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Zonal Measure Algebras on Isotropy Irreducible Homogeneous Spaces

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This paper analyzes the convolution algebra $M(K\backslash G/K)$ of zonal measures on a Lie group G, with compact subgroup K, primarily for the case when $M(K\backslash G/K)$ is commutative and G/K is isotropy irreducible. A basic result for such (G, K) is that the convolution of dim G/K continuous (on G/K) zonal measures is absolutely continuous. Using this, the spectrum (maximal ideal space) of $M(K\backslash G/K)$ is determined and shown to be in 1–1 correspondence with the bounded Borel spherical functions. Also, certain asymptotic results for the continuous spherical functions are derived. For the special case when G is compact, all the idempotents in $M(K\backslash G/K)$ are determined.

INTRODUCTION

In this paper we study the convolution and maximal ideal structure of the zonal measure algebra $M(K\backslash G/K)$ of K bi-invariant measures on a Lie group G with compact subgroup K. Most of our results relate to those pairs (G, K) for which $M(K\backslash G/K)$ is *commutative* and for which the linear isotropy representation of K_0 on $(G/K)_{eK}$ is irreducible. For such (G, K), we prove the basic result (2.5) which shows that the convolution of $n =$ dim G/K measures in $M(K\backslash G/K)$, which are completely nonatomic as measures on G/K, is absolutely continuous with respect to Haar measure on G. Using this, we are able, in Section 3, to show that the maximal ideal space of $M(K\backslash G/K)$ is just a finite-point compactification of the maximal ideal space of $L_1(K\backslash G/K)$ (see (3.5, 3.11)), at least when G is connected or compact. Section 3 also contains several other results relating to the harmonic analysis of zonal measures.

Examples of (G, K) to which our results apply are provided by the isotropy irreducible, Riemannian symmetric pairs since Gelfand [5]

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has shown $L_1(K\backslash G/K)$ is commutative for these pairs and we show in (1.2) that this implies $M(K\backslash G/K)$ is commutative. Thus this paper subsumes, and was inspired by, our earlier work on central measures on Lie groups [10] and rotation invariant measures on Euclidean space [11] as well as some of Dunkl's work [1] on zonal measures on spheres. The detailed relationship of the present paper to these examples is discussed in Section 5.

Section 4 contains some additional results for the case when G is compact. We compute all the idempotent measures in $M(K\backslash G/K)$, (4.4), which provides an analog of Cohen's theorem on idempotent measures. Our results are related to those of Rider [13, 14] in the compact group case.

1. Basic Definitions and Preliminary Results

Let G be any Lie group with $K \subseteq G$ a *compact* subgroup. Our basic object of study is the Banach Algebra $M(K\backslash G/K)$ of K bi-invariant finite Borel measures on G with convolution multiplication. If we denote the point mass at $g \in G$ by ϵ_g, then we have the equivalent definitions.

$$\begin{aligned} M(K\backslash G/K) &= \{\mu \in M(G) : \epsilon_k * \mu = \mu * \epsilon_k = \mu, \text{ all } k \in K\} \\ &= \{\mu \in M(G) : \mu(kSk') = \mu(S), \text{ all } k, k' \in K, \text{ all Borel } S \subseteq G\}. \end{aligned} \tag{1.1}$$

(We shall have occasion to use this same notation when G is only locally compact and, in fact, when G is only a semigroup.) It is often useful to identify $\mu \in M(K\backslash G/K)$ with the measure $\tilde{\mu}$ on G/K given on a Borel set $S \subseteq G/K$ by $\tilde{\mu}(S) = \mu(\pi^{-1}(S))$, where $\pi\colon G \to G/K$ is the natural projection. Henceforth we shall not distinguish these two measures, but we shall make it clear which identification we are using. Of course, the measure $\tilde{\mu}$ is invariant under the natural action of K on G/K on the left. Such K-invariant measures on G/K are called *zonal* measures on G/K and hence $M(K\backslash G/K)$ is called the *algebra of zonal measures* on G/K.

The algebra $M(K\backslash G/K)$ contains an identity — m_K, normalized Haar measure on K considered as a measure on G. In fact, we have $M(K\backslash G/K) = \{\mu \in M(G)\colon m_K * \mu * m_K = \mu\}$. It also contains the algebra $L_1(K\backslash G/K)$ of K bi-invariant L_1-functions when we identify f in $L_1(K\backslash G/K)$ with fm_G in $M(K\backslash G/K)$, where m_G is a fixed left Haar measure on G. Gelfand [5] (see [7, p. 408]) showed that when G/K

is a symmetric space, this L_1-algebra is *commutative*. For the most part we restrict our attention to the case when $M(K\backslash G/K)$ is commutative. This includes those pairs (G, K) which occur in Gelfand's paper in light of

PROPOSITION 1.2. *If G is locally compact, K compact, then $M(K\backslash G/K)$ is commutative if and only if $L_1(K\backslash G/K)$ is commutative.*

Proof. The only if direction is clear. So suppose $L_1(K\backslash G/K)$ is commutative. Since a measure μ in $M(K\backslash G/K)$ is determined by $\mu * f$ as f ranges over $L_1(K\backslash G/K)$ (or $C_c(K\backslash G/K)$), it suffices to show $\mu * \nu * f = \nu * \mu * f$ for μ, $\nu \in M(K\backslash G/K)$, $f \in L_1(K\backslash G/K)$. But $L_1(K\backslash G/K)$ has a two-sided approximate identity $\{\phi_\alpha\}$ (let $\{U_\alpha\}$ be a basis of compact neighborhoods at e and set $\phi_\alpha = \xi_{KU_\alpha K}/\| \xi_{KU_\alpha K} \|_1$, where ξ_S is the characteristic function of S) and also $L_1(K\backslash G/K)$ is an ideal in $M(K\backslash G/K)$. Thus the commutativity of $L_1(K\backslash G/K)$ shows that

$$\begin{aligned}\mu * \nu * f &= \lim_\alpha \phi_\alpha * \mu * \nu * f = \lim_\alpha (\nu * f) * (\phi_\alpha * \mu) \\ &= \lim_\alpha \nu * (\phi_\alpha * \mu * f) = \nu * \mu * f. \quad \blacksquare\end{aligned}$$

Several specific examples derived from Gelfand's result are listed in Section 5.

By analogy with the full measure algebra, we introduce a decomposition of zonal measures into continuous and discrete parts. Since a zonal measure naturally lives on G/K, the notions of continuous and discrete we adopt are relative to G/K. Thus we define the *continuous zonal measures*, $M_c(K\backslash G/K)$, by

$$M_c(K\backslash G/K) = \{\mu \in M(K\backslash G/K) : \mu(gK) = 0, \text{ all } g \in G\}. \tag{1.3}$$

Since $\{gK\}$ are the points of G/K, and μ can be considered as a measure on G/K, this is equivalent to

$$M_c(K\backslash G/K) = \{\mu \in M(K\backslash G/K) : |\mu|(gK) = 0, g \in G\}.$$

Also, since the point mass at gK on G/K corresponds to the measure on G given by

$$\mu_{gK} = \epsilon_g * m_K = \epsilon_{gk} * m_K, \tag{1.4}$$

we define the *discrete zonal measures*, $M_d(K\backslash G/K)$, by

$$M_d(K\backslash G/K) = \Big\{\mu \in M(K\backslash G/K) : \mu = \sum_{G/K} c_{gK}\mu_{gK}, c_{gK} \in \mathbb{C}\Big\}. \tag{1.5}$$

Warning. Not every measure μ on G, $\mu = \sum_{G/K} c_{gK}\mu_{gK}$, is in $M_d(K\backslash G/K)$ since such μ may not be *left* K-invariant (see (1.10) below).

Now for any measure $\mu \in M(K\backslash G/K)$, we set

$$\mu_d = \sum_{G/K} \mu(gK)\, \mu_{gK} \in M_d(K\backslash G/K), \tag{1.6}$$

(μ_d is K bi-invariant since $\mu(kgK) = \mu(gK)$ and $\epsilon_k * u_{gK} * \epsilon_{k'} = \mu_{kgK}$), and we set

$$\mu_c = \mu - \mu_d \in M_c(K\backslash G/K). \tag{1.7}$$

Thus we have a direct sum decomposition

$$M(K\backslash G/K) = M_d(K\backslash G/K) + M_c(K\backslash G/K).$$

Just as in the case of the full measure algebra, we have

PROPOSITION 1.8. *If G is locally compact and $M(K\backslash G/K)$ is commutative, then $M_c(K\backslash G/K)$ is a (two-sided) ideal.*

Proof. Let $\mu \in M(K\backslash G/K)$, $\nu \in M_c(K\backslash G/K)$. Then

$$\mu * \nu(gK) = \int \nu(h^{-1}gK)\, d\mu(h) = 0$$

as $\nu \in M_c(K\backslash G/K)$. Thus $\mu * \nu \in M_c(K\backslash G/K)$. Since $M(K\backslash G/K)$ is commutative, this shows $M_c(K\backslash G/K)$ is an ideal. ∎

Remark. When G is a Lie group, it can be shown that $M_c(K\backslash G/K)$ is a two-sided ideal without assuming commutativity. Whether this can be done in the general locally compact case is not known to us.

Now we develop a more concrete picture of the discrete zonal measures. First, observe that for any $\mu \in M(K\backslash G/K)$, we must have $\mu(gK) = 0$ unless gK has a finite K-orbit in G/K, since $\mu(kgK) = \mu(gK)$ by left K-invariance. Thus it is natural to try to describe the discrete zonal measures in terms of the set

$$N = \{g \in G : gK \text{ has a finite } K\text{-orbit in } G/K\}. \tag{1.9}$$

In fact we have

PROPOSITION 1.10. *Let G be locally compact and N be as in* (1.9). *Then*:

(i) *N is a subsemigroup of G with $K \subseteq N$.*

(ii) *$\mu \in M_d(K\backslash G/K)$ if and only if $\mu = \sum_{N/K} c_{nK}\mu_{nK}$ with $\sum_{N/K} | c_{nK} | < \infty$ and $c_{nK} = c_{knK}$ for all $k \in K$ and $n \in N$.*

(iii) *If $KnK = \bigcup_{i=1}^{p} n_iK$, then $\mu_{mK} * \mu_{nK} = 1/p(\sum_{i=1}^{p} \mu_{mn_iK})$. Hence $M_d(K\backslash G/K)$ is a subalgebra.*

Moreover, if N is retopologized so that nK has the relative topology from G, while each nK is open in N, then $M_d(K\backslash G/K) = M(K\backslash N/K)$.

Proof. (i) That $K \subseteq N$ is obvious. Now if $g, h \in N$ and $KhK = \bigcup_j h_jK$, $KgK = \bigcup_i g_iK$, then $KghK \subseteq \bigcup_i g_iKhK = \bigcup_i \bigcup_j g_ih_jK$. So $gh \in N$.

(ii) That $\mu \in M_d(K\backslash G/K)$ has the desired form follows from the observation preceding this proposition that $\mu(gK) = 0$ if $g \notin N$, together with the definition, (1.5), of $M_d(K\backslash G/K)$. Conversely, for the "if" part, suppose μ has the desired form. Then clearly μ is right K-invariant as $u_{gK} * \epsilon_k = \epsilon_g * m_K * \epsilon_k = \mu_{gK}$ by the right K-invariance of m_K. Also, μ is left K-invariant since $\epsilon_k * \mu = \sum_{N/K} c_{nK}\mu_{knK} = \sum_{N/K} c_{knK}\mu_{knK} = \mu$ because $c_{nK} = c_{knK}$.

(iii) If the first assertion is true, then clearly $M_d(K\backslash G/K)$ is a subalgebra in light of (ii). To prove the first assertion, it suffices to show that $m_K * \epsilon_n * m_K = (1/p) \sum_{i=1}^{p} \epsilon_{n_i} * m_K$. To do this, set $K_i = \{k \in K; knK = n_iK\}$, where $KnK = \bigcup n_iK$ and $n_1 = n$. Then K_1 is a subgroup of K, the K_i are the cosets of K_1 and, thus, $m_K(K_i) = 1/p$. Now for $f \in C_c(G)$ we get, writing dk for dm_K,

$$\begin{aligned} m_K * \epsilon_n * m_K(f) &= \iint_{K\times K} f(k'nk)\,dk\,dk' \\ &= \sum_{i=1}^{p} \int_{K_i} \left(\int_K f(k'nk)\,dk\right) dk' \\ &= \sum_{i=1}^{p} \int_{K_i} \left(\int_K f(n_ik)\,dk\right) dk', \qquad \text{as} \quad k' \in K_i \\ &= (1/p) \sum_{i=1}^{p} \int f(n_ik)\,dk = (1/p) \left(\sum_{i=1}^{p} \epsilon_{n_i} * m_K\right)(f). \end{aligned}$$

So we are done. ∎

In case G is a Lie group, we can further identify the semigroup N and in fact show that it is a closed subgroup. Specifically we have

PROPOSITION 1.11. *Let G be a Lie group, K a compact subgroup with identity component K_0, and let N be as in* (1.9). *Then*

(i) *N = normalizer of K_0 in $G = N(K_0)$.*

(ii) *If K is connected and $M(K\backslash G/K)$ is commutative, then $M_d(K\backslash G/K)$ is isomorphic to the group measure algebra $M((N(K)/K)_d)$, where $(N(K)/K)_d$ is the abelian group $N(K)/K$ with the discrete topology.*

Proof. (i) If $n \in N(K_0)$, then $k_0 nK = nK$ for $k_0 \in K_0$. So the K orbit of nK in G/K is actually the K/K_0 orbit which is finite as K_0 is open in K. Thus $n \in N$. Conversely, if $n \in N$, then the K_0 orbit of nK is connected and finite. Hence it is a single point. Thus $K_0 n \subseteq nK$. This implies $n^{-1}K_0 n \subseteq K$. So $n^{-1}K_0 n = K_0$ as $n^{-1}K_0 n$ is a connected subgroup of K of the same dimension as K_0. Hence $n \in N(K_0)$.

(ii) If K is connected, then $N(K_0) = N(K)$, so K is normal in N by (i). Thus $nK = knK$ for all $k \in K$ and $n \in N$, so (1.10(ii)) implies $M_d(K\backslash G/K) = l_1(N(K)/K)$. Moreover, since $KnK = nK$, (1.10(iii)) shows that $\mu_{mK} * \mu_{nK} = \mu_{mnK}$. Thus the convolution structure of $M_d(K\backslash G/K)$ is isomorphic to $M((N(K)/K)_d)$. Since $M_d(K\backslash G/K)$ is commutative, it follows that $N(K)/K$ is an abelian group. ∎

In the next section we examine the convolution structure of $M_c(K\backslash G/K)$.

2. Geometry and Convolutions in the Isotropy Irreducible Case

We now wish to study the convolution structure in $M_c(K\backslash G/K)$. Since the extreme points of the unit ball of this space are given by the measures $m_K * \epsilon_g * m_K$, $g \notin N$, we begin by examining some geometric properties of the double cosets KgK, $g \notin N$, which are the supports of these measures. For this investigation we restrict attention to Lie group pairs (G, K) satisfying

(2.1) *The linear isotropy representation of the identity component K_0 on the tangent space $(G/K)_{eK}$ is* (a) *irreducible and* (b) *nontrivial.*

Condition (a) can be rephrased to say that if $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively, then the representation of K_0 on $\mathfrak{g}/\mathfrak{k}$, induced by restricting the adjoint representation of G to K_0, is irreducible. Condition (b) guarantees that $N(K_0) \neq G_0$ and also rules out the case when G is a one-dimensional extension of K. In the setting of condition (2.1), we have the following geometric result.

Proposition 2.2. *Let (G, K) be a pair of connected Lie groups satisfying* (2.1). *For any $a_i \in G$, $i = 1,\ldots, n$ with $a_i \notin N(K)$ (equivalently, $\mathrm{Ad}(a_i)\mathfrak{k} \not\subseteq \mathfrak{k}$), define a map $f^n\colon K^{n+1} \to G$ by $f^n(k_1,\ldots, k_{n+1}) = k_1 a_1 k_2 \cdots a_n k_{n+1}$. Then rank $f^n \geqslant \min(\dim G,\ n + \dim K)$ except on a proper analytic subvariety of K^{n+1}.*

Proof. We identify the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ with the corresponding right-invariant vector fields, and we also identify the tangent space at any point of G or K with $\mathfrak{g}$ or $\mathfrak{k}$ via right translation to e. Then the differential $df_{\mathbf{k}}^n \colon \mathfrak{k}^{n+1} \to g$, where $\mathbf{k} = (k_1, \ldots, k_{n+1})$, is given by

$$\begin{aligned} df_{\mathbf{k}}^n(X_1, \ldots, X_{n+1}) &= (d/dt)(\exp - tX_1 \cdot k_1 a_1 \cdots a_n \exp - tX_{n+1} \cdot k_{n+1}) \\ &= X_1 + Ad(k_1 a_1)\, X_2 + \cdots + Ad(k_1 a_1 \cdots k_n a_n)\, X_{n+1} \,. \end{aligned} \tag{2.3}$$

This shows that the range, hence the rank, of $df_{\mathbf{k}}^n$ is independent of the choise of k_{n+1}. It also shows that our result is true when $n = 0$, since then the range of df^0 is $\mathfrak{k}$.

Now as f^n is an analytic map and the condition "rank of f^n at $\mathbf{k} \leqslant j$" is determined by the vanishing of analytic functions, it suffices to show that the "rank of f^n at $\mathbf{k} \geqslant \min(\dim G, n + \dim K)$" is true for just one $\mathbf{k} = (k_1, \ldots, k_{n+1})$ (since K is connected). Suppose, by induction, that this result is true for f^{n-1}. Then we can find $k_1, \ldots, k_{n-1}$ such that

$$\mathfrak{a} = \text{range } df^{n-1}_{(k_1, \ldots, k_{n-1}, *)}$$

has $\dim \mathfrak{a} \geqslant \min(\dim G, \dim K + n - 1)$. ($\mathfrak{a}$ is independent of the choice of $*$ in K by the remark following (2.3).) If $\dim \mathfrak{a} = \dim G$, then we are done. If not, then since (2.3) shows

$$\mathfrak{a} + Ad(k_1 a_1 \cdots k_n a_n)(\mathfrak{k}) = \text{range } df_{\mathbf{k}}^n,$$

it suffices to show that we can find $k_n \in K$ such that

$$Ad(k_n a_n)(\mathfrak{k}) \not\subseteq Ad((k_1 a_1 \cdots k_{n-1} a_{n-1})^{-1})(\mathfrak{a}) = \mathfrak{b}.$$

But since $Ad(a_n)(\mathfrak{k}) \not\subseteq \mathfrak{k}$, there exists $X \in \mathfrak{k}$ with $Ad(a_n)(X) \notin \mathfrak{k}$. Also, as $Ad \mid K$ induces an irreducible representation on $\mathfrak{g}/\mathfrak{k}$ and $\mathfrak{b} + \mathfrak{k}$ is *proper* in $\mathfrak{g}$, there is a $k_n \in K$ with $Ad(k_n)(Ad(a_n)(X)) \notin \mathfrak{b} + \mathfrak{k}$. So the proof is done. ▮

By using this geometric result, we can derive a very useful theorem illustrating the smoothing effects of convolution. But first we need the following

LEMMA 2.4. *Let the Lie group pair (G, K) satisfy (2.1). Then $N(K_0)/K$ is a discrete subset of G/K.*

Proof. We have already observed that the conditions of (2.1) imply $N(K_0) \neq G$. Moreover, since the representation of K_0 on $\mathfrak{g}/\mathfrak{k}$

is irreducible, $K_0 = N(K_0)_0$ as the Lie algebra of $N(K_0)$ contains $\mathfrak{k}$ and is K_0 invariant. Thus K_0, and hence K, is open in $N(K_0)$. So $N(K_0)/K$ is discrete in G/K as $N(K_0)$ is closed in G. ∎

Now we have the major result on convolutions in $M_c(K\backslash G/K)$.

THEOREM 2.5. *Let (G, K) be a Lie group pair satisfying* (2.1). *If $\mu_i \in M_c(K\backslash G/K)$, $i = 1,\ldots, n = \dim G/K$, then $\mu = \mu_1 * \mu_2 * \cdots * \mu_n$ is absolutely continuous*, i.e., $\mu \in L_1(K\backslash G/K)$.

Proof. It suffices to consider the case when K is connected since (2.1) depends only on K_0 and K bi-invariant measures are surely K_0 bi-invariant. Thus we assume K connected.

First we observe that since each μ_i is continuous as a measure on G/K and since $N(K)/K$ is discrete by (2.4), μ_i vanishes on all finite, hence all compact, subsets of $N(K)/K$. Thus, by regularity, μ_i vanishes on $N(K)/K$. Thus, back on G, we have $|\mu_i|(N(K)) = 0$. So μ_i is supported on the set $G^* = G - N(K)$.

Now let $S \subseteq G$ be a Borel set with ξ_S its characteristic function. (Later we shall assume $m_G(S) = 0$.) Then, using the K bi-invariance of each μ_i, the previous paragraph, and Fubini's theorem, we get

$$\mu(S) = \mu_1 * \cdots * \mu_n(S) = m_K * \mu_1 * m_K * \mu_2 * \cdots * m_K * \mu_n * m_K(S)$$

$$= \int_{G^*} \cdots \int_{G^*} \left(\int_K \cdots \int_K \xi_S(k_1 a_1 \cdots a_n k_{n+1})\, dk_1 \cdots dk_{n+1}\, d\mu_1(a_1) \cdots d\mu_n(a_n) \right)$$

Fixing $a_i \in G^*$, we let $f(k_1,\ldots, k_{n+1}) = k_1 a_1 \cdots a_n k_{n+1}$, as in (2.2). Then the inner integral is just

$$\int_{K^{n+1}} \xi_S(k_1 a_1 \cdots a_n k_{n+1})\, dk_1 \cdots dk_{n+1} = m_{K^{n+1}}(f^{-1}(S)).$$

We shall show that if $m_G(S) = 0$, then $m_{K^{n+1}}(f^{-1}(S)) = 0$ and hence that $\mu(S) = 0$. This will complete the proof.

Haar measure on any Lie group is mutually absolutely continuous with respect to Lebesgue measure in any coordinate patch. Thus, so far as questions of measure zero are concerned, we may restrict attention to a coordinate neighborhood and replace $m_{K^{n+1}}$ and m_G by Lebesgue measures. Now Proposition 2.2 implies that f has rank equal to $\dim G = \dim K + \dim G/K$, except on a proper subvariety of K^{n+1}. This subvariety must be of smaller dimension than K^{n+1}, hence it has Lebesgue measure zero (as a subset of K^{n+1}). Moreover, on the open complement of this subvariety, f has maximal rank, so the implicit function theorem shows that after proper choice of coordinates, near each point f is just an orthogonal projection between two Euclidean spaces. Then Fubini's theorem shows $f^{-1}(S)$ has Lebesgue measure zero as S has Lebesgue measure zero. ▌

In the next section we shall use (2.5) to investigate the maximal ideal structure of $M(K\backslash G/K)$.

3. The Spectrum of $M(K\backslash G/K)$

From here on we shall restrict attention to pairs (G, K) which, in addition to satisfying (2.1), are such that the algebra $M(K\backslash G/K)$ is commutative. For such pairs we use our previous results to determine the spectrum (or maximal ideal space) of $M(K\backslash G/K)$, which we denote by $\mathscr{S}(M(K\backslash G/K))$. (We shall use the notation $\mathscr{S}(A)$ to denote the spectrum of any commutative Banach algebra A.) In addition, we shall prove several results relating to the (zonal) Fourier–Stieltjes transform. We begin with a discussion of two easily defined subsets of $\mathscr{S}(M(K\backslash G/K))$.

First, since $L_1(K\backslash G/K)$ is an ideal in $M(K\backslash G/K)$, each complex homomorphism $\lambda \in \mathscr{S}(L_1(K\backslash G/K))$ can be uniquely extended to a homomorphism, λ', of $M(K\backslash G/K)$ by choosing $f \in L_1(K\backslash G/K)$ with $\lambda(f) \neq 0$ and defining

$$\lambda'(\mu) = \lambda(\mu * f)/\lambda(f). \tag{3.1}$$

From (3.1) and the definition of the topology on the spectrum of a Banach algebra, it is clear that the extension map from $\mathscr{S}(L_1(K\backslash G/K))$ to $\mathscr{S}(M(K\backslash G/K))$ is injective and is a homeomorphism onto its image. Thus we can identify $\mathscr{S}(L_1(K\backslash G/K))$ as a subspace of $\mathscr{S}(M(K\backslash G/K))$. In all the important examples of pairs (G, K) satisfying our assumptions, the spaces $\mathscr{S}(L_1(K\backslash G/K))$ has been quite explicitly determined. This is discussed in Section 5 in connection with these examples. A slightly different picture of this extension can be obtained from the work of Gelfand [5] (see [7, p. 410]) which shows that the spectrum $\mathscr{S}(L_1(K\backslash G/K))$ is in 1–1 correspondence with the *(elementary)* ***bounded continuous spherical functions***, that is, the bounded continuous functions ϕ: $G \to C$ which solve the functional equation

$$\phi(x)\,\phi(y) = \int_K \phi(xky)\,dk, \qquad x, y \in G. \tag{3.2}$$

In particular, if ϕ_λ denotes the solution to (3.2) corresponding to $\lambda \in \mathscr{S}(L_1(K\backslash G/K))$, then for $f \in L_1(K\backslash G/K)$,

$$\lambda(f) = \int_G \phi_\lambda(g^{-1}) f(x)\, dg.$$

By use of the Eq. (3.2) and the K bi-invariance of $\mu \in M(K\backslash G/K)$, it is easy to see that the extension λ' is given by the integral formula

$$\lambda'(\mu) = \int \phi(g^{-1})\, d\mu(g). \tag{3.3}$$

The second subset of $\mathscr{S}(M(K\backslash G/K))$ arises by duality from the surjective algebra homomorphism from $M(K\backslash G/K)$ to $M_d(K\backslash G/K)$ which maps $\mu = \mu_d + \mu_c$ to μ_d. The corresponding map of the spectra takes $\gamma \in \mathscr{S}(M_d(K\backslash G/K))$ and lifts it to $\gamma * \in \mathscr{S}(M(K\backslash G/K))$ given by

$$\gamma*(\mu) = \gamma(\mu_d), \qquad \mu = \mu_d + \mu_c . \tag{3.4}$$

This, too, gives a homeomorphic map from the compact space $\mathscr{S}(M_d(K\backslash G/K))$ onto its image.

In most of the examples satisfying (2.1), this second part of $\mathscr{S}(M(K\backslash G/K))$ is actually finite, as a result of

PROPOSITION 3.5. *Let* (G, K) *be a Lie group pair satisfying* (2.1).

(i) *If G is compact, then $N(K_0)/K$ is finite.*

(ii) *If G is connected but not compact, then $N(K_0) = K$.*

Thus under either of the conditions (i) *or* (ii), $M_d(K\backslash G/K))$ *is finite.*

Proof. (i) follows immediately from (2.4) as G/K is compact. Now for (ii) we claim that under its hypotheses, G/K has a G-invariant Riemannian metric with respect to which it is either a Euclidean space or an irreducible Riemannian symmetric space of noncompact type. To verify this, we may assume G acts effectively on G/K, for if K' is the compact kernel of the G-action, then $(G/K', K/K')$ still satisfies the hypotheses and $G/K \cong G/K'/K/K'$. Now (1.1) and (1.2) in Wolf's work [15] apply to give our claim. Thus G/K has the structure of a simply connected Riemannian manifold of nonpositive sectional curvature with K acting as isometries (see [7, p. 205]). In this setting, geodesics joining points in G/K are unique, so there can be no fixed points for the isometries of K_0 except for eK as otherwise K_0 would have a 1-dimensional invariant subspace in $\mathfrak{g}/\mathfrak{k}$. Thus $N(K_0) = K$ as $N(K_0)/K$ is the set of fixed points of K_0 in G/K.

The last part of the proposition follows from the identification of (1.10). ▮

We can get a description of this second part of $\mathscr{S}(M(K\backslash G/K))$ similar to (3.2) and (3.3) if we recall that in (1.10) we identified $M_d(K\backslash G/K)$ with $M(K\backslash N(K_0)/K)$, where $N(K_0)$ is given the topology which makes each K-coset open. Since we are assuming that (2.1) holds for the pair (G, K), Lemma 2.4 shows that this is the topology on $N(K_0)$ induced from G. Thus $M(K\backslash N(K_0)/K) = L_1(K\backslash N(K_0)/K)$. So the theory of spherical functions, mentioned in connection with (3.2) and (3.3), applies to the pair $(N(K_0), K)$ and shows that each $\gamma \in \mathscr{S}(M(K\backslash N(K_0)/K))$ corresponds to a bounded continuous ϕ_γ: $N(K_0) \to \mathbb{C}$ which satisfies (3.2) on $N(K_0)$ and is such that

$$\gamma^*(\mu) = \gamma(\mu_d) = \int_{N(K_0)} \phi_\gamma(y^{-1})\, d\mu_d(y), \tag{3.6}$$

where we have identified the spectra of $M_d(K\backslash G/K)$ and $M(K\backslash N(K_0)/K)$. Now if we extend ϕ_γ to a function on all of G by setting

$$\phi_\gamma(g) = \begin{cases} \phi_\gamma(g), & g \in N(K_0), \\ 0, & g \notin N(K_0), \end{cases} \tag{3.7}$$

then ϕ_γ is a bounded Borel function on G. Moreover, if $\mu = \mu_d + \mu_c$, then, since (2.1) holds, $N(K_0)$ is a set of μ_c-measure zero (see the second paragraph in the proof of (2.5)) and hence

$$\gamma^*(\mu) = \int_G \phi_\gamma(g^{-1})\, d\mu(g), \qquad \mu \in M(K\backslash G/K). \tag{3.8}$$

Since integration against the K bi-invariant function ϕ_γ gives a homomorphism of $M(K\backslash G/K)$, ϕ_γ must satisfy the functional equation (3.2) in light of

Lemma 3.9. *Let ϕ be a K bi-invariant (bounded) Borel function on G and suppose $\mu \to \int_G \phi(g^{-1})\, d\mu(g)$ gives a continuous homomorphism of $M(K\backslash G/K)$. Then ϕ satisfies* (3.2) *and* $|\phi| \leqslant 1$.

Proof. This follows by applying the homomorphism to the measures $m_K * \epsilon_x * m_K$ and $m_K * \epsilon_y * m_K$. Specifically, since ϕ is K bi-invariant, we have

$$\phi(x^{-1}) = \int_K \int_K \phi((kxk')^{-1})\, dk\, dk' = \int \phi(g^{-1})\, d(m_K * \epsilon_x * m_K)(g). \tag{3.10}$$

Thus, the homomorphic property and the K bi-invariance of ϕ show

$$\phi(x^{-1})\,\phi(y^{-1}) = \int \phi(g^{-1})\, d(m_K * \epsilon_x * m_K * m_K * \epsilon_y * m_K)(g)$$
$$= \int_K \phi(y^{-1}kx^{-1})\, dk,$$

as m_K is inversion invariant. Since any continuous homomorphism has norm 1, (3.10) shows $|\phi| \leqslant 1$ as $\| m_K * \epsilon_x * m_K \| = 1$. ▮

As a consequence of this, we see that under our assumptions on (G, K), all the homomorphisms of $M(K\backslash G/K)$ which arise from elements in $\mathcal{S}(L_1(K\backslash G/K))$ and $\mathcal{S}(M_d(K\backslash G/K))$ are given by integration against bounded Borel solutions to the functional equation for spherical functions, (3.3). In fact, these exhaust the spectrum of $M(K\backslash G/K)$ as we show in

THEOREM 3.11. *Let* (G, K) *be a Lie group pair satisfying* (2.1) *and assume* $M(K\backslash G/K)$ *is commutative. Then*:

(i) $\mathcal{S}(M(K\backslash G/K)) = \mathcal{S}(L_1(K\backslash G/K)) \cup \mathcal{S}(M(K\backslash N(K_0)/K))$. *If* K *is connected,* $\mathcal{S}(M(K\backslash N(K)/K))$ *is just* $(N(K)/K)\hat{}$, *the dual of the discrete abelian group* $N(K)/K$.

(ii) *Every homomorphism in* $\mathcal{S}(M(K\backslash G/K))$ *is given by integration against a unique nonzero bounded Borel solution to the functional equation* (3.2).

(iii) *The only nonzero bounded Borel solutions to* (3.2) *on* G *are the bounded continuous spherical functions on* G *and the bounded continuous spherical functions on* $N(K_0)$ *extended to be zero off* $N(K_0)$.

Proof. For (i), just as in (3.2) of [10], it is easy to show, using (2.5), that a homomorphism of $M(K\backslash G/K)$ is either determined by its restriction to $L_1(K\backslash G/K)$ and so is in $\mathcal{S}(L_1(K\backslash G/K))$ or else is zero on $M_c(K\backslash G/K)$ and is determined by its restriction to $M_d(K\backslash G/K)$ and so is in $\mathcal{S}(M_d(K\backslash G/K)) = \mathcal{S}(M(K\backslash N(K_0)/K))$. This proves the first sentence. The second follows from (1.11(ii)).

For (ii) only the uniqueness needs to be proved. So let ϕ, ψ be nonzero bounded Borel solutions to (3.2) with $\int \phi(g^{-1})\, d\mu(g) = \int \psi(g^{-1})\, d\mu(g)$ for all $\mu \in M(K\backslash G/K)$. Then from (3.2), it follows that ϕ, ψ are K bi-invariant since m_K is K bi-invariant. So (3.10) holds for both ϕ and ψ and shows they are equal, as $m_K * \epsilon_x * m_K \in M(K\backslash G/K)$ for all $x \in G$.

For (iii) we note that it is easy to show that any bounded Borel solution to (3.2) defines, via the appropriate integral, a homomorphism of $M(K\backslash G/K)$ to $\mathbb{C}$. So (iii) follows from (i), (ii), and the discussion preceding the theorem. ∎

These results give a complete description of the *points* in $\mathscr{S}(M(K\backslash G/K))$. In order to describe the *topology* of this spectrum, that is, how the pieces fit together, we introduce the *fiber mapping* Fb: $\mathscr{S}(M(L_1(K\backslash G/K)) \to \mathscr{S}(M_d(K\backslash G/K))$ given by

$$Fb(\lambda) = \lambda' \mid M_d(K\backslash G/K), \qquad \lambda \in \mathscr{S}(L_1(K\backslash G/K)) \tag{3.12}$$

(recall λ' is the extension of λ to $M(K\backslash G/K)$ given by (3.1)).

In terms of the mapping Fb and the topology of the two pieces of $\mathscr{S}(M(K\backslash G/K))$, we have the following description of the topology of the full spectrum.

THEOREM 3.13. *Let (G, K) be a Lie group pair satisfying* (2.1) *and assume $M(K\backslash G/K)$ is commutative. Then $\mathscr{S}(M(K\backslash G/K))$ is the compactification of $\mathscr{S}(L_1(K\backslash G/K))$ obtained by adjoining $\mathscr{S}(M(K\backslash N(K_0)/K)$ in such a way that a net $\{\lambda_\alpha\}$ in $\mathscr{S}(L_1(K\backslash G/K)$ converges to γ in $\mathscr{S}(M(K\backslash N(K_0)/K))$ if and only if $\lambda_\alpha \to \infty$* in *$\mathscr{S}(L_1(K\backslash G/K))$ and $Fb(\lambda_\alpha) \to \gamma$ in $\mathscr{S}(M(K\backslash N(K_0)/K)$. When $\mathscr{S}(M(K\backslash G/K))$ is realized as the space of bounded Borel solutions to* (3.2), *then its topology is just the topology of (bounded) pointwise convergence.*

Proof. For the "only if" direction, note that if $\lambda_\alpha' \to \gamma^*$ in $\mathscr{S}(M(K\backslash G/K))$ then (a) λ_α must eventually be outside of any compact set in $\mathscr{S}(L_1(K\backslash G/K))$ as otherwise a subnet would converge inside $\mathscr{S}(L_1(K\backslash G/K))$, and (b) the definition of the weak$*$-topology on $\mathscr{S}(M(K\backslash G/K))$ implies that for $\mu_d \in M_d(K\backslash G/K)$,

$$Fb(\lambda_\alpha)(\mu_d) = \lambda_\alpha'(\mu_d) \to \gamma^*(\mu_d) = \gamma(\mu_d),$$

i.e., $Fb(\lambda_\alpha) \to \gamma$ in $\mathscr{S}(M(K\backslash N(K_0)/K))$. For the other direction, suppose $\lambda_\alpha \to \infty$ and $Fb(\lambda_\alpha) \to \gamma$, then

$$\lambda_\alpha'(\mu) = \lambda_\alpha'(\mu_d) + \lambda_\alpha'(\mu_c) = Fb(\lambda_\alpha)(\mu_d) + \lambda_\alpha'(\mu_c).$$

But $\lambda_\alpha'(\mu_c) \to 0$ as $\lambda_\alpha(\mu_c{}^m) \to 0$, $m = \dim G/K$, by the standard Gelfand theory since $\mu_c{}^m \in L_1(K\backslash G/K)$ by (2.5). So $\lambda_\alpha'(\mu) \to \gamma(\mu_d) = \gamma^*(\mu)$. Hence $\lambda_\alpha' \to \gamma^*$ in $\mathscr{S}(M(K\backslash G/K))$.

To describe the topology in terms of the set of bounded Borel solutions to (3.2), let $\{\tau_\alpha\}$ be a net in $\mathscr{S}(M(K\backslash G/K))$ converging to τ with $\{\phi_\alpha\}$ and ϕ the uniquely associated Borel solutions to (3.2). Then from (3.10) it follows that for each $g \in G$,

$$\phi_\alpha(g^{-1}) = \tau_\alpha(m_K * \epsilon_g * m_K) \to \tau(m_K * \epsilon_g * m_K) = \phi(g^{-1}).$$

Since $|\phi_\alpha(g)| \leqslant 1$, this is actually *bounded* pointwise convergence. Conversely, let $\{\phi_\alpha\}$ be a net of bounded Borel solutions to (3.2) which converges pointwise to a bounded Borel solution ϕ of this equation, with $\{\tau_\alpha\}$ and τ the associated homomorphisms. Then any convergent subnet of $\{\tau_\alpha\}$ must converge to τ, since the associated solutions to (3.2) will converge pointwise to ϕ, by the first part of this paragraph. Since $\mathscr{S}(M(K\backslash G/K))$ is compact this implies $\tau_\alpha \to \tau$, and we are done. ∎

As a corollary we have a simple asymptotic result about the bounded continuous spherical functions on G.

COROLLARY 3.14. *Suppose (G, K) is as in (3.13). Let ϕ_λ be the spherical function corresponding to $\lambda \in \mathscr{S}(L_1(K\backslash G/K))$. Then $\phi_\lambda(g) \to 0$ for $g \notin N(K_0)$ as $\lambda \to \infty$ in $\mathscr{S}(L_1(K\backslash G/K))$.*

Proof. Suppose that for some $g \in G$, $\phi_\lambda(g) \not\to 0$ as $\lambda \to \infty$. Then we can pick a net $\lambda_\alpha \to \infty$ with $\lambda_\alpha' \to \gamma^*$ in $\mathscr{S}(M(K\backslash G/K))$, some $\gamma \in \mathscr{S}(M_d(K\backslash G/K))$ and such that $|\phi_{\lambda_\alpha}(g)| \geqslant \epsilon > 0$. But then (3.13) implies $|\phi_{\lambda_\alpha}(g)| \to |\phi_\gamma(g)| \geqslant \epsilon > 0$. Hence $g \in N(K_0)$ by the definition ϕ_γ, (3.7). ∎

In some of the examples of pairs (G, K) satisfying the assumptions of this corollary, the spherical functions are well-known special functions. Thus we have a rather interesting way of obtaining some (weak) asymptotic information for these functions. Of course, our soft techniques do not yield the detailed asymptotic behaviors of these functions which have been classically derived. Several examples of this situation are discussed in Section 5.

One natural question with regard to the structure of $\mathscr{S}(M(K\backslash G/K))$ is whether $\mathscr{S}(L_1(K\backslash G/K))$ is *dense*. The answer is trivially yes when G is connected, but not compact, since then (3.5(ii)) and (3.11) imply $\mathscr{S}(M(K\backslash G/K))$ is just the one-point compactification of $\mathscr{S}(L_1(K\backslash G/K))$. The answer is also yes when G is compact. To prove this we need some facts about the (*zonal*) *Fourier–Stieltjes transform* of a measure $\mu \in M(K\backslash G/K)$. This is the function $\hat{\mu}$ on $\mathscr{S}(L_1(K\backslash G/K))$ given by

$$\hat{\mu}(\lambda) = \int \phi_\lambda(g^{-1})\, d\mu(g) = \lambda'(\mu), \qquad \lambda \in \mathscr{S}(L_1(K\backslash G/K)). \qquad (3.15)$$

Since $L_1(K\backslash G/K)$ is semisimple (see [7, p. 453]), it follows that $\mu \to \hat{\mu}$ is an injective map, as $\mu = 0$ when $\mu * f = 0$ for all $f \in L_1(K\backslash G/K)$. This fact will help us complete the proof of

THEOREM 3.16. *Let (G, K) satisfy the hypotheses of* (3.11). *Moreover, suppose G is either connected or compact. Then $\mathscr{S}(L_1(K\backslash G/K))$ is dense in $\mathscr{S}(M(K\backslash G/K))$.*

Proof. The remarks at the beginning of the previous paragraph show that we need only consider the compact case. But then $\mathscr{S}(M_d(K\backslash G/K))$ is finite by (3.5(i)). So if $\mathscr{S}(L_1(K\backslash G/K))$ was not dense, there would exist a point $\gamma \in \mathscr{S}(M_d(K\backslash G/K))$ which is isolated in $\mathscr{S}(M(K\backslash G/K))$, by (3.13). The Šilov idempotent theorem then implies there exists an idempotent measure $\mu \in M(K\backslash G/K)$ with $\gamma * (\mu) = 1$ and $\hat{\mu}(\lambda) = \lambda'(\mu) = 0$, all $\lambda \in \mathscr{S}(L_1(K\backslash G/K))$. This is impossible since the Fourier–Stieltjes transform is injective. ∎

Remark. We can avoid appealing to the Silov theorem by actually constructing the idempotent μ. This approach is outlined at the end of Section 4a in connection with our discussion of idempotent measures in the compact case.

We close this section with one simple result which essentially characterizes the measures μ for which $\hat{\mu}$ vanishes at infinity.

THEOREM 3.17. *Let (G, K) satisfy the hypotheses of* (3.11). *If $\mu \in M_c(K\backslash G/K)$, then $\hat{\mu}$ vanishes at infinity. Conversely, if $\mathscr{S}(L_1(K\backslash G/K))$ is dense in $\mathscr{S}(M(K\backslash G/K))$ and $\hat{\mu}$ vanishes at infinity, then $\mu \in M_c(K\backslash G/K)$.*

Proof. If $\mu \in M_c(K\backslash G/K)$, then $\mu^m \in L_1(K\backslash G/K)$, $m = \dim G/K$, by (2.5). So $\hat{\mu}$ vanishes at infinity as $(\hat{\mu})^m$ does.

For the converse, by the density assumption for each γ in $\mathscr{S}(M_d(K\backslash G/K))$, we can find a net $\lambda_\alpha' \to \gamma *$, $\lambda_\alpha \in \mathscr{S}(L_1(K\backslash G/K))$. Now suppose $\hat{\mu}$ vanishes at infinity. Then $\lambda_\alpha'(\mu_d) = \hat{\mu}(\lambda_\alpha) - \hat{\mu}_c(\lambda_\alpha) \to 0$ as a consequence of the first part. Thus $\gamma(\mu_d) = 0$ for all $\gamma \in \mathscr{S}(M(K\backslash N(K_0)/K)$. So $\mu_d = 0$, as $M(K\backslash N(K_0)/K)$ is semisimple, and $\mu = \mu_c$ as desired. ∎

4. IDEMPOTENTS IN THE COMPACT CASE

In this section we determine explicitly all zonal idempotent measures for a *compact* pair (G, K) satisfying the hypotheses of (3.11). Our results are analogous to those of P. Cohen for abelian groups.

Most of the tools for discussing idempotent measures are in

PROPOSITION 4.1. *Suppose (H, K) is a pair with H compact and $M(K\backslash H/K)$ commutative. Let ϕ, ϕ_1, ϕ_2 be bounded continuous spherical functions on H. Then*

(i) $\phi(h^{-1}) = \bar{\phi}(h)$;

(ii) *if* $\phi_1 \neq \phi_2$, *then* $\int \bar{\phi}_1(h)\, \phi_2(h)\, dm_H(h) = 0$;

(iii) $\phi * \phi = (\int |\phi|^2\, dm_H)\phi$. *Thus* $(\int |\phi|^2\, dm_H)^{-1} m_H$ *is an idempotent measure.*

Proof. This is essentially (9.4.8) in [4]. Or it is an easy exercise to derive it from the functional equation (3.2). ∎

This result reminds us (via (ii)) that when G is compact, $\mathscr{S}(L_1(K\backslash G/K))$ is discrete. Also, it identifies some of the idempotents in $M(K\backslash G/K)$, namely, for each closed subgroup $H \supseteq K$ and each $\theta \in \mathscr{S}(L_1(K\backslash H/K))$ with associated spherical function ϕ_θ on H, there is an idempotent

$$\nu_{H,\theta} = c_\theta \phi_\theta m_H, \qquad c_\theta^{-1} = \int_H |\phi_\theta|^2\, dm_H. \tag{4.2}$$

Of course, given idempotents $\mu_1, \mu_2 \in M(K\backslash G/K)$, new idempotents are given by

$$\mu_1 * \mu_2, \qquad \mu_1 + \mu_2 - \mu_1 * \mu_2, \qquad m_K - \mu_1, \tag{4.3}$$

since we are assuming $M(K\backslash G/K)$ is commutative. Our main result says that under the hypotheses of (3.11), the basic idempotents $\{\nu_{H,\theta}: H \supseteq K,\ \theta \in \mathscr{S}(L_1(K\backslash H/K))\}$ together with the operations in (4.3) generate *all* the idempotents. In fact, we have an explicit description of all the idempotents since G and $N(K_0)$ are, essentially, the only closed subgroups which properly contain K.

THEOREM 4.4. *Let (G, K) be a pair of compact Lie groups satisfying the hypotheses of* (3.11). *Then $\mu \in M(K\backslash G/K)$ is an idempotent if and only if there exist disjoint sets A, B with $A \cup B = \mathscr{S}(M(K\backslash N(K_0)/K))$ and finite sets $E_\gamma \subseteq Fb^{-1}(\gamma)$, $\gamma \in \mathscr{S}(M(K\backslash N(K_0)/K))$, such that*

$$\mu = \sum_{\gamma \in A} \sum_{E_\gamma} \nu_{G,\lambda} + \sum_{\gamma \in B} \Big(\nu_{N,\gamma} - \sum_{E_\gamma} \nu_{G,\lambda}\Big) \tag{4.4(i)}$$

(where $N = N(K)$). Hence, $\{\nu_{G,\lambda}\}$, $\{\nu_{N,\gamma}\}$, and the operations (4.3) *generate all odempotents.*

Proof. Since E_γ and $\mathscr{S}(M(K\backslash G/K))$, hence A, B, are finite, μ is well-defined. To see that μ is idempotent, we use the Fourier–Stieltjes transform and verify that $\hat{\mu}(\lambda') = 0$ or 1 all λ'. In fact, since $\phi_{\lambda'}(g^{-1}) = \overline{\phi_{\lambda'}(g)}$ by (4.1.1), (4.1.(ii), (iii)) imply $\hat{\nu}_{G,\lambda}(\lambda') = \delta_{\lambda\lambda'}$. Also $\phi_{\lambda'} \upharpoonright N$ is spherical on N, so again (4.1) implies $\hat{\nu}_{N,\gamma}(\lambda') = \delta_{\gamma, Fb(\lambda')}$.

Using these two observations and the fact that the sets $E_\gamma \subseteq Fb^{-1}(\gamma)$, as γ varies, are disjoint, it follows that $\hat{\mu}(\lambda') = 0$ or 1 for all λ'.

Conversely, suppose μ is an idempotent, then so is μ_d as the map $\mu = \mu_c + \mu_d \to \mu_d$ is a homomorphism. So for each $\gamma \in \mathscr{S}(M(K\backslash N/K))$, $\gamma(\mu_d) = 0$ or 1. Let $A = \{\gamma : \gamma(\mu_d) = 0\}$, $B = \{\gamma : \gamma(\mu_d) = 1\}$. To define E_γ, notice that $\hat{\mu}_c(\lambda) = (\hat{\mu} - \hat{\mu}_d)(\lambda) = 0$, 1, or -1 for all λ in $\mathscr{S}(L_1(K\backslash G/K))$. Since $\hat{\mu}_c$ vanishes at infinity by (3.17), $\hat{\mu}_c$ has finite support so if we set $E_\gamma = \{\lambda \in Fb^{-1}(\gamma): \hat{\mu}_c(\lambda) \neq 0\}$, then E_γ is finite. With these definitions of A, B, and E_γ, μ is given by (4.4(i)) since both sides have the same Fourier–Stieltjes transform. ▮

This theorem can be recast, in a form more analogous to Cohen's theorem for abelian groups, as a characterization of the ring of sets $\mathscr{I} = \{S(\mu) = \{\lambda: \hat{\mu}(\lambda) = 1\}: \mu * \mu = \mu\}$. Since this would require extensive discussion of the hypergroup and hypercoset structure of $\mathscr{S}(L_1(K\backslash G/K))$ (see [2]), but does not provide much additional information about the idempotents, we have chosen to omit it.

Of course the determination of all idempotents in $M(K\backslash G/K)$ means that we have determined all the closed complemented G-invariant subspaces of $C(G/K)$ and $L_1(G/K)$, since the projection on such a space is given by (right) convolution with an idempotent in $M(K\backslash G/K)$. See [4, (9.4.10)] for details.

Before finishing with the topic of idempotent measures, we give a new proof of (3.16) which avoids the Šilov idempotent theorem.

Proof of (3.16) *in the Compact Case.* Suppose $\mathscr{S}(L_1(K\backslash G/K))$ is not dense in $\mathscr{S}(M(K\backslash G/K))$. Then since $\mathscr{S}(L_1(K\backslash G/K))$ is discrete, the description of the topology of $\mathscr{S}(M(K\backslash G/K))$ in (3.13) shows there must exist $\gamma \in \mathscr{S}(M(K\backslash N/K))$ with $Fb^{-1}(\gamma)$ *finite.* But then consider $\mu = \nu_{N,\gamma} - \sum_{Fb^{-1}(\gamma)} \nu_{G,\lambda}$. By the observations in the previous proof

$$\begin{aligned}\hat{\mu}(\lambda') &= \hat{\nu}_{N,\gamma}(\lambda') - \sum_{Fb^{-1}(\gamma)} \hat{\nu}_{G,\lambda}(\lambda') \\ &= \delta_{\gamma,Fb(\lambda')} - \sum_{Fb^{-1}(\gamma)} \delta_{\lambda\lambda'} \\ &\equiv 0.\end{aligned}$$

So $\nu_{N,\lambda} = \sum_{Fb^{-1}(\gamma)} \nu_{G,\lambda}$, which is impossible since $\nu_{N,\gamma}$ is discrete while $\nu_{G,\lambda}$ is absolutely continuous. ▮

Remark. This proof simplifies 3.3 and 5.2 of [10] since it eliminates the use of induced representation.

5. Examples and Connections with Other Work

In this section we discuss three classes of examples of measure algebras which can be studied using the results of the previous sections. These are the algebras of (I) central measures on compact simple Lie groups, (II) rotation invariant measures on Euclidean space, and (III) zonal measures on irreducible Riemannian symmetric spaces of compact or noncompact type. For (I) and (II) we indicate why these algebras are isomorphic to appropriate zonal measure algebras for pairs (G, K) satisfying all of our assumptions. For all of the examples we discuss the implication of our results and also briefly indicate the relation between other papers and our present work.

(I) *Central measures on compact simple Lie groups*

Let H be a compact simple Lie group (perhaps disconnected) with $\dim H \neq 0$. Then the pair (G, K) with $G = H \times H$, $K = \{(h, h)\colon h \in H\}$ satisfies all the hypotheses of (3.11). In fact, $K_0 \simeq H_0$ and the action of K_0 on $\mathfrak{g}/\mathfrak{k} = \mathfrak{h}$ is just the adjoint action of H_0 on $\mathfrak{h}$ which is irreducible since H is simple and is nontrivial as $\dim H \geqslant 3$. Moreover, the homeomorphism $G/K \to H$ given by $(h', h)K \to h'h^{-1}$ induces a Banach *space* isometry from $M(G/K)$ to $M(H)$. When restricted to $M(K\backslash G/K)$, this isometry is easily seen to be an *algebra* isomorphism onto $M^z(H)$—the center of $M(H)$. Since $M^z(H)$ is commutative, all the assumptions of (3.11) are satisfied for $M(K\backslash G/K)$.

The group $N(K_0)$ is easily identified since the map $G/K \to H$ carries $N(K_0)/K$ to the set of points in H with finite conjugacy class. In particular, when H (hence K) is connected, $N(K)/K$ is isomorphic to $Z(H)$—the center of H—and $M_d(K\backslash G/K)$ corresponds to the (discrete) measures on $Z(H)$.

In the present setting, $\mathscr{S}(L_1(K\backslash G/K))$ is essentially the discrete space, $\hat{H}$, of (classes of) irreducible unitary representations of H, and a spherical function ϕ_λ on G, when carried to H via the map $G/K \to H$, is identified with the normalized character, $\chi_\lambda/d(\lambda)$, $d(\lambda) = \chi_\lambda(e)$, of the corresponding representation (see [7, p. 408] for details). In case H is connected, then the spherical functions for $N(K)$ are just the characters of the finite abelian group $Z(H)$, so when we apply (3.11), we see that $\mathscr{S}(M^z(H)) = \mathscr{S}(M(K\backslash G/K)) = \hat{H} \cup Z(H)^\wedge$. Thus we recover Theorems 3.2 and 3.6 of [10].

The asymptotic relation (3.14) implies that if H is connected, then

$$\lim_{\lambda\to\infty} \chi_\lambda(h)/d(\lambda) = 0 \qquad \text{for} \quad h \notin Z(H). \tag{5.1}$$

This consequence of our work has been used by Rider in [13, 14].

(II) *Rotation invariant measures on Euclidean space*

Let K be any compact Lie group and $\pi: K \to O(n)$ any orthogonal representation of K on $\mathbb{R}^n$ such that $\pi \mid K_0$ is irreducible and nontrivial (so $n > 1$). Then the algebra $M^K(\mathbb{R}^n)$ of K-invariant measures on $\mathbb{R}^n$ is isomorphic as an algebra to $M(K\backslash G/K)$, where $G = K \times_\pi \mathbb{R}^n$ is a semidirect product. As in (I), this is seen by restricting to $M(K\backslash G/K)$ the natural Banach *space* isometry from $M(G/K)$ to $M(\mathbb{R}^n)$ induced by the homeomorphism of $K \times_\pi \mathbb{R}^n/K$ to $\mathbb{R}^n$ taking $(k, v)K \to \pi(k)v$. Since $M(\mathbb{R}^n)$, hence $M^K(\mathbb{R}^n)$, is abelian, the commutativity assumption on $M(K\backslash G/K)$ is satisfied.

In this case, $N(K_0) = K$ as the origin is the only point of $\mathbb{R}^n$ ($\simeq G/K$) with finite K-orbit (a simple case of (3.5(ii))). So (3.11) shows that $\mathscr{S}(M^K(\mathbb{R}^n))$ is the one-point compactification of $\mathscr{S}(L_1(K\backslash G/K)) = \mathscr{S}(L_1{}^K(\mathbb{R}^n))$. This special case was considered in [11], where we showed that $\mathscr{S}(L_1{}^K(\mathbb{R}^n)) = \mathbb{R}^n/K =$ the space of K-orbits in $\mathbb{R}^n$.

The reason for calling (3.14) a *weak* asymptotic estimate is best pointed up in this context. When $K = SO(n)$, $\pi = \mathrm{Id}$, the spherical functions ϕ_r, $r \in \mathbb{R}^+ = \mathbb{R}^n/SO(n)$, can be explicitly calculated in terms of Bessel functions (see [11, (A.2)]) from which it follows via classical estimates that $\phi_r(k, v) = O(r^{-(n-1)/2})$, $r \to \infty$, and $v \neq 0$. However, (3.14) only says $\phi_r(k, v) = o(1)$.

(III) *Zonal measures on irreducible Riemannian symmetric spaces*

Both of the previous examples considered algebras which were shown to be isomorphic to $M(K\backslash G/K)$ with (G, K) a Riemannian symmetric pair. The remaining examples are provided by the zonal measures algebras of *irreducible Riemannian symmetric pairs* (G, K). These are tabulated in [7, pp. 339–355]. (Strictly speaking, the connected Lie groups in example (I) fit under this general category.) By definition, K_0 acts irreducibly on $\mathfrak{g}/\mathfrak{k}$ in this case and $M(K\backslash G/K)$ is commutative by [5] and (1.2), so (3.11) applies. There are two distinct examples here.

(a) *The Noncompact Type.* In case (G, K) is an irreducible Riemannian symmetric pair of noncompact type, then from (3.5(ii)) we know $N(K_0) = K$, as G is connected by definition. Thus $\mathscr{S}(M(K\backslash G/K))$ is the one-point compactification of $\mathscr{S}(L_1(K\backslash G/K))$. Now the space $\mathscr{S}(L_1(K\backslash G/K))$ has been determined by Helgason and Johnson in [8] (also see [6, p. 66]). It consists of a certain unbounded domain with boundary in $\mathbb{C}^l$, $l = \operatorname{rank} G/K$, modulo a certain finite group of linear transformations. Thus $\mathscr{S}(M(K\backslash G/K))$ is a complex manifold with boundary, and the complex structure is reflected in the fact that the Gelfand transforms are holomorphic off the boundary.

A specific example is given by $G = SL(2, \mathbb{R})$, $K = SO(2)$. For this pair, we have

$$\mathscr{S}(L_1(K\backslash G/K)) = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant 1\}/z \sim -z,$$

where $z \in \mathbb{C}$ corresponds to the homomorphism given by integration against the spherical function ϕ_z:

$$\phi_z\left(k\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2}\end{pmatrix} n\right) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh r + \sinh r \cos u)^{(iz-1)/2}\, du,$$

$k \in K$, $r \in \mathbb{R}$, and $n = \left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right)$ some $t \in \mathbb{R}$ (see [7, p. 406]). Thus $\mathscr{S}(M(K\backslash G/K))$ is biholomorphically equivalent to the closed unit disc, and the space of Gelfand transforms of this algebra can be considered as a rather interesting Banach algebra of analytic functions on the disc with continuous extensions to the boundary. Corollary (3.14) says that $\phi_z(g)$ converges to 0 as $z \to \infty$ provided $r \neq 0$. Certainly this is not at all obvious from the integral expression above.

We should note that (1.1) and (1.2) of Wolf's work [15] imply that the examples in (II) and (IIIa) essentially exhaust all pairs (G, K) satisfying the assumptions of (3.11) with G *noncompact and connected.* This is strictly true if we rule out the trivial exceptions in which K contains some nondiscrete closed normal subgroup of G.

(b) *The Compact Type.* Our last class of examples is provided by the irreducible Riemannian symmetric pairs of *compact type*. For these pairs, the space $\mathscr{S}(L_1(K\backslash G/K))$ has also been explicitly determined. It is the discrete space of irreducible class-one representations of G. And by (3.5(i)), $\mathscr{S}(M(K\backslash G/K))$ is just a finite pointed compactification of this discrete space. Of course, all the results of Section 4 apply here.

In certain classical examples, $\mathscr{S}(L_1(K\backslash G/K))$ has more well-known descriptions. For instance, if $G = SO(m+1)$, $K = SO(m)$, so $G/K = S^m$, the m-sphere, then the continuous spherical functions which correspond to $\mathscr{S}(L_1(K\backslash G/K))$ are just the functions $\phi_n(g) = P_n^{\lambda}(\langle gv, v\rangle)$, $n = 0, 1,\ldots$, where P_n^{λ} is the ultrapherical polynomial of degree n (normalized so that $P_n^{\lambda}(1) = 1$), $\lambda = (m-1)/2$, and v is the north pole of S^m. (See [4, (9.6.7)]). Since $K = K_0$ has just two fixed points on S^m (the poles), we have $N(K)/K \simeq Z_2$ so $\mathscr{S}(M(K\backslash G/K)) = \mathbb{N} \cup Z_2\hat{}$. Since $P_n^{\lambda}(-1) = (-1)^n$, it follows from (3.13) that the sequence of even integers in $\mathbb{N}$ converges to the trivial character of

Z_2 while the sequence of odd integers converges to the nontrivial character of Z_2. Thus we have recovered many of the results in Dunkl's paper [1].

For the special case of S^m, Dunkl proves a stronger result than (2.5): the convolution of two (rather than m) continuous zonal measures on S^m is absolutely continuous. This comes about since in this case it is easy to see that in the geometric result (2.2) we have $\operatorname{rank} f^2 = \dim G$ (except on a small exceptional set). Perhaps some similar improvement of (2.2) can be obtained in the general case. The natural (minimal) guess is that $\operatorname{rank} f^k = \dim G$, where $k = [(\dim G/K - 1)/j] + 1$ and $j =$ minimum dimension of any nonfinite K orbit in G/K.

In connection with the compact examples, we should mention that Dunkl has observed that his general theory of commutative $P*$-hypergroups in [2], which came to our attention after this research was done, applies to the compact double coset space $K\backslash G/K$. When suitably interpreted, this general theory contains a few of the results of our paper. It is possible that all of the irreducible pairs (G, K) of compact type are such that $K\backslash G/K$ satisfy Dunkl's definition of an $SP*$-hypergroup, Definition 3.1 in [3], although this seems difficult to verify. If this were true, other of our results, notably the determination of $\mathscr{S}(M(K\backslash G/K))$ and, essentially, the characterization of the idempotents, would follow from his work, at least in the case when K is connected.

We close this paper by pointing out that we do not know if there are any pairs (G, K) with G compact and connected and K acting effectively on G/K which satisfy the assumptions of (3.11), but are distinct from those pairs discussed under (I) and (IIIb). Wolf [15] has given a complete list of all pairs (G, K) satisfying the irreducibility criterion (2.1) so the problem is to determine whether any of these pairs have $M(K\backslash G/K)$ or $L_1(K\backslash G/K)$ commutative.

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