, the normalised incomplete gamma function $Q(a,z) \equiv \Gamma(a,z)/\Gamma(a)$ exhibits a 'cut-off' type behaviour, since it is approximately unity when $z < a$ and decays algebraically to zero when $z > a$. The behaviour of the complementary normalised incomplete gamma function $P(a,z) \equiv \gamma(a,z)/\Gamma(a)$ follows from the relation $P(a,z) + Q(a,z) = 1$.

The earliest uniform expansions of $\Gamma(a,z)$ for large $a$ and $z$ were obtained by Mahler [9] and Tricomi [18]. These expansions $^1$ involve a series of inverse powers of $z - a$ of the form

$$
\Gamma(a,z) \sim z^a e^{-z} \sum_{k=0}^{\infty} \frac{b_k}{(z - a)^{k+1}}
$$

valid in certain sectors and consequently breakdown in a neighbourhood of the transition point $z = a$. The coefficients $b_k$ in these two types of expansion possess a different dependence on the variables $a$ and $z$: in Mahler’s expansion $b_k = b_k(z)$, while in Tricomi’s expansion $b_k = b_k(a)$—see Section 4 for details. Another uniform expansion of this type was subsequently given by Gautschi [6]—see [15, Section 4] for a discussion when the variables are complex—in which $b_{2k+1} = 0$ and the even-order coefficients depend on both $a$ and $\lambda = z/a$. This dependence is described by $b_{2k} = (-a)^k f_k(\lambda)$ where $f_k(\lambda)$ denotes a polynomial in $\lambda$ of degree $k$, with the first few polynomials being given by

$$
\begin{align*}
    f_0(\lambda) &= 1, \\
    f_1(\lambda) &= \lambda, \\
    f_2(\lambda) &= \lambda(2\lambda + 1), \\
    f_3(\lambda) &= \lambda(6\lambda^2 + 8\lambda + 1), \\
    f_4(\lambda) &= \lambda(24\lambda^3 + 58\lambda^2 + 22\lambda + 1), & \ldots .
\end{align*}
$$

In his 1950 paper, Tricomi also gave, among others, an expansion of a more uniform character of which the first two terms are

$$
Q(a+1,a + (2a)^{1/2}x) = \frac{1}{2} \text{erfc}(-x) + \frac{1}{3} \sqrt{\frac{2}{\pi a}} (1 + x^2) e^{-x^2} + \text{O}(a^{-1})
$$

for $a \to \infty$ in $|\arg a| < \frac{1}{2}\pi$ with $x$ real and bounded, where erfc denotes the complementary error function. This form describes the behaviour of the incomplete gamma function near the transition point. A major advance in the understanding of the uniform asymptotic structure of the incomplete gamma function was made by Temme [14], where it was shown that as $a \to \infty$

$$
Q(a,z) \sim \frac{1}{2} \text{erfc} \left( \eta \sqrt{\frac{1}{2}a} \right) + \frac{\text{e}^{-(1/2)\eta^2a^2}}{\sqrt{2\pi a}} \sum_{k=0}^{\infty} c_k(\eta)a^{-k};
$$

see also [13]. The auxiliary variable $\eta$, measuring transition through $z = a$, is defined by $\eta = \{2(\lambda - 1 - \log \lambda)\}^{1/2}$, $\lambda = z/a$ with the branch being chosen so that $\eta(\lambda)$ is analytic in the vicinity of

$^1$ Tricomi’s expansion is for the function $\Gamma(a + 1,z) = a\Gamma(a,z) + z^a e^{-z}$.
\( \lambda = 1 \) \((z = a)\) and \( \eta \sim \lambda - 1 \) as \( \lambda \rightarrow 1 \). The coefficients \( c_k(\eta) \) are defined recursively by

\[
c_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \quad c_k(\eta) = \frac{1}{\eta} \frac{d}{d\eta} c_{k-1}(\eta) + \frac{\gamma_k}{\lambda - 1} \quad (k \geq 1),
\]

where \( \gamma_k \) are the Stirling coefficients appearing in the well-known asymptotic expansion for \( \Gamma(z) \); see Section 4. Expansion (1.4) holds uniformly in \(|z| \in [0, \infty)\) in the domains \(|\arg a| \leq \pi - \varepsilon_1\) and \(|\arg(z/a)| \leq 2\pi - \varepsilon_2\), where \( \varepsilon_1, \varepsilon_2 \) are arbitrarily small positive numbers. This form of expansion encapsulates the change in asymptotic structure through the transition point \( z = a \) (where \( \eta = 0 \)) as well as containing the large-\( a \) asymptotics. However, it suffers from the inconvenience of having coefficients \( c_k(\eta) \) that possess a removable singularity at \( \eta = 0 \), thereby rendering their evaluation in the neighbourhood of the transition point difficult. Representations for these coefficients not having a removable singularity are given in [2] the form of power series expansions in \( \eta \).

Our aim in this paper is to present new uniform asymptotic expansions for \( \gamma(a,z) \) and \( \Gamma(a,z) \) for \( z \rightarrow \infty \) that are particularly suitable for computation in the neighbourhood of the transition point. We use the integrals in (1.1) expressed in terms of the new variable \( t \), where \( t = ze^{\pm \tau} \), to obtain the representations

\[
\gamma(a,z) = z^a \int_0^\infty e^{-at - ze^{-\tau}} d\tau, \quad \Gamma(a,z) = z^a \int_0^\infty e^{at - ze^\tau} d\tau \quad (1.5)
\]

valid under the restrictions \( \text{Re}(a) > 0 \) and \( \text{Re}(z) > 0 \), respectively. The method we employ consists of factorising the exponentials in the above integrands as described in [5] into an exponential factor containing only the linear and quadratic terms and another factor which is expanded in ascending powers of \( \tau \); see also [3, p. 113]. The expansions we obtain also contain a complementary error function as main approximant (multiplied by a series in inverse powers of \( z^{1/2} \)), but with the auxiliary variable \((z - a)/\sqrt{2z}\) as its argument. It will be seen that, although our expansions are not valid in as large a domain of \( a \) and \( z \) as that in (1.4), the coefficients have the advantage of not possessing a removable singularity at the transition point \( z = a \) and so are more straightforward to compute.

The above procedure was also used by Mahler and Tricomi (with only the linear term retained) to the integrals in (1.5) and (1.1), respectively, to produce uniform expansions of the form (1.2) valid away from the transition point. Dingle [3, p. 249] also applied the same method (retaining quadratic terms) to the integrals (1.1) to derive a uniform asymptotic expansion for \( \Gamma(a+1,z) \) for \( a \rightarrow \infty \) which, when the coefficients are regrouped in the manner described in Section 3, yields an expansion valid in the neighbourhood of the transition point similar to ours; see Section 4 for details. We show the connection that exists between our expansion and that of Dingle with the expansions of Mahler and Tricomi, respectively, as one moves away from the neighbourhood of the transition point. Some numerical examples are presented to illustrate the use of our expansions and indicate how the accuracy compares with that in (1.4).

2. Derivation of the large-\( z \) expansions

Consider first the integral for \( \Gamma(a,z) \) in (1.5), which we write in the factored form [5]

\[
\Gamma(a,z) = z^ae^{-z} \int_0^\infty e^{-(z-a)\tau - (1/2)\tau^2} H d\tau \quad (|\arg z| < \frac{1}{2} \pi), \quad (2.1)
\]
where
\[H = e^{-zh}, \quad h(\tau) = e^\tau - 1 - \tau - \frac{1}{2} \tau^2\] (2.2)
and in what follows we shall impose the additional restriction Re(z - a) ≥ 0. We divide the path of integration into the intervals [0, \mu] and [\mu, \infty), where \mu = |z|^{\varepsilon - 1/3} and \varepsilon denotes a constant satisfying 0 < \varepsilon < \frac{1}{3} that will be chosen later;\(^2\) it follows that \mu ≤ 1 whenever |z| ≥ 1. It is convenient at this point to note the following easily established inequalities concerning \(h'(\pm sFS)\) for 0 ≤ sSYN ≤ 1:
\[h'(sFS) = e^{sFS} - 1 - sFS \geq \frac{1}{2} sSYN^2 + \frac{1}{2} sSYN (sFS - sSYN) (sFS \geq sSYN)\] (2.3)
and
\[h'(-sFS) = e^{-sFS} - 1 + sFS \geq \begin{cases} \frac{1}{3} sSYN^2, & (sFS \leq 1), \\ \frac{1}{2} sSYN^2 + \frac{1}{2} sSYN (sFS - sSYN) (sFS \geq sSYN). \end{cases}\] (2.4)
The main contribution to the integral (2.1) as \(z \to \infty\) in |arg z| < \frac{1}{3}\pi then arises from the interval [0, sSYN], since the contribution from [sSYN, \infty) is exponentially small. This follows from the fact that, when Re(z - a) ≥ 0, the modulus of the contribution from [\mu, \infty) is bounded by
\[
\int_{\mu}^{\infty} e^{-Re(z)(e^{\tau} - 1 - \tau)} d\tau \leq \int_{0}^{\infty} e^{-Re(z)((1/2)\mu^2 + (1/2)\mu w) \phi} dw
\]
\[= O(|z|^{-2/3 - \varepsilon} e^{-T(z)}), \quad T(z) = \frac{1}{2} |z|^{1/3 + 2\varepsilon} \cos \theta, \]
where \(\theta = \arg z\) and we have made use of the inequality in (2.3).

We expand the factor \(H\) as the finite Taylor expansion
\[H = \sum_{k=0}^{n-1} \frac{c_k(z)}{k!} \tau^k + r_n(z, \tau) \quad (n = 1, 2, \ldots),\] (2.5)
where \(r_n(z, \tau)\) denotes the remainder term. The coefficients \(c_k(z)\) satisfy the recurrence relation
\[c_{k+1}(z) = -z \sum_{j=2}^{k} \binom{k}{j} c_{k-j}(z) \quad (k \geq 2),\] (2.6)
with \(c_0(z) = 1\) and \(c_1(z) = c_2(z) = 0\); their values up to \(k = 12\) are presented in Table 1. It is seen that the \(c_k(z)\) are polynomials in \(z\) of degree \([k/3]\) with the general form
\[c_k(z) = \sum_{j=0}^{[k/3]} \alpha_j(k) z^j.\] (2.7)
The first few values of the coefficients \(\alpha_j(k)\) follow from Table 1; we note that \(\alpha_0(k) = \delta_{0k}\), with \(\delta\) denoting the Kronecker symbol. Substitution of the expansion (2.5) into the integral (2.1) taken over
\(^2\) This choice of scaling is made to facilitate the determination of the bound for the remainder term in (2.5); see the appendix.
Table 1
The coefficients $c_k(z)$ for $3 \leq k \leq 12$, where $c_0(z) = 1$ and $c_1(z) = c_2(z) = 0$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c_k(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$-z$</td>
</tr>
<tr>
<td>4</td>
<td>$-z$</td>
</tr>
<tr>
<td>5</td>
<td>$-z$</td>
</tr>
<tr>
<td>6</td>
<td>$-z + 10z^2$</td>
</tr>
<tr>
<td>7</td>
<td>$-z + 35z^2$</td>
</tr>
<tr>
<td>8</td>
<td>$-z + 91z^2$</td>
</tr>
<tr>
<td>9</td>
<td>$-z + 210z^2 - 280z^3$</td>
</tr>
<tr>
<td>10</td>
<td>$-z + 456z^2 - 2100z^3$</td>
</tr>
<tr>
<td>11</td>
<td>$-z + 957z^2 - 10395z^3$</td>
</tr>
<tr>
<td>12</td>
<td>$-z + 1969z^2 - 42735z^3 + 15400z^4$</td>
</tr>
</tbody>
</table>

provided $z \to \infty$ in $|\arg z| < \frac{1}{2}\pi$ and Re$(z - a) \geq 0$, the upper limit of integration in the first term on the right-hand side of (2.8) can be extended to $\infty$, with the introduction of an exponentially small error term of $O(|z|^{-(1/2)k-1/2+2n\epsilon})$ for each nonnegative integer value of $k \leq n - 1$. Under the same conditions it is shown in (A.5) that the remainder term $R_n^+(a,z)$ satisfies the order estimate (for fixed $n$)

$$|R_n^+(a,z)| = O(|z|^{-(1/2)k-1/2+2n\epsilon}) \quad (z \to \infty \text{ in } |\arg z| < \frac{1}{2}\pi, \text{ Re}(z - a) \geq 0).$$

Hence, upon absorbing the exponentially small error term into the order estimate, we find

$$I = \sum_{k=0}^{n-1} \frac{c_k(z)}{k!} \int_0^\infty \tau^k e^{-(z-a)\tau-(1/2)\tau^2} \, d\tau + O(|z|^{-(1/2)k-1/2+2n\epsilon})$$

$$= \sum_{k=0}^{n-1} \frac{c_k(z)}{z^{(1/2)k+1/2}} d_k(\chi) + O(|z|^{-(1/2)k-1/2+2n\epsilon}),$$

(2.10)

where we have used the standard result in [7, Section 3.462] to evaluate the integral in terms of the parabolic cylinder function $D_n(z)$ in the form

$$\int_0^\infty \tau^k e^{-(z-a)\tau-(1/2)\tau^2} \, d\tau = z^{-(1/2)k-1/2} k! d_k(\chi),$$

(2.11)
with
\[ d_k(\chi) = e^{z^2/4}D_{-k-1}(\chi), \quad \chi = \frac{z-a}{\sqrt{z}}. \]

The treatment of the function \( \gamma(a, z) \) follows an analogous procedure. From (1.5) we have, when \( \text{Re}(a) > 0 \),
\[ \gamma(a, z) = z^a e^{-z} \int_0^\infty e^{-(a-z)\tau-(1/2)\tau z^2} H \, d\tau, \quad (2.12) \]
where now \( H = e^{-z\arg(\chi)} \). With \( \mu = |z|^1/3 \), as above, and use of inequality (2.4b), the contribution to (2.12) arising from the interval \([\mu, \infty)\) is similarly found to be exponentially small for \( z \to \infty \) in \( |\arg z| < \frac{1}{2}\pi \) when \( \text{Re}(z-a) \leq 0 \). Then, from (2.5) and (2.11), we consequently obtain
\[ \gamma(a, z) = z^a e^{-z} \left\{ \sum_{k=0}^{n-1} \left[ (-)^k c_k(z) \left( \frac{1}{z^{1/2}\tau' + 1/2} \right) d_k(-\chi) + R_n(a, z) \right] \right\}, \quad (2.13) \]
where the remainder term is given by
\[ R_n(a, z) = \int_0^\mu e^{-(a-z)\tau-(1/2)\tau z^2} r_n(z, -\tau) \, d\tau. \quad (2.14) \]
From (A.5), we find that \( |R_n(a, z)| = O(|z|^{-(1/6)n-1/2+2n\epsilon}) \) as \( z \to \infty \) in \( |\arg z| < \frac{1}{2}\pi \), when \( \text{Re}(z-a) \leq 0 \).

We now choose \( \epsilon = n^{-1/2} \) and let \( n = 3m + 4 \), \( m = 1, 2, 3, \ldots \). The terms in the sums in (2.10) and (2.13) corresponding to \( k = 3m + l \), \( l = 1, 2, 3 \) and the associated order terms are \( O(|z|^{-(1/2)m-1}) \) as \( z \to \infty \). Thus we obtain, for \( m = 1, 2, 3, \ldots \),
\[ I(a, z) = z^a e^{-z} \left\{ \sum_{k=0}^{3m} \left[ \frac{c_k(z)}{z^{1/2}\tau' + 1/2} d_k(\chi) + O(|z|^{-(1/2)m-1}) \right] \right\}, \quad \text{Re}(z-a) \geq 0, \]
\[ \gamma(a, z) = z^a e^{-z} \left\{ \sum_{k=0}^{3m} \left[ (-)^k c_k(z) \left( \frac{1}{z^{1/2}\tau' + 1/2} \right) d_k(-\chi) + O(|z|^{-(1/2)m-1}) \right] \right\}, \quad \text{Re}(z-a) \leq 0 \quad (2.15) \]
valid as \( z \to \infty \) in \( |\arg z| < \frac{1}{2}\pi \).

### 3. The modified expansions

The expansions in (2.15) have been obtained for \( z \to \infty \) in the sector \( |\arg z| < \frac{1}{2}\pi \) subject to the restrictions \( \text{Re}(z-a) \geq 0 \) for \( I(a, z) \) and \( \text{Re}(z-a) \leq 0 \) for \( \gamma(a, z) \). These expansions have been given previously in a little-known thesis by Dopper [4], who used a different method to derive the order terms in the truncated series. However, he proceeded to express the coefficients \( d_k(\pm\chi) \) as an infinite sum in ascending powers of \( \chi \) to yield asymptotic expansions in the form of double series. Here we shall show how expansions (2.15) can be rearranged to produce more useful asymptotic results valid in the vicinity of the transition point. The form that these expansions take when \( |\chi| \gg 1 \) is discussed in Section 4.

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\(^3\) The condition \( \text{Re}(a) > 0 \) is replaced by the more stringent condition \( \text{Re}(z-a) \leq 0 \) when \( |\arg z| < \frac{1}{2}\pi \).
The functions $d_k(\chi)$ ($k = 0, 1, 2, \ldots$) in (2.15) involve the parabolic cylinder function of negative integer order and so can be expressed by recurrence in terms of the complementary error function, since

$$d_0(\chi) = \sqrt{\frac{\pi}{2}} e^{\chi^2/2} \text{erfc}(\chi/\sqrt{2}).$$

(3.1)

From the recurrence relation $D_{v+1}(z) - zD_v(z) + vD_{v-1}(z) = 0$ satisfied by the parabolic cylinder function, we find that

$$d_{k+1}(\chi) = \frac{1}{k+1} \left\{ d_{k-1}(\chi) - \chi d_k(\chi) \right\} \quad (k = 0, 1, 2, \ldots),$$

(3.2)

where $d_{-1}(\chi) = e^{\chi^2/4} D_0(\chi) = 1$. This result shows that $d_k(\chi)$ can be expressed in the form

$$d_k(\chi) = p_k(\chi) d_0(\chi) + q_k(\chi),$$

where $p_k(\chi)$, $q_k(\chi)$ are polynomials in $\chi$ of degree $k$ and $k - 1$; see also [20, Section 3]. The coefficients $p_k(\chi)$ and $q_k(\chi)$ satisfy the recurrence relation (3.2) with the starting values $(p_{-1}, p_0) = (0, 1)$ and $(q_{-1}, q_0) = (1, 0)$. The first few coefficients are consequently given by

- $p_0(\chi) = 1$, $q_0(\chi) = 0$,
- $p_1(\chi) = -\chi$, $q_1(\chi) = 1$,
- $p_2(\chi) = \frac{1}{2} + \frac{1}{2} \chi^2$, $q_2(\chi) = -\frac{1}{2} \chi$,
- $p_3(\chi) = -\frac{1}{3} \chi - \frac{1}{6} \chi^3$, $q_3(\chi) = \frac{1}{3} + \frac{1}{6} \chi^2$,
- $p_4(\chi) = \frac{1}{8} + \frac{1}{4} \chi^2 + \frac{1}{24} \chi^4$, $q_4(\chi) = -\frac{5}{24} \chi - \frac{1}{24} \chi^3$

and so on. We now define the coefficients $C_k(\chi)$ by

$$C_k(\chi) = \sum_{j=0}^{k} \alpha_j(k+2j) d_{k+2j}(\chi) = A_k(\chi) d_0(\chi) + B_k(\chi),$$

where the $\alpha_j(k)$ are specified in (2.7) and

$$\left\{ \begin{array}{l} A_k(\chi) \\ B_k(\chi) \end{array} \right\} = \sum_{j=0}^{k} \alpha_j(k+2j) \left\{ \begin{array}{l} p_{k+2j}(\chi) \\ q_{k+2j}(\chi) \end{array} \right\}. $$

(3.3)

The coefficients $A_k(\chi)$ and $B_k(\chi)$ are polynomials in $\chi$ of degree $3k$ and $3k - 1$, respectively, with $B_0(\chi) \equiv 0$; their expressions for $0 \leq k \leq 8$ are presented in Table 2.

Since $c_k(z)$ are polynomials in $z$ of degree $[k/3]$, we substitute the representation (2.7) into (2.15) and collect terms involving like powers of $z$ together to obtain, for the function $I(a,z)$,

$$I(a,z) \sim z^{a-1/2} e^{-z} \sum_{k=0}^{\infty} z^{-(1/2)k} d_k(\chi) \sum_{j=0}^{[k/3]} \alpha_j(k) z^j$$

$$= z^{a-1/2} e^{-z} \sum_{k=0}^{\infty} z^{-(1/2)k} C_k(\chi).$$
Table 2
The coefficients $A_k(x)$ and $B_k(x)$ for $0 \leq k \leq 8$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(-)^k A_k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}x + \frac{1}{6}x^3$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{12} + \frac{3}{8}x^2 + \frac{1}{6}x^4 + \frac{1}{72}x^6$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{8}x + \frac{47}{144}x^3 + \frac{37}{240}x^5 + \frac{1}{48}x^7 + \frac{1}{1296}x^9$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{288} + \frac{5}{32}x^2 + \frac{347}{1152}x^4 + \frac{617}{4320}x^6 + \frac{23}{960}x^8 + \frac{1}{648}x^{10} + \frac{1}{31104}x^{12}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5}{576}x + \frac{79}{432}x^3 + \frac{367}{1280}x^5 + \frac{32353}{241920}x^7 + \frac{785}{31104}x^9 + \frac{37}{17280}x^{11} + \frac{5}{62208}x^{13} + \frac{1}{933120}x^{15}$</td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{139}{51840} + \frac{35}{2304}x^2 + \frac{953}{4608}x^4 + \frac{115319}{141720}x^6 + \frac{15311}{120960}x^8 + \frac{371809}{4515200}x^{10} + \frac{4793}{1866240}x^{12}$</td>
</tr>
<tr>
<td></td>
<td>$+\frac{163}{1244160}x^{14} + \frac{1}{311040}x^{16} + \frac{1}{33592320}x^{18}$</td>
</tr>
<tr>
<td>7</td>
<td>$-\frac{973}{103680}x + \frac{13897}{62208}x^3 + \frac{10553}{46080}x^5 + \frac{1584917}{5806080}x^7 + \frac{6319193}{5254720}x^9 + \frac{105949}{4147200}x^{11} + \frac{747767}{26127360}x^{13}$</td>
</tr>
<tr>
<td></td>
<td>$+\frac{3971}{22394880}x^{15} + \frac{1}{165888}x^{17} + \frac{1}{67184640}x^{19} + \frac{1}{1410877440}x^{21}$</td>
</tr>
<tr>
<td>8</td>
<td>$-\frac{571}{2488320} - \frac{973}{46080}x^2 + \frac{147029}{4976640}x^4 + \frac{3731897}{14929920}x^6 + \frac{8367811}{30965760}x^8 + \frac{8692181}{74649600}x^{10} + \frac{79208459}{3135283200}x^{12}$</td>
</tr>
<tr>
<td></td>
<td>$+\frac{532807}{174182400}x^{14} + \frac{1359989}{627056640}x^{16} + \frac{13673}{403107840}x^{18} + \frac{1}{49766400}x^{20} + \frac{1}{352719360}x^{22}$</td>
</tr>
<tr>
<td></td>
<td>$+\frac{1}{6772211720}x^{24}$</td>
</tr>
</tbody>
</table>

Then expansions (2.15) can be cast in the final form

$$ \Gamma(a, z) \sim z^{a-\frac{1}{2}} e^{-z} \left\{ d_0(\chi) \sum_{k=0}^{\infty} \frac{A_k(\chi)}{z^{(1/2)k}} + \sum_{k=1}^{\infty} \frac{B_k(\chi)}{z^{(1/2)k}} \right\}, \quad \text{Re}(z - a) \geq 0, $$

$$ \gamma(a, z) \sim z^{a-\frac{1}{2}} e^{-z} \left\{ d_0(-\chi) \sum_{k=0}^{\infty} \frac{A_k(\chi)}{z^{(1/2)k}} - \sum_{k=1}^{\infty} \frac{B_k(\chi)}{z^{(1/2)k}} \right\}, \quad \text{Re}(z - a) \leq 0 $$

(3.4)

for $z \to \infty$ in the sector $\mid \arg z \mid < \frac{1}{2} \pi$, where $d_0(\chi)$ is defined in (3.1)
Table 2 (Continued)

<table>
<thead>
<tr>
<th>(k)</th>
<th>((-)^{k+1} B_k(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{3} + \frac{1}{6} x^2)</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{4} x + \frac{11}{72} x^3 + \frac{1}{72} x^5)</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{4}{135} + \frac{241}{1080} x^2 + \frac{293}{2160} x^4 + \frac{13}{648} x^6 + \frac{1}{1296} x^8)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{241}{4320} x^3 + \frac{341}{1620} x^5 + \frac{6377}{51840} x^7 + \frac{389}{31104} x^9 + \frac{47}{1399680} x^{11})</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{8}{2835} + \frac{14297}{181440} x^2 + \frac{7403}{36288} x^4 + \frac{9179}{17280} x^6 + \frac{403}{51840} x^8 + \frac{37}{1937160} x^{10} + \frac{1}{933120} x^{12})</td>
</tr>
<tr>
<td>6</td>
<td>(-\frac{2099}{362880} x^3 + \frac{216679}{2177280} x^5 + \frac{4372733}{43545600} x^7 + \frac{9168673}{391910400} x^9 + \frac{13697}{5598720} x^{11})</td>
</tr>
<tr>
<td></td>
<td>(+\frac{179}{1939680} x^{13} + \frac{1}{33592320} x^{15} + \frac{1}{33592320} x^{17})</td>
</tr>
<tr>
<td>7</td>
<td>(-\frac{16}{8505} + \frac{18479}{2177280} x^2 + \frac{516407}{43545600} x^4 + \frac{3726229}{18662400} x^6 + \frac{26611129}{261273600} x^8 + \frac{1813519}{78382080} x^{10})</td>
</tr>
<tr>
<td></td>
<td>(+\frac{6351973}{2351462400} x^{12} + \frac{40349}{235146240} x^{14} + \frac{929}{105764160} x^{16} + \frac{73}{7054382720} x^{18} + \frac{1}{1410877440} x^{20})</td>
</tr>
<tr>
<td>8</td>
<td>(-\frac{25415}{3483648} x + \frac{193391}{17418240} x^3 + \frac{3394381}{24883200} x^5 + \frac{87005}{435456} x^7 + \frac{3682388147}{37623398400} x^9 + \frac{855530953}{37623398400} x^{11})</td>
</tr>
<tr>
<td></td>
<td>(+\frac{107840753}{37623398400} x^{13} + \frac{2613509}{12541132800} x^{15} + \frac{37201}{4180377600} x^{17} + \frac{391}{1791590400} x^{19} + \frac{1}{67722117120} x^{21})</td>
</tr>
<tr>
<td></td>
<td>(+\frac{1}{67722117120} x^{23})</td>
</tr>
</tbody>
</table>

4. Discussion

The expansions (3.4) have been derived for \(z \to \infty\) in the sector \(|\arg z| < \frac{1}{2}\pi\) uniformly in the parameter \(a\). The domain of the parameter \(a\) is controlled by the different restrictions on \(\Re(z - a)\) in (3.4). These do not present any computational difficulty: if \(\Re(z - a) > 0\) one computes \(Q(a,z)\) while if \(\Re(z - a) < 0\) one computes \(P(a,z)\), with the other function being determined from the identity \(P(a,z) + Q(a,z) = 1\). If \(\Re(z - a) = 0\), either function can be computed. In this way we see that all values of \(a\) satisfying \(|\arg a| \leq \pi\) can be covered by one expansion or the other in (3.4).

The expansions in (3.4) contain a complementary error function of argument \(\pm \chi/\sqrt{2}\), where the variable \(\chi = (z - a)/\sqrt{|z|}\) measures transition through the point \(z = a\), together with two asymptotic series in inverse powers of \(z^{1/2}\) with coefficients that depend on \(\chi\). In terms of the variable \(\chi\), we note that the sectors \(|\arg z| < \frac{1}{4}\pi\) and \(\Re(z - a) \geq 0\) or \(\Re(z - a) \leq 0\) correspond to the sectors \(|\arg(\pm \chi)| < \frac{3}{4}\pi\), respectively. The form of the coefficients \(A_k(\chi)\) and \(B_k(\chi)\) makes the expansions in (3.4) particularly well suited for numerical computation in a neighbourhood of the transition point, where \(|\chi|\) is not too large. For large values of \(\chi\), however, these coefficients in the form (3.3) become
increasingly difficult to compute on account of the severe cancellation that takes place. This is to be contrasted with Temme’s uniform expansion in (1.4), which is valid for \( a \to \infty \) and contains an isolated complementary error function of a more complicated argument measuring transition through \( z = a \), together with a single asymptotic series in inverse powers of \( a \). The coefficients in this expansion, however, present a removable singularity at \( z = a \) (where \( \eta = 0 \)) and are consequently more difficult to compute in a neighbourhood of the transition point.

The nature of expansions (3.4) for large \( \chi \) (that is, when |\( z - a | \gg |z|^{1/2} \)) can be seen by making use of the result [19, p. 347]

\[
d_s(z) \sim \chi^{-k-1} \left\{ 1 - \frac{(k+1)(k+2)}{2\chi^2} \frac{(k+1)(k+2)(k+3)(k+4)}{2!4\chi^4} - \cdots \right\}
\]

(4.1)

for \( \chi \to \infty \) in |\( \arg \chi | < \frac{3}{4}\pi \). Insertion of this expansion into (2.15) followed by routine algebra shows that the expansions in (3.4) reduce to the form (cf. (1.2))

\[
\Gamma(a, z) \sim z^a e^{-z} \sum_{k=0}^{\infty} \frac{b_k(z)}{(z - a)^{k+1}}, \quad \text{Re}(z - a) \geq 0,
\]

\[
\gamma(a, z) \sim -z^a e^{-z} \sum_{k=0}^{\infty} \frac{b_k(z)}{(z - a)^{k+1}}, \quad \text{Re}(z - a) \leq 0,
\]

(4.2)

where the \( b_k(z) \) are defined as the Taylor coefficients in the expansion (2.5) when \( h(\tau) \) is replaced by \( e^{i-1-\tau} \). Thus

\[
b_0(z) = 1, \quad b_1(z) = 0, \quad b_2(z) = b_3(z) = -z, \quad b_4(z) = -z + 3z^2,
\]

\[
b_5(z) = -z + 10z^2, \quad b_6(z) = -z + 25z^2 - 15z^3, \ldots
\]

with the higher coefficients given by a recurrence relation of type (2.6) when the lower limit in the summation is replaced by \( j = 1 \). The expansions in (4.2) are valid for \( z \to \infty \) in |\( \arg z | < \frac{1}{2}\pi \) when |\( z - a | \gg |z|^{1/2} \) and were first given by Mahler [9]. He employed the same approach as in Section 2, but retained only the linear term \( (z - a)\tau \) in the exponential factor in (2.1) and in the analogous expression for \( \gamma(a, z) \). It is worth pointing out that an expansion of this type had been obtained even earlier by Ramanujan [1] in the more general setting of the function defined by the series \( \sum_{n=0}^{\infty} (n + a)^{-s} x^n / n! \), which is equal to \( (xe^{\pm \pi i})^{-a} \gamma(a, xe^{\pm \pi i}) \) when \( s = 1 \). We also remark that the expansion for \( \gamma(a, z) \) in (4.2b) when \( \arg z = \pm \pi \) and |\( \arg a | < \pi \) has been given in [8] in the study of magnetised Bose plasmas. These authors gave a more detailed discussion of the coefficients \( b_k(z) \) using a methodology that involved tree diagrams for the determination of the high-order coefficients.

A different type of uniform expansion, which generalises Tricomi’s result in (1.3), has been given by Dingle [3, p. 249] who employed the same factorisation procedure, but applied to the integral representations in (1.1). This generates expansions valid for \( a \to \infty \) in |\( \arg a | < \frac{1}{2}\pi \) uniform in \( z \) which, when expressed in our notation, take the form

\[
\Gamma(a + 1, z) \sim z^{a+1} e^{-z} \sum_{k=0}^{\infty} \hat{c}_k(a) d_k(\zeta') \frac{e^{a/2}}{a^{1/2} k + 1/2}, \quad \text{Re}(z - a) \geq 0,
\]

(4.3)

where \( \zeta' = (z - a)/\sqrt{a} \), together with an analogous expression for \( \gamma(a + 1, z) \) valid when \( \text{Re}(z - a) \leq 0 \). The coefficients \( \hat{c}_k(a) \) are defined to be the \( k \)th derivative of \( (1 + \tau)^a \exp \{ a(\tau + \frac{1}{2} \tau^2) \} \) evaluated at
described in Section 3, we obtain the expansions for 
\[ s_{\text{CR}} \]
cf. (2.7). If the terms in (4.3) and in the analogous expansion for 
\[ z \]
at the transition point
shall present our results in terms of the normalised incomplete gamma functions
in (3.4) and make a comparison with the uniform expansion in (1.4). For numerical convenience we

5. Numerical results

values of the first few of these coefficients are given in Table 3.

For large values of \( \chi' \) the expansions in (4.4) can be shown by application of (4.1) to reduce to the expansions obtained by Tricomi [18] valid away from the transition point (cf. (1.2))

\[ \Gamma(a + 1, z) \sim z^{a+1} e^{-z} \sum_{k=0}^{\infty} \frac{\hat{b}_k(a)}{(z-a)^{k+1}}, \quad \text{Re}(z-a) \geq 0, \]

\[ \gamma(a + 1, z) \sim z^{a+1} e^{-z} \sum_{k=0}^{\infty} \frac{\hat{b}_k(a)}{(z-a)^{k+1}}, \quad \text{Re}(z-a) \leq 0, \]

(4.5)

for \( a \to \infty \) in \(|\arg a| < \frac{\pi}{2}\) when \(|z-a| \gg |a|^{1/2}\). The coefficients \( \hat{b}_k(a) \) are defined to be the \( k \)th derivative of the function \((1 + \tau)^a e^{-a\tau}\) evaluated at \( \tau = 0 \), so that

\[ \hat{b}_0(a) = 1, \quad \hat{b}_1(a) = 0, \quad \hat{b}_2(a) = -a, \quad \hat{b}_3(a) = -6a + 3a^2, \quad \hat{b}_4(a) = 24a - 20a^2, \]

\[ \hat{b}_5(a) = -120a + 130a^2 - 15a^3, \quad \hat{b}_6(a) = 720a - 924a^2 + 210a^3, \ldots. \]

By means of an intricate analytic continuation argument, Tricomi showed that (4.5a) holds in the wider domain \(|\arg \chi'| < \frac{3\pi}{4}\).

5. Numerical results

In this section we carry out a numerical investigation of the accuracy of the asymptotic expansions in (3.4) and make a comparison with the uniform expansion in (1.4). For numerical convenience we shall present our results in terms of the normalised incomplete gamma functions \( P(a, z) \) and \( Q(a, z) \).

First, as a simple verification of (3.4), we note that both these expansions are valid for large \( a \) at the transition point \( z = a \) (where \( \chi = 0 \)). Hence, substitution of these expansions into the sum

\[ P(a, a) + Q(a, a) \sim \frac{1}{\Gamma^*(a)} \sum_{k=0}^{\infty} a^{-(1/2)k} A_k(0) \]

(5.1)
Table 3

The coefficients $\hat{A}_k(x)$ and $\hat{B}_k(x)$ for $0 \leq k \leq 6$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(-)^k \hat{A}_k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$x + \frac{1}{3} x^3$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{12} + x^2 + \frac{7}{12} x^4 + \frac{1}{18} x^6$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{12} x + \frac{37}{36} x^3 + \frac{47}{60} x^5 + \frac{5}{36} x^7 + \frac{1}{162} x^9$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{288} + \frac{1}{12} x^2 + \frac{151}{144} x^4 + \frac{1031}{1080} x^6 + \frac{341}{1440} x^8 + \frac{13}{648} x^{10} + \frac{1}{1944} x^{12}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{288} x + \frac{73}{864} x^3 + \frac{767}{720} x^5 + \frac{16699}{15120} x^7 + \frac{13331}{38880} x^9 + \frac{539}{12960} x^{11} + \frac{1}{486} x^{13} + \frac{1}{29160} x^{15}$</td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{139}{51840} + \frac{1}{288} x^2 + \frac{295}{3456} x^4 + \frac{72977}{25920} x^6 + \frac{149699}{120960} x^8 + \frac{154219}{340200} x^{10} + \frac{6547}{93312} x^{12} + \frac{391}{116640} x^{14} + \frac{19}{524880} x^{16}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(-)^{k+1} \hat{B}_k(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{2}{3} + \frac{1}{3} x^2$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{7}{12} x + \frac{19}{36} x^3 + \frac{1}{18} x^5$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{4}{135} + \frac{307}{540} x^2 + \frac{179}{270} x^4 + \frac{43}{324} x^6 + \frac{1}{162} x^8$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{143}{4320} x + \frac{1445}{2592} x^3 + \frac{9983}{12960} x^5 + \frac{943}{4320} x^7 + \frac{19}{972} x^9 + \frac{1}{1944} x^{11}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{8}{2835} - \frac{575}{18144} x^2 + \frac{1661}{3024} x^4 + \frac{12989}{15120} x^6 + \frac{221}{720} x^8 + \frac{1541}{38880} x^{10} + \frac{59}{29160} x^{12} + \frac{1}{29160} x^{14}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1997}{362880} x - \frac{32503}{1088640} x^2 + \frac{6599}{12150} x^4 + \frac{2543713}{2721600} x^6 + \frac{19399451}{48988800} x^8 + \frac{91811}{1399680} x^{10} + \frac{1399680}{1399680} x^{11}$</td>
</tr>
</tbody>
</table>

as $a \to \infty$ in $|\arg a| < \frac{1}{4} \pi$, where $\Gamma^*(z) = (2\pi)^{-1/2} z^{-1/2} e^{-z} \Gamma(z)$ is the scaled gamma function. From numerical computations (see Table 2), it is found that $A_{2k}(0) = (-)^k \gamma_k$ and $A_{2k+1}(0) = 0$, where $\gamma_k$ are the Stirling coefficients appearing in the well-known asymptotic expansion [19, p. 253]

$$\Gamma^*(z) \sim \sum_{k=0}^{\infty} (-)^k \gamma_k z^{-k} \quad (z \to \infty \text{ in } |\arg z| < \pi)$$

with

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320} \ldots .$$
The modulus of the relative error in the computation of the incomplete gamma functions from (3.4) in the neighbourhood of the transition point and for two different values of the truncation index are presented in Tables 4–6. The absolute values of the relative error in the computation of $Q(a, z)$ when $z = 50e^{0.9}$ and the truncation index $m = 6$. The errors refer to the computation of $Q(a, z)$ when $0 \leq \phi \leq \frac{1}{2}\pi$ and to $P(a, z)$ when $\frac{1}{2}\pi < \phi \leq \pi$

| $|z|$ | $\phi/\pi$ | 0.25 | 0.50 | 0.75 | 1.00 |
|-----|----------|-----|-----|-----|-----|
| $\theta = 0$ | 0.1 | $9.232 \times 10^{-10}$ | $1.205 \times 10^{-9}$ | $1.875 \times 10^{-9}$ | $1.111 \times 10^{-9}$ | $7.976 \times 10^{-10}$ |
| | 1.0 | $1.305 \times 10^{-10}$ | $8.292 \times 10^{-10}$ | $1.982 \times 10^{-8}$ | $8.861 \times 10^{-10}$ | $1.290 \times 10^{-10}$ |
| | 2.0 | $1.269 \times 10^{-11}$ | $3.056 \times 10^{-10}$ | $8.547 \times 10^{-8}$ | $3.150 \times 10^{-10}$ | $1.450 \times 10^{-11}$ |
| $\theta = \frac{1}{2}\pi$ | 0.1 | $9.216 \times 10^{-10}$ | $1.191 \times 10^{-9}$ | $1.856 \times 10^{-9}$ | $1.097 \times 10^{-9}$ | $8.075 \times 10^{-10}$ |
| | 1.0 | $1.307 \times 10^{-10}$ | $8.108 \times 10^{-10}$ | $1.943 \times 10^{-8}$ | $8.571 \times 10^{-10}$ | $1.294 \times 10^{-10}$ |
| | 2.0 | $1.274 \times 10^{-11}$ | $2.965 \times 10^{-10}$ | $8.238 \times 10^{-8}$ | $3.041 \times 10^{-10}$ | $1.444 \times 10^{-11}$ |
| $\theta = \frac{3}{2}\pi$ | 0.1 | $9.164 \times 10^{-10}$ | $1.175 \times 10^{-9}$ | $1.842 \times 10^{-9}$ | $1.102 \times 10^{-9}$ | $8.326 \times 10^{-10}$ |
| | 1.0 | $1.311 \times 10^{-10}$ | $7.995 \times 10^{-10}$ | $1.924 \times 10^{-8}$ | $8.315 \times 10^{-10}$ | $1.303 \times 10^{-10}$ |
| | 2.0 | $1.296 \times 10^{-11}$ | $2.901 \times 10^{-10}$ | $8.065 \times 10^{-8}$ | $2.952 \times 10^{-10}$ | $1.427 \times 10^{-11}$ |
| $\theta = \frac{5}{2}\pi$ | 0.1 | $9.058 \times 10^{-10}$ | $1.158 \times 10^{-9}$ | $1.830 \times 10^{-9}$ | $1.118 \times 10^{-9}$ | $8.623 \times 10^{-10}$ |
| | 1.0 | $1.314 \times 10^{-10}$ | $7.958 \times 10^{-10}$ | $1.926 \times 10^{-8}$ | $8.121 \times 10^{-10}$ | $1.311 \times 10^{-10}$ |
| | 2.0 | $1.328 \times 10^{-11}$ | $2.866 \times 10^{-10}$ | $8.023 \times 10^{-8}$ | $2.892 \times 10^{-10}$ | $1.400 \times 10^{-11}$ |

This observation then enables us to see that the right-hand side of (5.1) correctly reduces asymptotically to unity. A similar remark applies to the modified Dingle expansions in (4.4) since the coefficients $\tilde{A}_k(0) = A_k(0)$; see Table 3.

The results of computations using the expansions in (3.4) truncated after $m + 1$ terms, where $m$ denotes a positive integer, are presented in Tables 4–6. The absolute values of the relative error in the evaluation of $P(a, z)$ and $Q(a, z)$ (with the exact values being determined with Mathematica) are shown in Table 4 when $z = 100$ for different positive values of the parameter $a$ in the neighbourhood of the transition point and for two different values of the truncation index $m$. When $a \leq 100$ we compute...
Table 6  
Values of \(Q(a,z)\) computed from (3.4) for different complex values of \(a\) and \(z\) in the sector \(\frac{1}{2}\pi \leq \arg z \leq \pi\) and truncation index \(m\)

<table>
<thead>
<tr>
<th>(z = 100i) (a = 80i) ((\chi = 1.4142 + 1.4142i)) (m = 6)</th>
<th>(\text{Asymptotic})</th>
<th>(-0.1524353801584 - 0.0644167542886i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Exact})</td>
<td>(-0.1524353801621 - 0.0644167542856i)</td>
<td></td>
</tr>
<tr>
<td>(\text{Relative error})</td>
<td>(2.865 \times 10^{-11})</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(z = 100e^{3\pi i/4}) (a = 80e^{7\pi i/8}) ((\chi = 3.8268 + 1.2388i)) (m = 6)</th>
<th>(\text{Asymptotic})</th>
<th>(-1.4904168479594 \cdot 10^{-5} + 3.6640552608423 \cdot 10^{-5}i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Exact})</td>
<td>(-1.4904168479597 \cdot 10^{-5} + 3.6640552608426 \cdot 10^{-5}i)</td>
<td></td>
</tr>
<tr>
<td>(\text{Relative error})</td>
<td>(9.371 \times 10^{-11})</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(z = 100e^{\pi i}) (a = 100e^{-7\pi i/8}) ((\chi = 3.8268 + 0.7612i)) (m = 6)</th>
<th>(\text{Asymptotic})</th>
<th>(+0.46959774950759 - 1.7488764552114 \cdot 10^{100}i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Exact})</td>
<td>(+0.46959774950759 - 1.7488764552114 \cdot 10^{100}i)</td>
<td></td>
</tr>
<tr>
<td>(\text{Relative error})</td>
<td>(5.677 \times 10^{-14})</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(z = 100e^{\pi i}) (a = -100.25) ((\chi = -0.025i)) (m = 10)</th>
<th>(\text{Asymptotic})</th>
<th>(+0.4767363802299 + 0.5232636197701i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Exact})</td>
<td>(+0.4767363802299 + 0.5232636197701i)</td>
<td></td>
</tr>
<tr>
<td>(\text{Relative error})</td>
<td>(3.767 \times 10^{-15})</td>
<td></td>
</tr>
</tbody>
</table>

\(Q(a,100)\), while when \(a \geq 100\) we compute \(P(a,100)\). The modulus of the relative error when \(z\) and \(a\) are complex is shown in Table 5 for \(m = 6\). Here the calculations are presented for \(z = 50e^{i\theta}\) and different values of \(\chi = (z-a)/\sqrt{z} = |\chi|e^{i\theta}\) in the neighbourhood of the transition point. Although expansions (3.4) have been established only in the sector \(|\arg z| < \frac{1}{2}\pi\), it is found numerically that they remain valid for values of \(z\) in the wider sector \(|\arg z| < \pi\), provided \(|\arg \chi| < \frac{3}{4}\pi\) for \(Q(a,z)\) and \(|\arg(\pm \chi)| < \frac{3}{4}\pi\) for \(P(a,z)\). Greater numerical accuracy is achieved, however, if these latter sectors are restricted to \(|\arg(\pm \chi)| < \frac{1}{2}\pi\). Further results demonstrating the accuracy of expansions (3.4) outside the sector \(|\arg z| < \frac{1}{2}\pi\) are presented in Table 6, where we show the values of \(Q(a,z)\) for different values of \(a\) and \(z\) when \(\frac{1}{2}\pi \leq \arg z \leq \pi\).

Finally, we make a comment on the accuracy of expansion (3.4) compared to the uniform expansions in (1.4) and (4.4). In the neighbourhood of the transition point, the accuracy of expansions (3.4) is found to be comparable to that of the modified Dingle expansions (4.4). This is because both expansions have a similar structure with the asymptotic variables \(z^{-1/2}\) and \(a^{-1/2}\), respectively, being comparable near the transition point. For a given truncation index, however, the Temme expansion (1.4) yields greater accuracy, since its asymptotic variable is \(a^{-1}\), but at the cost of greater computational effort in the evaluation of its coefficients near the transition point. This can be seen by comparing the result of the last entry in Table 6 with those obtained by Temme [16, p. 343] for the function \(\gamma^*(a,x) = x^{-a}P(a,x)\). The last entry in Table 6 yields a value for

\[
x^a\gamma^*(a, -x) = e^{-\pi i a}\{1 - Q(a, xe^{\pi i})\}
\]

when \(a = -100.25\) and \(x = 100\) given by \(0.740006507775256\), which is associated with an absolute error of \(3.767 \times 10^{-15}\) when the truncation index \(m = 10\). For the same values of \(a\) and \(x\), Temme's
result using the expansion (1.4) with a truncation index \( m = 6 \) is associated with the absolute error of \( 5.585 \times 10^{-18} \).

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Appendix A. A bound for the remainder terms \( R_n^\pm(a, z) \)

The remainder terms in (2.9) and (2.14) are defined by

\[
R_n^\pm(a, z) = \int_0^\mu e^{\pm(a-z)\tau-(1/2)\varepsilon^2} r_n(z, \pm \tau) \, d\tau, \tag{A.1}
\]

where \( \mu = |z|^{-1/3} \), with \( \varepsilon \) satisfying \( 0 < \varepsilon < \frac{1}{3} \). The functions \( r_n(z, \pm \tau) \) are the remainders after truncation of the expansion of \( e^{-zh(\pm \tau)} \) at the \( n \)th term, where \( h(\tau) \) is defined in (2.2); see (2.5). From [19, p. 95], an integral representation for \( r_n(z, \pm \tau) \) is given by

\[
\frac{d^n}{d\tau^n} e^{-zh(\pm \tau)} = e^{-zh(\pm \tau)} \sum_{k=1}^n (-z)^k \beta_{k,n}(\pm \tau), \tag{A.3}
\]

where

\[
\beta_{n,n}(\pm \tau) = (h'(\pm \tau))^n, \quad \beta_{n-1,n}(\pm \tau) = \frac{1}{2} n(n-1)(h'(\pm \tau))^{n-2} h''(\pm \tau), \ldots
\]

and the prime denotes differentiation with respect to \( \tau \).

On the interval \( 0 \leq \tau \leq \mu \) we have the order estimates

\[
h'(\pm \mu) = O(\mu^2), \quad h''(\pm \mu) = O(\mu), \quad h'''(\pm \mu) = h''''(\pm \mu) = \cdots = O(1)
\]

as \( |z| \to \infty \). Then, with the above scaling for \( \mu \), the dominant term in the sum in (A.3) (when \( 0 \leq \tau \leq \mu \)) as \( z \to \infty \) is easily shown to correspond to \( k = n \) (we omit these details) and to be \( O(z^n h'(\pm \mu))^n = O(|z|^{1/3} n + 2n\varepsilon) \). From (A.3), we consequently find that for \( |z| \to \infty \) and \( 0 \leq \tau \leq \mu \) (with \( 0 \leq u \leq 1 \)),

\[
\left| \frac{d^n}{du^n} e^{-zh(\pm ut)} \right| \leq \tau^n e^{-\text{Re}(z) h(\pm ut)} \sum_{k=1}^n |z|^k |\beta_{k,n}(\pm ut)|
\]

\[
= \tau^n e^{-\text{Re}(z) h(\pm ut)} O(|z|^{(1/3)n + 2n\varepsilon}).
\]
Upon noting that $h(\pm \tau)$ are monotonic increasing and decreasing functions, respectively, on $\tau \geq 0$, it then follows from (A.2) that

$$|r_n(z, \pm \tau)| = O(|z|^{(1/3)n+2n_0}) \frac{\tau^n}{\Gamma(n)} \int_0^1 (1 - u)^{n-1} e^{-\text{Re}(z)h(\pm u\tau)} \, du$$

$$= O(|z|^{(1/3)n+2n_0}) \frac{\tau^n}{n!} \left\{ \frac{1}{e^{-\text{Re}(z)h(-\tau)}} \right\} (0 \leq \tau \leq \mu) \quad (A.4)$$

for $z \to \infty$ in the sector $|\arg z| < \frac{1}{7} \pi$.

Use of (A.4) in (A.1), when $\text{Re}(z-a) \geq 0$ for the upper sign and $\text{Re}(z-a) \leq 0$ for the negative sign, then shows that

$$|R_{n}^{\pm}(a,z)| \leq \int_0^{\mu} e^{-(1/2)\text{Re}(z)\tau^2} |r_n(z, \pm \tau)| \, d\tau$$

$$= O(|z|^{(1/3)n+2n_0}) J_{\pm},$$

where, by inequality (2.4a) with $\theta = \arg z$,

$$J_{\pm} = \frac{1}{n!} \int_0^{\mu} \tau^n \left\{ \frac{e^{-(1/2)\text{Re}(z)\tau^2}}{e^{-\text{Re}(z)(e^{-\tau}-1+\tau)}} \right\} d\tau$$

$$\leq \frac{1}{n!} \int_0^{\mu} \tau^n \left\{ \frac{e^{-(1/2)\text{Re}(z)\tau^2}}{e^{-(1/3)\text{Re}(z)\tau^2}} \right\} d\tau = O(|z|\cos \theta)^{-\frac{1}{2}} (\text{A.5})$$

Hence

$$|R_{n}^{\pm}(a,z)| = O(|z|^{-(1/6)n-1/2+2n_0}) \quad (n = 1, 2, \ldots)$$

as $z \to \infty$ in $|\arg z| < \frac{1}{7} \pi$, when $\text{Re}(z-a) \geq 0$ or $\text{Re}(z-a) \leq 0$ for the remainder term with the upper or lower sign, respectively.

\section*{References}


