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Quantaloidal nuclei, the syntactic congruence and tree automata

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Introduction

In [3], Betti and Kasangian indicate how one can approach the study of tree automata from the perspective of enriched category theory. If $\mathscr{A}$ is an algebraic theory (in the sense of Lawvere [13]), then one can construct a bicategory $\mathscr{P}(\mathscr{A})$ so that a tree automaton can be realized as a category enriched in $\mathscr{P}(\mathscr{A})$ subject to certain conditions, together with an initial and final bimodule. The composition of these bimodules results in the forest (set of trees), which is called the behavior of the automaton. Kasangian and Rosebrugh have pursued this idea further in subsequent papers [10, 11].

The bicategory $\mathscr{P}(\mathscr{A})$ is an example of a quantaloid. A quantaloid is a category enriched in the category $\mathscr{SL}$ of sup-lattices and it is a natural generalization of the notion of a quantale. Quantales include frames, lattices of ideals, and relations as examples and their basic theory and some applications have been collected together in [18]. Quantaloids were studied by Pitts [17] in investigating distributive categories of relations and topos theory. Some of the basic features of quantaloids, viewed as generalized quantales, including a discussion of enriched categories, were developed by Rosenthal in [20] and some further aspects were examined in [21].

In this paper, we propose using the theory of quantaloidal nuclei (which generalize the notion of a nucleus on a quantale [18] or a frame [9]) to study syntactic congruences, which arise in the theory of automata and tree automata. If $\mathscr{Q}$ is a quantaloid and $f : c \to d$ is a morphism of $\mathscr{Q}$, one can construct a

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quantaloidal nucleus $j(f)$, the syntactic nucleus associated to f, so that the resulting quotient quantaloid $\mathcal{Q}_{j(f)}$ is the smallest quotient of $\mathcal{Q}$ by a nucleus containing the morphism f.

If M is a monoid and $\mathcal{Q} = \mathcal{P}(M)$ is the quantale of subsets of M, then if $B \subseteq M$, one can recover from the nucleus $j(B)$ the syntactic congruence on M associated with B. These congruences play a central role in automata theory in the study of recognizable sets [5, 16]. If $\mathcal{A}$ is an algebraic theory and $\mathcal{Q}$ is the associated free quantaloid $\mathcal{P}(\mathcal{A})$, then given a forest F, one can recover the syntactic congruence of F in the sense of tree automata (see [7]) from nucleus $j(F)$.

The theory of quantaloids and quantaloidal nuclei provides a unified treatment of syntactic congruences in both of these cases. Furthermore, the categorical approach clearly indicates the way in which the tree automata case generalizes that of automata; it is the passage from a one-object category enriched in sup-lattices (a quantale) to an enriched category with several objects (a quantaloid), thus clarifying the treatment in [7]. Finally, our construction applies not just to forests, but to more general morphisms in the quantaloid $\mathcal{P}(\mathcal{A})$ and the residuation operations of $\mathcal{P}(\mathcal{A})$ provide a natural, intrinsic way of understanding the calculations behind the syntactic congruence.

The first two sections of the paper provide the reader with the necessary background on quantales and quantic nuclei and more generally quantaloids and quantaloidal nuclei. Section 3 presents an adjunction between congruences on a locally small category $\mathcal{A}$ and nuclei on the free quantaloid $\mathcal{P}(\mathcal{A})$ generated by $\mathcal{A}$. This adjunction is the key to recovering the syntactic congruences from our syntactic nuclei in the later sections. The main construction of the syntactic nucleus and its basic properties are developed in Section 4. We then proceed to looking at some examples. The case of monoids is addressed via the quantales $\mathcal{P}(M)$ and the final section discusses the tree automata case in detail showing how one can recover the relevant constructions from [7] using the theory of quantaloids.

1. Quantales

We begin with a discussion of quantales, as an understanding of their properties as well as familiarity with some key examples form an essential background for the reader.

Definition 1.1. A *quantale* is a complete lattice $\mathcal{Q}$ together with an associative binary operation $\circ$ satisfying: $a \circ (\sup_\alpha b_\alpha) = \sup_\alpha (a \circ b_\alpha)$ and $\sup_\alpha (b_\alpha) \circ a = \sup_\alpha (b_\alpha \circ a)$ for all $a \in \mathcal{Q}$ and $\{b_\alpha\} \subseteq \mathcal{Q}$.

It follows that because $a \circ -$ and $- \circ a$ preserve sups, they have right adjoints, which are denoted $a \rightarrow_r -$ and $a \rightarrow_\ell -$ respectively.

Definition 1.2. A quantale $\mathcal{Q}$ is called *unital* iff it has an element 1 such that $1 \circ a = a = a \circ 1$ for all $a \in \mathcal{Q}$.

The following formulas are easily established (see [18]).

Lemma 1.3. *Let $\mathcal{Q}$ be a quantale and let $a,b,c \in \mathcal{Q}$.*
(1) $a \circ (a \rightarrow_r c) \leq c$.
(2) $(a \rightarrow_\ell c) \circ a \leq c$.
(3) $b \rightarrow_r (a \rightarrow_r c) = a \circ b \rightarrow_r c$.
(4) $a \rightarrow_\ell (b \rightarrow_\ell c) = a \circ b \rightarrow_\ell c$.
If $\mathcal{Q}$ is unital with unit 1, *then*
(5) $1 \rightarrow_r a = a = 1 \rightarrow_\ell a$ *for all* $a \in \mathcal{Q}$. □

We shall now present our main example of interest, as well as mention several others.

Examples. (1) Let S be a semigroup and consider $\mathcal{P}(S)$, the power set of S. Then, $\mathcal{P}(S)$ becomes a quantale via the operation $A \circ B = \{ab \mid a \in A,\ b \in B\}$. Unions play the role of sups and one can see that $A \rightarrow_r B = \{m \in S \mid am \in B$ for all $a \in A\}$ and $A \rightarrow_\ell B = \{m \in S \mid ma \in B$ for all $a \in A\}$.

If M is a monoid, then $\mathcal{P}(M)$ is unital, with $\{e\}$ being the unit, where e is the identity of M. In fact, $\mathcal{P}(M)$ is the free unital quantale on the monoid M.

Besides playing a central role in the semantics of linear logic via Girard quantales [8, 18, 19, 22], these power monoids are important in the study of formal languages and recognizable sets [16] and it is primarily this aspect that we shall be interested in.

(2) Other examples include frames (and hence complete Boolean algebras), various ideal lattices of rings and C^*-algebras, and Rel(X), the relations on a set X. For more on these and other examples, see [18].

Definition 1.4. If $\mathcal{Q}$ and $\mathcal{S}$ are quantales, a function $f : \mathcal{Q} \rightarrow \mathcal{S}$ is a *homomorphism* iff it preserves sups and $\circ$.

If $\mathcal{Q}$ and $\mathcal{S}$ are unital and f preserves identity elements, then f is called a *unital homomorphism*.

The category of quantales and homomorphisms will be denoted **Quant**.

There are other notions of morphism of quantales, which are more appropriate than homomorphisms in certain contexts, such as closed maps of quantales, but we shall not need to delve into these in this paper. For details, once again see [18].

Definition 1.5. Let $\mathcal{Q}$ be a quantale. A closure operator $j : \mathcal{Q} \rightarrow \mathcal{Q}$ is called a *quantic nucleus* iff $j(a) \circ j(b) \leq j(a \circ b)$ for all $a,b \in \mathcal{Q}$.

If j is a quantic nucleus on $\mathcal{Q}$, let $\mathcal{Q}_j = \{a \in \mathcal{Q} \mid j(a) = a\}$.

Proposition 1.6. *Let $\mathcal{Q}$ be a quantale. Then, $\mathcal{Q}_j$ is a quantale with operation $a \circ_j b = j(a \circ b)$ for all $a,b \in \mathcal{Q}$, and with sups calculated by $\sup_j a_\alpha = j(\sup a_\alpha)$ for all $\{a_\alpha\} \subseteq \mathcal{Q}$.* $\square$

These are precisely the quotients in **Quant**. For a thorough discussion of nuclei, see [15] or [18]. We have the following characterization:

Proposition 1.7. *Let $\mathcal{Q}$ be a quantale and let $S \subseteq \mathcal{Q}$. S is of the form $\mathcal{Q}_j$ for some quantic nucleus j on $\mathcal{Q}$ iff S is closed under infs and for all $s \in S$, $x \in \mathcal{Q}$, we have $x \rightarrow_r s \in S$ and $x \rightarrow_\ell s \in S$.* $\square$

Example. If M is a commutative monoid and $D \subseteq M$, on $\mathcal{P}(M)$ let $j(A) = (A \rightarrow D) \rightarrow D$ (we do not need subscripts on $\rightarrow$ because of the commutativity). Then j is a quantic nucleus on $\mathcal{P}(M)$. The resulting quotient quantale is denoted by $\mathcal{P}(M)_D$. It is quotients of this type that arise in linear logic.

We have the following characterization theorem for quantales, which indicates the central role played by power semigroups and nuclei (for a complete proof, see [18]):

Theorem 1.8. *Let Q be a quantale. Then, there is a semigroup S and a quantic nucleus j on $\mathcal{P}(S)$ such that $\mathcal{Q} \cong \mathcal{P}(S)_j$.* $\square$

In fact, we can take $S = \mathcal{Q}$ and for $A \subseteq \mathcal{Q}$ define j by $j(A) = \{b \in \mathcal{Q} \mid b \leq \sup A\}$.

2. The free quantaloid on a locally small category

Quantaloids are categorical generalizations of quantales. A quantaloid can be thought of as a quantale 'with many objects' or to turn it around, a unital quantale is a quantaloid with one object.

In their work on quantales and process semantics [1], Abramsky and Vickers need the generality of quantaloids, when introducing a notion of typing on the processes. In [3], Betti and Kasangian indicate how categories enriched in a certain quantaloid provide the appropriate categorical framework for considering tree automata. This was developed further by Kasangian and Rosebrugh in [10] and [11]. It is precisely this observation which motivates consideration of our general notion of syntactic congruence in Section 4.

For details of the following, the reader is referred to [20].

Definition 2.1. A *quantaloid* is a locally small category $\mathcal{Q}$ such that:
(1) for $a, b \in \mathcal{Q}$, the hom-set $\mathcal{Q}(a, b)$ is a complete lattice,
(2) composition of morphisms in $\mathcal{Q}$ preserves sups in both variables.

In the language of enriched category theory [2, 4, 12, 14] this says precisely that $\mathcal{Q}$ is enriched in the symmetric, monoidal, closed category $\mathcal{SL}$ of sup-lattices. Note that the hom-sets $\mathcal{Q}(a, a)$ are unital quantales for all $a \in \mathcal{Q}$.

From (2) above, it follows that we have left and right residuations analogous to those of quantale theory. If we are given morphisms in $\mathcal{Q}$, $f : a \to b$ and $g : a \to c$, then there exists a morphism $f \to_\ell g : b \to c$ such that $h \circ f \leq g$ iff $h \leq f \to_\ell g$ for all $h : b \to c$.

Similarly, given $g : a \to c$ and $h : b \to c$, there is $h \to_r g : a \to b$ such that $h \circ f \leq g$ iff $f \leq h \to_r g$.

Diagrammatically,

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"'] & c \\ b \arrow[ur, dashed, "f \to_\ell g"'] & \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "g"] \arrow[d, dashed, "h \to_r g"'] & c \\ b \arrow[ur, "h"'] & \end{tikzcd}$$

Formulas equivalent to those in Lemma 1.3 can also be established for the residuation operations in a quantaloid, although some care must be taken with the domains and codomains of the morphisms (see the above diagram).

Lemma 2.2. *Let $\mathcal{Q}$ be a quantaloid and suppose we have morphisms $f : a \to b$, $g : a \to c$, $h : b \to c$, $k : d \to b$, $j : b \to e$ in $\mathcal{Q}$. Then*:
(1) $h \circ (h \to_r g) \leq g$,
(2) $(f \to_\ell g) \circ f \leq g$,
(3) $k \to_r (h \to_r g) = h \circ k \to_r g$,
(4) $j \to_\ell (f \to_\ell g) = j \circ f \to_\ell g$.

If $i_a : a \to a$ and $i_b : b \to b$ denote the identity morphisms on a and b, then $i_b \to_r f = f = i_a \to_\ell f$. □

Definition 2.3. Let $\mathcal{Q}$ and $\mathcal{S}$ be quantaloids. A *quantaloid homomorphism* is a functor $F : \mathcal{Q} \to \mathcal{S}$ such that on hom-sets it induces a sup-lattice morphism $\mathcal{Q}(a, b) \to \mathcal{S}(F(a), F(b))$.

Thus, a quantaloid homomorphism is just an $\mathcal{SL}$-enriched functor. Let **Qtlds** denote the category of quantaloids and homomorphisms. We should point out that quantaloids and some of their properties were investigated by Pitts in [17]. Also, the notion of Girard quantale can be generalized to that of a Girard quantaloid. These are studied in detail in [21].

Let us now list some examples of quantaloids that will be of interest to us. For more examples, consult [20].

(1) A quantaloid with one object is just a unital quantale, as remarked above.

(2) $\mathcal{R}e\ell$, the category of sets and relations, is a quantaloid. (So is any locally complete, distributive allegory, in the terminology of [6].)

The following example generalizes the construction $\mathcal{P}(M)$ from monoids (one object categories) to locally small categories.

(3) *Our main example*: Let $\mathcal{A}$ be a locally small category. Define a quantaloid $\mathcal{P}(\mathcal{A})$ as follows. The objects of $\mathcal{P}(\mathcal{A})$ are precisely those of $\mathcal{A}$. If $a,b \in \mathcal{A}$, then $\mathcal{P}(\mathcal{A})(a, b) = \mathcal{P}(\mathcal{A}(a, b))$, the power set of the hom-set $\mathcal{A}(a, b)$. If $S : a \to b$ and $T : b \to c$ are sets of morphisms of $\mathcal{A}$, let $TS = \{g \circ f \mid g \in T,\ f \in S\}$. This operation preserves unions in each variable and thus we have a quantaloid.

We record the following results from [20]:

Theorem 2.4. *Let* **LSm** *denote the category of locally small categories and functors between them.*

(1) *Then* $\mathcal{P}$: **LSm** $\to$ **Qtlds** *is the left adjoint of the forgetful functor* **Qtlds** $\to$ **LSm.**

(2) $\mathcal{P}$ *defines a monad on* **LSm** *and the category* **Qtlds** *is equivalent to the category* $\mathcal{P}$-**Alg**, *of* $\mathcal{P}$*-algebras.* $\square$

Thus, we shall call $\mathcal{P}(\mathcal{A})$ the free quantaloid on the category $\mathcal{A}$. A particular example of interest is the following. Let $\mathcal{A}$ be an algebraic theory in the sense of Lawvere [13]. Then, the objects of $\mathcal{A}$ can be thought of as $[0], [1], [2], \ldots, [n], \ldots$ and a morphism $[n] \to [m]$ is an n-tuple of m-ary operations of the theory. Composition is by substitution of operations. Thus, in the free quantaloid $\mathcal{P}(\mathcal{A})$, we consider as morphisms $S : [n] \to [m]$ sets S of n-tuples of m-ary operations. $\mathcal{P}(\mathcal{A})$ is precisely the quantaloid which provides the proper categorical framework for considering tree automata via enriched category theory [3, 10, 11].

Much of the theory of quantales carries over to quantaloids, in particular the theory of quantic nuclei generalizes to quantaloidal nuclei.

Definition 2.5. Let $\mathcal{Q}$ be a quantaloid. A *quantaloidal nucleus* is a lax functor $j : \mathcal{Q} \to \mathcal{Q}$, which is the identity on objects and such that the maps $j_{a,b} : \mathcal{Q}(a, b) \to \mathcal{Q}(a, b)$ satisfy:

(1) $f \le j_{a,b}(f)$ for all $f \in \mathcal{Q}(a, b)$,

(2) $j_{a,b}(j_{a,b}(f)) = j_{a,b}(f)$ for all $f \in \mathcal{Q}(a, b)$,

(3) $j_{b,c}(g) \circ j_{a,b}(f) \le j_{a,c}(g \circ f)$ for all $f \in \mathcal{Q}(a, b)$, $g \in \mathcal{Q}(b, c)$.

(Note that (3) is the laxity condition, however we wish to single it out.)

Let $\mathcal{Q}_j$ be the bicategory with the same objects as $\mathcal{Q}$ and with morphisms $f : a \to b$ being those maps $f \in \mathcal{Q}(a, b)$ such that $j_{a,b}(f) = f$.

Proposition 2.6. *If $\mathcal{Q}$ is a quantaloid and j is a quantaloidal nucleus on $\mathcal{Q}$, then $\mathcal{Q}_j$ is a quantaloid, where if $f \in \mathcal{Q}_j(a, b)$ and $g \in \mathcal{Q}_j(b, c)$, then composition is defined by $g \circ_j f = j_{a,c}(g \circ f)$.* □

Furthermore, $j : \mathcal{Q} \to \mathcal{Q}_j$ is a quantaloid homomorphism. For more details, see [20]. The following result is the analogue of Theorem 1.8.

Theorem 2.7. *Let $\mathcal{Q}$ be a quantaloid. Then, there exists a locally small category $\mathcal{A}$ and a quantaloidal nucleus j on $\mathcal{P}(\mathcal{A})$ such that $\mathcal{Q} \cong \mathcal{P}(\mathcal{A})_j$.* □

We finish this section with the following lemma, which we shall need later on.

Lemma 2.8. *Let j be a quantaloidal nucleus on $\mathcal{Q}$ and suppose $f \in \mathcal{Q}(c, d)$ with $j_{c,d}(f) = f$. Then, if $x : b \to d$ is a morphism of $\mathcal{Q}$, $j_{c,b}(x \to_r f) = x \to_r f$. If $y : c \to a$ is a morphism of $\mathcal{Q}$, then $j_{a,d}(y \to_\ell f) = y \to_\ell f$.*

Proof. We must prove that $j_{c,b}(x \to_r f) \le x \to_r f$. This is true iff we have $x \circ j_{c,b}(x \to_r f) \le f$. The following string of inequalities holds:

$$\begin{aligned} x \circ j_{c,b}(x \to_r f) &\le j_{b,d}(x) \circ j_{c,b}(x \to_r f) \\ &\le j_{c,d}(x \circ (x \to_r f)) \le j_{c,d}(f) = f\,. \end{aligned}$$

The argument for the case of left residuation follows analogously. □

3. Congruences on categories and quantaloidal nuclei

We shall only be interested in congruences on morphisms, so that objects never get identified, only morphisms with the same domain and codomain.

Definition 3.1. Let $\mathcal{A}$ be a locally small category. A *congruence* ϑ on $\mathcal{A}$ consists of a family $\{\vartheta_{a,b}\}$, where

(1) $\vartheta_{a,b}$ is a congruence on the hom-set $\mathcal{A}(a, b)$,
(2) if $(f, g) \in \vartheta_{a,b}$ and $h \in \mathcal{A}(b, c)$, then $(h \circ f, h \circ g) \in \vartheta_{a,c}$,
(3) if $(f, g) \in \vartheta_{a,b}$ and $k \in \mathcal{A}(d, a)$, then $(f \circ k, g \circ k) \in \vartheta_{d,b}$.

We shall establish an adjunction between congruences on $\mathcal{A}$ and quantaloidal nuclei on $\mathcal{P}(\mathcal{A})$.

Let ϑ be a congruence on $\mathcal{A}$. Define j_ϑ on $\mathcal{P}(\mathcal{A})$ as follows. Suppose that $A \subseteq \mathcal{A}(a, b)$. Let $j_{\vartheta,a,b}(A) = \{f \in \mathcal{A}(a, b) \mid (g, f) \in \vartheta_{a,b} \text{ for some } g \in A\}$.

Lemma 3.2. *If ϑ is a congruence on $\mathcal{A}$, then $j_\vartheta : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ is a quantaloidal nucleus.*

Proof. The fact that j_ϑ is increasing follows from reflexivity and its idempotence comes from transitivity. If we let $S \subseteq \mathscr{A}(b, c)$ and $T \subseteq \mathscr{A}(a, b)$, then we must show $j_{\vartheta,b,c}(S) \circ j_{\vartheta,a,b}(T) \subseteq j_{\vartheta,a,c}(ST)$. But, if $(f, s) \in \vartheta_{b,c}$ and $(g, t) \in \vartheta_{a,b}$, then by (2) above $(f \circ g, f \circ t) \in \vartheta_{a,c}$ and by (3) above $(f \circ t, s \circ t) \in \vartheta_{a,c}$, hence by transitivity, $(f \circ g, s \circ t) \in \vartheta_{a,c}$, proving our desired result. □

Conversely, suppose j is a quantaloidal nucleus on $\mathscr{P}(\mathscr{A})$. Define ϑ_j on $\mathscr{A}$ by $(f, g) \in (\vartheta_j)_{a,b}$ iff $j_{a,b}(\{f\}) = j_{a,b}(\{g\})$. (We shall hereon write this as $j_{a,b}(f) = j_{a,b}(g)$.)

Lemma 3.3. *If j is a quantaloidal nucleus on $\mathscr{P}(\mathscr{A})$, then ϑ_j is a congruence on $\mathscr{A}$.*

Proof. The fact that each $(\vartheta_j)_{a,b}$ is a congruence is immediate. Now, suppose $j_{a,b}(f) = j_{a,b}(g)$ and $h : b \to c$ is a morphism of $\mathscr{A}$. Then, $h \circ f \in j_{b,c}(h) \circ j_{a,b}(f) = j_{b,c}(h) \circ j_{a,b}(g) \subseteq j_{a,c}(h \circ g)$. Thus, $j_{a,c}(h \circ f) \subseteq j_{a,c}(h \circ g)$. The opposite containment follows similarly, thus proving (2) in Definition 3.1 and (3) follows by analogous arguments. □

Let $\mathrm{Con}(\mathscr{A})$ denote the lattice of congruences on $\mathscr{A}$ and let $\mathscr{N}(\mathscr{P}(\mathscr{A}))$ denote the lattice of quantaloidal nuclei on $\mathscr{P}(\mathscr{A})$. Using Lemma 3.2 and Lemma 3.3, we obtain order preserving maps $F : \mathrm{Con}(\mathscr{A}) \to \mathscr{N}(\mathscr{P}(\mathscr{A}))$ and $G : \mathscr{N}(\mathscr{P}(\mathscr{A})) \to \mathrm{Con}(\mathscr{A})$ given by $F(\vartheta) = j_\vartheta$ and $G(j) = \vartheta_j$. We have the following adjunction:

Proposition 3.4. *The map $F : \mathrm{Con}(\mathscr{A}) \to \mathscr{N}(\mathscr{P}(\mathscr{A}))$ is the left adjoint of $G : \mathscr{N}(\mathscr{P}(\mathscr{A})) \to \mathrm{Con}(\mathscr{A})$.*

Proof. Let σ be a congruence on $\mathscr{A}$ and let j be a nucleus on $\mathscr{P}(\mathscr{A})$. We must show that $\sigma \subseteq \vartheta_j$ iff $j_\sigma \leq j$. Let $\sigma \subseteq \vartheta_j$ and let $S \subseteq \mathscr{A}(a, b)$. Then,

$$\begin{aligned} j_{\sigma,a,b}(S) &= \{ g \mid (g, s) \in \sigma_{a,b} \text{ for some } s \in S \} \\ &\subseteq \{ g \mid (g, s) \in \vartheta_{j,a,b} \text{ for some } s \in S \} \\ &= \{ g \mid j_{a,b}(g) = j_{a,b}(s) \text{ for some } s \in S \} \subseteq j_{a,b}(S) . \end{aligned}$$

Conversely, suppose $j_\sigma \leq j$ and $(f, g) \in \sigma_{a,b}$. Then, it follows that $f \in j_{\sigma,a,b}(g)$ and $g \in j_{\sigma,a,b}(f)$ and hence using $j_\sigma \leq j$ and the idempotence of nuclei, it follows that $j_{a,b}(f) \subseteq j_{a,b}(g)$ and vice versa, yielding $(f, g) \in \vartheta_{j,a,b}$. □

Notice that $G(F(\vartheta)) = \vartheta$ holds for all congruences ϑ on $\mathscr{A}$, since $(f, g) \in G(F(\vartheta))_{a,b}$ iff $j_{\vartheta,a,b}(f) = j_{\vartheta,a,b}(g)$ iff $(f, g) \in \vartheta_{a,b}$.

However, $F(G(j))$ may fail to equal j in dramatic fashion. For example, let $\mathscr{Q}$ be a quantale and consider the quantic nucleus j on $\mathscr{P}(\mathscr{Q})$ defined

$$j(A) = (\sup A)\downarrow = \{ b \in \mathscr{Q} \mid b \leq \sup A \} .$$

Then, $(x, y) \in \vartheta_j$ iff $j(x) = j(y)$ iff $x = y$, i.e. ϑ_j is the diagonal relation on $\mathscr{Q}$ and hence $F(\vartheta_j)$ is the identity nucleus on $\mathscr{P}(\mathscr{Q})$.

Thus, as one would expect, the notion of quantaloidal nucleus is more general than that of congruence.

Let $(\mathscr{A} \mid \vartheta)$ denote the quotient category of $\mathscr{A}$ by the categorical congruence ϑ. We shall use brackets [ ] to denote equivalence classes of the relations $\vartheta_{a,b}$. We shall not write the subscripts whenever the context is clear. The following proposition shows that the quantaloid quotient $\mathscr{P}(\mathscr{A})_{j_\vartheta}$ can be realized as the free quantaloid on $(\mathscr{A} \mid \vartheta)$.

Proposition 3.5. *Let ϑ be a congruence on a locally small category $\mathscr{A}$. Then, there is an equivalence $\mathscr{P}(\mathscr{A} \mid \vartheta) \cong \mathscr{P}(\mathscr{A})_{j_\vartheta}$ of quantaloids.*

Proof. If $S \subseteq (\mathscr{A} \mid \vartheta)(a, b)$, then S is a set of equivalence classes $[s]$ of morphisms of $\mathscr{A}$. Define $H : \mathscr{P}(\mathscr{A} \mid \vartheta) \to \mathscr{P}(\mathscr{A})_{j_\vartheta}$ by $H(S) = \bigcup S$, which equals

$$\bigcup\Big\{[s] \mid [s] \in S\Big\} = \{k \in \mathscr{A}(a, b) \mid (k, s) \in \vartheta_{a,b} \text{ for some } s \text{ with } [s] \in S\}$$
$$= j_{\vartheta,a,b}\Big(\bigcup S\Big)\,.$$

Thus, $H(S) = \bigcup S$ is in fact in $\mathscr{P}(\mathscr{A})_{j_\vartheta}$.

If $T \subseteq (\mathscr{A} \mid \vartheta)(b, c)$, then $T \circ_\vartheta S = \{[t \circ s] \mid [t] \in T,\ [s] \in S\}$. We have that

$$H[T] \circ_{j_\vartheta} H(S) = j_{\vartheta,a,c}(H(T) \circ H(S)) = j_{\vartheta,a,c}\Big(\bigcup T \circ \bigcup S\Big)$$
$$= j_{\vartheta,a,c}\Big(\bigcup TS\Big) = H(T \circ_\vartheta S)\,.$$

Here the penultimate equality follows since composition preserves sups. Thus, H is a functor, i.e. a quantaloid homomorphism. To see that H is an isomorphism, let $U \in \mathscr{P}(\mathscr{A})_{j_\vartheta}(a, b)$ and define $S_U = \{[f] \mid f \in U\}$. We obtain

$$H(S_U) = \{k \mid (k, f) \in \vartheta_{a,b} \text{ for some } f \in U\}$$
$$= j_{\vartheta,a,b}(U) = U\,.$$

This finishes the proof that H is an isomorphism of quantaloids. □

4. Syntactic nuclei on quantaloids

If M is a monoid and $A \subseteq M$, then the syntactic congruence ϑ_A on M corresponding to A gives rise to the smallest quotient of M such that A is saturated under ϑ_A. This is related to formal language theory, recognizable sets and automata, in that A is recognizable (by a finite automaton) iff the quotient monoid M/ϑ_A is finite and the syntactic congruence and resulting monoid are related to the minimal automaton on A in that the transition monoid of the minimal automaton is isomorphic to M/ϑ_A (see [5, 16]).

In the theory of tree automata, the notion of recognizable set is replaced by that of a recognizable forest [7] and one can define analogues of the syntactic congruence and the minimal recognizer. In [7] there is some discussion of attempts to generalize the notion of syntactic monoid to the setting of tree automata, but any such attempt retaining the notion of monoid is bound to fail to capture the entire picture as in moving from monoids to algebraic theories, we are passing from one-object categories to more general categories, and more precisely from the quantales $\mathcal{P}(M)$ to the quantaloids $\mathcal{P}(\mathcal{A})$. In the tree automata case, one really is interested in the syntactic category or syntactic quantaloid, not in monoids.

We shall develop a general construction of the syntactic nucleus associated to a morphism (or family of morphisms) in a quantaloid $\mathcal{Q}$. This will give rise to a general theory of syntactic congruences, which will simultaneously generalize both the monoid (automata) and algebraic theory (tree automata) cases, as well as apply to other quantaloids.

We shall be utilizing various parts of Lemma 2.2 throughout the following.

Let $\mathcal{Q}$ be a quantaloid and let $f \in \mathcal{Q}(c, d)$. Define $j(f): \mathcal{Q} \to \mathcal{Q}$, which is the identity on objects as follows: for $h \in \mathcal{Q}(a, b)$,

$$j(f)_{a,b}(h) = \sup\{g \in \mathcal{Q}(a, b) \mid h \to_r (x \to_r f) = g \to_r (x \to_r f) \\ \text{for all } x \in \mathcal{Q}(b, d)\}.$$

Theorem 4.1. *Let $\mathcal{Q}$ be a quantaloid and let $f \in \mathcal{Q}(c, d)$. Then, $j(f)$ is a quantaloidal nucleus on $\mathcal{Q}$.*

Proof. For notational convenience, we shall denote $j(f)_{a,b}(h)$ by $\bar{h}$. Now, clearly $h \le \bar{h}$, by the definition of $j(f)$. To see that $j(f)_{a,b} \circ j(f)_{a,b} = j(f)_{a,b}$, consider

$$\bar{h} \to_r (x \to_r f) = (\sup g) \to_r (x \to_r f) = \inf_g (g \to_r (x \to_r f)) \\ = \inf_g (h \to_r (x \to_r f)) = h \to_r (x \to_r f)$$

and thus

$$k \to_r (x \to_r f) = \bar{h} \to_r (x \to_r f) \quad \text{iff} \quad k \to_r (x \to_r f) = h \to_r (x \to_r f)$$

proving that $j(f)_{a,b}(\bar{h}) = \bar{h}$.

It remains to verify that $\bar{k} \circ \bar{h} \le \overline{k \circ h}$, for all $h \in \mathcal{Q}(a, b)$ and $k \in \mathcal{Q}(b, e)$, where we are writing $\bar{k}$ for $j(f)_{b,e}(k)$, $\bar{h}$ for $j(f)_{a,b}(h)$ and $\overline{k \circ h}$ for $j(f)_{a,e}(k \circ h)$.

Suppose $v \to_r (x \to_r f) = h \to_r (x \to_r f)$ for all $x \in Q(b, d)$ and suppose $w \to_r (y \to_r f) = k \to_r (y \to_r f)$ for all $y \in \mathcal{Q}(e, d)$. Then,

$$\begin{aligned} w \circ v \to_r (y \to_r f) &= v \to_r (w \to_r (y \to_r f)) \\ &= v \to_r (y \circ w \to_r f) \\ &= h \to_r (y \circ w \to_r f) . \end{aligned}$$

This, in turn equals

$$\begin{aligned} h \to_r (w \to_r (y \to_r f)) &= h \to_r (k \to_r (y \to_r f)) \\ &= k \circ h \to_r (y \to_r f) . \end{aligned}$$

Thus, $w \circ v \leq \overline{k \circ h}$ for all w, v as above. Since $\bar{k} = \sup w$ and also $\bar{h} = \sup v$ and composition preserves sups, we have $\bar{k} \circ \bar{h} \leq \overline{k \circ h}$ as desired, finishing the proof that $j(f)$ is a quantaloidal nucleus. □

Theorem 4.2. *Let* $\mathcal{Q}$ *be a quantaloid and let* $f \in \mathcal{Q}(c, d)$. *Then*:

(1) $j(f)_{c,d}(f) = f$.

(2) *If* j *is any quantaloidal nucleus on* $\mathcal{Q}$ *satisfying* $j_{c,d}(f) = f$, *then* $j \leq j(f)$.

Proof. (1) If $f \to_r (x \to_r f) = g \to_r (x \to_r f)$ for all $x : d \to d$, then letting $x = 1_d$, the identity morphism of d, and utilizing Lemma 2.2, we have

$$1_d \leq f \to_r f = f \to_r (1_d \to_r f) = g \to_r (1_d \to_r f) = g \to_r f .$$

Thus, $g \leq f$, proving that $j(f)_{c,d}(f) \leq f$, from which equality follows.

(2) Suppose j is a quantaloidal nucleus satisfying $j_{c,d}(f) = f$, and let $h \in \mathcal{Q}(a, b)$, and let $x : b \to d$. We shall show that

$$j_{a,b}(h) \to_r (x \to_r f) = h \to_r (x \to_r f) ,$$

which will prove that $j_{a,b}(h) \leq j(f)_{a,b}(h)$, and since h is arbitrary, this will prove that $j \leq j(f)$. First of all, since $h \leq j_{a,b}(h)$, we have

$$j_{a,b}(h) \to_r (x \to_r f) \leq h \to_r (x \to_r f) .$$

For the opposite inequality, note that

$$\begin{aligned} j_{a,b}(h) \circ (h \to_r (x \to_r f)) &\leq j_{a,b}(h) \circ j_{c,a}(h \to_r (x \to_r f)) \\ &\leq j_{c,b}(h \circ (h \to_r (x \to_r f))) \\ &\leq j_{c,b}(x \to_r f) = x \to_r f \end{aligned}$$

(by Lemma 2.8). Using adjointness, we get that

$$(h \to_r (x \to_r f)) \leq j_{a,b}(h) \to_r (x \to_r f)\,,$$

as desired. □

Analyzing this in terms of quantaloidal quotients, we have that if j is a quantaloidal nucleus satisfying $j_{c,d}(f) = f$, then $\mathcal{Q}_{j(f)} \subset \mathcal{Q}_j$. Thus, $\mathcal{Q}_{j(f)}$ is the smallest quantaloidal quotient of $\mathcal{Q}$ containing f.

Definition 4.3. If $\mathcal{Q}$ is a quantaloid and $f \in \mathcal{Q}(c, d)$ is a morphism of $\mathcal{Q}$, then $j(f)$ is called the *syntactic nucleus associated to f.*

The choice for using right residuation was arbitrary. Thus, a natural question to ask is what happens if one considers the following. As before, let $f \in \mathcal{Q}(c, d)$ and let $h \in \mathcal{Q}(a, b)$. Now, define

$$\mathcal{J}(f)_{a,b}(h) = \sup\{ g \in \mathcal{Q}(a, b) \mid h \to_\ell (y \to_\ell f) = g \to_\ell (y \to_\ell f) \text{ for all } y \in \mathcal{Q}(c, a)\}\,.$$

We record the following theorem, whose proof follows exactly as the proofs of Theorems 4.1 and 4.2 with suitable adjustments made for the transition from right residuations to left residuations.

Theorem 4.4. *Let $\mathcal{Q}$ be a quantaloid and let $f \in \mathcal{Q}(c, d)$. Then*:

(1) *$\mathcal{J}(f)$ is a quantaloidal nucleus on $\mathcal{Q}$.*

(2) *$\mathcal{J}(f)_{c,d}(f) = f$.*

(3) *If j is a quantaloidal nucleus on $\mathcal{Q}$ satisfying $j_{c,d}(f) = f$, then $j \leq \mathcal{J}(f)$.* □

Corollary. *Let $\mathcal{Q}$ be a quantaloid and $f \in \mathcal{Q}(c, d)$. Then, the quantaloidal nuclei $j(f)$ and $\mathcal{J}(f)$ are equal.* □

There is a duality at work here. This duality is evident in the monoid case, as indicated in [5] by looking at the dual monoid. The meaning of this duality in the tree automata case is much less evident and it will be briefly examined in Section 6, when we discuss this example in detail.

We should also point out that if $\mathcal{F}$ is a family of morphisms of $\mathcal{Q}$, we can form the nucleus $j(\mathcal{F}) = \bigcap \{ j(f) \mid f \in \mathcal{F}\}$. The resulting quotient $\mathcal{Q}_{j(\mathcal{F})}$ is the smallest one containing all the morphisms in $\mathcal{F}$.

5. Examples

(1) *Frames.* First of all, let us consider the case of frames. Let L be a frame and let $a \in L$. If j is a nucleus on L satisfying $j(a) = a$, then $b \to a \in L_j$ for all

$b \in L$. It is not hard to show that $S = \{b \to a \mid b \in L\}$ is a frame quotient of L, since it is closed under infs (we have $\inf_\alpha(b_\alpha \to a) = \sup_\alpha b_\alpha \to a)$ and also $c \to (b \to a) = c \wedge b \to a$.

S is clearly the smallest quotient of L containing a and since $S = L_j$, where $j(b) = (b \to a) \to a$, we have that $j(f)(b) = (b \to a) \to a$.

(2) *Quantales.* The case of quantales is more complicated. Let $\mathcal{Q}$ be a quantale. If S is a quotient of $\mathcal{Q}$ via a quantic nucleus, then not only is S closed under infs, but also if $f \in S$ and $x \in \mathcal{Q}$, then both $x \to_r f$ and $x \to_\ell f$ must be in S (see Proposition 1.7).

If $f \in \mathcal{Q}$ is fixed, then in general $(- \to_r f) \to_\ell f$ and $(- \to_\ell f) \to_r f$ are not quantic nuclei, without some additional assumptions on f. One can see [18] for details.

One case, which yields the same results as Example (1) is when f is *cyclic*, which means that $(- \to_r f) = (- \to_\ell f)$. In this case, we do not need the subscripts on residuation and the arguments from the frame case carry over. This notion of cyclicity is important in noncommutative linear logic [19, 22]. (It can also be generalized to quantaloids [21].)

(3) *Power monoids and automata.* Let M be a monoid and let $A \subseteq M$. The syntactic congruence of A, denoted by ϑ_A, is defined by $(s, t) \in \vartheta_A$ when $usv \in A$ iff $utv \in A$ for all $u,v \in M$. This is the largest congruence on M saturating A and one can define A to be recognizable iff ϑ_A is of finite index, that is M/ϑ_A is a finite quotient of M.

The syntactic nucleus on $\mathcal{P}(M)$ associated with A, $j(A)$, is defined by

$$j(A)(C) = \bigcup \{B \in \mathcal{P}(M) \mid B \to_r (X \to_r A) = C \to_r (X \to_r A) \text{ for all } X \subseteq M\}\ .$$

Recall that using Proposition 3.4, we have an adjunction between monoid congruences on M and nuclei on the quantale $\mathcal{P}(M)$. In order to justify our use of the terminology ‘syntactic nucleus’, it should be the case that we can recover the congruence ϑ_A from the nucleus $j(A)$. (Recall also that the notion of nucleus is more general than that of congruence and thus one would not necessarily expect to obtain $j(A)$ from ϑ_A.) Indeed, this is the case. Recall that in Section 3 we discussed the functor $G : \mathcal{N}(\mathcal{P}(M)) \to \mathrm{Con}(M)$ defined by $(s, t) \in G(j)$ iff $j(s) = j(t)$, which associates a congruence $G(j)$ on M to a quantic nucleus j on $\mathcal{P}(M)$.

Proposition 5.1. *Let M be a monoid and let $A \subseteq M$. Let ϑ_A denote the syntactic congruence of A on M and let $j(A)$ denote the syntactic nucleus on $\mathcal{P}(M)$. Then, $G(j(A)) = \vartheta_A$.*

Proof. Suppose $j(A)(s) = j(A)(t)$, where $s,t \in M$. We need to show that $usv \in A$ iff $utv \in A$. Letting $X = \{u\}$, then, $usv \in A$ iff $sv \in X \to_r A$ iff $v \in s \to_r (X \to_r A)$.

Since $j(A)(s) = j(A)(t)$, this is true iff $v \in t \rightarrow_r (X \rightarrow_r A)$ iff $utv \in A$, proving that $G(j(A)) \subseteq \vartheta_A$.

Conversely, suppose $(s, t) \in \vartheta_A$. Suppose given $X \subseteq M$, we have $B \rightarrow_r (X \rightarrow_r A) = s \rightarrow_r (X \rightarrow_r A)$. If $u \in X$, then given $v \in M$, $usv \in A$ iff $ubv \in A$ for all $b \in B$. Since $(s, t) \in \vartheta_A$, $usv \in A$ iff $utv \in A$ and hence $B \rightarrow_r (X \rightarrow_r A) = t \rightarrow_r (X \rightarrow_r A)$. From this it easily follows that $j(A)(s) = j(A)(t)$, finishing the proof that $G(j(A)) = \vartheta_A$. $\square$

6. Tree automata and the syntactic nucleus

In [3], Betti and Kasangian provided a category theoretical framework for considering tree automata. Let $\mathscr{A}$ be an algebraic theory in the sense of Lawvere [13]. The objects of $\mathscr{A}$ are represented by the nonnegative integers [0], [1], [2], . . . , $[n]$, . . . , with $[n]$ being the n-fold coproduct of [1]. A morphism $[1] \rightarrow [n]$ is interpreted as an n-ary operation of the theory and more generally a morphism $[m] \rightarrow [n]$ is an m-tuple of n-ary operations. Composition is given by substitution of operations.

Betti and Kasangian consider the quantaloid $\mathscr{P}(\mathscr{A})$ as their base bicategory and show that a tree automaton can be identified with a certain kind of $\mathscr{P}(\mathscr{A})$-enriched category E, equipped with an initial bimodule $I : E \mapsto [0]$ and a final bimodule $F : [1] \mapsto E$. The resulting composite $F \circ I : [1] \mapsto [0]$ is the behavior of the automaton E. Thus, behaviors arise as sets of terms (trees) $[1] \mapsto [0]$, which are called *forests* by Gecseg and Steinby [7] (and thus the free quantaloid $\mathscr{P}(\mathscr{A})$ comes into play). (For discussion of categories enriched in a bicategory and bimodules, see [2, 4]. For these notions in the context of quantaloids, see [20, 21].)

The ideas from [3] were developed further by Kasangian and Rosebrugh in [10], where they applied categorical ideas to decomposition of automata and in [11], where they indicate how the regular operations on recognizable sets in the monoid (automata) case can be realized in terms of glueing of bimodules.

There is discussion of the syntactic congruence for tree automata in [7], where they are limited by their universal algebraic, as opposed to categorical, perspective. The use of quantaloidal nuclei, which we advocate, first of all clearly indicates the way in which the monoid (automata) case is generalized; it is simply the passage from a one-object category to a category with many objects, thus providing a general framework including both examples in a natural way. Also, the construction is much more general in that it applies not just to forests (morphisms $[1] \rightarrow [0]$ in the quantaloid $\mathscr{P}(\mathscr{A})$), but to any morphism of $\mathscr{P}(\mathscr{A})$ and also the notion of nucleus is more general than that of congruence, as witnessed in Section 3. Finally, the residuation operations of a quantaloid are a very intrinsic part of their structure, being adjoint to composition, and hence an explanation of these constructions in terms of them is quite natural.

We shall now indicate exactly how the quantaloidal approach works with regard to forests and we shall show that the congruences discussed in [7] are recovered from our approach. We begin by presenting a detailed analysis of the residuation operations in $\mathscr{P}(\mathscr{A})$ relative to forests and how we can use them to arrive at an understanding of the syntactic nucleus in this setting.

Let $B : [1] \to [0]$ be a forest and let $X : [n] \to [0]$ be a morphism of $\mathscr{P}(\mathscr{A})$. Thus, X is a set of n-tuples $(x_1, x_2, \ldots, x_n)$ of terms $x_i : [1] \to [0]$.

$$\begin{tikzcd} {[1]} \arrow[r, "B"] \arrow[d, dashed, "X \to_r B"'] & {[0]} \\ {[n]} \arrow[ur, "X"'] & \end{tikzcd}$$

Then, $X \to_r B : [1] \to [n]$ is defined by

$$X \to_r B = \{ f \mid f \text{ is in an } n\text{-ary operation and } f(x_1, x_2, \ldots, x_n) \in B \text{ for all } (x_1, x_2, \ldots, x_n) \in X \} .$$

If $Y : [1] \to [n]$ is a set of n-ary operations and $H : [m] \to [n]$ is a set of m-tuples of n-ary operations, then $H \to_r Y : [1] \to [m]$ is defined by

$$H \to_r Y = \{ g \mid g \text{ is an } m\text{-ary operation and } g(h_1, h_2, \ldots, h_m) \in Y \text{ for all } (h_1, h_2, \ldots, h_m) \in H \} .$$

We have the following diagram:

$$\begin{tikzcd} {[1]} \arrow[r, "Y"] \arrow[d, dashed, "H \to_r Y"'] & {[n]} \\ {[m]} \arrow[ur, "H"'] & \end{tikzcd}$$

Thus, $g \in H \to_r (X \to_r B)$ iff g is an m-ary operation and given $(h_1, h_2, \ldots, h_m) \in H$ (where each h_i is n-ary), given $(x_1, x_2, \ldots, x_n) \in X$ (where each x_i is a term), we have

$$g(h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_m(x_1, x_2, \ldots, x_n)) \in B .$$

Therefore, given n-ary operations $k_1, k_2, \ldots, k_m$, we have that $(k_1, k_2, \ldots, k_m) \in j_B(H)$ iff for all $(h_1, h_2, \ldots, h_m) \in H$, for all $(x_1, x_2, \ldots, x_n) \in X$ and for all m-ary operations g,

$$g(h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_m(x_1, x_2, \ldots, x_n)) \in B$$

iff

$$g(k_1(x_1, x_2, \ldots, x_n), k_2(x_1, x_2, \ldots, x_n), \ldots, k_m(x_1, x_2, \ldots, x_n)) \in B .$$

If we look at the nucleus $\mathcal{J}_B$ instead, we arrive at the same end result, albeit via an alternate route, since we are utilizing left residuations instead of right ones. At first glance, we might not expect this since an algebraic theory $\mathscr{A}$ does not have a built-in duality (unlike monoids where one can consider the dual monoid), but we are in fact in the context of free quantaloids, and it is true that for most calculations involving right residuations, there is an analogous one using left residuations.

Now that we have an explicit calculation of $j_B(H)$, let us consider some special cases.

(1) Suppose $h,k : [1] \to [n]$ are n-ary operations. Then, using j_B we obtain $j_B(h) = j_B(k)$ iff for all unary operations f, for all n-tuples of terms (trees) $(x_1, x_2, \ldots, x_n)$:

$$f(h(x_1, x_2, \ldots, x_n)) \in B \quad \text{iff} \quad f(k(x_1, x_2, \ldots, x_n)) \in B .$$

(2) Let us simplify this first example to where $n = 0$, i.e. h and k are trees. If $\mathcal{T} = \mathscr{A}([1], [0])$ is the free $\mathscr{A}$-algebra of all $\mathscr{A}$-trees, then we obtain from j_B the congruence ϑ_B on $\mathcal{T}$ defined by $(h, k) \in \vartheta_B$ iff $j_B(h) = j_B(k)$ iff for all unary operations f,

$$f(h) \in B \quad \text{iff} \quad f(k) \in B .$$

This is precisely the congruence on $\mathcal{T}$ described in [7, pp. 89–90] which produces the minimal recognizer of the forest B.

With a little more effort, one can see that this also is the congruence described on pp. 94–95 of [7] and this congruence is of finite index iff the forest B is recognizable.

Thus, as witnessed in Proposition 5.1 and the above example, the same residuation calculations in the theory of quantaloids are behind the construction of the syntactic congruence in the theory of automata as well as the theory of tree automata.

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