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Uniqueness of the Limit Cycle for Gause-Type Predator–Prey Systems*

Tzy-Wei Hwang

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In this paper we derive some results to ensure that the number of the limit cycle of a generalized Gause-type predator–prey system does not exceed one. Two examples are presented in the final section. © 1999 Academic Press

Key Words: predator–prey system; uniqueness limit cycle; Dulac criterion.

1. INTRODUCTION

The main purpose of this paper is to establish conditions to ensure that the number of the limit cycle of the following system (1.1) does not exceed one:

$$\begin{aligned} x'(t) &= g(x) - \varphi(x)\pi(y) = F(x,y) \\ y'(t) &= \psi(x)\rho(y) = G(x,y) \qquad (1.1) \\ x(0) &> 0, \quad y(0) > 0. \end{aligned}$$

We make the following assumptions:

(A1) $\pi, \rho \in C^1[0,\infty)$, $\pi(0) = \rho(0) = 0$, and $\pi'(y) > 0$, $\rho'(y) > 0$ for $y \geq 0$.

(A2) $\varphi \in C^1[0,\infty)$, $\varphi(0) = 0$ and $\varphi'(x) > 0$ for $x \geq 0$.

(A3) $g \in C^1[0,\infty)$, $g(0) = 0$, and there exists $K > 0$ such that $g'(K) < 0$ and $(x - K)g(x) < 0$ for $x \in (0,\infty) - \{K\}$.

(A4) $\psi \in C^1[0,\infty)$, and there exists $\lambda \in (0,K)$ such that $\psi(\lambda) = 0$, $\psi'(\lambda) > 0$, and $(x - \lambda)\psi(x) > 0$ for $x \in [0,\infty) - \{\lambda\}$.

(A5) $h = g/\varphi \in C^1(0,\infty)$ and $h((0,K]) \subseteq \pi([0,\infty))$.

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A large class of biological and bioeconomic models are special cases of (1.1) satisfying assumptions (A1)–(A5). For the detailed biological meaning, the reader may consult [6, 17–19]. The method used to prove the nonexistence of periodic solution of (1.1) is the Dulac criterion. For the part of "uniqueness," Cheng [2] was the first one to prove the uniqueness of the limit cycle for a predator–prey system with a Holling type 2 functional response by using the symmetry of the prey isocline. Subsequently, Liou and Cheng [14] further developed a method of reflection to extend the class of predator–prey model for which the results are valid. With some restrictions on φ, Kuang and Freedman [13] and Huang and Merrill [11] transform a class of predator–prey model with Gause-type to a generalized Lienard system where the results of uniqueness of the limit cycle are available. Ding [4] studied a kind of predator–prey system and shows that the results hold. Hsu and Hwang [10] give a sufficient condition for the uniqueness of the limit cycle of system (1.1), the prey isocline of which has two humps. In this paper we employ the techniques in [4] to prove the results.

The main results and their consequences are given in Section 2. In Section 3, two examples are provided to show the applicability of the main theorems.

2. MAIN RESULTS

It is clear that system (1.1) has equilibria at $E_0(0,0)$, $E_1(K,0)$, and $E^*(\lambda, y^*)$, where $y^* = \pi^{-1}(h(\lambda)) > 0$. Both E_0 and E_1 are saddle points. Since the Jacobian of system (1.1) at E^* is

$$J = \begin{bmatrix} \varphi(\lambda)h'(\lambda) & -\varphi(\lambda)\pi'(y^*) \\ \rho(y^*)\psi'(\lambda) & 0 \end{bmatrix},$$

the eigenvalues are given by

$$\tfrac{1}{2}\left\{\varphi(\lambda)h'(\lambda) \pm \sqrt{[\varphi(\lambda)h'(\lambda)]^2 - 4\varphi(\lambda)\pi'(y^*)\rho(y^*)\psi'(\lambda)}\right\}.$$

Hence E^* is stable if $h'(\lambda) < 0$, and E^* is unstable if $h'(\lambda) > 0$. The case where $h'(\lambda) = 0$ is undecided, as E^* could be either a center or a focus, since the set $\Omega = (0, K) \times R_+ \subseteq R_+^2$ is positively invariant and any trajectory must intersect it from the exterior to the interior provided $x(0) \geq K$. Therefore we will restrict our attentions to the open region Ω in the following discussion.

THEOREM 2.1. *Let the assumptions* $(A1)$–$(A5)$ *hold. Assume that there exists* $a, b \in R$ *such that*

$$0 \neq \varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x) \le 0 \qquad \text{for } x \in (0, K).$$

Then system (1.1) *has no periodic solution in* R_+^2.

Proof. It is sufficient to prove that $\Omega = (0, K) \times R_+$ contains no periodic solution of system (1.1). Let $H(x, y) = l(x)r(y)$, where $l(x)$ and $r(y)$ will be determined. Then

$$\begin{aligned}\Delta &= \frac{\partial}{\partial x}(FH) + \frac{\partial}{\partial y}(GH)\\ &= H(x,y)\left\{\varphi(x)h'(x) + h(x)\left[\varphi'(x) + \varphi(x)\frac{l'(x)}{l(x)}\right]\right.\\ &\qquad \left. -\pi(y)\left[\varphi'(x) + \varphi(x)\frac{l'(x)}{l(x)}\right] + \psi(x)\left[\rho'(y) + \rho(y)\frac{r'(y)}{r(y)}\right]\right\}.\end{aligned}$$

Let

$$r(y) = [\rho(y)]^{-1}\exp\left(\int_{y^*}^{y}\frac{a + b\pi(\xi)}{\rho(\xi)}\,d\xi\right)$$

and

$$l(x) = [\varphi(x)]^{-1}\exp\left(b\int_{\lambda}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\,d\xi\right);$$

then

$$\rho'(y) + \rho(y)\frac{r'(y)}{r(y)} = a + b\pi(y)$$

$$\varphi'(x) + \varphi(x)\frac{l'(x)}{l(x)} = b\psi(x),$$

and

$$\Delta = H(x,y)\{\varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x)\} \le 0 \qquad \text{for } (x,y) \in \Omega.$$

Now the assertion follows by Dulac criterion.

THEOREM 2.2. *Suppose that* $h'(\lambda) > 0$ *and* $(A1)$–$(A5)$ *hold and, moreover, there exist* $\alpha, \beta \ge 0$ *and not all zero such that*

$$\frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))}\right) \le 0$$

for all $x \in (0, K) - \{\lambda\}$. *Then system* (1.1) *possesses at most one limit cycle, and if it exists then it is stable.*

Proof. Without loss of generality, we may assume that system (1.1) has nontrivial periodic orbits. Let $\Gamma(t) = (x(t), y(t))$ be any periodic solution of (1.1) and $a, b \in R$; one obtains

$$\psi(x)(a + bh(x)) = \frac{a + b\pi(y)}{\rho(y)} y'(t) + b\frac{\psi(x)}{\varphi(x)} x'(t),$$

$$\varphi'(x)(h(x) - \pi(y)) = \frac{\varphi'(x)}{\varphi(x)} x'(t), \quad \psi(x)\rho'(y) = \frac{\rho'(y)}{\rho(y)} y'(t)$$

and

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \varphi(x)h'(x) + \varphi'(x)(h(x) - \pi(y)) + \psi(x)\rho'(y).$$

Since Γ is a periodic orbit of (1.1), then

$$\oint_\Gamma \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dt = \oint_\Gamma \left[\varphi(x)h'(x) + \psi(x)(a + bh(x)) \right] dt.$$

Since E^* is unstable, there must be a periodic orbit Γ_1 which is the nearest one around E^*. It follows that Γ_1 must be stable from inside, and by Poincaré criterion of stability, we get

$$\oint_{\Gamma_1} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dt \le 0.$$

Let $x_1 = \min\{x \mid (x, y) \in \Gamma_1\}$ and $z = \max\{x \mid (x, y) \in \Gamma_1\}$. Define

$$a = -\frac{\varphi(x_1)h'(x_1)}{\psi(x_1)(\alpha + \beta h(x_1))}\alpha$$
$$b = -\frac{\varphi(x_1)h'(x_1)}{\psi(x_1)(\alpha + \beta h(x_1))}\beta \tag{2.1}$$

and

$$w(x) = \varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x).$$

Clearly $x_1 \in (0, \lambda)$ and $w(x_1) = 0$. Since $w(x)/\psi(x)[\alpha + \beta h(x)]$ is non-increasing in $(0, K) - \{\lambda\}$ and $\psi(x) < 0$ in $(0, \lambda)$, we have $w(x) < 0$ as $x \in (0, x_1)$ and $w(x) > 0$ as $x \in (x_1, \lambda)$. If $w(x) \geq 0$ as $x \in (\lambda, z)$ then

$$0 \geq \oint_{\Gamma_1} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dt = \oint_{\Gamma_1} w(x)\, dt > 0,$$

a contradiction. Hence there must exist an $x_2 \in (\lambda, z)$ such that $w(x) > 0$ as $x \in (x_1, x_2)$ and $w(x) < 0$ as $x \in (0, x_1) \cup (x_2, K)$. Suppose there exists another periodic orbit Γ_2 that is outside and closest to Γ_1. The vertical line $x = x_1$ intersects the orbit Γ_2 at points A_2 and D (see Fig. 1). The vertical line $x = x_2$ intersects Γ_1 and Γ_2 at points B_1, C_1 and B_2, C_2, respectively. Then

$$\oint_{\Gamma_1} w(x)\, dt = \left(\int_{A_1 B_1} + \int_{B_1 C_1} + \int_{C_1 A_1} \right) w(x)\, dt,$$

$$\oint_{\Gamma_2} w(x)\, dt = \left(\int_{A_2 B_2} + \int_{B_2 C_2} + \int_{C_2 D} + \int_{D A_2} \right) w(x)\, dt.$$

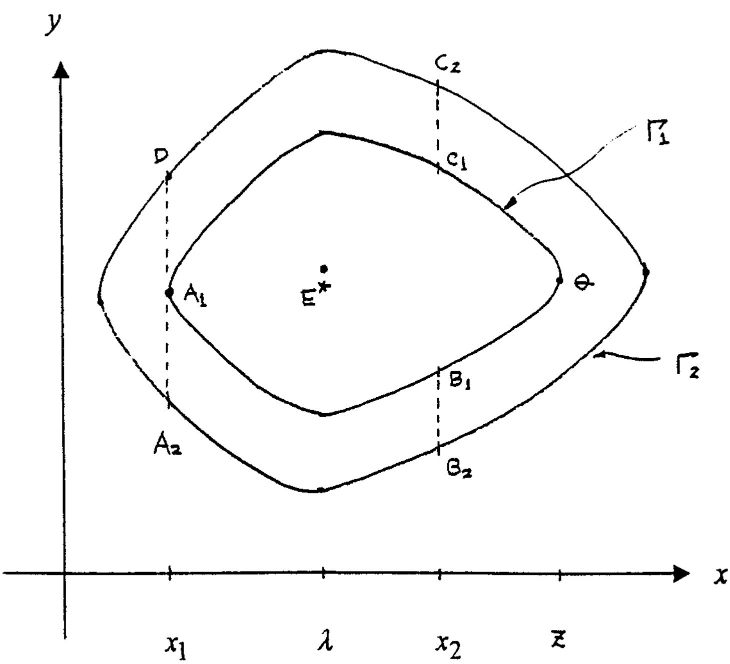

FIGURE 1

Let $y = y_1(x)$ and $y = y_2(x)$ denote the functions of curves A_1B_1 and A_2B_2, respectively. Then

$$\int_{A_2B_2} w(x)\,dt - \int_{A_1B_1} w(x)\,dt$$

$$= \int_{x_1}^{x_2} \frac{w(x)}{\varphi(x)[h(x) - \pi(y_2(x))]}\,dx - \int_{x_1}^{x_2} \frac{w(x)}{\varphi(x)[h(x) - \pi(y_1(x))]}\,dx$$

$$= \int_{x_1}^{x_2} \frac{w(x)}{\varphi(x)} \frac{\pi(y_2(x)) - \pi(y_1(x))}{[h(x) - \pi(y_1(x))][h(x) - \pi(y_2(x))]}\,dx < 0.$$

Similarly, we can prove

$$\int_{C_2D} w(x)\,dt - \int_{C_1A_1} w(x)\,dt < 0.$$

Since

$$\frac{d}{dx}\left(\frac{w(x)}{\psi(x)(\alpha + \beta h(x))}\right) = \frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))}\right) < 0$$

as $x \in (0, K) - \{\lambda\}$ and $w(x_1) = w(x_2) = 0$, one has

$$\int_{DA_2} w(x)\,dt = \left(\int_{DA_2} + \int_{A_2D}\right) w(x)\,dt$$

$$= \int_{DA_2A_1D} \frac{w(x)}{\psi(x)(\alpha + \beta h(x))}\left[\beta\frac{\psi(x)}{\varphi(x)}\,dx + \frac{\alpha + \beta\pi(y)}{\rho(y)}\,dy\right]$$

$$= \iint_{\Omega_1} \frac{\alpha + \beta\pi(y)}{\rho(y)} \frac{d}{dx}\left(\frac{w(x)}{\psi(x)(\alpha + \beta h(x))}\right) dx\,dy < 0$$

and

$$\int_{B_2C_2} w(x)\,dt - \int_{B_1C_1} w(x)\,dt$$

$$= \int_{B_2C_2C_1QB_1B_2} \frac{w(x)}{\psi(x)(\alpha + \beta h(x))}\left[\beta\frac{\psi(x)}{\varphi(x)}\,dx + \frac{\alpha + \beta\pi(y)}{\rho(y)}\,dy\right]$$

$$= \iint_{\Omega_2} \frac{\alpha + \beta\pi(y)}{\rho(y)} \frac{d}{dx}\left(\frac{w(x)}{\psi(x)(\alpha + \beta h(x))}\right) dx\,dy < 0,$$

where Ω_1 and Ω_2 are two regions bounded by the above two closed paths, respectively. Thus

$$\oint_{\Gamma_2}\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)dt<\oint_{\Gamma_1}\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)dt\leq 0.$$

Since two periodic orbits with the same stability cannot exist side by side, we conclude that Γ_1 is externally unstable. To obtain a contradiction, let

$$H(x)=\begin{cases} h(x)+\frac{a}{b}-\left(h(x_2)+\frac{a}{b}\right)\exp\left(-b\int_{x_2}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\,d\xi\right) & \text{if } b\neq 0\\ h(x)+a\int_{x_2}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\,d\xi-h(x_2) & \text{if } b=0\end{cases}$$

for $x\in[0,\infty)$; here a and b are given in (2.1). Hence $H(x)$ is the solution to the initial value problem,

$$\begin{aligned} H'(x)+b\frac{\psi(x)}{\varphi(x)}H(x)&=\frac{w(x)}{\varphi(x)}\\ H(x_2)&=0. \end{aligned}\tag{2.2}$$

Note that

$$H'(x_2)=-b\frac{\psi(x_2)}{\varphi(x_2)}H(x_2)+\frac{w(x_2)}{\varphi(x_2)}=0.$$

Since

$$\frac{d}{dx}\left\{\exp\left(b\int_{x_2}^{x}\frac{\psi(x)}{\varphi(\xi)}\right)H(x)\right\}=\exp\left(b\int_{x_2}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\right)\frac{w(x)}{\varphi(x)}<0$$

for $x\in(x_2,K)$, we have $H(x)<0$ for $x\in(x_2,K]$. Let $\theta=\varphi(K)H(K)$, $\eta=\varphi'(K)H(K)+\varphi(K)H'(K)$, and $p(x)=(\eta^2/4\theta-1)(x-K)^2+\eta(x-K)+\theta$ for $x\in R$. Clearly, since $\theta<0$, $\eta^2/4\theta-1<0$, and $\eta^2-4\theta(\eta^2/4\theta-1)=4\theta<0$, the polynomial $p(x)<0$ on R. Now consider the new system

$$\begin{aligned} x'(t)&=g_\varepsilon(x)-\varphi(x)\pi(y)=F_\varepsilon(x,y), \qquad \varepsilon\geq 0\\ y'(t)&=\psi(x)\rho(y), \end{aligned}\tag{2.3}$$

where

$$g_\varepsilon(x) = \begin{cases} g(x) & \text{if } x \in [0.x_2] \\ g(x) + \varepsilon\varphi(x)H(x) & \text{if } x \in (x_2, K] \\ g(x) + \varepsilon p(x) & \text{if } x > K. \end{cases}$$

Clearly, $g_\varepsilon(x) = g(x)$ if $x \in [0, x_2]$ and $g_\varepsilon(x) < g(x)$ if $x > x_2$. Using that $H'(x_2) = 0$, one can show that $g_\varepsilon \in C^1[0, \infty)$ for $\varepsilon \geq 0$. As ε varies, (2.3) is a rotated vector field, and it is (1.1) when $\varepsilon = 0$. According to the theory of the rotated vector field [5], as $0 < \varepsilon \ll 1$ the semistable limit cycle Γ_1 will split into at least two limit cycles Γ_1' and Γ_1'', where Γ_1' is enclosed by Γ_1'' and, moreover, Γ_1'' is at least unstable on the outside and Γ_1' is at least stable from the inside, i.e.,

$$\oint_{\Gamma_1''} \left(\frac{\partial F_\varepsilon}{\partial x} + \frac{\partial G}{\partial y} \right) dt \geq 0 \geq \oint_{\Gamma_1'} \left(\frac{\partial F_\varepsilon}{\partial x} + \frac{\partial G}{\partial y} \right) dt.$$

On the other hand, the system (2.3) satisfies the assumptions (A1)–(A5). To see this, let us show that

For each $0 < \varepsilon \ll 1$, there exists $K_\varepsilon \in (\lambda, K)$ such that

$$g_\varepsilon(K_\varepsilon) = 0, g_\varepsilon'(K_\varepsilon) < 0 \quad \text{and} \quad (x - K_\varepsilon)g_\varepsilon(x) < 0$$
$$\text{for all } x \in (0, \infty) - \{K_\varepsilon\}.$$

Since $g \in C^1[0, \infty)$ and $g'(K) < 0$, there exists a positive, small δ such that $K - \delta \in (x_2, K)$, $g'(x) < \frac{1}{2}g'(K) < 0$ for $x \in [K - \delta, K]$ and $g(x) > g(K - \delta) > g(y)$ for $x \in [x_2, K - \delta)$ and $y \in (K - \delta, K]$. Let M^* be the C^1-norm of φH on $[x_2, K]$, $\varepsilon_0 = \frac{1}{M^*}\min\{-\frac{1}{2}g'(K), g(K - \delta)\}$ and $\varepsilon \in (0, \varepsilon_0)$. Then we have, for $x \in [x_2, K - \delta]$, $g_\varepsilon(K) = g(K) + \varepsilon\varphi(K)H(K) = \varepsilon\varphi(K)H(K) < 0$ and

$$\begin{aligned} g_\varepsilon(x) &= g(x) + \varepsilon\varphi(x)H(x) \geq g(K - \delta) + \varepsilon\varphi(x)H(x) \\ &\geq \varepsilon_0 M^* + \varepsilon\varphi(x)H(x) > \varepsilon(M^* + \varphi(x)H(x)) \geq 0. \end{aligned}$$

This means that there is a $K_\varepsilon \in (K - \delta, K)$ such that $g_\varepsilon(K_\varepsilon) = 0$ for $\varepsilon \in (0, \varepsilon_0)$. Moreover, for $x \in [K - \delta, K]$, we have

$$\begin{aligned} g_\varepsilon'(x) &= g'(x) + \varepsilon(\varphi(x)H(x))' < \tfrac{1}{2}g'(K) + \varepsilon(\varphi(x)H(x))' \\ &\leq -\varepsilon_0 M^* + \varepsilon(\varphi(x)H(x))' < \varepsilon[-M^* + \varepsilon(\varphi(x)H(x))'] < 0. \end{aligned}$$

Hence, for $x \in [K - \delta, K] - \{K_\varepsilon\}$, $(x - K_\varepsilon)g_\varepsilon(x) < 0$ and $g_\varepsilon'(K_\varepsilon) < 0$. Notice that $g_\varepsilon(x) = g(x) > 0$ if $x \in (0, x_2)$ and $g_\varepsilon(x) < 0$ if $x > K$; thus we have $(x - K_\varepsilon)g_\varepsilon(x) < 0$ for $x \in (0,\infty) - \{K_\varepsilon\}$. Clearly, $\lambda < x_2 < K - \delta < K_\varepsilon$, $h_\varepsilon = g_\varepsilon/\varphi \in C^1(0,\infty)$ for $\varepsilon \in (0, \varepsilon_0)$, and since $0 \le g_\varepsilon \le g$ on $[0, K_\varepsilon]$, we have $h_\varepsilon((0, K_\varepsilon]) \subseteq h((0, K]) \subseteq \pi([0,\infty))$. So the system (2.3) satisfies assumptions (A1)–(A5). Next let

$$\Delta_\varepsilon(x) = \frac{\varphi(x)h'_\varepsilon(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]}$$

$$w_\varepsilon(x) = \varphi(x)h'_\varepsilon(x) + \psi(x)[a + bh_\varepsilon(x)]$$

and

$$q_\varepsilon(x) = \frac{w_\varepsilon(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]} = \Delta_\varepsilon(x) + \frac{b}{\beta}$$

for $x \in (0, K_\varepsilon) - \{\lambda\}$ and $0 \le \varepsilon \ll 1$. Since $h_\varepsilon = g_\varepsilon/\varphi = g/\varphi = h$ on $(0, x_2]$ for $0 \le \varepsilon \ll 1$, one obtains that $w_\varepsilon(x) = w(x)$ and $\Delta_\varepsilon(x) = \frac{\varphi(x)h'(x)}{\psi(x)[\alpha + \beta h(x)]}$ if $x \in (0, x_2] - \{\lambda\}$ and $0 \le \varepsilon \ll 1$. Thus $\Delta'_\varepsilon(x) = \frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)[\alpha + \beta h(x)]}\right) < 0$ on $(0, \lambda) \cup (\lambda, x_2)$ and $0 \le \varepsilon \ll 1$. If $x \in (x_2, K_\varepsilon)$ and $0 \le \varepsilon \ll 1$, then, since $h_\varepsilon(x) = h(x) + \varepsilon H(x)$ and because of (2.2), we have

$$\begin{aligned} w_\varepsilon(x) &= \varphi(x)[h'(x) + \varepsilon H'(x)] + \psi(x)[a + b(h(x) + \varepsilon H(x))] \\ &= w(x) + \varepsilon\varphi(x)\left[H'(x) + b\frac{\psi(x)}{\varphi(x)}H(x)\right] = (1 + \varepsilon)w(x) \end{aligned}$$

and

$$\begin{aligned} q_\varepsilon(x) &= \frac{w_\varepsilon(x)}{\psi(x)[\alpha + \beta h(x) + \varepsilon\beta H(x)]} \\ &= (1 + \varepsilon)\frac{w(x)}{\psi(x)[\alpha + \beta h(x)]}\,\frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)}. \end{aligned}$$

Hence,

$$\begin{aligned} \Delta'_\varepsilon(x) &= q'_\varepsilon(x) \\ &= (1 + \varepsilon)\left[\frac{d}{dx}\left(\frac{w(x)}{\psi(x)[\alpha + \beta h(x)]}\right)\frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)}\right. \\ &\qquad \left. + \varepsilon\beta\frac{w(x)}{\psi(x)[\alpha + \beta h(x)]}\,\frac{w(x)[\beta H(x) - \alpha - \beta h(x)]}{\varphi(x)[\alpha + \beta h(x) + \varepsilon\beta H(x)]^2}\right]. \end{aligned}$$

Since

$$\frac{d}{dx}\left(\frac{w(x)}{\psi(x)[\alpha+\beta h(x)]}\right)=\frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)[\alpha+\beta h(x)]}\right)<0$$

and $H(x)<0$ on (x_2, K_ε), we have $\Delta'_\varepsilon(x)<0$ if $x\in(x_2, K_\varepsilon)$. Moreover, Δ'_ε has a jumped discontinuity at x_2 and can be extended continuously on $(\lambda, x_2]$ and $[x_2, K_\varepsilon)$, respectively. Let $x'_1=\min\{x \mid (x,y)\in\Gamma'_1\}$, $z'=\max\{x \mid (x,y)\in\Gamma'_1\}$, and

$$W(x)=\varphi(x)h'_\varepsilon(x)-\Delta_\varepsilon(x'_1)\psi(x)(\alpha+\beta h_\varepsilon(x)).$$

Clearly, $x'_1\in(0,\lambda)$ and $W(x'_1)=0$. Since $\frac{W(x)}{\psi(x)[\alpha+\beta h_\varepsilon(x)]}=\Delta_\varepsilon(x)-\Delta_\varepsilon(x'_1)$ is nonincreasing in $(0,K_\varepsilon)$, there must exist an $x'_2\in(\lambda, z')$ such that $W(x)>0$ as $x\in(x'_1, x'_2)$ and $W(x)<0$ as $x\in(0,x'_1)\cup(x'_2,K_\varepsilon)$. Moreover, we have $x_2\in(\lambda, z')$. If not, then, since $h_\varepsilon=h$ on $(0,x_2]$, Γ'_1 is a periodic orbit of system (1.1) and is enclosed by Γ_1, a contradiction. To show that

$$\oint_{\Gamma''_1}\left(\frac{\partial F_\varepsilon}{\partial x}+\frac{\partial G}{\partial y}\right)dt<\oint_{\Gamma'_1}\left(\frac{\partial F_\varepsilon}{\partial x}+\frac{\partial G}{\partial y}\,dt\right), \tag{2.4}$$

there are two cases to be considered:

Case 1. $x'_2\ge x_2$. By applying similar arguments in the beginning of the proof, we have inequality (2.4).

Case 2. $x'_2\in(\lambda, x_2)$ (see Fig. 2). Let the vertical line $x=x'_2$ intersect Γ'_1 and Γ'_2 at points B'_1, E'_1 and B'_2, E'_2, respectively, and the vertical line $x=x_2$ intersect Γ'_1 and Γ'_2 at points C'_1, D'_1 and C'_2, D'_2, respectively. Again, using similar arguments in the beginning of the proof, one can show that

$$\int_{E'_1A'_1B'_1}W(x)\,dt>\int_{E'_2A'_2B'_2}W(x)\,dt.$$

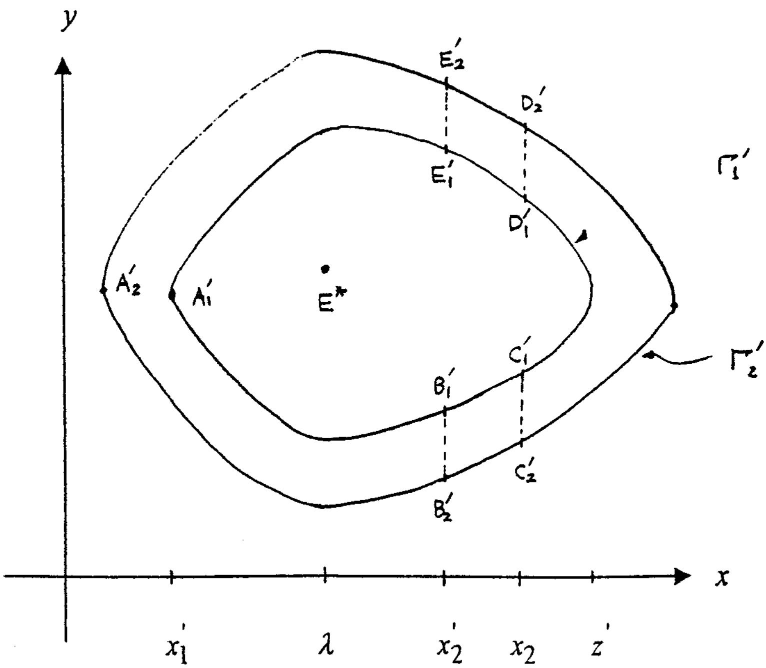

FIGURE 2

Note that

$$\int_{B_2'C_2'D_2'E_2'} W(x)\,dt - \int_{B_1'C_1'D_1'E_1'} W(x)\,dt$$

$$= \left(\int_{B_1'B_2'} + \int_{B_2'C_2'} + \int_{C_2'C_1'} - \int_{B_1'C_1'}\right) W(x)\,dt$$

$$+ \left(\int_{C_1'C_2'} + \int_{C_2'D_2'} + \int_{D_2'D_1'} - \int_{C_1'D_1'}\right) W(x)\,dt$$

$$+ \left(\int_{D_1'D_2'} + \int_{D_2'E_2'} + \int_{E_2'E_1'} - \int_{D_1'E_1'}\right) W(x)\,dt$$

$$= \left(\oint_{B_1'B_2'C_2'C_1'B_1'} + \oint_{C_1'C_2'D_2'D_1'C_1'} + \oint_{D_1'D_2'E_2'E_1'D_1'} \right)$$

$$\frac{W(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]} \left(\beta \frac{\psi(x)}{\varphi(x)} dx + \frac{\alpha + \beta\pi(y)}{\rho(y)} dy \right)$$

$$\equiv I + II + III.$$

Since Δ_ε' can be extended continuously to $(\lambda, x_2]$ and $[x_2, K_\varepsilon)$, respectively, and because of Green's Theorem, we have

$$I = \iint_{\Omega_1'} \frac{d}{dx} \left(\frac{W(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]} \right) \cdot \frac{\alpha + \beta\pi(y)}{\rho(y)} \, dx\, dy$$

$$= \iint_{\Omega_1'} \Delta_\varepsilon'(x) \cdot \frac{\alpha + \beta\pi(y)}{\rho(y)} \, dx\, dy < 0,$$

where Ω_1' is the region bounded by the closed curve $B_1'B_2'C_2'C_1'B_1'$. Similarly, we have $II < 0$ and $III < 0$. This shows

$$\int_{B_1'C_1'D_1'E_1'} W(x)\, dt > \int_{B_2'C_2'D_2'E_2'} W(x)\, dt,$$

and hence the inequality (2.4) holds. This implies that

$$\oint_{\Gamma_1''} \left(\frac{\partial F_\varepsilon}{\partial x} + \frac{\partial G}{\partial y} \right) dt < \oint_{\Gamma_1'} \left(\frac{\partial F_\varepsilon}{\partial x} + \frac{\partial G}{\partial y} \right) dt \le 0,$$

a contradiction. Hence system (1.1) possesses at most one limit cycle, and if it exists it is stable. This completes the proof of Theorem 2.2.

Remark. If $\alpha = 1$ and $\beta = 0$, then Theorem 2.2 is the criterion for the uniqueness of the limit cycle given by Huang and Merrill [11] and Ding [4].

THEOREM 2.3. *Let assumptions* $(A1)$–$(A5)$ *hold. Assume*

(1) *There exist* $\alpha, \beta \ge 0$ *and* $\alpha^2 + \beta^2 > 0$ *such that the function* $q(x) = \frac{\varphi(x)h'(x) - \varphi(\lambda)h'(\lambda)}{\psi(x)(\alpha + \beta h(x))}$ *is* C^1 *and* $q'(x) \le 0$ *as* $x \in (0, K)$.

(2) $\frac{d}{dx}(\psi(x)(\alpha + \beta h(x))) > 0$ *as* $x \in (0, K)$.

Then system (1.1) *possesses at most one limit cycle in* R_+^2.

Proof. The proof will be divided into two cases.

Case 1. $h'(\lambda) \le 0$. Since $q \in C^1(0, K)$ and $q'(x) \le 0$ as $x \in (0, K)$, we have

$$\varphi(x)h'(x) - \varphi(\lambda)h'(\lambda)$$
$$= q(x)\psi(x)(\alpha + \beta h(x)) \le q(\lambda)\psi(x)(\alpha + \beta h(x))$$

as $x \in (0, K)$. Hence $\varphi(x)h'(x) \leq q(\lambda)\psi(x)(\alpha + \beta h(x)) + \varphi(\lambda)h'(\lambda)$ as $x \in (0, K)$. Thus Theorem 2.1 implies that system (1.1) has no periodic orbit in R_+^2.

Case 2. $h'(\lambda) > 0$. Since $q(x) = \frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))} - \frac{\varphi(\lambda)h'(\lambda)}{\psi(x)(\alpha + \beta h(x))}$ as $x \in (0, K) - \{\lambda\}$, we have

$$\frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))}\right) = q'(x) + \varphi(\lambda)h'(\lambda)\frac{d}{dx}\left(\frac{1}{\psi(x)(\alpha + \beta h(x))}\right) < 0$$

for all $x \in (0, K) - \{\lambda\}$. Thus the assertion follows from Theorem 2.2.

Remark. If E^* is locally asymptotically stable, then condition (2) in Theorem 2.3 is not necessary, and in this case, Theorem 2.3 is a generalization of Lemmas 2.4 and 3.3 in Ardito and Ricciardi [1] and Theorem 2 in Cheng et al. [3].

3. EXAMPLES

EXAMPLE 1. Let us consider the model

$$x'(t) = x(1 - x) - \frac{x^2}{(a + x)(b + x)}y$$

$$y'(t) = c\left[\frac{x^2}{(a + x)(b + x)} - \frac{\lambda^2}{(a + \lambda)(b + \lambda)}\right]y,$$

where $a, b, c \in R_+$, $\lambda \in (0, 1)$, and $a > b$. Then the system possesses at most one limit cycle in R_+^2.

Let $q(x)$ be the function given in Theorem 2.3 with $\alpha = 1$ and $\beta = 0$. It is sufficient to show that q is C^1 and $q'(x) \leq 0$ for $x \in (0, 1)$. Let $\varphi(x) = x^2/(a + x)(b + x)$, $h(x) = (1 - x)(a + x)(b + x)/x$, and $\psi(x) = c[\varphi(x) - \varphi(\lambda)]$. Then

$$\varphi(x) - \varphi(\lambda) = (x - \lambda)\frac{((a + b)\lambda + ab)x + ab\lambda}{(\lambda + a)(\lambda + b)(x + a)(x + b)}$$

and

$$\varphi(x)h'(x) - \varphi(\lambda)h'(\lambda) = \frac{(x - \lambda)p(x)}{(\lambda + a)(\lambda + b)(x + a)(x + b)},$$

where

$$p(x) = -2(\lambda + a)(\lambda + b)(x + a)(x + b) + a(1 + a)(\lambda + b)(x + b) + b(1 + b)(\lambda + a)(x + a).$$

Thus $cq(x) = \frac{p(x)}{((a+b)\lambda + ab)x + ab\lambda}$ is C^1. To show that $q'(x) \le 0$ for $x \in (0,1)$, it is equivalent to show that $\Delta(x) = p'(x)(((a+b)\lambda + ab)x + ab\lambda) - p(x)((a+b)\lambda + ab) \le 0$ for $x \in (0,1)$.

With some calculations one can show that

$$\begin{aligned}
\Delta(x) = & -2((a+b)\lambda + ab)(\lambda + a)(\lambda + b)x^2 - 4ab\lambda(\lambda + a)(\lambda + b)x \\
& + ab\lambda(a(1+a)(\lambda + b) + b(1+b)(\lambda + a) \\
& \qquad -2(\lambda + a)(\lambda + b)(a + b)) \\
& -((a+b)\lambda + ab)ab((1+a)(\lambda + b) + (1+b)(\lambda + a) \\
& \qquad -2(a + \lambda)(b + \lambda))) \\
= & -2((a+b)\lambda + ab)(\lambda + a)(\lambda + b)x^2 - 4ab\lambda(\lambda + a)(\lambda + b)x \\
& - ab(\lambda b(1+a)(\lambda + b) + \lambda a(1+b)(\lambda + a) \\
& \qquad + ab(1 - \lambda)(2\lambda + a + b)).
\end{aligned}$$

Thus $\Delta(x) < 0$ for $x \in (0,1)$. Hence the assertion follows from Theorem 2.3.

Remark. The conclusion in Example 1 also holds if $a = b$. Indeed,

$$cq(x) = \frac{2(\lambda + a)}{a} \frac{(x + a)(a(1 - \lambda) - (\lambda + a)x)}{(2\lambda + a)x + a\lambda}$$

is C^1, and

$$\begin{aligned}
& \frac{ac}{2(\lambda + a)} q'(x) \\
& \quad = \frac{-(2\lambda + a)(\lambda + a)x^2 - 2a\lambda(\lambda + a)x - a^2(\lambda + a)}{((2\lambda + a)x + a\lambda)^2} < 0.
\end{aligned}$$

EXAMPLE 2. Consider a model, in which the mortality rate of the predator population is a function of the density of the prey population (see [16]),

$$x'(t) = x(1 - x) - \frac{x}{a + x} y$$

$$y'(t) = D\left[\frac{x}{a + x} - \frac{bx + c}{dx + e}\right] y,$$

where $a, b, c, d, e, D > 0$ and

$$a \in (0,1)$$
$$d - b > 0$$
$$cd - be > 0$$
$$d + e - (a+1)(b+c) > 0.$$

Then, since $\frac{d}{dx}(\frac{x}{a+x} - \frac{bx+c}{dx+e}) > 0$ in $R - \{-a, -\frac{e}{d}\}$ and the polynomial $(d-b)x^2 + (e-c-ab)x - ac$ has exactly two roots, there exist $\Lambda > \max\{a, \frac{e}{d}\}$ and $\lambda \in (0,1)$, such that $\Lambda - \lambda = \frac{e-ab-c}{d-b}$, $\Lambda\lambda = \frac{ac}{d-b}$, and the problem can be rewritten as

$$x'(t) = \frac{x}{a+x}[(1-x)(x+a) - y] \equiv \varphi(x)[h(x) - y]$$
$$y'(t) = D(d-b)\frac{(x-\lambda)(x+\Lambda)}{(x+a)(dx+e)}y \equiv \psi(x)y.$$

THEOREM 3.1. *If $\lambda \in (0, \frac{1-a}{2})$, then the system has exactly one periodic orbit in R^2_+.*

Proof. Since $h'(\lambda) = 2(\frac{1-a}{2} - \lambda) > 0$, the equilibrium point $E^* = (\lambda, h(\lambda))$ is unstable. Hence, the existence of a periodic orbit follows from the Poincaré–Bendixson Theorem and the fact that the solutions of this system are positive and bounded (one can find a proof in Hsu [8]). Now if we can show that $\frac{d}{dx}\frac{\varphi(x)h'(x)}{\psi(x)h(x)} < 0$ for $x \in (0,1) - \{\lambda\}$, then Theorem 2.2 (choose $\alpha = 0$, $\beta = 1$) implies that the system has exactly one periodic orbit in R^2_+. Since

$$\frac{\varphi(x)h'(x)}{\psi(x)h(x)} = -\frac{2}{D(d-b)}\frac{x}{a+x}\frac{dx+e}{x+\Lambda}\frac{x-(1-a)/2}{(1-x)(x-\lambda)}$$

for $x \in (0,1) - \{\lambda\}$, $\frac{d}{dx}\frac{\varphi(x)h'(x)}{\psi(x)h(x)} < 0$ is equivalent to

$$\frac{x}{a+x}\frac{dx+e}{x+\Lambda}\frac{(x-(1-a)/2)^2 + ((1-a)/2-\lambda)(1+a)/2}{(x-\lambda)^2(1-x)^2}$$
$$> \left(\frac{a}{(x+a)^2}\frac{dx+e}{x+\Lambda} + \frac{x}{x+a}\frac{d\Lambda - e}{(x+\Lambda)^2}\right)\frac{(1-a)/2 - x}{(x-\lambda)(1-x)}$$

for $x \in (0,1) - \{\lambda\}$, or equivalent to

$$\left(x - \frac{1-a}{2}\right)^2 + \frac{1}{2}\left(\frac{1-a}{2} - \lambda\right)(1+a)$$
$$> \left(\frac{1-a}{2} - x\right)(1-x)\frac{x-\lambda}{x}\left(\frac{a}{x+a} + \frac{(d\Lambda - e)x}{(dx+e)(x+\Lambda)}\right). \tag{2.5}$$

for $x \in (0,1) - \{\lambda\}$. Clearly, the left-hand side of (2.5) is positive, and $((1-a)/2 - x)(1-x)(x-\lambda) < 0$ if $x \in (0, \lambda)$; hence (2.5) holds on $(0, \lambda)$.

For $x \in (\lambda, 1)$, since $0 < \frac{x-\lambda}{x} < 1$ and

$$\left(x - \frac{1-a}{2}\right)^2 + \left(\frac{1-a}{2} - \lambda\right)\frac{1+a}{2} - \left(\frac{1-a}{2} - x\right)(1-x)$$
$$= \frac{1+a}{2}(x-\lambda) > 0,$$

the inequality (2.5) holds, provided that $p(x) = \frac{a}{a+x} + \frac{(d\Lambda - e)x}{(x+\Lambda)(dx+e)} < 1$. Now since

$$p(x) = 1 - x\left[\frac{1}{x+a} + \frac{1}{x+\Lambda} - \frac{1}{x+e/d}\right]$$
$$= 1 - \frac{x\left[x^2 + 2(e/d)x + (\Lambda + a)(e/d) - a\Lambda\right]}{(x+a)(x+\Lambda)(x+e/d)}$$
$$< 1 - \frac{x\left[\lambda^2 + 2(e/d)\lambda + (\Lambda + a)(e/d) - a\Lambda\right]}{(x+a)(x+\Lambda)(x+(e/d))}$$

and

$$\lambda^2 + 2\frac{e}{d}\lambda + (\Lambda + a)\frac{e}{d} - a\Lambda$$
$$= \left(\lambda + \frac{e}{d}\right)^2 + \left(\Lambda - \frac{e}{d}\right)\left(\frac{e}{d} - a\right)$$
$$= (\lambda + \Lambda)\left(\frac{2e}{d} + \lambda - \Lambda\right) + \left(\Lambda - \frac{e}{d}\right)(\Lambda - a)$$
$$> (\lambda + \Lambda)\frac{(d-b)e + (dc - be) + abd}{d(d-b)} > 0,$$

$p(x) < 1$ for $x \in (\lambda, 1)$. Hence, $\frac{d}{dx}\left(\frac{\varphi(x)h'(x)}{\psi(x)h(x)}\right) < 0$ on $(0,1) - \{\lambda\}$.

Remark. If $\lambda \in [\frac{1-a}{2}, 1)$, then E^* is globally asymptotically stable in R_+^2 (see [1]).

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