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On some inequalities of Hermite–Hadamard type via m-convexity

M. Emin Özdemir [a], Merve Avcı [b], Erhan Set [b,*]

[a] Graduate School of Natural and Applied Sciences, Ağrı İbrahim Çeçen University, Ağrı, Turkey

[b] Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey

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ABSTRACT

In this paper we give some estimates to the right-hand side of Hermite–Hadamard inequality for functions whose absolute values of second derivatives raised to positive real powers are m-convex.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [1]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

For recent results, generalizations and new inequalities related to the inequality presented above see [2–5].

In [6] G.Toader defined the concept of m-convexity as the following:

Definition 1. The function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) \leq 0$.

Some interesting and important inequalities for m-convex functions can be found in [7,8].

Using the classical results of Hermite and Hadamard on convex functions, S.S. Dragomir, P. Cerone and A. Sofo obtained the following result. (see [9]).

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on (a, b) and suppose that $-\infty < k \leq f''(x) \leq K < \infty$ for all $x \in (a, b)$. Then the following inequality holds:*

$$k\frac{(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq K\frac{(b-a)^2}{12}. \tag{1.1}$$

* Corresponding author.
E-mail addresses: emos@atauni.edu.tr (M.E. Özdemir), merveavci@ymail.com (M. Avcı), erhanset@yahoo.com (E. Set).

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In order to prove our main results we need the following lemma (see [10]).

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\mathrm{d}t.$$

The main purpose of this paper is to establish some new inequalities like those given in [10], but now for the class of m-convex functions.

2. Hermite–Hadamard type inequalities

We will start with the following theorem containing Hermite–Hadamard type inequality.

Theorem 2. *Let $f : I^\circ \to \mathbb{R}$, where $I^\circ \subset [0, \infty)$ be a twice differentiable function on I° such that $f'' \in L[a, b]$, where $a, b \in I,\ a < b$. If $|f''|^q$ is m-convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \geq 1$ then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Proof. First suppose that $q = 1$. From Lemma 1 we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|\mathrm{d}t. \tag{2.1}$$

Since $\left|f''\right|$ is m-convex on $[a, b]$ we know that for any $t \in [0, 1]$

$$\left|f''(ta+(1-t)b)\right| \leq t\left|f''(a)\right| + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|.$$

Therefore,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(t\left|f''(a)\right| + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)\mathrm{d}t \\ &= \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right| + m\left|f''\left(\frac{b}{m}\right)\right|}{2}\right]\end{aligned}$$

which completes the proof for this case.

Suppose now that $q > 1$. Using Lemma 1 and the well-known Hölder's inequality (see for example [11]) for $q,\ p = \frac{q}{q-1}$ we obtain

$$\begin{aligned}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|\mathrm{d}t &= \int_0^1 (t-t^2)^{1-\frac{1}{q}}(t-t^2)^{\frac{1}{q}}\left|f''(ta+(1-t)b)\right|\mathrm{d}t \\ &\leq \left[\int_0^1 (t-t^2)\mathrm{d}t\right]^{\frac{q-1}{q}}\left[\int_0^1 (t-t^2)\left|f''(ta+(1-t)b)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned} \tag{2.2}$$

Since $\left|f''\right|^q$ is m-convex on $[a, b]$ we know that for every $t \in [0, 1]$

$$\left|f''(ta+(1-t)b)\right|^q \leq t\left|f''(a)\right|^q + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q. \tag{2.3}$$

Hence, from (2.1) and (2.3) we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\leq \frac{(b-a)^2}{2}\left[\int_0^1 (t-t^2)\mathrm{d}t\right]^{\frac{q-1}{q}}\left[\int_0^1 (t-t^2)\left|f''(ta+(1-t)b)\right|^q\mathrm{d}t\right]^{\frac{1}{q}} \\ &\leq \frac{(b-a)^2}{2}\left[\int_0^1 (t-t^2)\mathrm{d}t\right]^{\frac{q-1}{q}}\left[\int_0^1 (t-t^2)\left\{t\left|f''(a)\right|^q + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right\}\right]^{\frac{1}{q}}\end{aligned}$$

$$= \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{\frac{q-1}{q}}\left[\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{12}\right]^{\frac{1}{q}}$$

$$= \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}$$

which completes the proof. □

Remark 1. If in Theorem 2 we choose $m = 1$ and if $\left|f''(x)\right| \le K$ on $[a, b]$ we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)^2}{12}\left\{\frac{\left|f''(a)\right| + \left|f''(b)\right|}{2}\right\} = K\frac{(b-a)^2}{12},$$

which is the right-hand side of (1.1).

A similar result is embodied in the following theorem.

Theorem 3. *Let $f : I^\circ \to \mathbb{R}$, where $I^\circ \subset [0, \infty)$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a, b]$. If $\left|f''\right|^q$ is m-convex on $[a, b]$ for some fixed $q > 1$ and $m \in (0, 1]$ then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $p = \frac{q}{q-1}$.

Proof. From Lemma 1 and using the well-known Hölder's inequality we have successively

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|\mathrm{d}t$$

$$\le \frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\le \frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\left|f''(a)\right|^q\int_0^1 t\mathrm{d}t + m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 (1-t)\mathrm{d}t\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\,\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^2}{2}\frac{\left(\pi^{\frac{1}{2}}\right)^{\frac{1}{p}}}{2^{\frac{1}{p}}2^2}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}},$$

and since $\sqrt{\pi} < 2$, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

We note that, the Beta and the Gamma function (see [12], pp 908–910),

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y > 0, \qquad \Gamma(x) = \int_0^\infty \mathrm{e}^{-t}t^{x-1}\mathrm{d}t, \quad x > 0,$$

are used to evaluate the integral

$$\int_0^1 (t-t^2)^p\mathrm{d}t = \int_0^1 t^p(1-t)^p\mathrm{d}t = \beta(p+1, p+1),$$

where,

$$\beta(x,x)=2^{1-2x}\beta\left(\frac{1}{2},x\right),\quad \text{and}\quad \beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \text{thus we can obtain that}$$

$$\beta(p+1,p+1)=2^{1-2(p+1)}\beta\left(\frac{1}{2},p+1\right)=2^{1-2(p+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)},$$

and, $\Gamma(\frac{1}{2})=\sqrt{\pi}$, which completes the proof. □

Corollary 1. *With the above assumptions given that* $\left|f''(x)\right|\le K$ *on* $[a,b]$, *and* $0<m\le 1$, *we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le K\frac{(b-a)^2}{8}\left(\frac{1+m}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}.$$

Corollary 2. *From Theorems 2 and 3 we have the inequality for* $q>1$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\min\{K_1,K_2\}$$

where

$$K_1=\frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}$$

$$K_2=\frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Another Hermite–Hadamard type inequality for powers in terms of the second derivatives is obtained as following:

Theorem 4. *With the assumptions of Theorem 3 we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)^2}{2}\left(\left|f''(a)\right|^q+m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

Proof. From Lemma 1 and Hölder's inequality we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|\mathrm{d}t\\ &\le\frac{(b-a)^2}{2}\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^q\left|f''(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\ &\le\frac{(b-a)^2}{2}\left(\int_0^1 t^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+m(1+q)\left|f''\left(\frac{b}{m}\right)\right|}{(q+1)(q+2)}\right)^{\frac{1}{q}}.\end{aligned}$$

Since

$$\lim_{p\to\infty}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}=1\quad\text{and}\quad\lim_{p\to 1^+}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}=\frac{1}{2},$$

we have

$$\frac{1}{2}<\left(\frac{1}{1+p}\right)^{\frac{1}{p}}<1,\quad q\in(1,\infty),$$

hence for $q\in(1,\infty)$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)^2}{2}\left(\left|f''(a)\right|^q+m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.\quad\square$$

The following result also holds:

Theorem 5. *Let $f : I^\circ \to \mathbb{R}$, where $I^\circ \subset [0,\infty)$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a,b]$. If $\left|f''\right|^q$ is m-convex on $[a,b]$ for some fixed $q \geq 1$ and $m \in (0,1]$ then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{(b-a)^2}{4}\left(2\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

Proof. From Lemma 1 and the well-known power-mean inequality we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| &\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|\mathrm{d}t \\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^q\left|f''(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^q\left[\left|f''(a)\right|^q t + \left|f''\left(\frac{b}{m}\right)\right|^q m(1-t)\mathrm{d}t\right]\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 t\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\left|f''(a)\right|^q\int_0^1 t^2(1-t)^q\mathrm{d}t + m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 t(1-t)^{q+1}\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{2}{(q+1)(q+2)(q+3)}\left|f''(a)\right|^q + m\frac{1}{(q+2)(q+3)}\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{4}\left(\frac{2}{(q+1)(q+2)(q+3)}\right)^{\frac{1}{q}}\left[2\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $\left(\frac{2}{(q+1)(q+2)(q+3)}\right)^{\frac{1}{q}} \leq 1, q \in [1,\infty)$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{(b-a)^2}{4}\left[2\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

which completes the proof. □

Remark 2. From Theorems 3–5, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \min\{E_1, E_2, E_3\}$$

where

$$\begin{aligned}
E_1 &= \frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \\
E_2 &= \frac{(b-a)^2}{2}\left(\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \\
E_3 &= \frac{(b-a)^2}{4}\left(2\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

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