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ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 320–339
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Rota–Baxter algebras and dendriform algebras

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Received 20 November 2006; received in revised form 31 May 2007; accepted 31 May 2007
Available online 30 June 2007

Communicated by C.A. Weibel

Abstract

In this paper we study the adjoint functors between the category of Rota–Baxter algebras and the categories of dendriform dialgebras and trialgebras. In analogy to the well-known theory of the adjoint functor between the category of associative algebras and Lie algebras, we first give an explicit construction of free Rota–Baxter algebras and then apply it to obtain universal enveloping Rota–Baxter algebras of dendriform dialgebras and trialgebras. We further show that free dendriform dialgebras and trialgebras, as represented by binary planar trees and planar trees, are canonical subalgebras of free Rota–Baxter algebras.

MSC: 16A06; 47B99

1. Introduction

It is well-known that the natural functor from the category of associative algebras to that of Lie algebras and the adjoint functor play a fundamental role in the study of these algebraic structures and their applications. This paper establishes a similar relationship between Rota–Baxter algebras and dendriform dialgebras and dendriform trialgebras by using free Rota–Baxter algebras.

A Rota–Baxter algebra is an algebra A with a linear endomorphism R satisfying the *Rota–Baxter equation*:

$$R(x)R(y) = R\left(R(x)y + xR(y) + \lambda xy\right), \quad \forall x, y \in A. \tag{1}$$

Here λ is a fixed element in the base ring and is sometimes denoted by $-\theta$. This equation was introduced by the mathematician Glen E. Baxter [9] in 1960 in his probability study, and was popularized mainly by the work of Rota [58–60] and his school.

Linear operators satisfying Eq. (1) in the context of Lie algebras were introduced independently by Belavin and Drinfeld [10], and Semenov-Tian-Shansky [61] in the 1980s and were related to solutions, called r-matrices, of the (modified) classical Yang–Baxter equation, named after the physicists Chen-ning Yang and Rodney Baxter. Recently, there have been several interesting developments of Rota–Baxter algebras in theoretical physics and mathematics, including quantum field theory [12,13], Yang–Baxter equations [1–3], shuffle products [16,36,37], operads [4,14,17, 45–47], Hopf algebras [8,16,28], combinatorics [33] and number theory [16,20,34,40,54,55,62]. The most prominent

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.05.025

of these is the work [12,13] of Connes and Kreimer in their Hopf algebraic approach to renormalization theory in perturbative quantum field theory, continued in a series of papers [15,19,22–28].

A dendriform dialgebra is a module D with two binary operations $\prec$ and $\succ$ that satisfy three relations between them (see Eq. (18)). This concept was introduced by Loday [48] in 1995 with motivation from algebraic K-theory, and was further studied in connection with several areas in mathematics and physics, including operads [49], homology [31, 32], Hopf algebras [11,42,53,57,29], Lie and Leibniz algebras [32], combinatorics [6,7,30,52], arithmetic [50] and quantum field theory [30,41].

A few years later Loday and Ronco defined dendriform trialgebras in their study [53] of polytopes and Koszul duality. Such a structure is a module T equipped with binary operations $\prec$, $\succ$ and $\cdot$ that satisfy seven relations that will be recalled in Eq. (19).

The dendriform dialgebra and trialgebra share the property that the sum of the binary operations $\prec + \succ$ (for dialgebra) or $\prec + \succ + \cdot$ (for trialgebra) is associative. Other dendriform algebra structures have the similar property of "splitting associativity" in the sense that an associative product decomposes into a linear combination of several binary operations. Many such structures have been obtained lately, such as the quadri-algebra of Loday and Aguiar [4] and the ennea- and NS-algebra of Leroux [45,46]. In [17] (see also [51]), we showed how these more complex structures, equipped with large numbers of compositions and relations, can be derived from an operadic point of view in terms of products. Further examples and developments can be found in [18,49].

The first link between Rota–Baxter algebras and dendriform algebras was given by Aguiar [1] who showed that a Rota–Baxter algebra of weight $\lambda = 0$ carries a dendriform dialgebra structure, resembling the Lie algebra structure on an associative algebra. This has been extended to further connections between linear operators and dendriform type algebras [14,46,4,17], in particular to dendriform trialgebras by the first named author. See Theorem 3.1 for details.

Consequently, there are natural functors from the category of Rota–Baxter algebras of weight λ to the categories of dendriform dialgebras and trialgebras. We study the adjoint functors in this paper.

As a preparation, we first construct in Section 2 free Rota–Baxter algebras (Theorem 2.6) which play a central role in the study of the adjoint functors. This is in analogy to the central role played by the free associative algebras in the study of the adjoint functor from the category of Lie algebras to the category of associative algebras. As we will see, free Rota–Baxter algebras can be defined in various generalities, such as over a set or over another algebra, in various contexts, such as unitary or nonunitary algebras, and they can be constructed in various terms, such as by words or by trees, either explicitly or recursively. For the purpose of our application to adjoint functors, we only consider a special case of free Rota–Baxter algebras, namely free nonunitary Rota–Baxter algebras $Ш^{NC,0}(A)$ generated by another algebra A that possesses a basis over the base ring. Further studies of free Rota–Baxter algebras can be found in [5, 21,28,35,38,39].

Then in Section 3, we use these free Rota–Baxter algebras to obtain adjoint functors of the functors from Rota–Baxter algebras to dendriform dialgebras (Theorem 3.5) and trialgebras (Theorem 3.4) by proving the existence of the corresponding universal enveloping Rota–Baxter algebras. In the case of dendriform trialgebras, let $D = (D, \prec, \succ, \cdot)$ be a dendriform trialgebra. Let $Ш^{NC,0}(D)$ be the free nonunitary Rota–Baxter algebra over the nonunitary algebra $(D, \cdot)$ constructed in Theorem 2.6. Let I_R be a suitable Rota–Baxter ideal of $Ш^{NC,0}(D)$ generated by relations from $\prec$ and $\succ$. Theorem 3.5 shows that the quotient Rota–Baxter algebra $Ш^{NC,0}(D)/I_R$ is the universal enveloping Rota–Baxter algebra of D in the sense of Definition 3.3.

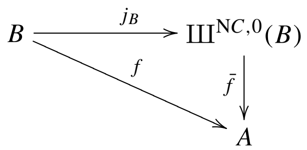
The special case of free dendriform algebras is considered in Section 4 where we realize the free dendriform dialgebra and trialgebra of Loday and Loday–Ronco in terms of decorated planar rooted trees as canonical subalgebras of free Rota–Baxter algebras.

Notations. In this paper, $\mathbf{k}$ is a commutative unitary ring which will be further assumed to be a field in Sections 3 and 4. Let **Alg** be the category of unitary $\mathbf{k}$-algebras A whose unit is identified with the unit $\mathbf{1}$ of $\mathbf{k}$ by the structure homomorphism $\mathbf{k} \to A$. Let $\mathbf{Alg}^0$ be the category of nonunitary $\mathbf{k}$-algebras. Similarly let $\mathbf{RB}_\lambda$ (resp. $\mathbf{RB}^0_\lambda$) be the category of unitary (resp. nonunitary) Rota–Baxter $\mathbf{k}$-algebras of weight λ. The subscript λ will be suppressed if there is no danger of confusion.

2. Free nonunitary Rota–Baxter algebras on an algebra

We now construct free nonunitary Rota–Baxter algebras over another nonunitary algebra. Other than its theoretical significance, our main purpose is for the application in later sections to study universal enveloping Rota–Baxter algebras of dendriform dialgebras and trialgebras. The reader can regard such free Rota–Baxter algebras over another algebra as the Rota–Baxter analog of the tensor algebra over a module. It is well-known that such tensor algebras are essential in the study of enveloping algebras of Lie algebras [56]. Because of the nonunitariness of Lie algebras, it is the free nonunitary, instead of unitary, associative algebras that are used in the study of the adjoint functor from Lie algebra to associative algebras. For the similar reason, free nonunitary Rota–Baxter algebras are convenient in the study of the adjoint functor from dendriform algebras to Rota–Baxter algebras. As remarked earlier, other cases of free Rota–Baxter algebras are considered elsewhere [21].

Let B be a nonunitary $\mathbf{k}$-algebra. Recall [36,37] that a free nonunitary Rota–Baxter algebra over B is a nonunitary Rota–Baxter algebra $Ш^{NC,0}(B)$ with a Rota–Baxter operator R_B and a nonunitary algebra homomorphism $j_B : B \to Ш^{NC,0}(B)$ such that, for any nonunitary Rota–Baxter algebra A and any nonunitary algebra homomorphism $f : B \to A$, there is a unique nonunitary Rota–Baxter algebra homomorphism $\bar{f} : Ш^{NC,0}(B) \to A$ such that $\bar{f} \circ j_B = f$.



We assume that the nonunitary algebra B possesses a basis over the base ring $\mathbf{k}$. This is no restriction if the base ring is a field as is customarily taken to be the case in the study of dendriform algebras/operads and therefore in our later sections.

We first display a $\mathbf{k}$-basis of the free Rota–Baxter algebra in terms of words in Section 2.1. The product on the free Rota–Baxter algebra is given in 2.2 and the universal property of the free Rota–Baxter algebra is proved in 2.3.

2.1. A basis of a free Rota–Baxter algebra as words

Let B be a nonunitary $\mathbf{k}$-algebra with a $\mathbf{k}$-basis X. We first display a $\mathbf{k}$-basis $\mathfrak{X}_\infty$ of $Ш^{NC,0}(B)$ in terms of words from the alphabet set X.

Let $\lfloor$ and $\rfloor$ be symbols, called brackets, and let $X' = X \cup \{\lfloor, \rfloor\}$. Let $M(X')$ be the free semigroup generated by X'.

Definition 2.1. Let Y, Z be two subsets of $M(X')$. Define the *alternating product* of Y and Z to be

$$\Lambda_X(Y,Z) = \left(\bigcup_{r\geq 1} (Y\lfloor Z\rfloor)^r\right) \bigcup \left(\bigcup_{r\geq 0} (Y\lfloor Z\rfloor)^r\, Y\right) \bigcup \left(\bigcup_{r\geq 1} (\lfloor Z\rfloor Y)^r\right) \bigcup \left(\bigcup_{r\geq 0} (\lfloor Z\rfloor Y)^r \lfloor Z\rfloor\right). \tag{2}$$

We construct a sequence $\mathfrak{X}_n$ of subsets of $M(X')$ by the following recursion. Let $\mathfrak{X}_0 = X$ and, for $n \geq 0$, define

$$\mathfrak{X}_{n+1} = \Lambda_X(X, \mathfrak{X}_n).$$

More precisely,

$$\mathfrak{X}_{n+1} = \left(\bigcup_{r\geq 1} (X\lfloor \mathfrak{X}_n\rfloor)^r\right) \bigcup \left(\bigcup_{r\geq 0} (X\lfloor \mathfrak{X}_n\rfloor)^r\, X\right) \bigcup \left(\bigcup_{r\geq 1} (\lfloor \mathfrak{X}_n\rfloor X)^r\right) \bigcup \left(\bigcup_{r\geq 0} (\lfloor \mathfrak{X}_n\rfloor X)^r \lfloor \mathfrak{X}_{n-1}\rfloor\right). \tag{3}$$

Further, define

$$\mathfrak{X}_\infty = \bigcup_{n\geq 0} \mathfrak{X}_n = \varinjlim \mathfrak{X}_n. \tag{4}$$

Here the second equation in Eq. (4) follows since $\mathfrak{X}_1 \supseteq \mathfrak{X}_0$ and, assuming $\mathfrak{X}_n \supseteq \mathfrak{X}_{n-1}$, we have

$$\mathfrak{X}_{n+1} = \Lambda_X(X, \mathfrak{X}_n) \supseteq \Lambda_X(X, \mathfrak{X}_{n-1}) \supseteq \mathfrak{X}_n.$$

Definition 2.2. A word in $\mathfrak{X}_\infty$ is called a *(strict) Rota–Baxter (bracketed) word (RBWs).*

A similar concept of parenthesized words has appeared in the work of Kreimer [44] to represent Hopf algebra structure on Feynman diagrams in pQFT, with a different set of restrictions on the words. We use the brackets $\lfloor$ and $\rfloor$ instead of (and) to avoid confusion with the usual meaning of parentheses.

The verification of the following properties of RBWs are quite easy and is left to the reader.

Lemma 2.3. (a) *For each $n \geq 1$, the union of $\mathfrak{X}_n = \Lambda_X(X, \mathfrak{X}_{n-1})$ expressed in Eq. (3) is disjoint:*

$$\mathfrak{X}_n = \left(\dot{\bigcup_{r\geq 1}} (X\lfloor \mathfrak{X}_{n-1}\rfloor)^r\right) \dot{\bigcup} \left(\dot{\bigcup_{r\geq 0}} (X\lfloor \mathfrak{X}_{n-1}\rfloor)^r X\right)$$
$$\dot{\bigcup} \left(\dot{\bigcup_{r\geq 1}} (\lfloor \mathfrak{X}_{n-1}\rfloor X)^r\right) \dot{\bigcup} \left(\dot{\bigcup_{r\geq 0}} (\lfloor \mathfrak{X}_{n-1}\rfloor X)^r \lfloor \mathfrak{X}_{n-1}\rfloor\right). \tag{5}$$

(b) *We further have the disjoint union*

$$\mathfrak{X}_\infty = \left(\dot{\bigcup_{r\geq 1}} (X\lfloor \mathfrak{X}_{\infty}\rfloor)^r\right) \dot{\bigcup} \left(\dot{\bigcup_{r\geq 0}} (X\lfloor \mathfrak{X}_{\infty}\rfloor)^r X\right)$$
$$\dot{\bigcup} \left(\dot{\bigcup_{r\geq 1}} (\lfloor \mathfrak{X}_{\infty}\rfloor X)^r\right) \dot{\bigcup} \left(\dot{\bigcup_{r\geq 0}} (\lfloor \mathfrak{X}_{\infty}\rfloor X)^r \lfloor \mathfrak{X}_{\infty}\rfloor\right). \tag{6}$$

(c) *Every RBW $\mathbf{x} \neq \mathbf{1}$ has a unique decomposition*

$$\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b, \tag{7}$$

where $\mathbf{x}_i, 1 \leq i \leq b$, is alternatively in X or in $\lfloor \mathfrak{X}_\infty \rfloor$. This decomposition will be called the standard decomposition *of* $\mathbf{x}$.

For a RBW $\mathbf{x}$ in $\mathfrak{X}_\infty$ with standard decomposition $\mathbf{x}_1 \cdots \mathbf{x}_b$, we define b to be the *breadth* $b(\mathbf{x})$ of $\mathbf{x}$, we define the *head* $h(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_1$ is in X (resp. in $\lfloor \mathfrak{X}_\infty \rfloor$). Similarly define the *tail* $t(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_b$ is in X (resp. in $\lfloor \mathfrak{X}_\infty \rfloor$). In terms of the decomposition (5), the head, tail and breadth of a word $\mathbf{x}$ are given in the following table.

$\mathbf{x}$	$(X\lfloor \mathfrak{X}_{n-1}\rfloor)^r$	$(X\lfloor \mathfrak{X}_{n-1}\rfloor)^r X$	$(\lfloor \mathfrak{X}_{n-1}\rfloor X)^r$	$(\lfloor \mathfrak{X}_{n-1}\rfloor X)^r \lfloor \mathfrak{X}_{n-1}\rfloor$
$h(\mathbf{x})$	0	0	1	1
$t(\mathbf{x})$	1	0	0	1
$b(\mathbf{x})$	$2r$	$2r+1$	$2r$	$2r+1$

Finally, define the *depth* $d(\mathbf{x})$ to be

$$d(\mathbf{x}) = \min\{n \mid \mathbf{x} \in \mathfrak{X}_n\}.$$

So, in particular, the depth of elements in X is 0 and depth of elements in $\lfloor X \rfloor$ is one.

Example 2.4. For $x_1, x_2, x_3 \in X$, the word $\lfloor \lfloor x_1 \rfloor x_2 \rfloor x_3$ has head 1, tail 0, breadth 2 and depth 2.

2.2. The product in a free Rota–Baxter algebra

Let

$$Ш^{NC,0}(B) = \bigoplus_{\mathbf{x}\in\mathfrak{X}_\infty} \mathbf{k}\mathbf{x}.$$

We now define a product $\diamond$ on $Ш^{NC,0}(B)$ by defining $\mathbf{x} \diamond \mathbf{x}' \in Ш^{NC,0}(B)$ for $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ and then extending bilinearly. Roughly speaking, the product of $\mathbf{x}$ and $\mathbf{x}'$ is defined to be the concatenation whenever $t(\mathbf{x}) \neq h(\mathbf{x}')$. When $t(\mathbf{x}) = h(\mathbf{x}')$, the product is defined by the product in B or by the Rota–Baxter relation in Eq. (8).

To be precise, we use induction on the sum $n := d(\mathbf{x}) + d(\mathbf{x}')$ of the depths of $\mathbf{x}$ and $\mathbf{x}'$. Then $n \geq 0$. If $n = 0$, then $\mathbf{x}, \mathbf{x}'$ are in X and so are in B and we define $\mathbf{x} \diamond \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}' \in B \subseteq Ш^{NC,0}(B)$. Here $\cdot$ is the product in B.

Suppose $\mathbf{x} \diamond \mathbf{x}'$ have been defined for all $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n \geq k \geq 0$ and let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n = k + 1$.

First assume the breadth $b(\mathbf{x}) = b(\mathbf{x}') = 1$. Then $\mathbf{x}$ and $\mathbf{x}'$ are in X or $\lfloor \mathfrak{X}_\infty \rfloor$. We accordingly define

$$\mathbf{x} \diamond \mathbf{x}' = \begin{cases} \mathbf{x} \cdot \mathbf{x}', & \text{if } \mathbf{x}, \mathbf{x}' \in X, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in X, \mathbf{x}' \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}' \in X, \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \diamond \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \diamond \lfloor \overline{\mathbf{x}}' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}} \diamond \overline{\mathbf{x}}' \rfloor, & \text{if } \mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor \in \lfloor \mathfrak{X}_\infty \rfloor. \end{cases} \tag{8}$$

Here the product in the first case is the product in B, in the second and third case are by concatenation and in the fourth case is by the induction hypothesis since for the three products on the right-hand side we have

$$\begin{aligned} d(\lfloor \overline{\mathbf{x}} \rfloor) + d(\overline{\mathbf{x}}') &= d(\lfloor \overline{\mathbf{x}} \rfloor) + d(\lfloor \overline{\mathbf{x}}' \rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 1, \\ d(\overline{\mathbf{x}}) + d(\lfloor \overline{\mathbf{x}}' \rfloor) &= d(\lfloor \overline{\mathbf{x}} \rfloor) + d(\lfloor \overline{\mathbf{x}}' \rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 1, \\ d(\overline{\mathbf{x}}) + d(\overline{\mathbf{x}}') &= d(\lfloor \overline{\mathbf{x}} \rfloor) - 1 + d(\lfloor \overline{\mathbf{x}}' \rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 2 \end{aligned}$$

which are all less than or equal to k.

Now assume $b(\mathbf{x}) > 1$ or $b(\mathbf{x}') > 1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_{b'}$ be the standard decompositions from Lemma 2.3. We then define

$$\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}_1 \cdots \mathbf{x}_{b-1} (\mathbf{x}_b \diamond \mathbf{x}'_1)\, \mathbf{x}'_2 \cdots \mathbf{x}'_{b'} \tag{9}$$

where $\mathbf{x}_b \diamond \mathbf{x}'_1$ is defined by Eq. (8) and the rest is given by concatenation. The concatenation is well-defined since by Eq. (8), we have $h(\mathbf{x}_b) = h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $t(\mathbf{x}'_1) = t(\mathbf{x}_b \diamond \mathbf{x}'_1)$. Therefore, $t(\mathbf{x}_{b-1}) \neq h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $h(\mathbf{x}'_2) \neq t(\mathbf{x}_b \diamond \mathbf{x}'_1)$.

We record the following simple properties of $\diamond$ for later applications.

Lemma 2.5. *Let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$. We have the following statements:*

(a) $h(\mathbf{x}) = h(\mathbf{x} \diamond \mathbf{x}')$ *and* $t(\mathbf{x}') = t(\mathbf{x} \diamond \mathbf{x}')$.
(b) *If* $t(\mathbf{x}) \neq h(\mathbf{x}')$, *then* $\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}\mathbf{x}'$ *(concatenation).*
(c) *If* $t(\mathbf{x}) \neq h(\mathbf{x}')$, *then for any* $\mathbf{x}'' \in \mathfrak{X}_\infty$,

$$(\mathbf{x}\mathbf{x}') \diamond \mathbf{x}'' = \mathbf{x}(\mathbf{x}' \diamond \mathbf{x}''), \qquad \mathbf{x}'' \diamond (\mathbf{x}\mathbf{x}') = (\mathbf{x}'' \diamond \mathbf{x})\mathbf{x}'.$$

Extending $\diamond$ bilinearly, we obtain a binary operation

$$Ш^{NC,0}(B) \otimes Ш^{NC,0}(B) \to Ш^{NC,0}(B).$$

For $\mathbf{x} \in \mathfrak{X}_\infty$, define

$$R_B(\mathbf{x}) = \lfloor \mathbf{x} \rfloor. \tag{10}$$

Obviously $\lfloor \mathbf{x} \rfloor$ is again in $\mathfrak{X}_\infty$. Thus R_B extends to a linear operator R_B on $Ш^{NC,0}(B)$. Let

$$j_X : X \to \mathfrak{X}_\infty \to Ш^{NC,0}(B)$$

be the natural injection which extends to an algebra injection

$$j_B : B \to Ш^{NC,0}(B). \tag{11}$$

The following is our first main result which will be proved in the next subsection:

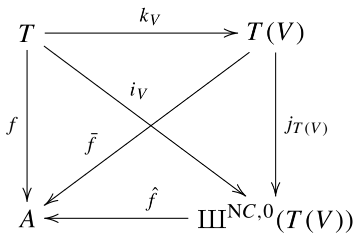
Theorem 2.6. *Let B be a nonunitary* **k***-algebra with a* **k***-basis X.*

(a) *The pair* $(Ш^{NC,0}(B), \diamond)$ *is a nonunitary associative algebra.*
(b) *The triple* $(Ш^{NC,0}(B), \diamond, R_B)$ *is a nonunitary* Rota–Baxter algebra *of weight* λ.
(c) *The quadruple* $(Ш^{NC,0}(B), \diamond, R_B, j_B)$ *is the free nonunitary* Rota–Baxter algebra *of weight λ on the algebra B.*

The following corollary of the theorem will be used later in the paper.

Corollary 2.7. *Let V be a* **k***-module and let $T(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ be the tensor algebra over V. Then $Ш^{\mathrm{NC},0}(T(V))$, together with the natural injection $i_V : V \to T(V) \xrightarrow{j_{T(V)}} Ш^{\mathrm{NC},0}(T(V))$, is a free nonunitary Rota–Baxter algebra over V, in the sense that, for any nonunitary Rota–Baxter algebra A and* **k***-module map $f : V \to A$ there is a unique nonunitary Rota–Baxter algebra homomorphism $\hat{f} : Ш^{\mathrm{NC},0}(T(V)) \to A$ such that $k_V \circ \bar{f} = f$.*

Proof. The maps in the corollary and in this proof are organized in the following diagram:



For the given **k**-module V, note that $T(V)$, together with the natural injection $k_V : V \to T(V)$, is the free nonunitary **k**-algebra over V. So for the given **k**-algebra A and **k**-module map $f : V \to A$, there is a unique nonunitary **k**-algebra homomorphism $\tilde{f} : T(V) \to A$ such that $\tilde{f} \circ k_V = f$. Then by the universal property of the free Rota–Baxter algebra $Ш^{\mathrm{NC},0}(T(V))$, there is a unique $\bar{\tilde{f}} : Ш^{\mathrm{NC},0}(T(V)) \to A$ such that $\bar{\tilde{f}} \circ j_{T(V)} = \tilde{f}$. Since $i_V = j_{T(V)} \circ k_V$, we have $\bar{\tilde{f}} i_V = \tilde{f} \circ k_V = f$. So we have proved the existence of $\hat{f} = \bar{\tilde{f}}$.

For the uniqueness of $\hat{f}$. Suppose there is another $\hat{f}' : Ш^{\mathrm{NC},0}(T(V)) \to A$ such that $\hat{f}' \circ i_V = f$. Then we have

$$\hat{f}' \circ j_{T(V)} \circ k_V = \hat{f}' \circ i_V = f = \hat{f} \circ i_V = \hat{f} \circ j_{T(V)} \circ k_V.$$

By the universal property of the free algebra $T(V)$, we have $\hat{f}' \circ j_{T(V)} = \hat{f} \circ j_{T(V)}$. Then by the universal property of the free Rota–Baxter algebra $Ш^{\mathrm{NC},0}(T(V))$, we have $\hat{f}' = \hat{f}$, as needed. □

2.3. The proof of Theorem 2.6

Proof. (a) We just need to verify the associativity. For this we only need to verify

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') \tag{12}$$

for $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$. We will do this by induction on the sum of the depths $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''')$. If $n = 0$, then all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have depth zero and so are in X. In this case the product $\diamond$ is given by the product $\cdot$ in B and so is associative.

Assume the associativity holds for $n \leq k$ and assume that $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ have $n = d(\mathbf{x}')+d(\mathbf{x}'')+d(\mathbf{x}''') = k+1$. If $t(\mathbf{x}') \neq h(\mathbf{x}'')$, then by Lemma 2.5,

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}'\mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'(\mathbf{x}'' \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

Similarly if $t(\mathbf{x}'') \neq h(\mathbf{x}''')$.

Thus we only need to verify the associativity when $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$. We next reduce the breadths of the words.

Lemma 2.8. *If the associativity*

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$$

holds for all $\mathbf{x}'$, $\mathbf{x}''$ and $\mathbf{x}'''$ in $\mathfrak{X}_\infty$ of breadth one, then it holds for all $\mathbf{x}'$, $\mathbf{x}''$ and $\mathbf{x}'''$ in $\mathfrak{X}_\infty$.

Proof. We use induction on the sum of breadths $m := b(\mathbf{x}')+b(\mathbf{x}'')+b(\mathbf{x}''')$. Then $m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume the associativity holds for $3 \leq m \leq j$ and take $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have breadth greater than or equal to 2.

First assume $b(\mathbf{x}') \geq 2$. Then $\mathbf{x}' = \mathbf{x}'_1\mathbf{x}'_2$ with $\mathbf{x}'_1,\ \mathbf{x}'_2 \in \mathfrak{X}_\infty$ and $t(\mathbf{x}'_1) \neq h(\mathbf{x}'_2)$. Thus

$$\begin{aligned}(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' &= ((\mathbf{x}'_1\mathbf{x}'_2) \diamond \mathbf{x}'') \diamond \mathbf{x}''' \\ &= (\mathbf{x}'_1(\mathbf{x}'_2 \diamond \mathbf{x}'')) \diamond \mathbf{x}''' \quad \text{by Lemma 2.5.(c)} \\ &= \mathbf{x}'_1((\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''') \quad \text{by Lemma 2.5.(a) and (c).}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') &= (\mathbf{x}'_1\mathbf{x}'_2) \diamond (\mathbf{x}'' \diamond \mathbf{x}''') \\ &= \mathbf{x}'_1(\mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')).\end{aligned}$$

Thus

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$$

whenever

$$(\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$$

which follows from the induction hypothesis.

A similar proof works if $b(\mathbf{x}''') \geq 2$.

Finally if $b(\mathbf{x}'') \geq 2$, then $\mathbf{x}'' = \mathbf{x}''_1\mathbf{x}''_2$ with $\mathbf{x}''_1,\ \mathbf{x}''_2 \in \mathfrak{X}_\infty$ and $t(\mathbf{x}''_1) \neq h(\mathbf{x}''_2)$. So using Lemma 2.5 repeatedly, we have

$$\begin{aligned}(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' &= (\mathbf{x}' \diamond (\mathbf{x}''_1\mathbf{x}''_2)) \diamond \mathbf{x}''' \\ &= ((\mathbf{x}' \diamond \mathbf{x}''_1)\mathbf{x}''_2) \diamond \mathbf{x}''' \quad \text{by Lemma 2.5.(a) and (c)} \\ &= (\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''') \quad \text{by Lemma 2.5.(a) and (c).}\end{aligned}$$

In the same way, we have

$$(\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

This again proves the associativity. □

To summarize, our proof of the associativity has been reduced to the special case when $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \mathfrak{X}_\infty$ are chosen so that

(a) $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''') = k + 1 \geq 1$ with the assumption that the associativity holds when $n \leq k$.
(b) the elements are of breadth one and
(c) $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$.

By item (b), the head and tail of each of the elements are the same. Therefore by item (c), either all the three elements are in X or they are all in $\lfloor \mathfrak{X}_\infty \rfloor$. If all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ are in X, then as already shown, the associativity follows from the associativity in B.

So it remains to consider $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ all in $\lfloor \mathfrak{X}_\infty \rfloor$. Then $\mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor, \mathbf{x}'' = \lfloor \overline{\mathbf{x}}'' \rfloor, \mathbf{x}''' = \lfloor \overline{\mathbf{x}}''' \rfloor$ with $\overline{\mathbf{x}}', \overline{\mathbf{x}}'', \overline{\mathbf{x}}''' \in \mathfrak{X}_\infty$. Using Eq. (8) and bilinearity of the product $\diamond$, we have

$$\begin{aligned}(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}'' &= \Big\lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' + \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor + \lambda \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \Big\rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\ &= \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\ &= \lfloor \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \\ &\quad + \lfloor \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\ &\quad + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda^2 \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor.\end{aligned}$$

Applying the induction hypothesis in n to the fifth term $(\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor$ and then use Eq. (8) again, we have

$$\begin{aligned}(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}'' &= \lfloor \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \\ &\quad + \lfloor \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \rfloor \\ &\quad + \lambda \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\ &\quad + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda^2 \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') &= \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''' \rfloor) \\
&= \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''') \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''') \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor) \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor) \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''') \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda^2 \lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''') \rfloor \\
&= \lfloor \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''') \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor) \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}''' \rfloor) \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''') \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda^2 \lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''') \rfloor.
\end{aligned}$$

Now by induction, the i-th term in the expansion of $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}'''$ matches with the $\sigma(i)$-th term in the expansion of $\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$. Here the permutation $\sigma \in \Sigma_{11}$ is

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 6 & 9 & 2 & 4 & 7 & 10 & 5 & 3 & 8 & 11 \end{pmatrix}. \tag{13}$$

This completes the proof of the first part of Theorem 2.6.

(b) The proof is immediate from the definition $R_B(\mathbf{x}) = \lfloor \mathbf{x} \rfloor$ and Eq. (8).

(c) Let (A, R) be a unitary Rota–Baxter algebra of weight λ. Let $f : B \to A$ be a nonunitary $\mathbf{k}$-algebra morphism. We will construct a $\mathbf{k}$-linear map $\bar{f} : Ш^{NC}(B) \to A$ by defining $\bar{f}(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{X}_\infty$. We achieve this by defining $\bar{f}(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{X}_n,\ n \geq 0$, using induction on n. For $\mathbf{x} \in \mathfrak{X}_0 := X$, define $\bar{f}(\mathbf{x}) = f(\mathbf{x})$. Suppose $\bar{f}(\mathbf{x})$ has been defined for $\mathbf{x} \in \mathfrak{X}_n$ and consider $\mathbf{x}$ in $\mathfrak{X}_{n+1}$ which is, by definition and Eq. (5),

$$\begin{aligned}
\Lambda_X(X, \mathfrak{X}_n) &= \left(\dot{\bigcup_{r \geq 1}} (X \lfloor \mathfrak{X}_n \rfloor)^r \right) \dot{\bigcup} \left(\dot{\bigcup_{r \geq 0}} (X \lfloor \mathfrak{X}_n \rfloor)^r X \right) \\
&\quad \times \dot{\bigcup} \left(\dot{\bigcup_{r \geq 0}} \lfloor \mathfrak{X}_n \rfloor (X \lfloor \mathfrak{X}_n \rfloor)^r \right) \dot{\bigcup} \left(\dot{\bigcup_{r \geq 0}} \lfloor \mathfrak{X}_n \rfloor (X \lfloor \mathfrak{X}_n \rfloor)^r X \right).
\end{aligned}$$

Let $\mathbf{x}$ be in the first union component $\dot{\bigcup}_{r \geq 1} (X \lfloor \mathfrak{X}_n \rfloor)^r$ above. Then

$$\mathbf{x} = \prod_{i=1}^{r} (\mathbf{x}_{2i-1} \lfloor \mathbf{x}_{2i} \rfloor)$$

for $\mathbf{x}_{2i-1} \in X$ and $\mathbf{x}_{2i} \in \mathfrak{X}_n$, $1 \leq i \leq r$. By the construction of the multiplication $\diamond$ and the Rota–Baxter operator R_B, we have

$$\mathbf{x} = \diamond_{i=1}^{r} (\mathbf{x}_{2i-1} \diamond \lfloor \mathbf{x}_{2i} \rfloor) = \diamond_{i=1}^{r} (\mathbf{x}_{2i-1} \diamond R_B(\mathbf{x}_{2i})).$$

Define

$$\bar{f}(\mathbf{x}) = *_{i=1}^{r} \left(\bar{f}(\mathbf{x}_{2i-1}) * R\left(\bar{f}(\mathbf{x}_{2i}) \right) \right) \tag{14}$$

where the right-hand side is well defined by the induction hypothesis. Similarly define $\bar{f}(\mathbf{x})$ if $\mathbf{x}$ is in the other union components. For any $\mathbf{x} \in \mathfrak{X}_\infty$, we have $R_B(\mathbf{x}) = \lfloor \mathbf{x} \rfloor \in \mathfrak{X}_\infty$, and by definition (Eq. (14)) of $\bar{f}$, we have

$$\bar{f}(\lfloor \mathbf{x} \rfloor) = R(\bar{f}(\mathbf{x})). \tag{15}$$

So $\bar{f}$ commutes with the Rota–Baxter operators. Combining this equation with Eq. (14) we see that if $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ is the standard decomposition of $\mathbf{x}$, then

$$\bar{f}(\mathbf{x}) = \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_b). \tag{16}$$

Note that this is the only possible way to define $\bar{f}(\mathbf{x})$ in order for $\bar{f}$ to be a Rota–Baxter algebra homomorphism extending f.

We remain to prove that the map $\bar{f}$ defined in Eq. (14) is indeed an algebra homomorphism. For this we only need to check the multiplicity

$$\bar{f}(\mathbf{x} \diamond \mathbf{x}') = \bar{f}(\mathbf{x}) * \bar{f}(\mathbf{x}') \tag{17}$$

for all $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$. For this we use induction on the sum of depths $n := d(\mathbf{x}) + d(\mathbf{x}')$. Then $n \geq 0$. When $n = 0$, we have $\mathbf{x}, \mathbf{x}' \in X$. Then Eq. (17) follows from the multiplicity of f. Assume the multiplicity holds for $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n \geq k$ and take $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}_\infty$ with $n = k + 1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_{b'}$ be the standard decompositions. By Eq. (8),

$$\bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \begin{cases} \bar{f}(\mathbf{x}_b \cdot \mathbf{x}'_1), & \text{if } \mathbf{x}_b, \mathbf{x}'_1 \in X, \\ \bar{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in X, \mathbf{x}'_1 \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \bar{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}'_1 \in X, \\ \bar{f}\left(\lfloor \lfloor \bar{\mathbf{x}}_b \rfloor \diamond \bar{\mathbf{x}}'_1 \rfloor + \lfloor \bar{\mathbf{x}}_b \diamond \lfloor \bar{\mathbf{x}}'_1 \rfloor \rfloor + \lambda \lfloor \bar{\mathbf{x}}_b \diamond \bar{\mathbf{x}}'_1 \rfloor\right), & \text{if } \mathbf{x}_b = \lfloor \bar{\mathbf{x}}_b \rfloor, \mathbf{x}'_1 = \lfloor \bar{\mathbf{x}}'_1 \rfloor \in \lfloor \mathfrak{X}_\infty \rfloor. \end{cases}$$

In the first three cases, the right-hand side is $\bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1)$ by the definition of $\bar{f}$. In the fourth case, we have, by Eq. (15), the induction hypothesis and the Rota–Baxter relation of R,

$$\begin{aligned} &\bar{f}(\lfloor \lfloor \bar{\mathbf{x}}_b \rfloor \diamond \bar{\mathbf{x}}'_1 \rfloor + \lfloor \bar{\mathbf{x}}_b \diamond \lfloor \bar{\mathbf{x}}'_1 \rfloor \rfloor + \lambda \lfloor \bar{\mathbf{x}}_b \diamond \bar{\mathbf{x}}'_1 \rfloor) = \bar{f}(\lfloor \lfloor \bar{\mathbf{x}}_b \rfloor \diamond \bar{\mathbf{x}}'_1 \rfloor) + \bar{f}(\lfloor \bar{\mathbf{x}}_b \diamond \lfloor \bar{\mathbf{x}}'_1 \rfloor \rfloor) + \bar{f}(\lambda \lfloor \bar{\mathbf{x}}_b \diamond \bar{\mathbf{x}}'_1 \rfloor) \\ &\quad = R(\bar{f}(\lfloor \bar{\mathbf{x}}_b \rfloor \diamond \bar{\mathbf{x}}'_1)) + R(\bar{f}(\bar{\mathbf{x}}_b \diamond \lfloor \bar{\mathbf{x}}'_1 \rfloor)) + \lambda R(\bar{f}(\bar{\mathbf{x}}_b \diamond \bar{\mathbf{x}}'_1)) \\ &\quad = R(\bar{f}(\lfloor \bar{\mathbf{x}}_b \rfloor) * \bar{f}(\bar{\mathbf{x}}'_1)) + R(\bar{f}(\bar{\mathbf{x}}_b) * \bar{f}(\lfloor \bar{\mathbf{x}}'_1 \rfloor)) + \lambda R(\bar{f}(\bar{\mathbf{x}}_b) * \bar{f}(\bar{\mathbf{x}}'_1)) \\ &\quad = R(R(\bar{f}(\bar{\mathbf{x}}_b)) * \bar{f}(\bar{\mathbf{x}}'_1)) + R(\bar{f}(\bar{\mathbf{x}}_b) * R(\bar{f}(\bar{\mathbf{x}}'_1))) + \lambda R(\bar{f}(\bar{\mathbf{x}}_b) * \bar{f}(\bar{\mathbf{x}}'_1)) \\ &\quad = R(\bar{f}(\bar{\mathbf{x}}_b)) * R(\bar{f}(\bar{\mathbf{x}}'_1)) \\ &\quad = \bar{f}(\lfloor \bar{\mathbf{x}}_b \rfloor) * \bar{f}(\lfloor \bar{\mathbf{x}}'_1 \rfloor) \\ &\quad = \bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1). \end{aligned}$$

Therefore $\bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1)$. Then

$$\begin{aligned} \bar{f}(\mathbf{x} \diamond \mathbf{x}') &= \bar{f}(\mathbf{x}_1 \cdots \mathbf{x}_{b-1} (\mathbf{x}_b \diamond \mathbf{x}'_1) \mathbf{x}'_2 \cdots \mathbf{x}'_{b'}) \\ &= \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_{b-1}) * \bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) * \bar{f}(\mathbf{x}'_2) \cdots \bar{f}(\mathbf{x}'_{b'}) \\ &= \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_{b-1}) * \bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1) * \bar{f}(\mathbf{x}'_2) \cdots \bar{f}(\mathbf{x}'_{b'}) \\ &= \bar{f}(\mathbf{x}) * \bar{f}(\mathbf{x}'). \end{aligned}$$

This is what we need. □

3. Universal enveloping algebras of dendriform trialgebras

3.1. Dendriform dialgebras and trialgebras

We recall the following definitions. A dendriform dialgebra [48] is a module D with two binary operations $\prec$ and $\succ$ such that

$$\begin{aligned} &(x \prec y) \prec z = x \prec (y \prec z + y \succ z), \qquad (x \succ y) \prec z = x \succ (y \prec z), \\ &(x \prec y + x \succ y) \succ z = x \succ (y \succ z) \end{aligned} \tag{18}$$

for $x, y, z \in D$.

A dendriform trialgebra [53] is a module T equipped with binary operations $\prec$, $\succ$ and $\cdot$ that satisfy the relations

$$\begin{aligned} &(x \prec y) \prec z = x \prec (y \star z), \qquad (x \succ y) \prec z = x \succ (y \prec z), \\ &(x \star y) \succ z = x \succ (y \succ z), \qquad (x \succ y) \cdot z = x \succ (y \cdot z), \\ &(x \prec y) \cdot z = x \cdot (y \succ z), \qquad (x \cdot y) \prec z = x \cdot (y \prec z), \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z). \end{aligned} \tag{19}$$

Here $\star = \prec + \succ + \cdot$. The category of dendriform trialgebras $(D, \prec, \succ, \cdot)$ is denoted by **DT**. Recall that $\cdot$, as well as $\star$, is an associative product. The category **DD** of dendriform dialgebras can be identified with the subcategory of **DT** of objects with $\cdot = 0$.

These algebras are related to Rota–Baxter algebras by the following theorem.

Theorem 3.1 (*Aguiar [2], Ebrahimi-Fard [14]*).

(a) *A Rota–Baxter algebra* (A, R) *of weight zero defines a dendriform dialgebra* $(A, \prec_R, \succ_R)$*, where*

$$x \prec_R y = xR(y), \qquad x \succ_R y = R(x)y. \tag{20}$$

(b) *A Rota–Baxter algebra* (A, R) *of weight* λ *defines a dendriform trialgebra* $(A, \prec_R, \succ_R, \cdot_R)$*, where*

$$x \prec_R y = xR(y), \qquad x \succ_R y = R(x)y, \qquad x \cdot_R y = \lambda xy. \tag{21}$$

(c) *A Rota–Baxter algebra* (A, R) *of weight* λ *defines a dendriform dialgebra* $(A, \prec'_R, \succ'_R)$*, where*

$$x \prec'_R y = xR(y) + \lambda xy, \qquad x \succ'_R y = R(x)y. \tag{22}$$

We note that (22) specializes to (20) when $\lambda = 0$. The same can be said of (21) since when $\lambda = 0$, the product $\cdot_R$ is zero and the relations of the trialgebra reduces to the relations of a dialgebra.

It is easy to see that the maps between objects in the categories $\mathbf{RB}^0_\lambda$, $\mathbf{DD}$ and $\mathbf{DT}$ in Theorem 3.1 are compatible with the morphisms. Thus we obtain functors

$$\mathcal{E} : \mathbf{RB}^0_\lambda \to \mathbf{DT}, \qquad \mathcal{F} : \mathbf{RB}^0_\lambda \to \mathbf{DD}.$$

We will study their adjoint functors. The two functors $\mathcal{E}$ and $\mathcal{F}$ are related by the following simple observation:

Proposition 3.2. (a) *Let* $(D, \prec, \succ, \cdot)$ *be in* $\mathbf{DT}$. *Then* $(D, \prec', \succ')$ *is in* $\mathbf{DD}$. *Here* $\prec' = \prec + \cdot$ *and* $\succ' = \succ$.
(b) *Let* $\mathcal{G} : \mathbf{DT} \to \mathbf{DD}$ *be the functor obtained from (a). Then we have* $\mathcal{F} = \mathcal{G} \circ \mathcal{E}$.
(c) *Fix a* $\lambda \in \mathbf{k}$. *If the adjoint functors* $\mathcal{E}' : \mathbf{DT} \to \mathbf{RB}^0_\lambda$ *and* $\mathcal{G}' : \mathbf{DD} \to \mathbf{DT}$ *exist, then the adjoint functor* $\mathcal{F}' : \mathbf{DD} \to \mathbf{RB}^0_\lambda$ *exists and* $\mathcal{F}' = \mathcal{E}' \circ \mathcal{G}'$.

Proof. (a) Let $\star' = \prec' + \succ$. Then we have $\star' = \star$. We have

$$\begin{aligned}(a \prec' b) \prec' c &= (a \cdot b + a \prec b) \prec' c \\ &= (a \cdot b + a \prec b) \cdot c + (a \cdot b + a \prec b) \prec c \\ &= (a \cdot b) \cdot c + (a \prec b) \cdot c + (a \cdot b) \prec c + (a \prec b) \prec c \\ &= a \cdot (b \cdot c) + a \cdot (b \succ c) + a \cdot (b \prec c) + a \prec (b \star c) \quad \text{(by Eq. (19))} \\ &= a \prec' (b \star' c).\end{aligned}$$

This verifies the first relation for the dendriform dialgebra. The other two relations are also easy to verify:

$$(a \succ' b) \succ' c = (a \succ b) \succ c = a \succ (b \star c) = a \succ' (b \star' c).$$
$$(a \succ' b) \prec' c = (a \succ b) \cdot c + (a \succ b) \prec c = a \succ (b \cdot c) + a \succ (b \prec c) = a \succ' (b \prec' c).$$

(b) For $(A, R) \in \mathbf{RB}^0_\lambda$, by Theorem 3.1 and item (a), we have

$$\begin{aligned}\mathcal{G}(\mathcal{E}((A, R))) &= \mathcal{G}((A, \prec_R, \succ_R, \cdot_R)) \\ &= (A, \prec_R + \cdot_R, \succ_R) \\ &= \mathcal{F}((A, R)).\end{aligned}$$

It is easy to check that the composition is also compatible with the morphisms. So we get the equality of functors.

(c) is standard: for any $C \in \mathbf{DD}$ and $A \in \mathbf{RB}^0_\lambda$, we have

$$\begin{aligned}\mathrm{Hom}(C, \mathcal{G}(\mathcal{F}(A))) &\cong \mathrm{Hom}(\mathcal{G}'(C), \mathcal{F}(A)) \\ &\cong \mathrm{Hom}(\mathcal{F}'(\mathcal{G}'(C)), A).\end{aligned}$$

So $\mathcal{F}'(\mathcal{G}'(C)) = \mathcal{E}'(C)$. $\square$

3.2. Universal enveloping Rota–Baxter algebras

Motivated by the enveloping algebra of a Lie algebra, we are naturally led to the following definition:

Definition 3.3. Let $D \in \mathbf{DT}$ (resp. $\mathbf{DD}$) and let $\lambda \in \mathbf{k}$. A *universal enveloping Rota–Baxter algebra* of weight λ of D is a Rota–Baxter algebra $\mathrm{RB}(D) := \mathrm{RB}_\lambda(D) \in \mathbf{RB}^0_\lambda$ with a morphism $\rho : D \to \mathrm{RB}(D)$ in $\mathbf{DT}$ (resp. $\mathbf{DD}$) such that for any $A \in \mathbf{RB}^0_\lambda$ and morphism $f : D \to A$ in $\mathbf{DT}$ (resp. $\mathbf{DD}$), there is a unique $\check{f} : \mathrm{RB}(D) \to A$ in $\mathbf{RB}^0_\lambda$ such that $\check{f} \circ \rho = f$.

By the universal property of $\mathrm{RB}(D)$, it is unique up to isomorphisms in $\mathbf{RB}^0_\lambda$.

3.3. The existence of enveloping algebras

We will separately consider the enveloping algebras for dialgebras and trialgebras.

3.3.1. The trialgebra case

Let $D = (D, \prec, \succ, \cdot) \in \mathbf{DT}$. Then $(D, \cdot)$ is a nonunitary $\mathbf{k}$-algebra. Let $\lambda \in \mathbf{k}$ be given. Let $\text{Ш}^{NC,0}(D) := \text{Ш}^{NC,0}{}_\lambda(D)$ be the free nonunitary Rota–Baxter algebra over D of weight λ constructed in Section 2.2. Identify D as a subalgebra of $\text{Ш}^{NC,0}(D)$ by the natural injection j_D in Eq. (11). Let I_R be the Rota–Baxter ideal of $\text{Ш}^{NC,0}(D)$ generated by the set

$$\{x \prec y - x\lfloor y\rfloor, x \succ y - \lfloor x\rfloor y \mid x, y \in D\}. \tag{23}$$

Here a Rota–Baxter ideal of $\text{Ш}^{NC,0}(D)$ is an ideal I of $\text{Ш}^{NC,0}(D)$ such that $R_B(I) \subseteq I$, and the Rota–Baxter ideal of $\text{Ш}^{NC,0}(D)$ generated by a subset of $\text{Ш}^{NC,0}(D)$ is the intersection of all Rota–Baxter ideals of $\text{Ш}^{NC,0}(D)$ that contain the subset. Let $\pi : \text{Ш}^{NC,0}(D) \to \text{Ш}^{NC,0}(D)/I_R$ be the quotient map.

Theorem 3.4. *The quotient Rota–Baxter algebra $\text{Ш}^{NC,0}(D)/I_R$, together with $\rho := \pi \circ j_D$, is the universal enveloping Rota–Baxter algebra of D.*

The theorem provides the adjoint functor $\mathcal{E}' : \mathbf{DT} \to \mathbf{RB}^0$ of the functor $\mathcal{E} : \mathbf{RB}^0 \to \mathbf{DT}$.

Proof. Let $(A, R) \in \mathbf{RB}^0_\lambda$. It gives an object in $\mathbf{DT}$ by Theorem 3.1 which we still denote by A. Let $f : D \to A$ be a morphism in $\mathbf{DT}$. We will complete the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{\;j_D\;} & \text{Ш}^{NC,0}(D) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \downarrow{\scriptstyle \pi} \\ A & \xleftarrow{\;\check{f}\;} & \text{Ш}^{NC,0}(D)/I_R \end{array} \tag{24}$$

By the freeness of $\text{Ш}^{NC,0}(D)$, there is a morphism $\bar{f} : \text{Ш}^{NC,0}(D) \to A$ in $\mathbf{RB}^0$ such that the upper left triangle commutes. So for any $x, y \in D$, by Eq. (14), we have

$$\begin{aligned} \bar{f}(x \prec y - x\lfloor y\rfloor) &= \bar{f}(x \prec y) - \bar{f}(x\lfloor y\rfloor) \\ &= \bar{f}(x \prec y) - \bar{f}(x)R(\bar{f}(y)) \\ &= f(x \prec y) - f(x)R(f(y)) \\ &= f(x \prec y) - f(x) \prec_R f(y) \\ &= f(x \prec y) - f(x \prec y) = 0. \end{aligned}$$

Therefore, $x \prec y - x\lfloor y\rfloor$ is in $\ker(\bar{f})$. Similarly, $x \succ y - \lfloor x\rfloor y$ is in $\ker(\bar{f})$. Thus I_R is in $\ker(\bar{f})$ and there is a morphism $\check{f} : \text{Ш}^{NC,0}(D)/I_R \to A$ in $\mathbf{RB}^0$ such that $\bar{f} = \check{f} \circ \pi$. Then

$$\check{f} \circ \rho = \check{f} \circ \pi \circ j_D = \bar{f} \circ j_D = f.$$

This proves the existence of $\check{f}$.

Suppose $\check{f}' : Ш^{NC,0}(D)/I_R \to A$ is a morphism in $\mathbf{RB}^0$ such that $\check{f}' \circ \rho = f$. Then

$$(\check{f}' \circ \pi) \circ j_D = f = (\check{f} \circ \pi) \circ j_D.$$

By the universal property of the free Rota–Baxter algebra $Ш^{NC,0}(D)$ over D, we have $\check{f}' \circ \pi = \check{f} \circ \pi$ in $\mathbf{RB}^0$. Since π is surjective, we have $\check{f}' = \check{f}$. This proves the uniqueness of $\check{f}$. □

3.3.2. The dialgebra case

Now let $D = (D, \prec, \succ) \in \mathbf{DD}$. Let $T(D) = \bigoplus_{n \geq 1} D^{\otimes n}$ be the tensor product algebra over D. Then $T(D)$ is the free nonunitary algebra generated by the $\mathbf{k}$-module D [43, Prop. II.5.1]. By Corollary 2.7, $Ш^{NC,0}(T(D))$, with the natural injection $i_D : D \to T(D) \to Ш^{NC,0}(T(D))$, is the free Rota–Baxter algebra over the vector space D.

Let J_R be the Rota–Baxter ideal of $Ш^{NC,0}(T(D))$ generated by the set

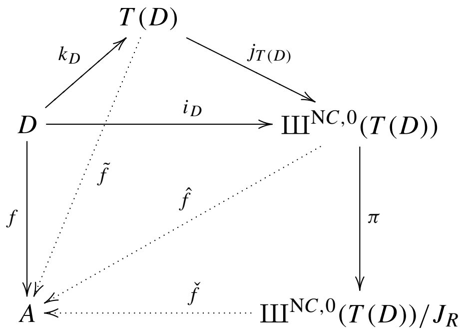
$$\{x \prec y - x\lfloor y\rfloor - \lambda x \otimes y,\ x \succ y - \lfloor x\rfloor y \mid x, y \in D\}. \tag{25}$$

Let $\pi : Ш^{NC,0}(T(D)) \to Ш^{NC,0}(T(D))/J_R$ be the quotient map.

Theorem 3.5. *The quotient Rota–Baxter algebra $Ш^{NC,0}(T(D))/J_R$, together with $\rho := \pi \circ i_D$, is the universal enveloping Rota–Baxter algebra of D of weight λ.*

Proof. Let (A, R) be a Rota–Baxter algebra of weight λ and let $f : D \to A$ be a morphism in $\mathbf{DD}$. More precisely, we have $f : D \to \mathcal{G}A$ where $\mathcal{G}A = (A, \prec'_R, \succ'_R)$ is the dendriform dialgebra in Theorem 3.1. We will complete the following commutative diagram, using notations from Corollary 2.7.

(26)



By the universal property of the free algebra $T(D)$ over D, there is a unique morphism $\tilde{f} : T(D) \to A$ in $\mathbf{Alg}^0$ such that $\tilde{f} \circ k_D = f$ and so $\tilde{f}(x_1 \otimes \cdots \otimes x_n) = f(x_1) * \cdots * f(x_n)$. Here $*$ is the product in A. Then by the universal property of the free Rota–Baxter algebra $Ш^{NC,0}(T(D))$ over $T(D)$, there is a unique morphism $\bar{\tilde{f}} : Ш^{NC,0}(T(D)) \to A$ in $\mathbf{RB}^0$ such that $\bar{\tilde{f}} \circ j_{T(D)} = \tilde{f}$. By Corollary 2.7, $\bar{\tilde{f}} = \hat{f}$. Then

$$\hat{f} \circ i_D = \hat{f} \circ j_{T(D)} \circ k_D = \tilde{f} \circ k_D = f. \tag{27}$$

So for any $x, y \in D$, we have

$$\begin{aligned}
\hat{f}(x \prec y - x\lfloor y\rfloor - \lambda x \otimes y) &= \hat{f}(x \prec y) - \hat{f}(x) * R(\hat{f}(y)) - \lambda \hat{f}(x \otimes y)\\
&= \hat{f}(x \prec y) - \hat{f}(x) * R(\hat{f}(y)) - \lambda \tilde{f}(x \otimes y)\\
&= f(x \prec y) - f(x) * R(f(y)) - \lambda f(x) * f(y)\\
&= f(x \prec y) - f(x) \prec'_R f(y)\\
&= f(x \prec y) - f(x \prec y) = 0.
\end{aligned}$$

Therefore, $x \prec y - x\lfloor y\rfloor - \lambda x \otimes y$ is in $\ker(\hat{f})$. Similarly, $x \succ y - \lfloor x\rfloor y$ is in $\ker(\hat{f})$. Thus J_R is in $\ker(\hat{f})$ and there is a morphism $\check{f} : Ш^{NC,0}(T(D))/J_R \to A$ in $\mathbf{RB}^0$ such that $\hat{f} = \check{f} \circ \pi$. Then by the definition of $\rho = \pi \circ i_D$ in the theorem and Eq. (27), we have

$$\check{f} \circ \rho = \check{f} \circ \pi \circ i_D = \hat{f} \circ i_D = f.$$

This proves the existence of $\check{f}$.

Suppose $\check{f}' : Ш^{NC,0}(T(D))/J_R \to A$ is also a morphism in $\mathbf{RB}^0$ such that $\check{f}' \circ \rho = f$. Then

$$(\check{f}' \circ \pi) \circ i_D = f = (\check{f} \circ \pi) \circ i_D.$$

By Corollary 2.7, the free Rota–Baxter algebra $Ш^{NC,0}(T(D))$ over the algebra $T(D)$ is also the free Rota–Baxter algebra over the vector space D with respect the natural injection i_D. So we have $\check{f}' \circ \pi = \check{f} \circ \pi$ in $\mathbf{RB}^0$. Since π is surjective, we have $\check{f}' = \check{f}$. This proves the uniqueness of $\check{f}$. □

4. Free dendriform di- and trialgebras and free Rota–Baxter algebras

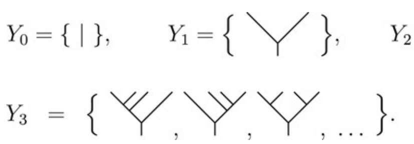
The results in this section can be regarded as more precise forms of results in Section 3 in special cases. Our emphasis here is to interpret free dendriform dialgebras and free dendriform trialgebras as natural subalgebras of free Rota–Baxter algebras. This interpretation also suggests a planar tree structure on free Rota–Baxter algebras which will be made precise in [21].

4.1. The dialgebra case

4.1.1. Free dendriform dialgebras

Let $\mathbf{k}$ be a field. We briefly recall the construction of free dendriform dialgebra $\mathbf{DD}(V)$ over a $\mathbf{k}$-vector space V as coloured planar binary trees. For details, see [48,57].

Let X be a basis of V. For $n \geq 0$, let Y_n be the set of planar binary trees with $n+1$ leaves and one root such that the valence of each internal vertex is exactly two. Let $Y_{n,X}$ be the set of planar binary trees with $n+1$ leaves and with vertices decorated by elements of X. The unique tree with one leave is denoted by $|$. So we have $Y_0 = Y_{0,X} = \{|\}$. Let $\mathbf{k}[Y_{n,X}]$ be the $\mathbf{k}$-vector space generated by $Y_{n,X}$. Here are the first few of them without decoration.



For $T \in Y_{m,X}$, $U \in Y_{n,X}$ and $x \in X$, the grafting of T and U over x is $T \vee_x U \in Y_{m+n+1,X}$. Let $\mathbf{DD}(V)$ be the graded vector space $\bigoplus_{n\geq 1} \mathbf{k}[Y_{n,X}]$. Define binary operations $\prec$ and $\succ$ on $\mathbf{DD}(V)$ recursively by

(a) $| \succ T = T \prec | = T$ and $| \prec T = T \succ | = 0$ for $T \in Y_{n,X}$, $n \geq 1$;
(b) For $T = T^\ell \vee_x T^r$ and $U = U^\ell \vee_y U^r$, define

$$T \prec U = T^\ell \vee_x (T^r \prec U + T^r \succ U), \qquad T \succ U = (T \prec U^\ell + T \succ U^\ell) \vee_y U^r.$$

Since $| \prec |$ and $| \succ |$ is not defined, the binary operations $\prec$ and $\succ$ are only defined on $\mathbf{DD}(V)$ though the operation $\star := \prec + \succ$ can be extended to $H_{\mathrm{LR}} := \mathbf{k}[Y_0] \oplus \mathbf{DD}(V)$ by defining $| \star T = T \star | = T$. By [48] $(\mathbf{DD}(V), \prec, \succ)$ is the free dendriform dialgebra over V.

Theorem 4.1. *Let V be a $\mathbf{k}$-vector space. The free dendriform dialgebra over V is a sub dendriform dialgebra of the free Rota–Baxter algebra $Ш^{NC,0}(V)$ of weight zero.*

The proof will be given in the next subsection.

4.1.2. Proof of Theorem 4.1

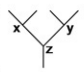
For the given vector space V, make V into a $\mathbf{k}$-algebra without identity by given V the zero product. Let $Ш^{NC,0}(V)$ be the free nonunitary Rota–Baxter algebra of weight zero over V constructed in Theorem 2.6. Since $Ш^{NC,0}(V)$ is a dendriform dialgebra, the natural map $j_V : V \to Ш^{NC,0}(V)$ extends uniquely to a dendriform dialgebra morphism $D(j) : \mathbf{DD}(V) \to Ш^{NC,0}(V)$. We will prove that this map is injective and identifies $\mathbf{DD}(V)$ as a subalgebra of $Ш^{NC,0}(V)$ in the category of dendriform dialgebras. We first define a map

$$\phi : \mathbf{DD}(V) \to Ш^{NC,0}(V)$$

and then show in Theorem 4.3 below that it agrees with $D(j)$. We construct ϕ by defining $\phi(T)$ for $T \in Y_{n,X}, n \geq 1$, inductively on n. Any $T \in Y_{n,X}, n \geq 1$ can be uniquely written as $T = T^\ell \vee_x T^r$ with $x \in X$ and $T^\ell, T^r \in \cup_{0\leq i<n} Y_{i,X}$. We then define

$$\phi(T) = \begin{cases} \lfloor \phi(T^\ell)\rfloor x \lfloor \phi(T^r)\rfloor, & T^\ell \neq 1, T^r \neq 1, \\ x\lfloor \phi(T^r)\rfloor, & T^\ell = 1, T^r \neq 1, \\ \lfloor \phi(T^\ell)\rfloor x, & T^\ell \neq 1, T^r = 1, \\ x, & T^\ell = 1, T^r = 1. \end{cases} \tag{28}$$

For example,

$\phi\Big($  $\Big) = x,\qquad \phi\Big($  $\Big) = \lfloor x\rfloor z \lfloor y\rfloor.$

We recall [48] that $\mathbf{DD}(V)$ with the operation $\star := \prec + \succ$ is an associative algebra.

We now describe a submodule of $Ш^{\mathrm{NC},0}(V)$ to be identified with the image of ϕ in Theorem 4.3.

Definition 4.2. A $y \in \mathfrak{X}_\infty$ is called a *dendriform diword (DW)* if it satisfies the following *additional* properties.

(a) y is not in $\lfloor \mathfrak{X}_\infty \rfloor$;
(b) There is no subword $\lfloor\lfloor \mathbf{x}\rfloor\rfloor$ with $\mathbf{x} \in \mathfrak{X}_\infty$ in the word;
(c) There is no subword of the form $\mathbf{x}_1\lfloor \mathbf{x}_2\rfloor \mathbf{x}_3$ with $\mathbf{x}_1, \mathbf{x}_3 \in X$ and $\mathbf{x}_2 \in \mathfrak{X}_\infty$.

We let $\mathrm{DW}(V)$ be the subspace of $Ш^{\mathrm{NC},0}(V)$ generated by the dendriform diwords.

For example

$$x_0\lfloor x_1\lfloor x_2\rfloor\rfloor, \qquad \lfloor x_0\rfloor x_1 \lfloor x_2\rfloor$$

are dendriform diwords while

$$\lfloor\lfloor x_1\rfloor\rfloor, \qquad \lfloor\lfloor x_1\rfloor x_2 \lfloor x_3\rfloor\rfloor, \qquad x_1\lfloor x_2\rfloor x_3$$

are in $\mathfrak{X}_\infty$ but not dendriform diwords.

Equivalently, $\mathrm{DW}(V)$ can be characterized in terms of the decomposition (6). For subsets Y, Z of $\mathfrak{X}_\infty$, define

$$D(Y,Z) = (Y\lfloor Z\rfloor) \bigcup (\lfloor Z\rfloor Y) \bigcup \lfloor Z\rfloor Y \lfloor Z\rfloor.$$

Then define $D_0(V) = X$ and, for $n \geq 0$, inductively define

$$D_{n+1}(V) = D(X, D_n(V)) = (X\lfloor D_n(V)\rfloor) \bigcup (\lfloor D_n(V)\rfloor X) \bigcup \lfloor D_n(V)\rfloor X \lfloor D_n(V)\rfloor. \tag{29}$$

Then $D_\infty := \cup_{n\geq 0} D_n(V)$ is the set of dendriform diwords and $\mathrm{DW}(V) = \oplus_{\mathbf{x}\in D_\infty} \mathbf{kx}$.

Theorem 4.1 follows from the following theorem.

Theorem 4.3. (a) $\phi : \mathbf{DD}(V) \to Ш^{\mathrm{NC},0}(V)$ *is a homomorphism of dendriform dialgebras.*
(b) $\phi = D(j)$*, the morphism of dendriform dialgebras induced by* $j : V \to Ш^{\mathrm{NC},0}(V)$.
(c) $\phi(\mathbf{DD}(V)) = \mathrm{DW}(V)$.
(d) ϕ *is injective.*

Proof. (a) we first note that the operations $\prec$ and $\succ$ can be equivalently defined as follows. Let $T \in Y_{m,X}, U \in Y_{n,X}$ with $m \geq 1, n \geq 1$. Then $T = T^\ell \vee_x T^r, U = U^\ell \vee_y U^r$ with $x, y \in X$ and $T^\ell, T^r, U^\ell, U^r \in \cup_{i\geq 0} Y_{i,X}$. Define

$$T \prec U := \begin{cases} T^\ell \vee_x (T^r \prec U + T^r \succ U), & \text{if } T^r \neq |, \\ T^\ell \vee_x U, & \text{if } T^r = |. \end{cases} \tag{30}$$

$$T \succ U := \begin{cases} (T \prec U^\ell + T \succ U^\ell) \vee_y U^r, & \text{if } U^\ell \neq |, \\ T \vee_y U^r, & \text{if } U^\ell = |. \end{cases} \tag{31}$$

Thus we have

$$\phi(T \prec U) = \begin{cases} \phi(T^\ell \vee_x (T^r \prec U + T^r \succ U)), & \text{if } T^r \neq |, \\ \phi(T^\ell \vee_x U), & \text{if } T^r = |. \end{cases}$$

$$= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r \prec U + T^r \succ U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r \prec U + T^r \succ U) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases}$$

(by definition of ϕ)

$$= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \prec_R \phi(U) + \phi(T^r) \succ_R \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor (\phi(T^r) \prec_R \phi(U) + \phi(T^r) \succ_R \phi(U)) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases}$$

(by induction hypothesis)

On the other hand, we have

$$\phi(T) \prec_R \phi(U) = \phi(T^\ell \vee_x T^r) \lfloor \phi(U) \rfloor$$

$$= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \rfloor \lfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r) \rfloor \lfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases}$$

(by definition of ϕ)

$$= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \lfloor \phi(U) \rfloor + \lfloor \phi(T^r) \rfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r) \lfloor \phi(U) \rfloor + \lfloor \phi(T^r) \rfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases}$$

(by Rota–Baxter relation of $R(T) = \lfloor T \rfloor$).

This proves $\phi(T \prec U) = \phi(T) \prec_R \phi(U)$. We similarly prove $\phi(T \succ U) = \phi(T) \succ_R \phi(U)$. Thus ϕ is a homomorphism in **DD**.

(b) follows from the uniqueness of the dendriform dialgebra morphism $\mathbf{DD}(V) \to Ш^{NC,0}(V)$ extending the map $j_V : V \to Ш^{NC,0}(V)$.

(c) We only need to prove $\mathrm{DW}(V) \subseteq \phi(\mathbf{DD}(V))$ and $\phi(\mathbf{DD}(V)) \subseteq \mathrm{DW}(V)$. To prove the former, we prove $D_n \subseteq \phi(\mathbf{DD}(V))$ by induction on n.

When $n = 0$, $D_n = X$ so the inclusion is clear. Suppose the inclusion holds for n. Then by the definition of $D_{n+1}(V)$ in Eq. (29), an element of $D_{n+1}(V)$ is of the following three forms:

(i) It is $\mathbf{x}\lfloor \mathbf{x}' \rfloor$ with $\mathbf{x} \in X$, $\mathbf{x}' \in D_n(V)$. Then it is $\mathbf{x} \prec_R \mathbf{x}'$ which is in $\phi(\mathbf{DD}(V))$ by the induction hypothesis and the fact that $\phi(\mathbf{DD}(V))$ is a subdendriform algebra.

(ii) It is $\lfloor \mathbf{x} \rfloor \mathbf{x}'$ with $\mathbf{x} \in D_n(V)$ and $\mathbf{x}' \in X$. Then the same proof works.

(iii) It is $\lfloor \mathbf{x} \rfloor \mathbf{x}' \lfloor \mathbf{x}'' \rfloor$ with $\mathbf{x}, \mathbf{x}'' \in D_n(V)$ and $\mathbf{x}' \in X$. Then it is

$$(\mathbf{x} \succ_R \mathbf{x}') \prec_R \mathbf{x}'' = \mathbf{x}' \succ_R (\mathbf{x}' \prec_R \mathbf{x}'').$$

By induction, $\mathbf{x}$ and $\mathbf{x}''$ are in the sub dendriform dialgebra $\phi(\mathbf{DD}(V))$. So the element itself is in $\phi(\mathbf{DD}(V))$.

The second inclusion follows easily by induction on degrees of trees in $\mathbf{DD}(V)$.

(d) By the definition of ϕ and part (c), ϕ gives a one–one correspondence between $\cup_{n \geq 0} Y_{n,X}$ as a basis of $\mathbf{DD}(V)$ and $\mathrm{DW}(V)$ as a basis of $\phi(\mathbf{DD}(V))$. Therefore ϕ is injective. □

4.2. The trialgebra case

4.2.1. Free dendriform trialgebras

We describe the construction of free dendriform trialgebra $\mathbf{DT}(V)$ over a vector space V as colored planar trees. For details when V is of rank one over $\mathbf{k}$, see [52].

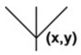
Let Ω be a basis of V. For $n \geq 0$, let T_n be the set of planar trees with $n+1$ leaves and one root such that the valence of each internal vertex is at least two. Let $T_{n,\Omega}$ be the set of planar trees with $n+1$ leaves and with vertices **valently decorated** by elements of Ω, in the sense that if a vertex has valence k, then the vertex is decorated by a vector in Ω^{k-1}. For example the vertex of  is decorated by $x \in \Omega$ while the vertex of  is decorated by $(x, y) \in \Omega^2$. The unique tree with one leaf is denoted by $|$. So we have $T_0 = T_{0,\Omega} = \{|\}$. Let $\mathbf{k}[T_{n,\Omega}]$ be the $\mathbf{k}$-vector space generated by $T_{n,\Omega}$.

Here are the first few of them without decoration.

For $T^{(i)} \in T_{n_i,\Omega}$, $0 \leq i \leq k$, and $x_i \in \Omega$, $1 \leq i \leq k$, the grafting of $T^{(i)}$ over $(x_1, \ldots, x_k)$ is

$$T^{(0)} \vee_{x_1} T^{(1)} \vee_{x_2} \cdots \vee_{x_k} T^{(k)}.$$

Any tree can be uniquely expressed as such a grafting of lower degree trees. For example,



$$= | \vee_x | \vee_y |.$$

Let $\mathbf{DT}(V)$ be the graded vector space $\bigoplus_{n\geq 1} \mathbf{k}[T_{n,\Omega}]$. Define binary operations $\prec$, $\succ$ and $\cdot$ on $\mathbf{DT}(V)$ recursively by:

(a) $| \succ T = T \prec | = T$, $| \prec T = T \succ | = 0$ and $| \cdot T = T \cdot | = 0$ for $T \in T_{n,\Omega}$, $n \geq 1$;
(b) For $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}$ and $U = U^{(0)} \vee_{y_1} \cdots \vee_{y_n} U^{(n)}$, define

$$T \prec U = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U),$$
$$T \succ U = (T \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)},$$
$$T \cdot U = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)}.$$

Here $\star := \prec + \succ + \cdot$ Since $| \prec |$, $| \succ |$ and $| \cdot |$ are not defined, the binary operations $\prec$, $\succ$ and $\cdot$ are only defined on $\mathbf{DT}(V)$ though the operation $\star$ can be extended to $H_{\mathbf{DT}} := \mathbf{k}[T_0] \oplus \mathbf{DT}(V)$ by defining $| \star T = T \star | = T$.

Theorem 4.4. $(\mathbf{DT}(V), \prec, \succ, \cdot)$ *is the free dendriform trialgebra over* V.

Proof. The proof is given by Loday and Ronco in [52] when V is of dimension one. The proof for the general case is the same. □

Our goal is to prove

Theorem 4.5. *Let* V *be a* $\mathbf{k}$*-vector space. The free dendriform trialgebra over* V *is a canonical subdendriform trialgebra of the free Rota–Baxter algebra* $Ш^{\mathrm{NC},0}(T(V))$ *of weight one.*

We restrict the weight of the Rota–Baxter algebra to one to ease the notations. The proof will be given in the next subsection.

4.2.2. Proof of Theorem 4.5

Let V be the given $\mathbf{k}$-vector space with basis Ω. Let $T(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ be the tensor product algebra over V. Then $T(V)$ is the free nonunitary algebra generated by the $\mathbf{k}$-space V. A basis of $T(V)$ is $X := M(\Omega)$, the free semigroup generated by Ω. By Theorem 2.6, $Ш^{\mathrm{NC},0}(T(V)) := Ш^{\mathrm{NC},0}{}_{\mathbf{1}}(T(V))$ is the free nonunitary Rota–Baxter algebra over $T(V)$ of weight $\mathbf{1}$ constructed in Section 2.2.

Since $Ш^{\mathrm{NC},0}(T(V))$ is a dendriform trialgebra, the natural map $j_V : V \to Ш^{\mathrm{NC},0}(T(V))$ extends uniquely to a dendriform trialgebra morphism $T(j) : \mathbf{DT}(V) \to Ш^{\mathrm{NC},0}(T(V))$. We will prove that this map is injective and identifies $\mathbf{DT}(V)$ as a subalgebra of $Ш^{\mathrm{NC},0}(T(V))$ in the category of dendriform trialgebras. We first define a map

$$\psi : \mathbf{DT}(V) \to Ш^{\mathrm{NC},0}(T(V))$$

and then show in Theorem 4.7 below that it agrees with $T(j)$. We construct ψ by defining $\psi(T)$ for $T \in T_{n,\Omega}, n \geq 1$, inductively on n. Any $T \in T_{n,\Omega}, n \geq 1$, can be uniquely written as $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_k} T^{(k)}$ with $x_i \in \Omega$ and $T^{(i)} \in \cup_{0 \leq i < n} T_{i,\Omega}$. We then define

$$\psi(T) = \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} \cdots \overline{\lfloor \psi(T^{(k-1)}) \rfloor} x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}, \tag{32}$$

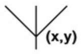
where $\overline{\lfloor \psi(T^{(i)}) \rfloor} = \lfloor \psi(T^{(i)}) \rfloor$ if $\psi(T^{(i)}) \neq |$. If $\psi(T^{(i)}) = |$, then the factor $\lfloor \psi(T^{(i)}) \rfloor$ is dropped when $i = 0$ or k, and is replaced by $\otimes$ when $0 < i < k$. For example,

$$\overline{\lfloor \psi(|) \rfloor} x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} x_2 \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor} = x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} x_2 \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}$$

and

$$\overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \overline{\lfloor \psi(|) \rfloor} x_2 \overline{\lfloor \psi(T^{(2)}) \rfloor} \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor} = \overline{\lfloor \psi(T^{(0)}) \rfloor} (x_1 \otimes x_2) \overline{\lfloor \psi(T^{(2)}) \rfloor} \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}.$$

In particular,

$\psi\Big($  $\Big) = \psi(| \vee_x | \vee_y |) = \overline{\lfloor \psi(|) \rfloor} \vee_x \overline{\lfloor \psi(|) \rfloor} \vee_y \overline{\lfloor \psi(|) \rfloor} = x \otimes y.$

We now describe a submodule of $Ш^{NC,0}(T(V))$ to be identified with the image of ψ in Theorem 4.7.

Definition 4.6. Let $X = M(\Omega)$. A $y \in \mathfrak{X}_\infty$ is called a *dendriform triword (TW)* if it satisfies the following *additional* properties:

(a) y is not in $\lfloor \mathfrak{X}_\infty \rfloor$;
(b) There is no subword $\lfloor \lfloor \mathbf{x} \rfloor \rfloor$ with $\mathbf{x} \in \mathfrak{X}_\infty$ in the word;

We let $TW(V)$ be the subspace of $Ш^{NC,0}(T(V))$ generated by the dendriform triwords.

For example

$$x_0 \lfloor x_1 \lfloor x_2 \rfloor \rfloor, \qquad \lfloor x_0 \rfloor x_1 \lfloor x_2 \rfloor, \qquad \lfloor x_0 \rfloor x_1 \lfloor x_2 \rfloor x_3 \lfloor x_4 \rfloor, \qquad x_0 \otimes x_1$$

are dendriform triwords while

$$\lfloor \lfloor x_1 \rfloor \rfloor, \qquad \lfloor x_1 \lfloor x_2 \rfloor x_3 \rfloor$$

are in $\mathfrak{X}_\infty$ but not dendriform triwords.

Equivalently, TWs can be characterized in terms of the decomposition (6). For subsets Y, Z of $\mathfrak{X}_\infty$, define

$$S(Y,Z) = \left(\bigcup_{r\geq 1} (Y\lfloor Z \rfloor)^r\right) \bigcup \left(\bigcup_{r\geq 0} (Y\lfloor Z \rfloor)^r Y\right) \bigcup \left(\bigcup_{r\geq 1} \lfloor Z \rfloor (Y\lfloor Z \rfloor)^r\right) \bigcup \left(\bigcup_{r\geq 0} \lfloor Z \rfloor (Y\lfloor Z \rfloor)^r Y\right). \tag{33}$$

Then define $S_0(V) = M(X)$. For $n \geq 0$, inductively define

$$S_{n+1}(V) = S(M(X), S_n(V)). \tag{34}$$

Then $S_\infty := \cup_{n\geq 0} S_n(V)$ is the set of dendriform triwords and $TW(V) = \oplus_{\mathbf{x} \in S_\infty} \mathbf{kx}$.

Theorem 4.5 follows from the following theorem:

Theorem 4.7. (a) $\psi : \mathbf{DT}(V) \to Ш^{NC,0}(T(V))$ *is a homomorphism of dendriform trialgebras.*
(b) $\psi = T(j)$*, the morphism of dendriform trialgebras induced by* $j : V \to Ш^{NC,0}(T(V))$.
(c) $\psi(\mathbf{DT}) = DT(V)$.
(d) ψ *is injective.*

Proof. The proof is similar to Theorem 4.3. For the lack of a uniform approach for both cases, we give some details.

(a) we first note that the operations $\prec$ and $\succ$ can be equivalently defined as follows without using $| \prec T$, etc. Let $T \in T_{i,X}$, $U \in T_{j,X}$ with $i \geq 1$, $j \geq 1$. Then $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}$ and $U = U^{(0)} \vee_{y_1} \cdots \vee_{y_n} U^{(n)}$, define

$$T \prec U = \begin{cases} T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U), & \text{if } T^{(m)} \neq |, \\ T^{(0)} \vee_{x_1} \cdots \vee_{x_m} U, & \text{if } T^{(m)} = | \end{cases}$$

$$T \succ U = \begin{cases} (T \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, & \text{if } U^{(0)} \neq |, \\ T \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, & \text{if } U^{(0)} = | \end{cases}$$

$$T \cdot U = \begin{cases} T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, & \text{if } T^{(m)} \neq |, U^{(0)} \neq |, \\ T^{(0)} \vee_{x_1} \cdots \vee_{x_m} U^{(0)} \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, & \text{if } T^{(m)} = |, U^{(0)} \neq |, \\ T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)} \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, & \text{if } T^{(m)} \neq |, U^{(0)} = |. \end{cases}$$

Now we use induction on $i + j$ to prove

$$\psi(T \prec U) = \psi(T) \prec_R \psi(U), \qquad \psi(T \succ U) = \psi(T) \succ_R \psi(U), \tag{35}$$

$$\psi(T \cdot U) = \psi(T) \cdot_R \psi(U). \tag{36}$$

Here $R := R_{T(V)}$ is the Rota–Baxter operator on $\text{Ш}^{\text{NC},0}(T(V))$. Since $i + j \geq 2$, we can first take $i + j = 2$. Then $T = | \vee_x |$, $U = | \vee_y |$. So by Eq. (32),

$$\psi(T \prec U) = \psi((| \vee_x |) \prec U) = \psi(| \vee_x U|) = x\lfloor \psi(U) \rfloor = x\lfloor y \rfloor = x \prec_R y.$$

We similarly have $\psi(T \succ U) = x \succ_R y$ and

$$\psi(T \cdot U) = \psi((| \vee_x |) \cdot (| \vee_y |)) = \psi(| \vee_x | \vee_y |) = x \otimes y = x \cdot_R y.$$

Assume Eq. (36) hold for $T \in T_{i,X},\ U \in T_{j,X}$ with $i + j \geq k \geq 2$. Then we also have

$$\begin{aligned} \psi(T \star U) &= \psi(T \prec U + T \succ U + T \cdot U) \\ &= \psi(T) \prec_R \psi(U) + \psi(T) \succ_R \psi(U) + \psi(T) \cdot_R \psi(U) \\ &= \psi(T) \star_R \psi(U). \end{aligned} \tag{37}$$

Here $\star_R = \prec_R + \succ_R + \cdot_R$. Consider T, U with $m+n = k+1$. We consider two cases of $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}$. Since $U \neq |$, we have $\overline{\lfloor T^{(m)} \star U \rfloor} = \lfloor T^{(m)} \star U \rfloor$ if $T^{(m)} \neq |$, and $\overline{\lfloor U \rfloor} = \lfloor U \rfloor$ if $T^{(m)} = |$.

Case 1. If $T^{(m)} \neq |$, then

$$\begin{aligned} \psi(T \prec U) &= \psi(T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U)) \quad \text{(definition of } \prec) \\ &= \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \cdots x_m \lfloor \psi(T^{(m)} \star U) \rfloor \quad \text{(definition of } \psi) \\ &= \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \cdots x_m \lfloor \psi(T^{(m)}) \star_R \psi(U) \rfloor \quad \text{(induction hypothesis (37))} \\ &= \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \cdots x_m \lfloor \psi(T^{(m)}) \rfloor \lfloor \psi(U) \rfloor \quad \text{(relation (1))} \\ &= \psi(T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}) \prec_R \psi(U) \quad \text{(defintion of } \psi) \\ &= \psi(T) \prec_R \psi(U). \end{aligned}$$

Case 2. If $T^{(m)} = |$, then

$$\begin{aligned} \psi(T \prec U) &= \psi(T^{(0)} \vee_{x_1} \cdots \vee_{x_m} U) \quad \text{(definition of } \prec) \\ &= \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \cdots x_m \lfloor \psi(U) \rfloor \quad \text{(definition of } \psi) \\ &= \psi(T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}) \lfloor \psi(U) \rfloor \quad \text{(defintion of } \psi) \\ &= \psi(T) \prec_R \psi(U). \end{aligned}$$

This proves $\psi(T \prec U) = \psi(T) \prec_R \psi(U)$. We similarly prove $\psi(T \succ U) = \psi(T) \succ_R \psi(U)$ and $\psi(T \cdot U) = \psi(T) \cdot_R \psi(U)$. Thus ψ is a homomorphism in **DT**.

(b) follows from the uniqueness of the morphism $\mathbf{DT}(V) \to \text{Ш}^{\text{NC},0}(T(V))$ of dendriform trialgebra extending the map $i : V \to \text{Ш}^{\text{NC},0}(T(V))$.

(c) We only need to prove $TW(V) \subseteq \psi(\mathbf{DT}(V))$ and $\psi(\mathbf{DT}(V)) \subseteq TW(V)$. To prove the former, we prove $S_n(V) \subseteq \psi(\mathbf{DT}(V))$ by induction on n.

When $n = 0$, $S_n(V) = X$ so the inclusion is clear. Suppose the inclusion holds for $1 \leq n \leq k$. Then by the definition of $S_{k+1}(V)$ in Eq. (34), an element of $S_{k+1}(V)$ has length greater or equal to 2. We apply induction on its length. If the length is 2, then it is one of the following two cases:

(i) It is $\mathbf{x}\lfloor\mathbf{x}'\rfloor$ with $\mathbf{x} \in X$, $\mathbf{x}' \in S_k(V)$. Then it is $\mathbf{x} \prec_R \mathbf{x}'$ which is in $\psi(\mathbf{DT}(V))$ by the induction hypothesis and the consequence from part (a) that $\psi(\mathbf{DT}(V))$ is a sub dendriform algebra.

(ii) It is $\lfloor\mathbf{x}\rfloor\mathbf{x}'$ with $\mathbf{x} \in S_k(V)$ and $\mathbf{x}' \in X$. Then the same proof works.

Suppose all elements of S_{k+1} with length $\leq q$ and ≥ 2 are in $\psi(\mathbf{DT}(V))$. Consider an element $\mathbf{x}$ of S_{k+1} with length $q + 1$. Then $q + 1 \geq 3$. If $q + 1 = 3$, we again have two cases.

(i) $\mathbf{x} = \lfloor\bar{\mathbf{x}}_1\rfloor\mathbf{x}_2\lfloor\bar{\mathbf{x}}_3\rfloor$ with $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2 \in S_n(V)$ and $\mathbf{x}_1 \in X$. Then it is $(\bar{\mathbf{x}}_1 \succ_R \mathbf{x}_2) \prec_R \bar{\mathbf{x}}_3$. By induction hypothesis on n, $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_3$ are in the subdendriform dialgebra $\psi(\mathbf{DT}(V))$. So the element itself is in $\psi(\mathbf{DT}(V))$.

(ii) $\mathbf{x} = \mathbf{x}_1\lfloor\bar{\mathbf{x}}_2\rfloor\mathbf{x}_3$ with $\mathbf{x}_1, \mathbf{x}_3 \in X$ and $\bar{\mathbf{x}}_2 \in S_n(V)$. Then $\mathbf{x} = \mathbf{x}_1 \cdot_R (\bar{\mathbf{x}}_2 \succ \mathbf{x}_3)$ which is in $\psi(\mathbf{DT}(V))$.

If $q + 1 \geq 4$, then $\mathbf{x}$ can be expressed as the concatenation of $\mathbf{x}_1$ and $\mathbf{x}_2$ of lengths at least two and hence are in $TW(V)$. By induction hypotheses, $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $\psi(\mathbf{DT}(V))$. Therefore $\mathbf{x} = \mathbf{x}_1 \cdot_R \mathbf{x}_2$ is in $\psi(\mathbf{DT}(V))$.

This completes the proof of the first inclusion. The proof of the second inclusion follows from a similar induction on the degree of trees in $\mathbf{DT}(V)$.

(d) By the definition of ψ and part (c), ψ gives a one–one correspondence between $\cup_{n\geq 0} T_{n,X}$ as a basis of $\mathbf{DT}(V)$ and $TW(V)$ as a basis of $\psi(\mathbf{DT}(V))$. Therefore ψ is injective. □

Acknowledgements

We thank M. Aguiar, J.-L. Loday and M. Ronco for helpful discussions. The first named author was supported by a Ph.D. grant from the Ev. Studienwerk e.V., and would like to thank the people at the Theory Department of the Physics Institute at Bonn University for encouragement and help. The second named author acknowledges support from NSF grant DMS 0505643 and a Research Council grant from the Rutgers University. Both authors acknowledge the warm hospitality of I.H.É.S. (LG) and L.P.T.H.E. (KEF) where this work was completed.

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