Partially Ordered Sets Associated with Permutations

RODICA SIMION

Given a set of permutations $\Sigma, S_n \supseteq \Sigma$, a poset $P = P(\Sigma)$ is chain-permutational with respect to $\Sigma$ if it is of minimum cardinality, with the property that its covering relations can be labeled with $1, 2, \ldots, n$, so that along the maximal chains the labels produce precisely the permutations in $\Sigma$, each one exactly once.

General properties of chain-permutational posets are discussed, including construction, uniqueness up to isomorphism (in view of which we may use the notation $P(\Sigma)$), and the fact that $P(\Sigma)$ is a distributive lattice iff $\Sigma$ is the collection of linear extensions of a poset.

Order-theoretic properties, amongst others, regarding rank cardinalities and shellability are discussed for the special choices of $\Sigma$ being the set of involutions in $S_n$, and $\Sigma$ being the pattern restricted permutations in $S_n$.

0. NOTATION, DEFINITIONS AND PRELIMINARIES

Let $\Sigma$ be a set of permutations on the set $[n] = \{1, 2, \ldots, n\}$. We will regard permutations as words over the alphabet $[n]$, and write $\sigma \in S_n$ as the $n$-tuple $\sigma(1) \sigma(2) \cdots \sigma(n)$. If a partially ordered set (poset) has unique maximal and minimal elements, these will be denoted $\hat{1}$ and $\hat{0}$, respectively. By a maximal chain in $P$ we mean a saturated $\hat{0}$-$\hat{1}$ chain. If $x, y$ are elements of a poset, and if $x \geq y$, we will write $x \triangleright y$. For further general definitions and results concerning posets we refer the reader to [1]. As usual, $|A|$ will denote the cardinality of the set $A$, and glb (lub, resp.) will mean the greatest lower (lowest upper, resp.) bound.

A poset $P$ is chain-permutational with respect to $\Sigma$ iff the covering relations (i.e. the edges in the Hasse diagram) of $P$ can be labeled with labels from $[n]$ so that the sequence of labels along each maximal chain of $P$ is a permutation in $\Sigma$, and each permutation in $\Sigma$ arises exactly once in this way. Furthermore, $P$ is required to be minimal with respect to the number of elements. Figure 1 shows two posets for $\Sigma = (123, 132)$; the first one fails to satisfy the minimality requirement.

Chain-permutational posets were introduced in [2] as a tool for answering an enumerative question about certain classes $\Sigma$ of permutations. In [3], the connection between chain-permutational posets and finite automata is pointed out, and some general results are proved: an explicit construction for $P(\Sigma)$ is given, which shows that its number of elements and covering relations, indeed its order-isomorphism type, are well determined by an equivalence relation of ‘compatibility’ defined on $\Sigma$. Briefly, start with a poset $\mathcal{P}$ consisting of $|\Sigma|$ disjoint chains, each labeled by a different $\sigma \in \Sigma$; the compatibility relation establishes all the possible identifications among elements of

![Figure 1](image-url)
THEOREM 0.1 [3; 1.1 and theorem 1.2]. Let $\Sigma$ be a subset of $S_n$. If $P_1$ and $P_2$ are two chain-permutational posets with respect to $\Sigma$, then $P_1 \equiv P_2$.

Also in [3], the relation between $\Sigma$ and order-theoretic properties of $P(\Sigma)$ is further explored, and the following is obtained:

THEOREM 0.2 [3; theorem 1.3]. Let $P(\Sigma)$ be the chain-permutational poset associated with a set of permutations $\Sigma$. Then $P(\Sigma)$ is a distributive lattice iff $P(\Sigma)$ is the family of linear extensions of some poset $Q$, using a labeling of the elements of $Q$ which may not be itself order-preserving.

Indeed, if $Q$ is any poset then the lattice of order ideals of $Q$, denoted $I(Q)$, is distributive, and this characterizes distributive lattices (see, e.g., [1]). Moreover, observe that if the elements of $Q$ are labeled in some manner, and if the covering $I_1 \prec I_2$ between two ideals is labeled by the unique element in $I_2 - I_1$, then the maximal chains in $I(Q)$ give the linear extensions of $Q$. The extensions are expressed in terms of the labeling put on $Q$, and each extension arises precisely once.

In particular, the Boolean lattice $B_n$ is the chain-permutational poset with respect to the symmetric group $S_n$.

This paper takes a closer look at three infinite classes of chain-permutational posets, corresponding to families of permutations of independent combinatorial interest.

In Section 1 we examine the chain-permutational poset for the case when $\Sigma = \Sigma_n$ is the set of all the involutions in $S_n$, i.e. the permutations $\sigma \in S_n$ satisfying $\sigma(\sigma(x)) = x$ for all $x \in [n]$. We construct recursively posets $P_n = P(\Sigma_n)$, and show that they are rank symmetric and rank unimodal.

The second and third sections contain a discussion of the chain-permutational posets for permutations in $S_n$ subject to certain pattern restrictions. Given any two permutations, $\rho \in S_3$ and $\sigma \in S_n$, we say that $\sigma$ avoids the pattern $\rho$ iff for no $1 \leq i_1 < i_2 < i_3 \leq n$ do $\sigma(i_1), \sigma(i_2), \sigma(i_3)$ bear the same magnitude relationships as $\rho(1), \rho(2), \rho(3)$. For example, the permutations which avoid $\rho = 123$ are those having no increasing subsequence of length three; as a second example, $\sigma = 25314$ contains the pattern $\rho = 123$ twice, as 214 and as 314; in fact, this $\sigma$ does not avoid any of the six 3-letter patterns.

Let $S_n(\rho)$ be the set of permutations in $S_n$ which avoid the pattern $\rho$. It is known, see for instance [4], that

$$|S_n(\rho)| = C_n, \quad \text{the } n\text{th Catalan number},$$

independently of the specific choice of $\rho \in S_3$. In [2], the chain-permutational posets $P(S_n(132))$ and $P(S_n(123))$ proved helpful in obtaining a direct bijection between $S_n(132), S_n(123)$, which explains and also refines (0.3). Let us remark that there are easy bijections between any two of $S_n(132), S_n(213), S_n(231)$ and $S_n(312)$, as well as a simple bijection between $S_n(123)$ and $S_n(321)$. Also, these bijections yield isomorphisms or anti-isomorphisms between the corresponding chain-permutational posets. Thus, we need only consider the chain-permutational posets associated, say, with $\Sigma = S_n(132)$ and $\Sigma = S_n(123)$. This remark and (0.3) imply that $P(S_n(123))$ and $P(S_n(132))$ have the same number of maximal chains. Here we further compare the two families of posets $P(S_n(123))$ and $P(S_n(132))$.

In Section 2 we distinguish between $P(S_n(123))$ and $P(S_n(132))$ structurally, through
Posets and permutations

FIGURE 2.

supersolvability and modularity properties. On the other hand, both classes of posets are shown to be EL-shellable, and hence, Cohen–Macaulay.

In Section 3, using the EL-shellability and results of Stanley [5], we compute the Möbius function of \( P(S_n(123)) \) and \( P(S_n(132)) \), and show that they have equal Zeta polynomials. Thus, we strengthen (0.3) by establishing that for each value of \( n \), the two posets have not just the same number of maximal chains, but the same number of chains of each length.

Some of the facts proved for \( P(S_n(132)) \) and \( P(S_n(123)) \) hold in more general situations, when similar constructions are made (see results 2.6 and 2.8).

Before specializing to involutions and pattern-restricted permutations, let us make a few general observations which will be useful in what follows.

If \( x \in P(\Sigma) \) lies on several \( \bar{0} \rightarrow x \) saturated chains, then the set of labels along any one such chain is the same. In particular, if \( x \triangleright a \triangleright y \) and \( x \triangleright b \triangleright y \), then opposite edges of the parallelogram \( xayb \) have the same label.

It is not true in general that if \( I' \) is a subset of \( I \), then \( P(I') \) is a subposet of \( P(I) \). For example, the chain-permutational poset for the involutions in \( S_n \) (see Section 2) is not a subposet of the Boolean lattice \( B_n \).

In view of Theorem 0.1, the following question suggests itself: If \( I_1 \) and \( I_2 \) are subsets of \( S_n \) and if \( P(I_1) \) and \( P(I_2) \) are isomorphic, what can be said about the relationship between \( I_1 \) and \( I_2 \)? The natural candidate answer would be that there exists \( \alpha \in S_n \) such that \( \Sigma_2 = \alpha \Sigma_1 \alpha^{-1} \). However, this is not true in general. Figure 2 shows the chain-permutational posets for \( \Sigma_1 = \{1234, 2431\} \) and \( \Sigma_2 = \{1234, 3142\} \); no renaming of 1, 2, 3, 4 will produce \( \Sigma_2 \) from \( \Sigma_1 \), because of the difference in cycle structures.

1. The Chain-permutational Poset for the Involutions in \( S_n \)

   We now consider \( \Sigma_n = \{\sigma \in S_n, \sigma^2 = \text{identity}\} \), and we will construct \( P(\Sigma_n) \).

   Let \( \sigma \in \Sigma_n \); if \( \sigma(k) = n \), where \( 1 \leq k \leq n \), then \( \sigma(n) = k \) is forced and \( \sigma(1) \sigma(2) \cdots \sigma(k - 1)\sigma(k + 1) \cdots \sigma(n - 1) \) is an involution on \( n - 2 \) elements if \( k \neq n \), and an involution on \( n - 1 \) elements if \( k = n \). Since \( \sigma(n) \) is determined by \( k \), the chains in \( P(\Sigma_n) \) corresponding to \( \sigma(k + 1) \cdots \sigma(n) \) are disjoint for different values of \( k \). Therefore we have the following recursive construction for \( \sigma(n) \):

   For \( n = 1 \), \( P(\Sigma_1) \) is a two-element chain, the edge of which is labeled 1.

   For \( n = 2 \), \( P(\Sigma_2) \equiv B_2 \), the Boolean lattice with two atoms, with the two maximal chains labeled 1, 2 and 2, 1, as usual.

   Suppose \( P(\Sigma_m) \) is constructed for each \( m < n \). Then \( P(\Sigma_n) \) is obtained from \( P(\Sigma_{n-1}) \) and \( P(\Sigma_{n-2}) \) as follows: add a new element, \( \hat{1} \), covering the maximum element of
FIGURE 3. The chain-permutational poset for the involutions in $S_4$.

$P(\Sigma_{n-1})$, and label this covering with label $n$. For each value of $k$, $1 \leq k < n$, take the rank selected subposet $P_k$ of $P(\Sigma_{n-2})$ consisting of the ranks $n-2$, $n-3$, $\ldots$, $k-1$; join $\hat{1}$ to the maximum element of $P_k$, label this covering with $k$; modify the labeling inherited by $P_k$ from $P(\Sigma_{n-2})$ by augmenting by one unit each label which is $\geq k$. For each minimal element of $P_k$ (which now has rank $k$ in the new poset) there is a unique element of rank $k-1$ in the copy of $P(\Sigma_{n-1})$, which can cover it; introduce and label with $n$ each of these coverings.

It is easy to see that the maximal chains of the poset $P(\Sigma_n)$ thus constructed produce precisely the involutions in $S_n$.

We now prove that this poset is minimal with respect to the number of elements and we count its covering relations. To each element $x$ in our poset, we will attach the set of labels appearing on the $0-x$ chains.

Let $W_n(r)$ be the number of elements of rank $r$, and let $E_n(r-1, r)$ be the number of covering relations between ranks $r-1$ and $r$ in the poset $P(\Sigma_n)$ constructed above. It is easy to see that we have the following recurrences:

$$W_n(r) = W_{n-1}(r) + rW_{n-2}(r-1);$$

$$E_n(r-1, r) = E_{n-1}(r-1, r) + (r-1)E_{n-2}(r-2, r-1) + W_{n-2}(r-1);$$

for $n \geq 3$, $1 \leq r \leq n-1$, and $W_n(0) = W_n(n) = 1$, $E_n(n) = n$, for all $n$. The values for $n = 1$ and 2 can be easily computed.

Now, let $A$ be a subset of $[n]$, say $|A| = r \geq 2$ and let $A$ contain $i$ elements larger than $r$, where $0 \leq i \leq r$ and $0 \leq i \leq n - r$. There are

$$\binom{r}{i}\binom{n-r}{i} = \binom{r}{i}\binom{n-r}{i}$$

such sets $A$. Let $\sigma$ be an involution with the property that $\{\sigma(1), \sigma(2), \ldots, \sigma(r)\} = A$. The permutation of the $i$ elements larger than $r$ determines completely the values of $\sigma(m)$, for each $m \in A$, $m > r$. Thus there must exist in $P(\Sigma_n)$ (at least) $i!$ elements of rank $r$ the edge-labels beneath which form the same set $A$. Thus,

$$W_n(r) \geq \sum_{i=0}^{r} \binom{r}{i}\binom{n-r}{i}i!$$

Similarly, we examine the number of elements which must be covered by an element in $P(\Sigma_n)$, corresponding to a copy of the set $A$. 
If \( r \in A \), then \( \sigma(r) \leq r \) for any involution corresponding to a chain passing through \( A \), and all choices for \( \sigma(r) \leq r \) are possible since the elements of \( A \) less than or equal to \( r \) can form any involution among themselves. Thus such elements \( A \) cover \( r - i \) elements of rank \( r - 1 \), and there are

\[
\binom{r-i}{r-i-1}\binom{n-r}{i}
\]

such \( A \)'s, with at least \( i! \) copies of each.

If \( r \in A \), then \( \sigma(r) > r \) and only one choice for the value of \( \sigma(r) \) is possible for each copy of \( A \). Therefore, each copy of such \( A \) covers only one element and there are

\[
\binom{r-1}{r-i}\binom{n-r}{i} = \binom{r-1}{i-1}\binom{n-r}{i}
\]

such \( A \)'s, at least \( i! \) copies of each. This yields

\[
E_n(r-1, r) = \sum_{i=0}^{\infty} \binom{n-r}{i}i!\left[\binom{r-1}{i}(r-i) + \binom{r-1}{i}ight].
\]  

(4)

In the case of both (3) and (4), equality holds for \( n = 1, 2 \) and all \( r \), as well as for all \( n \) and \( r = n \). Furthermore, some manipulations with binomial coefficients verify that the lower bound from (3) satisfies the recurrence (1). Hence (3) is actually an equality. Using this fact and, again, some computation, one obtains the result that recurrence (2) is satisfied by the lower bound in (4). Hence (2) also is an equality.

Hence our construction of \( P(\Sigma_n) \) provides the chain-permutational poset for the involutions in \( S_n \).

**Proposition 1.1.** The chain-permutational poset associated with the set of involutions in \( S_n \) is rank symmetric and rank unimodal.

**Proof.** From the formula obtained above for \( W_n(r) \), it follows immediately that \( W_n(r) = W_n(n-r) \), i.e. \( P(\Sigma_n) \) is rank symmetric.

Using the recurrence (1), we have

\[
W_n(r+1) - W_n(r) = [W_{n-1}(r+1) - W_{n-1}(r)] + r[W_{n-2}(r) - W_{n-2}(r-1)] + W_{n-2}(r).
\]  

(5)

If \( n \) is even, say, \( n = 2m + 2 \), and \( 0 \leq r \leq m \), or if \( n \) is odd, say, \( n = 2m + 1 \), and \( 0 \leq r \leq m - 1 \), then, inductively, each term on the right-hand side of (5) is non-negative; hence, \( W_n(r+1) - W_n(r) \geq 0 \). Using now the symmetry of the sequence \( W_n(0), W_n(1), \ldots, W_n(n) \), it follows that this sequence is unimodal, with a peak when \( n \) is even, and a plateau of two terms when \( n \) is odd. \( \square \)

Note that the elements marked \( a \) and \( b \) in \( P(\Sigma_4) \), show in Figure 3, do not have a least upper bound. Therefore, due to the recursive construction, \( P(\Sigma_n) \) is not a lattice for \( n \geq 4 \).

2. The Chain-permutational Posets Associated with Restricted Permutations

With the definitions given in the Introduction, we will now construct and discuss properties of the posets \( P_n = P(S_n(123)) \) and \( P(S_n(132)) \). The recursive construction was first given in [2], and is included below in the interest of completeness.
2.1. Construction of $P(S_n(132))$. First, for $n = 1$, the poset is a 2-element chain, the edge of which is labeled 1.

Having constructed $P(S_{n-1}(132))$, $P(S_n(132))$ is obtained from two copies of $P(S_{n-1}(132))$, by introducing the following additional $n$ covering relations: each element on the unique 0–1 chain $D$ labeled as $n−1$, $n−2$, . . . , 1 in one copy of $P(S_{n-1}(132))$ covers its counterpart from the other copy of $P(S_{n-1}(132))$; each of these coverings is labeled $n$.

We now verify the validity of this construction. Let $C$ be any 0–1 chain from $P(S_n(132))$ constructed as above. Let $x$ be the maximum element in $C$ which belongs to the lower copy of $P(S_{n-1}(132))$, and $x'$ be the element of $C$ covering $x$. The covering $x \prec x'$ is labeled $n$; since $x$ lies on $D$, the 0–$x$ subchain of $C$ carries the labels $n−1$, $n−2$, . . . , $k$ (where $n−k = rk(x)$) and, inductively, these form a 132-avoiding permutation. Similarly, the $x'–1$ subchain of $C$ carries the labels 1, 2, . . . , $k+1$ permuted in some 132-avoiding manner. Furthermore, since all labels preceding $n$ are larger than all labels following $n$, $C$ gives a 132-avoiding permutation of $[n]$. Thus, each maximal chain in $P(S_n(132))$ is labeled by a different 132-avoiding permutation. Furthermore, the number of maximal chains in $P(S_n(132))$ satisfies the same recurrence as do the Catalan numbers, namely, $C_n = \Sigma C_k C_{n−k−1}$, with $C_0 = 1$; in this context, $k$ designates the rank where originates the edge between the lower and the upper copies of $P(S_{n−1}(132))$. Hence, the maximal chains in $P(S_n(132))$ do give rise to the permutations in $S_n(132)$ in bijective fashion.

Based on the recursive construction, it is easy to verify that $P(S_n(132))$ has $2^n$ elements and $2^{n+1}−n−2$ covering relations. The fact that every subset $A$ of $[n]$, $|A| = k$, $0 \leq k \leq n$, is realizable as $A = \{\sigma(i), 1 \leq i \leq k\}$ for some $\sigma \in S_n$, e.g. arrange $A$ decreasingly followed by $[n]−A$ ordered increasingly, proves that $2^n$ is the minimum number of elements for $P(S_n(132))$. The minimality of the number of covering relations can be proved by examining the necessary number of coverings labeled $n$ and using induction.

2.2. Construction of $P(S_n(123))$. $P(S_1(123))$ is a two-element chain labeled 1. Assuming $P(S_{n−1}(123))$ is constructed, we obtain $P(S_n(123))$ by first forming the product poset of $P(S_{n−1}(123))$ with a two-element chain, keeping the original labels in
each copy of $P(S_{n-1}(123))$, and labeling the other coverings as $n$; now delete from the ‘lower’ copy of $P(S_{n-1}(123))$ all coverings except those in the subposet $T_{n-1}$ consisting of the saturated chains in $P(S_{n-1}(123))$ whose labels, starting from 0, are decreasing. Note that in graph-theoretic terms, $T_{n-1}$ is a spanning tree of $P(S_{n-1}(123))$.

Each maximal chain $C$ in $P(S_n(123))$ gives a different permutation from $S_n(123)$. Indeed, let $x$ and $x'$ be the elements of $C$ such that the covering $x < x'$ is labeled $n$. Then the labels preceding $n$ are along the unique 0–$x$ path in $T_{n-1}$ and are decreasing, hence no 123-pattern occurs up to $x$. Since $x'$ has a copy of the 0–$x$ chain under it in the ‘upper’ copy of $P(S_{n-1}(123))$, there are no occurrences of the pattern 123 after $n$, nor involving labels from $C - \{x < x'\}$. It is easy to see, inductively, that different chains $C$ give rise to different permutations in $S_n(123)$.

Conversely, let $\sigma \in S_n(123)$. If $\sigma(1) < n$, let $k$ be defined by $\sigma(1) > \sigma(2) > \cdots > \sigma(k) < \sigma(k+1) = n$. Then a unique path in $T_{n-1}$ corresponds to $\sigma(1) \cdots \sigma(k)$ and, inductively, there exists a chain in the upper copy of $P(S_{n-1}(123))$ corresponding to $\sigma(1) \cdots \sigma(k) \sigma(k+2) \cdots \sigma(n)$. A maximal chain in $P(S_n(123))$ corresponding to $\sigma$ is now easily determined. If $\sigma(1) = n$, then $\sigma' = \sigma(2) \cdots \sigma(n) \in S_n(123)$ and, inductively, there is a chain in $P(S_{n-1}(123))$, the upper half of $P(S_n(123))$, labeled by $\sigma'$. Preceded by the 0–$n$ covering, this chain yields a chain in $P(S_n(123))$ labeled by $\sigma$.

As in the case of $\Sigma = S_n(132)$, $P(S_n(123))$ has $2^n$ elements and $2^{n+1} - n - 2$ covering relations. The proof is easy and is omitted. The proof that these values are minimum possible is similar to that for the analogous result for $P(S_n(132))$.

Of course, each of $P(S_n(123))$, $P(S_n(132))$ can be regarded as a subposet of the Boolean lattice $B_n$. We have:

**Proposition 2.3.** Each of $P(S_n(123))$, $P(S_n(132))$ is a lattice.

**Proof.** In both cases the result is trivially true for $n = 1$.

For $P(S_n(123))$, let $x, y$ be any two elements. If both lie in the upper half, $P(S_{n-1}(123))$, or if both lie in $T_{n-1}$, then glb$(x, y)$ exists. Suppose $x \in T_{n-1}$ and $y \in P(S_{n-1}(123))$. Then all lower bounds for $x$ and $y$ belong to $T_{n-1}$. All such elements $Z$ are in fact less than $y_0 = \text{the element in } T_{n-1}$ which is covered by $y$. Finally,
glb(x, y) = glb(x, y₀) in Tₙ₋₁. Since P(Sₙ(123)) has a ı, and since every two elements have a greatest lower bound, P(Sₙ(123)) is a lattice.

Similarly, the proof for the fact that P(Sₙ(132)) is a lattice reduces to the existence of glb(x, y) when x and y belong to different ‘halves’ of the poset. If x lies in the lower copy of P(Sₙ₋₁(132)) and if y lies in the upper copy of P(Sₙ₋₁(132)), let Z be the largest element belonging to the decreasingly labeled 0–ı maximal chain, satisfying Z ≤ y, and let y₀ < Z belong to the lower copy of P(Sₙ₋₁(132)). Then it can be verified that glb(x, y₀) is the greatest lower bound of x and y. □

Note however, that the chain-permutational lattices under discussion are subposets but not sublattices of the Boolean lattice.

We have, at this point, two families of lattices, \{P(Sₙ(123))\}_{n≥1} and \{P(Sₙ(132))\}_{n≥1}, which share a number of features: they have the same number of elements, in fact, the same number of elements of each rank, the same number of covering relations, and the same number of maximal chains. However:

**Proposition 2.2.** P(Sₙ(132)) and P(Sₙ(123)) are not isomorphic for n ≥ 4.

**Proof.** Isomorphism is easily seen to hold for n = 1, 2, 3, but for n ≥ 4, P(Sₙ(123)) has an atom, {n} (we are denoting the elements of our lattices as subsets of [n]), with the property that it is covered by every rank 2 element which is not join-irreducible, i.e. \{i, n\}, 1 ≤ i < n. On the other hand, in P(Sₙ(132)) the rank 2 elements which are not join-irreducible are of the form \{i, i + 1\}, 1 ≤ i ≤ n – 1; hence no three of them cover the same atom. □

In view of Proposition 2.4 and the comments preceding it, it is natural to ask how far goes the similarity between our two families of lattices. The next results will lead to the conclusion that, for n ≥ 3, neither P(Sₙ(123)) nor P(Sₙ(132)) is a semimodular lattice; only the former is supersolvable; both are edge-wise lexicographically shellable. The definitions of these terms can be found below.

It is easy to see that for n ≥ 3 neither P(Sₙ(123)) nor P(Sₙ(132)) is a distributive lattice, since each has height n and more than n join-irreducible elements. This remark, combined with Theorem 0.2, yields the conclusion that there is no n-element poset the linear extensions of which are the permutations in Sₙ(123) or those in Sₙ(132) or, therefore, Sₙ(ρ), ρ ∈ S₃.

**Claim 2.5.** For n ≥ 3, neither P(Sₙ(123)) nor P(Sₙ(132)) is semimodular.

**Proof.** Figure 6 shows P(S₃(123)) ≠ P(S₃(132)). The elements denoted x and y cover 0 = glb(x, y), but neither is covered by ı = lub(x, y). Therefore, for n = 3, the

![Figure 6](image-url)
lattices are not semimodular. Now, since the lattice for \( n = 3 \) appears as a sublattice in \( P(S_n(123)) \) and \( P(S_n(132)) \) for every \( n \geq 4 \), the claim follows. \( \square \)

A lattice is called *supersolvable* (see [5]) if it contains a maximal chain, called an \( M \)-chain, with the property that together with any other chain it generates a distributive sublattice.

Toward showing that \( P(S_n(123)) \) is supersolvable for every \( n \), we begin with the following:

**Lemma 2.6.** Let \( L \) be a distributive lattice and \( x_0, \ldots, x_m \in L \) with the properties:

(i) \( \emptyset = x_0 < x_1 < x_2 < \cdots < x_m \);
(ii) \( x_i \) is join-irreducible, for each \( 1 \leq i \leq m \).

Then \( L(y) \) obtained from \( L \) by adjoining the elements \( y_0 < y_1 < \cdots < y_m \) and the covering relations \( y_i < x_i \) for \( 1 \leq i \leq m \), is also a distributive lattice.

**Proof.** Let \( Q \) be the set of join-irreducible elements in \( L \), and let \( a_1, \ldots, a_k \) be the minimal elements in \( Q - \{ x_1, x_2, \ldots, x_m \} \). Let \( Q(y) \) be the poset obtained from \( Q \) by adding \( a_0 < a_i \), \( 1 \leq i \leq k \). We will show that \( L(y) \) is isomorphic to the lattice of order ideals of \( Q(y) \) and, thus, \( L(y) \) is a distributive lattice.

Each ideal \( I \) from \( Q \) which contains some minimal element of \( Q - \{ x_1, x_2, \ldots, x_m \} \) corresponds to an ideal in \( Q(y) \) or cardinality bigger by one unit, since the latter will also contain \( a_0 \). Every other ideal in \( Q \) is of the form \( I = \{ x_1, x_2, \ldots, x_i \}, 1 \leq i \leq m \), and yields two ideals in \( Q(y) \), namely, \( I \) and \( I \cup \{ a_0 \} \). The former corresponds to the element \( y_i \) in \( L(y) \), and the latter corresponds to \( x_i \). \( \square \)

**Remark.** Condition (i) in the hypothesis of Lemma 2.6 is indeed necessary; otherwise \( L(y) \) contains a sublattice isomorphic to the lattice shown in Figure 7.

The necessity of condition (ii) is illustrated by the lattice in Figure 8, where the original
L is drawn in heavy line. The elements \( y_0, y_1, y_2, x_2, Z \) form a sublattice isomorphic to that of Figure 7.

**Proposition 2.7.** For every \( n \geq 1 \), the lattice \( P_n = P(S_n(123)) \) is supersolvable.

**Proof.** For \( n = 1, 2 \) the lattice is actually distributive. In the case \( n = 3 \), the only choice for the distinguished \( M \)-chain required for supersolvability is \( M = \hat{0} < a < b < \hat{1} \), as shown in Figure 9. This can be verified directly, or using the fact that the \( M \)-chain must consist of modular elements [6].

Now, in the labeling of \( P_3 \), this chain is the one corresponding to the permutation \( \sigma = 321 \).

Assume that the chain corresponding to \( \sigma \in S_k(123) \), \( \sigma(i) = k + 1 - i, 1 \leq i \leq k \), is an \( M \)-chain for \( P_k \), where \( k < n \). We will show that \( D \), the \( \hat{0} - \hat{1} \) chain in \( P_n \) labeled \( n, n-1, \ldots, 2, 1 \), along with any other chain in \( P_n \), generates a distributive sublattice.

Let \( C \) be any chain in \( P_n \). If all elements of \( C \) lie in the ‘upper half’ of \( P_n \), i.e. the copy of \( P_{n-1} \) embedded in \( P_n \) via \( \gamma \in S_n(123) \rightarrow \gamma \in S_n(123), \gamma(1) = n, \gamma(i) = \sigma(i-1) \) for \( 2 \leq i \leq n \), then, inductively, \( C \) and \( D - \{ \hat{0} \} \) generate a distributive sublattice in \( P_{n-1} \).

What \( C \) and \( D \) generate in \( P_n \) is the same sublattice, with an additional minimum element adjoined; hence this is still a distributive lattice (apply Lemma 2.6 with \( m = 0 \)).

Let now \( C' \) be the intersection of \( C \) with \( T_{n-1} \), and suppose it is non-empty. If, in fact, the first edge in \( C \) is the one labeled \( n-1 \), then an argument similar to the one above applies. Otherwise, \( C' \) is attached to \( C'' \), its counterpart in \( T_{n-2} \), present in \( P_{n-1} \), in a manner satisfying the conditions of Lemma 2.6. Inductively, \( D - \{ \hat{0} \} \) and \( C'' \cup (C \cap P_{n-1}) \) generate a distributive lattice. Now the conclusion follows from Lemma 2.6. \( \square \)

It is known, see [5], that supersolvability implies EL-shellability.

A (ranked) poset is *edge-wise lexicographically shellable* (El-shellable) if its covering relations admit an *EL-labeling*; that is, an integer labeling with the following two properties:

(a) for every \( x < y \), there exists a unique chain \( x = x_0 < x_1 < \cdots < x_m = y \) along which the labels form a (weakly) increasing sequence;

(b) the label sequence along the chain in (a) precedes lexicographically the label sequence of every other \( x-y \) chain.

Further related concepts and results, some of which will be used here later, appear in the beautiful survey article [5].

We give below a direct proof of the EL-shellability of \( P(S_n(123)) \), in part because

![Figure 9](attachment:image.png)
FIGURE 10. $P(S_3(123))$ with EL-labeling.

Proposition 2.8 will exhibit a general poset operation which preserves EL-shellability, and also because the explicit labeling will be useful later in this paper.

**Proposition 2.8.** Let $P$ be an EL-shellable poset which has a $\emptyset$ and $\hat{1}$. Let $T$ be the subposet of $P$ consisting of the $\emptyset$-x increasingly labeled chains in $P$, for all $x \in P$. Consider the poset $P \circ T$ formed from a copy of $P$, a copy of $T$ and letting each $x \in P$ cover $x \in T$. Then the poset $P \circ T$ is EL-shellable.

**Proof.** If, in the EL-labeling of $P$ the labels are numbers from $[n]$, augment the original labels in $P$ and $T$ by one unit, and label the other $|P|$ coverings in $P \circ T$ with 1.

If $x < y$ in $P \circ T$, the conditions (a) and (b) for EL-shellability are clearly satisfied when $x, y \in T$ and when $x, y \in P$. Suppose $x \in T$ and $y \in P$. Then $x < x' < y$, where $x' \in P$ is the copy of $x$ in $P$. The unique increasing $x' - y$ chain in $P$ extends to an increasing $x - y$ chain, since $x < x'$ is labeled 1. There is no other increasingly labeled $x - y$ chain, since either a covering labeled 1 will occur not in the first place, or else the $x' - y$ subchain is not increasingly labeled. Because it starts with label 1 and has the least label lexicographically from $x'$ to $y$, this $x - y$ chain via $x'$ is lexicographically the least among all $x - y$ chains in $P \circ T$. □

**Corollary 2.9.** For each $n \geq 1$, the chain-permutational poset $P(S_n(123))$ is EL-shellable.

**Proof.** The natural edge-labeling of $P(S_n(123))$ need only be complemented, i.e. replace label $\lambda$ by $n + 1 - \lambda$, to give an EL-labeling. The recursive construction of $\{P(S_n(123))\}_{n \geq 1}$ is precisely the type of construction described in Proposition 2.8. □

We now turn to $P(S_n(132))$ and show that it, too, is EL-shellable, but not supersolvable.

**Claim 2.10.** For $n \geq 4$, the lattice $P_n = P(S_n(132))$ is not supersolvable.

**Proof.** Since $P(S_3(132)) \cong P(S_3(123))$, we already know that $P_3$ is supersolvable and that the $M$-chain is unique, as shown in Figure 11; denote it by $M_3$. 


Using the uniqueness of the $M$-chain in $P_3$ it can be checked directly that $P_4$, $P_5$, $P_6$ are not supersolvable. Assume $P_k$ is not supersolvable for $4 \leq k < n$, and suppose that $M_n$ is an $M$-chain in $P_n$, $n \geq 7$. Let $x$ be the largest element on $M_n$ which belongs to the lower half, $P_{n-1}$, of $P_n$. Then the interval $[x, \hat{1}]$, being a sublattice of $P_n$, must be supersolvable with $M$-chain $M_n \cap [x, \hat{1}]$. On the other hand, $[x, \hat{1}] \cong P_{n-k}$, where $k = $ rank of $x$. Thus, $n - k \leq 3$. Also, the interval $[\hat{0}, x] \cong P_k$ is a sublattice of $P_n$ and must be supersolvable with $M$-chain $M_n \cap [\hat{0}, x]$. Hence, $k \leq 3$. But then $n \leq 6$, contradicting the above assumption that $n \geq 7$.

Although, for $n \geq 4$, $P(S_n(132))$ is neither semimodular nor supersolvable, it is EL-shellable, but not with the labeling which proves it to be the chain-permutational poset of $S_n(132)$. For example, with this labeling, there is no increasingly labeled chain from $\hat{0}$ to the element corresponding to the subset $\{1, 3\}$ of $[n]$, $n \geq 3$. It is also the case that neither one of two bijections $S_n(123) \leftrightarrow S_n(132)$ discussed in [2] helps convert the EL-labeling of $P(S_n(123))$ to an EL-labeling of $P(S_n(132))$. For instance, for $n = 3$, the bijection involving standard Young tableaux relabels the chains 123 and 213 as 321 and 213, respectively, therefore inconsistently relabeling the last covering of the chains. The more direct bijection given in [2] transforms the same two chains into 132 and 213, respectively, therefore having the same shortcoming. We now construct an EL-labeling for $P(S_n(132))$.

**Proposition 2.11.** For each $n \geq 1$, the lattice $P_n = P(S_n(132))$ is edge-wise lexicographically shellable.

**Proof.** We define a spanning tree $T_n$ of the Hasse diagram of $P_n$, rooted at $\hat{0}$, through the following recursive process: in each of the lower and upper half of $P_n$, consider $T_{n-1}$; join the root of the upper $T_{n-1}$ to $\hat{0}$ in $P_n$. For $n = 1$, $T_1 = P_1$, naturally; its edge is labeled 1.

Label the edges in $T_n$ as follows: each of the two copies of $T_{n-1}$ get their labels increased by one unit; the remaining edge is labeled 1. In this manner all $\hat{0}$--$x$ paths in $T_n$ are increasingly labeled.

The remaining edges in the upper and lower copies of $P_{n-1}$ are labeled as they were in $P_{n-1}$; each of the remaining $n - 1$ edges which cross from the lower to the upper copy of $P_{n-1}$ is labeled by the rank of the smaller element incident to it. In $P_2$, the unique edge not in $T_2$ is labeled 1. We will refer to the coverings in $P_n$ as 'tree-' or 'non-tree-' edges, depending on whether they belong or not to $T_n$.

Next we will verify that the labeling defined above is EL. We will denote by $\lambda(x, y)$ the label assigned to the covering $x \ll y$. 
Claim 2.12. \( \lambda(x, y) > \lambda(y, z) \) iff \( x < y \) is a tree-edge and \( y < z \) is a non-tree-edge.

Proof. The assertion holds for \( P(S_2(132)) \). Suppose it holds for \( P_{n-1} = P(S_{n-1}(132)) \). Then it also holds in the two copies of \( P_{n-1} \) which are the lower and the upper halves of \( P_n \), since the only modification of the labeling of \( P_{n-1} \) is a further uniform increase of the labels for tree-edges.

Thus it suffices to consider \( x < y < z \) in the following cases:

Case (1). \( x < y \) is an edge on the increasingly labeled \( 0-1 \) chain in the lower copy of \( P_{n-1} \). Then \( \lambda(x, y) = \text{rk}(y) + 1 \), while \( \lambda(y, z) = \text{rk}(y) \) if \( z \) lies in the upper copy of \( P_{n-1} \), and \( \lambda(x, y) > \lambda(y, z) \) otherwise. Hence, the claim is true.

Case (2). \( y < z \) is an edge on the increasingly labeled \( 0-1 \) chain in the upper copy of \( P_{n-1} \), and \( x \) lies in the lower copy of \( P_{n-1} \). Then \( \lambda(x, y) = \text{rk}(y) - 1 < \text{rk}(y) + 1 = \lambda(y, z) \).

Case (3). \( x < y \) and \( y < z \) are non-tree edges and \( y < z \) runs between the lower and the upper copy of \( P_{n-1} \). Then \( \lambda(x, y) < \lambda(y, z) \) from the construction of the labeling.

This proves Claim 2.12. Observe also that, if \( D \) is the \( 0-1 \) chain in \( T_n \), then for every \( x \in P_n - D \) there exists a unique non-tree-edge \( x < y \), and this has the smallest label among all \( \lambda(x, z) \) for \( z > x \).

On the basis of this observation and Claim 2.12 it is now easy to verify that the labeling constructed recursively for \( P(S_n(132)) \) is an EL-labeling.

3. Zeta Polynomial and Möbius Function

Our final goal is to prove that \( P(S_n(123)) \) and \( P(S_n(132)) \) have the same number of chains of each length. Let \( Z_n(k, 123) \) and \( Z_n(k, 132) \) denote the number of multichains \( 0 = x_0 \leq x_1 \leq \cdots \leq x_k = 1 \) in \( P(S_n(123)) \) and \( P(S_n(132)) \), respectively. The generating function \( \sum Z_n(k, P) t^k \) is known (see, e.g., [5]) to be a rational function of the form \( p(t)/(1 - t)^{n+1} \), for any ranked poset \( P \) of rank \( n \). In the situation when \( P \) is an EL-shellable poset, the coefficients of the polynomial \( p(t) \) have a combinatorial interpretation; namely, if we write

\[
p(t) = \sum \beta(S) t^{\text{S}\text{S}+1},
\]

where the sum ranges over all subsets \( S \subseteq [n] \), then \( \beta(S) \) is the number of maximal chains \( 0 = x_0 < x_1 < \cdots < x_n = 1 \) in \( P \) for which \( \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1}) \) iff \( i \in S \) [5; Theorem 2.2].

Theorem 3.1. If \( P_n \) denotes the chain-permutational poset of either \( S_n(123) \) or \( S_n(132) \), then the generating function of the Zeta polynomial of \( P_n \) is

\[
\sum Z(m, P_n) t^m = (1 - t)^{-n-1} \sum_{j=0}^{\text{C}_n} \left( \begin{array}{c} n-j \hfill \text{C}_n \text{C}_n \\
 \text{j} \hfill \text{C}_n \text{C}_n \end{array} \right) (t - 1)^j,
\]

where \( \text{C}_k \) is the \( k \text{th} \) Catalan number. In particular, \( P(S_n(123)) \) and \( P(S_n(132)) \) have the same number of (multi)chains of each length.

For \( n \leq 6 \) we give below the sum \( s_n(t) \) in the numerator of (7). As expected, the
constant terms and the coefficients of the linear terms are Eulerian numbers [9]:

<table>
<thead>
<tr>
<th>n</th>
<th>s_n(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 + t</td>
</tr>
<tr>
<td>3</td>
<td>1 + 4t</td>
</tr>
<tr>
<td>4</td>
<td>1 + 11t + 2t^2</td>
</tr>
<tr>
<td>5</td>
<td>1 + 26t + 15t^2</td>
</tr>
<tr>
<td>6</td>
<td>1 + 57t + 69t^2 + 5t^3</td>
</tr>
</tbody>
</table>

**Proof.** Using (6) and the EL-labeling constructed for \( P(S_n(132)) \) in Proposition 2.11, we will first show that (7) holds when \( P_n = P(S_n(132)) \).

A 0–1 chain in \( P(S_n(132)) \) having at least \( j \) descents in its label sequence can be constructed as follows: among the first \( n - 1 \) coverings, choose \( j \) to be followed by a covering with lower label. These \( j \) coverings cannot be consecutive, because decreases in labels occur only from a tree-edge to a non-tree edge. Hence, there are

\[
(\binom{n-1}{j})
\]

choices. Since at most one covering \( x < y \) is non-tree for each \( x \), the choice of the \( j \) ranks will determine the actual descending edges of the chain. The chain projects to a maximal 0–1 chain in \( P_\omega \), hence a total of

\[
\binom{n-j}{j}
\]

maximal chains in \( P(S_n(132)) \) have at least \( j \) descents. By inclusion–exclusion, the coefficient of \( t^k \) in

\[
\sum_{j \geq 0} C_{n-j}\binom{n-j}{j}(t-1)^j
\]

will be the number of maximal chains having precisely \( k \) descents. Thus, (7) holds for \( P_n = P(S_n(132)) \).

In order to prove that \( P(S_n(123)) \) has the same Zeta polynomial, we first recall the bijection \( S_n(123) \rightarrow S_n(132) \) established in [2]. Let \( \sigma \in S_n(123) \). We will construct the permutation \( \rho \in S_n(132) \) corresponding to \( \sigma \). Let \( k \) be defined by \( \sigma(1) > \sigma(2) > \cdots > \sigma(k) < \sigma(k + 1) \). If no such \( k \) exists, then \( \rho = \sigma \). Otherwise, let \( \rho(i) = \sigma(i) \) for \( 1 \leq i \leq k \) and let \( \rho(k + 1) \) be the minimum element in \( [n] - \{ \sigma(i), 1 \leq i \leq k \} \) which is larger than \( m = \sigma(k) \). If \( \sigma(k + 2) < m \), let \( \rho(k + 2) = \sigma(k + 2) \), replace \( m \) by \( \sigma(k + 2) \) and continue. If \( \sigma(k + 2) > m \), again let \( \rho(k + 2) \) be the minimum element in \( [n] - \{ \rho(i), 1 \leq i \leq k + 1 \} \) which is larger than \( m \), and continue.

For example, \( \sigma = 536142 \in S_6(123) \) yields \( \rho = 534126 \in S_6(132) \).

The proof that this correspondence is indeed bijective appears in [2] where the inverse map is also constructed.

For the purpose of the proof of Theorem 3.1, the important property of the bijection (which, in fact, suggested the correspondence) is that it preserves the tree–non-tree succession of the coverings in the chains. With the natural labeling of \( P(S_n(123)) \) and \( P(S_n(132)) \), i.e. the labeling which shows they are the appropriate chain-permutational posets, this property is not particularly important. However, with the EL-labelings put on \( P(S_n(123)) \) and \( P(S_n(132)) \), it shows that the numerator \( p(t) \) in (6) is the same for both posets. This is because—as proved in Claim 2.12 and as is obvious for
In the case of $P(S_n(132))$, the results on counting chains by descents double as results on counting $\sigma \in S_n(132)$ by the number of ascents. The same is not true, of course, for $S_n(132)$. For completeness, we give the following:

**Theorem 3.2.** Let $d_n(k)$ be the number of 132-avoiding permutations having precisely $k$ descents. Then the polynomials $q_n(t) = \sum d_n(k)t^k$ satisfy:

(a) $q_n(t) = q_{n-1}(t) + t \sum q_r(t)q_{n-r-1}(t)$ for $n \geq 2$; $q_0(t) = q_1(t) = 1$;

(b) $q_n(t)$ is a reciprocal polynomial;

(c) $q_n(-1) = 0$ if $n$ is even, and $(-1)^m C_m$ if $n = 2m + 1$.

**Proof.** (a) Let $\sigma \in S_n(132)$. If $\sigma(n) = n$ then the descents of $\sigma$ are precisely those of $\sigma(1) \sigma(2) \cdots \sigma(n-1) \in S_{n-1}(132)$. If $\sigma(r) = n, 1 \leq r < n$, then $\alpha = \sigma(1) \cdots \sigma(r-1)$ and $\beta = \sigma(r+1) \cdots \sigma(n)$, each is a 132-avoiding permutation and the number of descents in $\sigma$ is $1 + \#(\text{descents in } \alpha) + \#(\text{descents in } \beta)$. Hence,

$$d_n(k) = d_{n-1}(k) + \sum_{1 \leq r < n, 0 \leq s < k} d_{r-1}(s) \cdot d_{n-r}(k-s-1).$$

This yields (a). Note that by setting $t = 1$ in (a) we obtain, as we should, a well known recurrence for the Catalan numbers, since $q_n(1) = C_n$.

Now, (b) is true for $n = 0$ and 1. For $n \geq 2$, use induction and compute

$$t^{n+1}q_n(1/t) = t^n q_{n-1}(1/t) + t^n - 2 \sum q_r(1/t)q_{n-r-1}(1/t)$$

$$= t q_{n-1}(1/t) + t^{n-2} \left[ q_{n-1}(1/t) + \sum q_r(1/t)q_{n-r-1}(1/t) \right]$$

$$= q_{n-1}(1/t) + t \left[ q_{n-1}(1/t) + \sum q_r(1/t)q_{n-r-1}(1/t) \right]$$

$$= q_n(t).$$

(c) can also be proved by induction. Its combinatorial interest rests with Corollary 3.3. □

For values of $n \leq 6$, $q_n(t)$ are:

$$q_0(t) = q_1(r) = 1, \quad q_2(t) = 1 + t$$

$$q_3(t) = 1 + 3t + t^2, \quad q_4(t) = 1 + 6t + 6t^2 + t^3,$$

$$q_5(t) = 1 + 10t + 20t^2 + 10t^3 + t^4, \quad q_6(t) = 1 + 15t + 50t^2 + 50t^3 + 15t^4 + t^5.$$
The characteristic polynomial of \( P(\mathcal{S}_n) \) is given by

\[
\chi_n(t) = t^n - nt^{n-1} + (n-1)t^{n-2}.
\]

One can completely determine the values of the Möbius functions of the two chain-permutational posets, by further using \( (8) \) with \( \hat{0} \) replaced by an arbitrary poset element.

**Remark.** After this work was completed, Björner's paper [8] was brought to the author's attention. The notion of a chain-permutational poset appears in [8] under the name of universal (simple) representation, or the most economical (simple) representation; therefore, it recommends itself as an interesting poset representation to be studied.

Our paper addresses specific questions different from those treated in [8]. It also provides an answer to the following question raised in Björner's Remark 5.7:

Given an EL-shellable poset \( P \), consider the language \( L(P) \), the words of which are the label sequences of all \( \hat{0}-x \) chains in \( P \). When \( P \) is a distributive lattice this language is a greedoid (i.e. it is closed under word prefixes, and every word not of maximum length is extendable to a word in \( L(P) \) by the addition as a suffix of some letter of each longer word). Is \( L(P) \) a greedoid for all \( P \) belonging to some other class of EL-shellable posets? Specifically, Björner asks whether \( L(P) \) is a greedoid for all supersolvable lattices \( P \).

Our chain-permutational posets \( P = P(\mathcal{S}_n) \) provide (for \( n \geq 3 \)) an infinite family of supersolvable lattices for which \( L(P) \) is not a greedoid; the words \( v = n \) and \( w = n - 1 \), \( n \) violate the second (exchange) condition for a greedoid, since \( v = n \) can only be extended to \( n \), 1.

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RODICA SIMION†
Department of Mathematics, Southern Illinois University, and Bryn Mawr College, Bryn Mawr, PA 19010, U.S.A.

† Current address: Department of Mathematics, George Washington University, Washington, D.C. 20052, U.S.A.