A Characterization of Kempf Varieties by Means of Standard Monomials and the Geometric Consequences

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1. INTRODUCTION

Let G be a semi-simple, simply connected Chevalley group over a field K. Fix a maximal K-split torus T in G, a Borel subgroup $B \supset T$. Let W be the Weyl group of G relative to T. Let Q be a parabolic subgroup ($\supseteq B$) of classical type (cf. [12] or [13]), say $Q = \bigcap_{t=1}^{d} P_{k_t}$, where P_{k_t}, $1 \leqslant t \leqslant d$, is a maximal parabolic subgroup of classical type. For $w \in W/W_Q$, let $X(w)$ ($= \overline{BwQ} \pmod{Q}$ with the canonical reduced structure of a scheme) denote the Schubert variety in G/Q, associated to w. Given $\underline{m} = (m_{k_1}, m_{k_2}, ..., m_{k_d}) \in (\mathbf{Z}^+)^d$, the notion of "standard Young diagrams" on $X(w)$ of type $\underline{m}$ (or degree $\underline{m}$) was introduced in [13] (also see [12] and Section 2) and an explicit basis for $H^0(X(w), L)$ (where $L = \bigotimes_{t=1}^{d} L_{k_t}^{m_{k_t}}$, L_{k_t}, $1 \leqslant t \leqslant d$, being the ample generator of $\mathrm{Pic}(G/P_{k_t})$), indexed by standard Young diagrams of type $\underline{m}$, was constructed in [13] (see also [12]). If $G = SL_n$ and $Q = B$, then this notion, in fact, coincides with the classical Hodge–Young notion of standard Young diagrams on the flag manifold SL_n/B (cf. [8]). In [13], the classical notion was generalized to the notion of "weakly standard Young diagrams" on $X(w)$

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(cf. Section 2). It should be remarked that Young diagrams standard on $X(w)$ are weakly standard (cf. [13, Remark 12.2]). It turns out that the two notions coincide on $X(w) = G/B$ (or G/Q), G being of type A_n, B_n, or C_n. But for G of type D_n, one finds that even for the big cell $X(w) = G/B$, the two notions are different. In fact even for G of type D_4, if $\underline{m} = (1, 0, 1, 1)$, one finds that dim $H^0(G/B, L) = 350$, while there 385 weakly standard Young diagrams of type $(1, 0, 1, 1)$. Again, one observes that even though for SL_n/B, the two notions coincide, it is not so for Schubert varieties in SL_n/B. As an example, if one considers $X(w) \subseteq SL_3/B$, where $w = (312)(\in S_3)$, then one finds that if $\underline{m} = (1, 1)$, then dim $H^0(X(w), L) = 5$, while there are six weakly Standard Young diagrams of type $(1, 1)$ on $X(w)$.

Now, one natural question that arises is the following:

(*) Let G be of type A_n, B_n, or C_n. Is it possible to characterize the Schubert varieties in G/B (or G/Q) for which the two notions of Young diagrams being standard coincide? (here it is irrelevant to consider D_n, since, as remarked above, even for the big cell (G being of type D_n) the two notions are different).

One part of this paper is to answer (*). In [K], Kempf gives a desingularization of Schubert varieties in the Grassmannian by means of a certain class of smooth Schubert varieties in the flag manifold SL_n/B. These Schubert varieties were generalized in [10] to the case of G/B, G being a classical group and B a Borel subgroup, and were called Kempf varieties. These were defined in [10] by giving an explicit description of the corresponding w's. These may be geometrically described as follows: Let $P = P_{\hat{\alpha}_1}$ be the maximal parabolic subgroup of G obtained by omitting α_1 (note that P is of the same type as G). A Schubert variety X in G/B is a Kempf variety if and only if under the canonical morphism $G/B \to G/P$, the morphism $\pi \mid_X: X \to \text{Im } X$ is equidimensional and $X \cap P/B$ is a Kempf variety in lower rank. In Section 4, we prove (cf. Theorems 4.5 and 4.10) that G being of type A_n, B_n, or C_n, on a given Schubert variety $X(w)$ (in G/B), the two notions of Young diagrams being standard coincide if and only if $X(w)$ is a Kempf variety. This leads to the notion of a Kempf variety in G/Q (Q being a parabolic subgroup), namely, call $X(w)$ (in G/Q) a Kempf variety, if the two notions of Young diagrams being standard coincide (cf. Definitions 4.6 and 4.11).

The other part of this paper deals with proving the lexicographic shellability of certain partially ordered sets associated to Schubert varieties. Given a graded partially ordered set H (i.e., a finite partially ordered set which has an unique maximal (resp. minimal) element and in which all maximal chains have the same length) the notion of lexicographic shellability for H (cf. [1] or [2]) consists in labelling the maximal chains

in H by elements of some partially ordered set Ω such that certain properties hold (cf. Section 3 or [1] or [2]). Given $w \in W$ and a d-tuple,

$$(k) = (k_1, k_2, ..., k_d), \quad 1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant n \ (= \text{rank of } G),$$

let

$$Z_w^{(k)} = \{\tau \in W^{(k_t)}, \quad 1 \leqslant t \leqslant d/\tau \leqslant \text{projection of } w \text{ on } W/W_{k_t}\}$$

(here $W^{(k_t)}$ denotes the set of minimal representations (cf. Section 2) of W/W_{k_t}, where W_{k_t} is the Weyl group of the parabolic subgroup P_{k_t}). We then define a partial order on $Z_w^{(k)}$ (cf. Definition 2.7) and prove (cf. Theorems 3.12 and 3.14) that $Z_w^{(k)}$ is lexicographic shellable. As an important consequence we obtain the result that the multicones over Kempf varieties in G/Q (G being of type A_n, B_n or C_n) are Cohen–Macaulay (cf. Theorem 6.8). To prove this, one first deforms the ring $R_w = \bigoplus_{L \geqslant 0} H^0(X(w), L)$ (successively by flat deformations) using the explicit basis for R_w as given in [13] (see also Section 2) and one arrives at R_w^{def}, which (for $X(w)$ being a Kempf variety) turns out to be an algebra with straightening law (cf. [5] or [7]) on a partially ordered set $H_w^{(k)}$ (cf. Definition 6.4) (here (k) is given by $(k) = (k_1, ..., k_d)$, where $Q = \bigcap_{t=1}^d P_{k_t}$) such that the simplicial complex $\Delta(H_w^{(k)})$ (of chains in $H_w^{(k)}$) is a subdivision of $\Delta(Z_w^{(k)})$. Thus the problem of Cohen–Macaulayness for R_w is reduced to the problem of Cohen–Macaulayness for $K\{Z_w^{(k)}\}$ (cf. Section 2; given a finite partially ordered set H, $K\{H\}$ stands for $K[x_\alpha, \alpha \in H]/(x_\alpha x_\beta, \alpha$ and β not comparable) and it is a general result (cf. [5] or [7]) that a K-algebra B with straightening law over H is Cohen–Macaulay if $K\{H\}$ is). Now one concludes the Cohen–Macaulayness for $K\{Z_w^{(k)}\}$ using the lexicographic shellability property of $Z_w^{(k)}$.

For an arbitrary $X(w)$ (in G/Q), even though $Z_w^{(k)}$ turns out to be lexicographic shellable (cf. Theorems 3.12 and 3.14), this information does not help in concluding the Cohen–Macaulayness for R_w^{def}. In fact, in [9], we tackle this problem by studying the ideal theoretic unions and intersections in R_w^{def}.

The paper is organized as follows.

In Section 2, we deal with preliminaries, wherein we recall results concerning the Weyl group, reduced expressions for elements of W, the two notions of Standard Young diagrams on $X(w)$, algebras with straightening laws and lexicographic shellability. In Section 2, we also introduce the set $Z_w^{(k)}$, $w \in W$.

In Section 3, we prove the lexicographic shellability for $Z_w^{(k)}$ (cf. Theorems 3.12 and 3.14).

In Section 4, we prove that on a given $X(w)$ in G/B, the two notions of Young diagrams being standard coincide if and only if w is a Kempf

element (G being of type A_n, B_n, or C_n) (cf. Theorems 4.5 and 4.10). Then we introduce the definition of a Kempf variety in G/Q, where Q is any parabolic subgroup (cf. Definitions 4.6 and 4.11).

In Section 5, the deformation is carried out and R_w is deformed into R_w^{def} (by successive flat deformations).

In Section 6, we define the partially ordered set $H_w^{(k)}$ and we prove that when $X(w)$ is a Kempf variety in G/Q, R_w^{def} is an algebra with straightening law over $H_w^{(k)}$ ($(k) = (k_1, ..., k_d)$ being given by $Q = \bigcap_{t=1}^{d} P_{k_t}$), and using the results of Section 3, we conclude that R_w^{def} (and hence R_w) is Cohen–Macaulay.

2. PRELIMINARIES

Let G be a semi-simple, simply connected Chevalley group over K (K being the base field), T, a maximal K-split torus, and B a Borel subgroup containing T. Let

$$W = \text{Weyl group of } G \text{ relative to } T$$

$$R = \text{Root system of } G \text{ relative to } T$$

$$S = \text{System of simple roots of } R \text{ relative to } B.$$

Throughout the paper we shall order the simple roots as in [3].

We first start with recalling some generalities on W.

The set W^J. For a subset $J \subseteq S$, let W_J denote the subgroup of W generated by the reflections with respect to the simple roots belonging to J. Then the set of representatives of W/W_J given by $\{w \in W/l(ws_\alpha) > l(w), \alpha \in J\}$ shall be called the *set of minimal representatives of W/W_J* and shall be denoted by W^J. When $J = S - \{\alpha_i\}$, for some $\alpha_i \in S$, we shall denote also W^J by $W^{(i)}$ and W_J by W_i.

Reduced expressions. For $\alpha_i \in S$, let s_i denote the reflection with respect to α_i. Then, W is generated by $\{s_i\}_{1 \leq i \leq n}$, where $n = rk(G)$ (cf. [3]). For $w \in W$, an expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is called *reduced* if w cannot be expressed as product of s (simple) reflections with $s < r$. The number of reflections in a reduced expression for w is called the *length of w*, denoted by $l(w)$.

Schubert varieties. Given a parabolic subgroup $Q \supset B$ and $w \in W/W_Q$, let $X(w)$ denote $\overline{BwQ}$ (mod Q) with the canonical reduced scheme structure. The variety $X(w)$ is called the *Schubert variety* in G/Q, associated to $w \in W/W_Q$.

Partial order on W/W_Q.   Given w, $\tau \in W/W_Q$, call $w \geqslant \tau$, if $X(w) \supseteq X(\tau)$ (in G/Q).

It can be easily seen that given w, $\tau \in W/W_Q$, $w \geqslant \tau$ if and only if projection of w on W/W_P is $\geqslant$ projection of τ on W/W_P, for every maximal parabolic subgroup P containing Q.

Next, we want to interpret these results for $W = W(SL_n)$, $W(Sp_{2n})$, etc.

Weyl group of SL_n.   It is well known that Weyl group of SL_n may be identified with S_n. Ordering the set of simple roots as in [3], it can be easily seen that

$$W^{(d)} = \left\{ (i_1,...,\, i_d,\, i_{d+1},...,\, i_n) \in S_n \middle| \begin{array}{l} (1)\ \ 1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n \\ (2)\ \ (i_{d+1},...,\, i_n) = \complement\ \{i_1,...,\, i_d\} \\ \quad\ \text{in } \{1,...,\, n\} \text{ arranged in} \\ \quad\ \text{ascending order} \end{array} \right\}$$
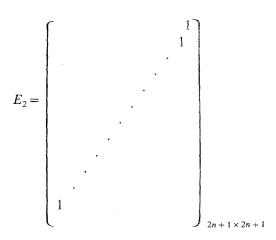
(observe that for a w in $W^{(d)}$ as above, $w(\alpha_j) > 0$, $j \neq d$; in fact, $w(e_j - e_{j+1}) = e_{i_j} - e_{i_{j+1}}$ (cf. [14], for instance)). In particular, we see that $W^{(d)}$ could be identified with $\{(i_1,...,\, i_d)/1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n\}$. Under this identification the partial order in $W^{(d)}$ is given by

$$(i_1,...,\, i_d) \geqslant (j_1,...,\, j_d)$$

if and only if $i_k \geqslant j_k$, $1 \leqslant k \leqslant d$. In particular, for two permutations $w = (a_1 \cdots a_n)$, $\tau = (b_1 \cdots b_n)$ in S_n, we have $w \geqslant \tau$ if and only if, for every d, $1 \leqslant d \leqslant n - 1$, the d-tuple $(a_1,...,\, a_d$, arranged in ascending order) $\geqslant$ the d-tuple $(b_1,...,\, b_d$ arranged in ascending order) (as elements of $W^{(d)}$). (Observe that for $1 \leqslant d \leqslant n - 1$, the d-tuple $(a_1,...,\, a_d)$ gives the projection of w on W/W_d under $W \to W/W_d$.)

Weyl groups of Sp_{2n} and SO_{2n+1}.   If $G = Sp_{2n}$ or SO_{2n+1}, then $W(G)$ can be identified with a subgroup of S_{2n} (resp. S_{2n+1}) as follows. Let

$$E_1 = \begin{bmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & 1 & & & & \\ & & -1 & & & & \\ & -1 & & & & & \\ \cdot & & & & & & \\ & \cdot & & & & & \\ -1 & & & & & & \end{bmatrix}_{2n \times 2n}$$

and

$$
E_2 = \begin{bmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ & \cdot & & & & & \\ 1 & & & & & & \end{bmatrix} \quad {\scriptstyle 2n+1 \times 2n+1}
$$

Then Sp_{2n} can be identified with $\{A \in SL_{2n} / {}^t A E_1 A = E_1\}$ and SO_{2n+1} can be identified with $\{A \in SL_{2n+1} / {}^t A E_2 A = E_2\}$ (cf. [14]). Also, $W(Sp_{2n})$ can be identified with

$$
\{(a_1 a_2 \cdots a_{2n}) \in S_{2n} / a_{2n+1-i} = 2n+1-a_i, \ 1 \leqslant i \leqslant n\}
$$

and $W(SO_{2n+1})$ can be identified with

$$
\left\{(a_1 \cdots a_{2n+1}) \in S_{2n+1} \left/ \begin{array}{l} (1) \ a_{n+1} = n+1 \\ (2) \ a_{2n+2-i} = 2n+2-a_i, \ 1 \leqslant i \leqslant n \end{array} \right. \right\} \quad \text{(cf. [14])}
$$

In particular we see immediately that $W(Sp_{2n}) \approx W(SO_{2n+1})$. If $w = (a_1 \cdots a_{2n}) \in W(Sp_{2n})$ then it is obvious that $\{a_{n+1}, ..., a_{2n}\}$ is uniquely determined by $\{a_1, ..., a_n\}$. So, we may as well denote w by just $(a_1 \cdots a_n)$. Further we have

PROPOSITION 2.1 (Cf. [15, Theorems 5A and 5BC]). *Under the above identification of $W(Sp_{2n})$ (resp. $W(SO_{2n+1})$) with a subgroup of $W(SL_{2n})$, the partial order on $W(Sp_{2n})$ (resp. $W(SO_{2n+1})$) is that induced from the partial order on $W(SL_{2n})$ (resp. $W(SL_{2n+1})$).*

Remark 2.1'. (a) From Proposition 2.1, we see that $W(SO_{2n+1})$ together with the canonical partial order, may be identified with $W(Sp_{2n})$ together with the natural partial order and in the following we shall use this identification. To be very precise, since our results (cf. the sets Z_w, $Z_w^{(k)}$, the lexicographic shellability of Z_w, $Z_w^{(k)}$, etc.) relate only to the partial order in W, proving them for $W(Sp_{2n})$ would imply proving them for $W(SO_{2n+1})$.

(b) If $\tau = (a_1,..., a_n)$, $\phi = (b_1,..., b_n)$ are two elements of $W(Sp_{2n})$, then $\tau \leqslant \phi$ if and only if, for every $1 \leqslant i \leqslant n$, $\{a_1,..., a_i,$ arranged in ascending order$\} \leqslant \{b_1,..., b_i,$ arranged in ascending order$\}$.

Remark 2.2. Under the above identification of $W = W(Sp_{2n})$ with

$$\{(a_1 a_2 \cdots a_{2n}) \in S_{2n}/a_{2n+1-i} = 2n + 1 - a_i, \ 1 \leqslant i \leqslant n\}$$

it can be easily seen that $W^{(d)}$ (for $1 \leqslant d \leqslant n$) can be identified with

$$\left\{(i_1,...,\ i_d)\middle/\begin{array}{l}(1)\ \ 1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant 2n \\ (2)\ \ \text{If } k \in \{i_1, i_2,..., i_d\}, \text{ then } 2n + 1 - k \notin \{i_1, i_2,..., i_d\}\end{array}\right\}$$

(this is because this set can be identified with

$$\{(i_1 \cdots i_d j_{d+1} \cdots j_n) \in W(Sp_{2n}), \qquad \text{where} \quad (i_1,..., i_d)$$

is as above and $(j_{d+1},..., j_n) =$ the first $(n-d)$ elements in $\complement (i_1,..., i_d,$ $i'_1,..., i'_d)$ in $(1, 2,..., 2n)$ arranged in ascending order (where for $1 \leqslant r \leqslant 2n$, r' denotes $2n + 1 - r$). In particular, we have $j_{d+2} < \cdots < j_n \leqslant n$. Now if $w \in$ the latter set, then we have

(1) $w(e_k - e_{k+1}) > 0$,   if $k + 1 \leqslant d$ or $k \geqslant d + 1$
(2) $w(2e_n)(= 2e_{j_n}) > 0$,   if $d < n$

(note that if $d < n$, then $j_n \leqslant n$ so that $w(2e_n) > 0$ (refer to [14, Sect. 3] for details); also note that if $W^J = W^{(d)}$, then $J = S - \{\alpha_d\}$, where $\alpha_t = e_t - e_{t+1}$, $t \leqslant n - 1$ and $\alpha_n = 2e_n$ (cf. [3] or [14])). Thus such w's belong to $W^{(d)}$ and conversely.

Next we want to recall some results from [10]. Let G be of type A_n, B_n, or C_n and that

$$\begin{array}{ll}u_i = s_n \cdots s_{i+1} s_i, & \text{if } G \text{ is of type } A_n \\ \ \ = s_i \cdots s_{n-1} s_n \cdots s_{i+1} s_i, & \text{if } G \text{ is of type } B_n \text{ or } C_n.\end{array}$$

Then recall (cf. [10, Propositions A.1, B.1, C.1, and A.18]).

PROPOSITION 2.3. *Let $w \in W$. Then a reduced expression for w is given by* $w = w_1 w_2 \cdots w_n$, *where w_i, $1 \leqslant i \leqslant n$ is either* Id *or a right (hand)-end segment of u_i.*

Remark 2.4. Given $w = w_1 w_2 \cdots w_n$ (as in Proposition 2.3 above), the projection of w on W/W_i, $1 \leqslant i \leqslant n$, is given by $w_1 w_2 \cdots w_i$ (obvious).

PROPOSITION 2.5   (i)   *Let G be of type A_n. Then for $1 \leqslant d \leqslant n$,*

$$W^{(d)} = \left\{ w \middle/ \begin{array}{ll} (1) \ w_j = \mathrm{Id}, & j > d \\ (2) \ l(w_j) \leqslant l(w_{j+1}), & 1 \leqslant j \leqslant d-1 \end{array} \right\}.$$

(ii)   *Let G be of type B_n or C_n. Then, for $1 \leqslant d \leqslant n$,*

$$W^{(d)} = \left\{ w \middle/ \begin{array}{l} (1) \ w_j = \mathrm{Id}, \quad j > d \\ (2) \ l(w_j) \leqslant l(w_{j+1}) \ (or) \ l(w_{j+1}) + 1 \quad \textit{according to whether} \\ \quad w_{j+1} < s_n \cdots s_{j+1} \ (or) \geqslant s_n \cdots s_{j+1}, \textit{respectively}, \\ \quad 1 \leqslant j \leqslant d-1. \end{array} \right\}$$

Proof.   (i)   Let $w \in W^{(d)}$, say $w = (a_1,..., a_d)$, where $1 \leqslant a_1 < a_2 < \cdots < a_d \leqslant n$. Then it is easily seen that $w = w_1 w_2 \cdots w_d$, where $w_k = s_{j_k} \cdots s_{k+1} s_k$ (where $j_k = a_k - 1$) if $a_k > k$ and $w_k = \mathrm{Id}$, if $a_k = k$, from which (i) follows.

(ii)   Let $w \in W^{(d)}$, say $w = (a_1 \cdots a_d)$ (as in Remark 2.2). Then $w = w_1 w_2 \cdots w_d$, where w_t, $1 \leqslant t \leqslant d$, is given as follows.

Let k be the largest integer $\leqslant d$ such that $a_k \leqslant n$. Then (as in (i)) it is easily seen that for $t \leqslant k$, $w_t = s_{a_t - 1} \cdots s_t$ or Id according to whether $a_t > t$ or $a_t = t$. Now let m be such that $k < m \leqslant d$, so that $a_m = r'$ for some $r \leqslant n$ (where $r' = 2n + 1 - r$). Then it can be easily seen that, if $a_{m-1} < (r+1)'$, then $w_m = s_r \cdots s_n \cdots s_m$; if $a_{m-1} = (r+1)'$, $a_{m-2} = (r+2)',..., a_{m-p} = (r+p)'$, $a_{m-p-1} < (r+p+1)'$, then $w_t = s_{r+p} \cdots s_n \cdots s_t$, $m - p \leqslant t \leqslant m$, from which (ii) follows.

Remark 2.6.   (i)   Let $w \in W^{(d)}$, say $w = w_1 w_2 \cdots w_d$ (as in Proposition 2.5 above, G being of type A_n, B_n, or C_n). Then for $t \leqslant d$, $w_1 w_2 \cdots w_t \in W^{(t)}$. (This following from Proposition 2.5.)

(ii)   Let $w \in W^{(d)}$ and assume $W = W(SL_n)$. Let $w = w_1 w_2 \cdots w_d$, where $w_r = s_{i_r} \cdots s_{r+1} s_r$ or Id, $1 \leqslant r \leqslant d$. Then as a d-tuple, w is given by $w = (j_1 j_2 \cdots j_d)$, where

$$j_r = i_r + 1, \quad \text{if} \quad w_r \neq \mathrm{Id}$$

$$= r, \quad \text{if} \quad w_r = \mathrm{Id}.$$

(A similar result can be stated for $W(Sp_{2n})$ also.)

The sets $Z^{(k)}$ and $Z_w^{(k)}$.   Let G be of type A_n, B_n, or C_n and let the simple roots (or the maximal parabolic subgroups) be ordered as in [3]. Given a d-tuple, $(k) = (k_1,..., k_d)$, $1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant n$, let

$$Z^{(k)} = \bigcup_{t=1}^{d} W^{(k_t)}.$$

For $w \in W$, let

$$Z_w^{(k)} = \{\tau \in W^{(k_t)}, \ 1 \leqslant t \leqslant d/\tau \leqslant \text{projection of } w \text{ on } W/W_{k_t}$$
$$\text{under } W \to W/W_{k_t}\}.$$

When $(k) = (1, 2,..., n)$, we shall denote $Z^{(k)}$ (resp. $Z_w^{(k)}$) by just Z (resp. Z_w).

DEFINITION 2.7. Given τ, $\phi \in Z$, say $\tau = (m_1,..., m_r)$, $\phi = (t_1,..., t_s)$, call $\tau \geqslant \phi$, if $r \leqslant s$ and $m_l \geqslant t_l$, $1 \leqslant l \leqslant r$ (cf. [8]). This obviously defines a partial order on Z. In terms of reduced expressions for τ and ϕ (cf. Proposition 2.5), if $\tau = \tau_1 \cdots \tau_r$ and $\phi = \phi_1 \cdots \phi_s$, then $\tau \geqslant \phi$ (in Z) if and only if $\tau_1 \cdots \tau_r \geqslant \phi_1 \cdots \phi_r$ (as elements of W). In particular, if $r = s$, this partial order is just the (canonical) partial order in $W^{(r)}$. Note that this partial order on Z gives rise to a partial order on $Z^{(k)}$ and $Z_w^{(k)}$ (since $Z_w^{(k)} \subseteq Z^{(k)} \subseteq Z$).

Standard monomials on Schubert varieties (cf. [12] or [13]). Let Q be a parabolic subgroup of classical type in G (cf. [13], for definition of classical type parabolic subgroups). Further, let $Q = P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_r}$ $(1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n)$, where P_{k_t} is a maximal parabolic subgroup of classical type. (Recall (cf. [13]) that if G is a classical group, then every maximal parabolic (and hence every parabolic) subgroup of G is of classical type.)

DEFINITION 2.8 (cf. [12] or [13]). Given $\underline{m} = (m_{k_1}, m_{k_2},..., m_{k_r}) \in (\mathbb{Z}^+)^r$, by a *Young diagram of type $\underline{m}$ or multidegree $\underline{m}$ on G/Q (or W/W_Q)*, we mean a pair (θ, δ), where $\theta = (\theta_{ij})$, $\delta = (\delta_{ij})$ and $(\theta_{ij}, \delta_{ij})$, $i \in \{k_1,..., k_r\}$, $1 \leqslant j \leqslant m_i$, is an admissible pair in W/W_i (cf. [12] or [13] for definition of admissible pairs). (If $m_t = 0$ for any $t \in \{k_1,..., k_r\}$, then the family θ_{t_-}, δ_{t_-} is assumed to be empty).

DEFINITION 2.9. A Young diagram (θ, δ) is said to be a *Young diagram on $X(w)$ (or just w)*, where $w \in W/W_Q$ if

$$w^{(i)} \geqslant \theta_{ij}, \ i \in \{k_1,..., k_r\}, \ 1 \leqslant j \leqslant m_i,$$

where $w^{(i)}$ is the projection of w on W/W_i under $W/W_Q \to W/W_i$.

DEFINITION 2.10. A Young diagram (θ, δ) is said to be *weakly standard* if

$$\theta_{k_1 1} \geqslant \delta_{k_1 1} \geqslant \theta_{k_1 2} \geqslant \cdots \geqslant \theta_{k_1 m_1} \geqslant \delta_{k_1 m_1} \geqslant \theta_{k_2 1} \geqslant \delta_{k_2 1} \geqslant \cdots.$$

(as elements of Z (cf. Definition 2.7)).

DEFINITION 2.11.   A Young diagram (θ, δ) is said to be *weakly standard on w*, $w \in W/W_Q$, if (θ, δ) is weakly standard and (θ, δ) is a Young diagram on w.

DEFINITION 2.12.   A Young diagram (θ, δ) is said to be *standard* if there exists a pair (α, β), which we call a *defining pair for* (θ, δ) such that

(1) $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$, α_{ij}, $\beta_{ij} \in W/W_Q$, $i \in \{k_1, \ldots, k_r\}$, $1 \leqslant j \leqslant m_i$.

(2)   Each α_{ij} (resp. β_{ij}) is a lift for θ_{ij} (resp. δ_{ij}) under $W/W_Q \rightarrow W/W_i$.

(3) $\alpha_{k_1 1} \geqslant \beta_{k_1 1} \geqslant \alpha_{k_1 2} \geqslant \cdots \geqslant \alpha_{k_1 m_1} \geqslant \beta_{k_1 m_1} \geqslant \alpha_{k_2 1} \geqslant \cdots$ (in W/W_Q).

DEFINITION 2.13.   A Young diagram (θ, δ) is said to be *standard on w*, $w \in W/W_Q$, if there exists a defining pair (α, β) for (θ, δ) with $w \geqslant \alpha_{k_1 1}$ (in W/W_Q).

THEOREM 2.14 (cf. [12] or [13]).   *Let* $L = L_{k_1}^{m_1} \otimes L_{k_2}^{m_2} \otimes \cdots \otimes L_{k_r}^{m_r}$, *where* L_{k_t}, $1 \leqslant t \leqslant r$, *is the ample generator of* $\mathrm{Pic}(G/P_{k_t})$. *Given a Young diagram* (θ, δ) *standard on w*, $w \in W/W_Q$, *one can associate an element* $p_{\theta, \delta} \in H^0(X(w), L)$ ($p_{\theta, \delta}$ *will be called a* standard monomial *on* $X(w)$). *Further, the standard monomials of* $\deg \underline{m}$ *on* $X(w)$ *form a K-basis for* $H^0(X(w), L)$.

Remark 2.15.   Recall (cf. [13]) the following:

(a)   A Young diagram standard on w is weakly standard on w.

(b)   For $X(w) = G/Q$, G being of type A_n, B_n, C_n, the two notions of being standard coincide. Also, for $G = SL_n$, the weakly standard Young diagrams on G/B are nothing but the classical Hodge–Young standard diagrams (cf. [8]).

(c)   For G of type D_n, even for the big cell G/B, the two notions are different. For instance, if G is of type D_4, one finds that for $\underline{m} = (1, 0, 1, 1)$, $\dim(H^0(G/B, L)) = 350$, while there are 385 weakly standard Young diagrams of type $(1, 0, 1, 1)$ on G/B.

(d)   Although for SL_n/B, the two notions of being standard coincide, the same is not true for Schubert varieties in SL_n/B. For instance, if one considers $X(w)$ (where $w = (312)$) in SL_3/B, then one finds that if $\underline{m} = (1, 1)$, then $\dim H^0(X(w), L) = 5$, while there are six weakly standard Young diagrams of type $(1, 1)$ on $X(w)$. In fact one of the main results in this paper is the result (cf. Theorems 4.5 and 4.10) that G being of type A_n, B_n, or C_n, on a given Schubert variety $X(w)$ in G/B, the two notions of being standard coincide if and only if $X(w)$ is a Kempf variety.

Remark 2.16.   Let (θ, δ) be a Young diagram on w. Further let (θ, δ) be standard on G/Q. Let (α^-, β^-) denote the (absolute) minimal defining pair

for (θ, δ) (cf. [13, Corollary 11.2']). Then it can be easily seen that (θ, δ) is standard on w if and only if $w \geqslant \alpha_{11}^-$ (note that $X(\alpha_{11}^-)$ is the smallest Schubert variety on which (θ, δ) is standard).

Algebras with straightening laws (cf. [5] or [7]). Given a finite partially ordered set H and a base ring R, an R-algebra B is said to be *an algebra with straightening laws over H*, if

(1) B is $\mathbb{Z}^+$-graded with $B_0 = R$.

(2) B has a set of algebra generators $\{x_\alpha\}_{\alpha \in H}$

(3) The monomials $x_\alpha x_\beta x_\gamma \cdots$, where $\alpha \geqslant \beta \geqslant \gamma \geqslant \cdots$ (called the *standard monomials*) form an R-basis for B.

(4) Given a non-zero, non-standard monomial $x_{\tau_1} x_{\tau_2} \cdots x_{\tau_r}$, let

$$(*) \quad x_{\tau_1} x_{\tau_2} \cdots x_{\tau_r} = \sum_{(\alpha)} (\ ) x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_r}$$

be the expression for $x_{\tau_1} x_{\tau_2} \cdots x_{\tau_r}$ as sum of standard monomials. Then, for $\sigma \in S_r$ and for each (α) on the RHS of $(*)$, $(\alpha_1, \alpha_2, ..., \alpha_r)$ is lexicographically $\geqslant (\sigma(\tau_1), \sigma(\tau_2), ..., \sigma(\tau_r))$.

Whenever we are given an R-algebra B which is an algebra with straightening laws, then using the relations $(*)$ in (4) (referred to as *straightening relations*), we can reduce B to the discrete algebra

$$R\{H\} = R[x_\alpha, \alpha \in H]/(x_\alpha x_\beta, \alpha \text{ and } \beta \text{ not comparable})$$

by successive flat deformations (cf. [5] or [7]), so that it may be concluded that algebraic properties like B are normal, Cohen–Macaulay, etc., by knowing the same for $R\{H\}$. For instance, if the base ring is Cohen–Macaulay and H has some nice properties (like shellability), one may conclude that B is Cohen–Macaulay.

3. LEXICOGRAPHIC SHELLABILITY

Given a finite partially ordered set H which is graded (i.e., which has an unique maximal and an unique minimal element and in which all maximal chains (i.e., maximal totally ordered subsets of H) have the same length) the notion of lexicographic shellability for H may be defined as follows (cf. [1] or [2]). The lexicographic shellability consists in labelling the maximal chains $\underline{m}$ in H, say $\lambda(\underline{m}) = (\lambda_1(\underline{m}), \lambda_2(\underline{m}), ..., \lambda_r(\underline{m}))$ (where r is the length of any maximal chain in H) by elements $\lambda_i(\underline{m})$ belonging to some partially ordered set Ω in such a way that the following two conditions hold:

(L1) If two maximal chains $\underline{m}$ and $\underline{m}'$ coincide along their first d edges, where d is an integer, $1 \leqslant d \leqslant r$, then $\lambda_i(\underline{m}) = \lambda_i(\underline{m}')$, $1 \leqslant i \leqslant d$.
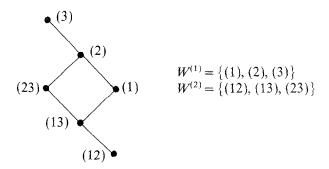
(L2)   For any interval $[x, y] (= \{\tau \in H \mid x \leqslant \tau \leqslant y\})$, together with a chain $\underline{c}$, going down from the unique maximal element in H to y, there is an unique maximal chain $\underline{m}_0$ in $[x, y]$ whose label is increasing, namely $\lambda_1(\underline{m}_0) \leqslant \lambda_2(\underline{m}_0) \leqslant \cdots \leqslant \lambda_t(\underline{m}_0)$ (where $t = \text{length of } [x, y]$) (here the label for $\underline{m}_0$ is induced from the maximal chain of H consisting of $\underline{c}$, followed by $\underline{m}_0$, followed by an arbitrary path from x to the unique minimal element of H) and if $\underline{m}$ is any other maximal chain in $[x, y]$, then $\lambda(\underline{m}_0)$ is lexicographically $< \lambda(\underline{m})$.

Our main result in this section is that the set $Z_w^{(k)}$ (cf. Section 2) is lexicographic shellable. We first prove this result for[1] the case $G = SL_n$, $(k) = (1, 2, ..., n-1)$, $w = $ the unique element of maximal length in W. Next, we prove it for $G = Sp_{2n}$, $(k) = (1, 2, ..., n)$ and $w = $ unique element of maximal length in W. Then we prove it for a general w and a general (k); the proof in the general case is quite analogous (but a little messy) to the case of $w = $ the unique element of maximal length, the proof in the latter case being very explicit.

THEOREM 3.1.   *Let $G = SL_n$. Then the set $Z = \bigcup_{i=1}^{n-1} W^{(i)}$ is lexicographic shellable (for the partial order on Z, as defined in Definition 2.7).*

Proof.   Before proceeding to the proof of the above theorem, we shall show that Z looks like in the case of SL_3. We shall exhibit Z by means of vertices and edges; the vertices are just the elements of Z and an edge is obtained by joining two vertices ϕ, τ such that ϕ *covers* τ, i.e., if $\phi > \tau$ and there does not exist $\lambda \in Z$ such that $\phi > \lambda > \tau$.

Z *(in the case of SL_3).*

$$W^{(1)} = \{(1), (2), (3)\}$$
$$W^{(2)} = \{(12), (13), (23)\}$$

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[1] As was pointed out by K. Baclowski, the proof of the lexicographic shellability in this case may be seen as a consequence of the corresponding poset being a distributive lattice. But, nevertheless, we do give the details of the proof in this case, since the philosophy of the proof (for other cases) is explicit in this case.

Covers in Z. Following [2], we shall denote a cover by $\phi \to \tau$ (a cover will also be called an *edge*). A cover $\phi \to \tau$ in Z is either a cover in $W^{(d)}$ for some d (namely, when ϕ and τ both $\in W^{(d)}$) or of the form $\phi = (a_1 \cdots a_d) \to \tau = (a_1 \cdots a_d n)$, where $1 \leqslant a_1 < a_2 < \cdots < a_d < n$. If $\phi \to \tau$ is a cover in Z and if $\phi, \tau \in W^{(d)}$ for some d, $1 \leqslant d \leqslant n-1$, then this is equivalent to the condition that $\phi \geqslant \tau$ (in $W^{(d)}$) and $l(\tau) = l(\phi) - 1$. (where for any $\theta \in W^{(d)}$, $1 \leqslant d \leqslant n-1$, $l(\theta)$ denotes the length of θ considered as an element of W. Observe that $l(\theta) = \dim X(\theta)$ (where $X(\theta) \subseteq G/P_d$)). Now we embed Z inside $W(SL_{n+1})$ as follows.

Define $i: Z \to W(SL_{n+1})$ by sending

$$(a_1 \cdots a_d) \to (a_1 \cdots a_d(n+1) \, b_1 \cdots b_r)$$

where $(b_1, ..., b_r) = \complement \, (a_1, ..., a_d)$ in $(1, ..., n)$ arranged in descending order. In terms of reduced expressions, this could be described as follows. If $w = w_1 w_2 \cdots w_d \in W^{(d)}$, then $i(w) = w_1 w_2 \cdots w_d u_{d+1} \cdots u_n$, where u_i, $1 \leqslant i \leqslant n$, is given by $u_i = s_n \cdots s_{i+1} s_i$ (in $W(SL_{n+1})$) (cf. Section 2). The fact that the partially ordered set Z is bounded is clear, the unique maximal (resp. minimal) element in Z being (n) (resp. $(123 \cdots n-1)$. The fact that any two maximal chains in Z have the same length can be easily seen. For instance, this fact may be concluded from the fact that a cover in Z continues to be a cover in $i(Z)$ and the fact that any two maximal chains in the interval $[i((12 \cdots n-1)), i((n))]$ have the same length. ("A cover $\phi \to \tau$ in Z continues to be a cover in $i(Z)$" is obvious if $\phi, \tau \in W^{(d)}$ for some d, $1 \leqslant d \leqslant n-1$; if $\phi \in W^{(d)}$ and $\tau \in W^{(d+1)}$, say $\phi = (a_1 \cdots a_d)$, $\tau = (a_1 \cdots a_d n)$, then $i(\phi) = i(\tau) s_{d+1}$, where s_{d+1} is the transposition $(d+1, \, d+2)$ (in S_{n+1}) and obviously $i(\phi)$ covers $i(\tau)$ in $i(Z)$.) Now, proving lexicographic shellability for Z is equivalent to proving the same for $i(Z)$.

Lexicographic shellability for $i(Z)$. Let w_0 be the unique element of maximal length in $W(SL_{n+1})$ and let us take a reduced expression for w_0 as $w_0 = u_1 u_2 \cdots u_n$. Now if v_0 denotes $i((n))$, then it is clear that $v_0 = s_n w_0$, so that a reduced expression for v_0 may be taken to be $v_0 = (s_{n-1} \cdots s_2 s_1) u_2 u_3 \cdots u_n$. Starting with this reduced expression for v_0, the maximal chains in $i(Z)$ shall be labelled by the rule prescribed in [2] so that (L1) is easily seen to be satisfied.

Verification of (L2). Let $([\tau, \phi], \underline{c})$ be a rooted interval in $i(Z)$. Further let $\tau = i(\tau_0)$, $\phi = i(\phi_0)$, where $\tau_0 = (a_1 \cdots a_j)$, $\phi_0 = (b_1 \cdots b_q)$ for some $q \leqslant j \leqslant n-1$ and $a_t \leqslant b_t$, $1 \leqslant t \leqslant q$ (in view of the partial order in Z, cf. Section 2). Then we have $\tau = \tau_1 \cdots \tau_j u_{j+1} \cdots u_n$, $\phi = \phi_1 \cdots \phi_q u_{q+1} \cdots u_n$, where

$$\tau_k = \mathrm{Id}(\text{namely if } a_k = k) \text{ or}$$

$$\tau_k = s_{a_k-1} \cdots s_{k+1} s_k \ (\text{if } a_k > k), \ 1 \leqslant k \leqslant j$$

and

$$\phi_k = \mathrm{Id}(\text{namely if } b_k = k) \text{ or}$$

$$\phi_k = s_{b_k - 1} \cdots s_{k+1} s_k \text{ if } b_k > k), \ 1 \leqslant k \leqslant q.$$

(cf. Remark 2.6(ii)). In particular we have $\tau_k \leqslant \phi_k$, $1 \leqslant k \leqslant n$; in fact τ_k is a right-end segment of ϕ_k, $1 \leqslant k \leqslant n$. Let then $\phi_r = s_{i_r}^{(r)} \cdots s_2^{(r)} s_1^{(r)} \tau_r$, $1 \leqslant r \leqslant j$.

Now going down from v_0 to ϕ through the elements of $i(Z)$ (in particular, through the elements in $\underline{c}$) it is easily seen that we end up with the above reduced expression for ϕ. The required maximal chain in $[\tau, \phi]$ with increasing label is given by

$$\phi \xrightarrow{s_{i_1}^{(1)}} \tau_1^{(1)} \xrightarrow{s_{i_1-1}^{(1)}} \tau_2^{(1)} \longrightarrow \cdots \xrightarrow{s_1^{(1)}} \tau_{i_1}^{(1)}$$

$$\xrightarrow{s_{i_2}^{(2)}} \tau_1^{(2)} \cdots \xrightarrow{s_1^{(j)}} \tau$$

where by $w_1 \xrightarrow{s_i} w_2$ we mean w_1 covers w_2 and $w_1 = s_i w_2$. (Here one should observe that all $\tau_{\{\}} \in Z$, in view of Proposition 2.5. Also, following [2], and edge $\theta \xrightarrow{s_x} \rho$ is labelled by the integer m, where m denotes the position of s_x (in v_0) that is dropped out in getting ρ from θ.) The fact that the above chain has increasing label is obvious. Now suppose k denotes the least integer such that $\tau_k < \phi_k$ (observe that $k \leqslant q + 1$), then in any other maximal chain, the first reflection that is dropped out occurs in ϕ_r for some $r \geqslant k$. If $r > k$, then at some point, one has to work with dropping out the reflections in ϕ_k and thus the corresponding label is not increasing and the corresponding label is (clearly) lexicographically $>$ the above increasing label. Suppose $r = k$ (we may assume, the corresponding chain has the first edge to be different from the first edge of the above chain with increasing label, by using induction on $l([\tau, \phi])$, the proof for the starting point of induction, namely $l(\tau) = l(\phi) - 1$ (note that $l([\tau, \phi]) = l(\phi) - l(\tau)$) being trivial); then it can be easily seen that the element corresponding to the first edge has length $< l(\phi) - 1$ and hence is not covered by ϕ. To make it very precise, if $\phi_0 = (b_1 \cdots b_k \cdots b_q)$ and $\tau_0 = (a_1 \cdots a_j)$, where $a_t = b_t$, $1 \leqslant t \leqslant k - 1$, $a_k < b_k$ (in the case $k \leqslant q$) so that $\phi_k = s_{b_k - 1} \cdots s_k$, $\tau_k = s_{a_k - 1} \cdots s_k$ or Id (depending on whether $a_k > k$ or $a_k = k$), the corresponding element is obtained by dropping out s_m in ϕ_k for some m, $k \leqslant m < b_k - 1$ and is not covered by ϕ (observe (see Remark 3.2 below) that a cover $\theta \to \rho$ in $W^{(r)}$, $1 \leqslant r \leqslant n - 1$, looks like $\theta = (c_1 \cdots m \cdots c_r)$, $\rho = (c_1 \cdots (m-1) \cdots c_r)$ for some $m \leqslant n$. If $k = q + 1$, the resulting element looks like $(b_1 \cdots b_q x)$, where $x < n$ (observe that $b_{q+1} = n + 1$) and hence is not covered by ϕ. Thus the possibility that $r = k$ does not exist.

This completes the verification of (L2) and hence the proof of Theorem 3.1.

Remark 3.2. If $\phi_0 = (b_1 \cdots b_q)$ and $\tau_0 = (a_1 \cdots a_j)$ then the unique maximal chain with increasing label is obtained by first reducing b_1 one by one till we arrive at a_1, then reducing b_2 to a_2, etc., and b_i to a_i (recall (cf. [15], for example) that a cover $\theta \to \rho$ in $W(QL_n)$ is given by $\theta = (c_1 \cdots c_k \cdots c_l \cdots c_n)$, $\rho = (c_1 \cdots c_l \cdots c_k \cdots c_n)$, where $c_k > c_l$, for every j, $k < j < l$, c_j is either $> c_k$ or $< c_l$ and $\rho = \theta(k, l)$. Hence a cover $\theta \to \rho$ in $W^{(i)}$ for some $i \leqslant n - 1$ is given by $\theta = (c_1 \cdots c_k \cdots c_i)$, $\rho = (c_1 \cdots c_l \cdots c_i)$, where $c_k > c_l$, $l > i$ ($l \leqslant i$ would imply $c_k \not> c_l$) and $\rho = \theta(k, l)$). (Note that θ as an element of W is given by $\theta = (c_1 \cdots c_i c_{i+1} \cdots c_n)$, where $\{c_{i+1}, \ldots, c_n\} = C(c_1, \ldots, c_i)$ in $(1, \ldots, n)$ thrown in ascending order). Now, if $c_k = r$ and $c_l = p$, where $p < r < n$, then any t, $p < t < r$, $t \notin \{c_{i+1}, \ldots, c_{l-1}\}$ (since $\phi \in W^{(i)}$, $c_j < c_m$ for $j < m < i$ or $i < j < m$), $t \notin \{c_{l+1}, \ldots, c_n\}$ (since $\tau \in W^{(i)}$ and the lth entry in τ is r), $t \notin \{c_{k+1}, \ldots, c_n\}$ (since $c_k = r$ and $\theta \in W^{(i)}$) $t \notin \{c_1, \ldots, c_{k-1}\}$ (since $\tau \in W^{(i)}$ and the kth entry in τ is p). Thus we obtain any t, $p < t < r$, $t \notin \{c_1, \ldots, c_n\}$. This is impossible unless $p = r - 1$. Thus a cover $\theta \to \rho$ in $W^{(i)}$ looks like $\theta = (a_1 \cdots r \cdots a_i)$ and $\rho = (a_1 \cdots r - 1 \cdots a_i)$ for some $r \leqslant n$ (r being replaced by $r - 1$ in θ)). Returning to the description of the unique maximal chain in $[\tau_0, \phi_0]$ with increasing label, as described above one first reduces b_1 to a_1 (one by one), then b_2 to a_2, etc., b_i to a_i; then $(a_1 \cdots a_i)$ is allowed by $(a_1 \cdots a_i n)$ ($\in W^{(i+1)}$) (observe that $a_i < n$, since $a_i < a_{i+1} < \cdots < a_j \leqslant n$. Also observe that $(a_1 \cdots a_i) \to (a_1 \cdots a_i n)$ is a cover in Z). Then $(a_1 \cdots a_i n)$ is followed by $(a_1 \cdots a_i n - 1)$, $(a_1 \cdots a_i n - 2), \ldots,$ $(a_1 \cdots a_i a_{i+1})$, which in turn is followed by $(a_1 \cdots a_i n - 1)$, $(a_1 \cdots a_i n - 2), \ldots,$ $(a_1 \cdots a_i a_{i+1})$, which in turn is followed by $(a_1 \cdots a_i a_{i+1} n)$ and so on.

Remark 3.3. If $\phi \to \tau$ is a cover in $i(Z)$, say $\phi = i(\phi_0)$, $\tau = i(\tau_0)$, where $\phi = (b_1 \cdots b_q)$ for some $q \leqslant n - 1$, then the label for the cover $\phi \to \tau$ depends only on $(b_1 \cdots b_q)$ and not on any maximal chain $\underline{c}$ of which $\phi \to \tau$ is an edge. In other words, the covers $\phi \to \tau$ in $i(Z)$ have been given a global labelling.

Next we want to prove the lexicographic shellability for Z, in the case of $W = W(Sp_{2n})$ or $W(SO_{2n+1})$. We start toward proving this with the discussion of covers in W, $W^{(i)}$ and Z.

Covers in $W(G)$ (G being of type B_n or C_n). For the results on covers in $W(G)$, one may refer [15] (cf. Corollaries 5ABCD). Since the terminology in [15] is different from our terminology, we want to discuss (and state the results on) the covers in $W(G)$ in our terminology. We shall carry out the discussion for G being of type C_n. The discussion for G of type B_n is completely analogous. Let then $\phi \to \tau$ be a cover in W, say $\tau = \phi s_\alpha$, where α is a positive root and $l(\tau) = l(\phi) - 1$ (cf. [6, Proposition 5]). Now the positive roots in G are given by (cf. [3]) $e_k - e_l$, $e_k + e_l$, $1 \leqslant k < l \leqslant n$, and $2e_k$, $1 \leqslant k \leqslant n$; the corresponding reflections are given by

(cf. [14]) $s_\alpha = (k, l)(k', l')$, $(k, l')(l, k')$, (k, k'), respectively (here, for any $1 \leqslant j \leqslant n$, j' denotes $2n + 1 - j$ and for any $1 \leqslant j < r \leqslant 2n$, (j, r) denotes the transposition (in S_{2n}) of the jth and the rth entries). Since $\phi \to \tau$ is a cover, we have $l_{SL}(\tau) = l_{SL}(\phi) + m(\phi) - 2 - m(\tau)$ (recall (cf. [14, Proposition 3.1]) that for any $w \in W(Sp_{2n})$, $l_{Sp}(w) = \frac{1}{2}(l_{SL}(w) + m(w))$, where $m(w) = \#\{1 \leqslant i \leqslant n/$ if $w = (d_1 \cdots d_{2n})$, then $d_i > n\}$). Let $\phi = (a_1 \cdots a_n)$ (recall (cf. Section 2), any $w \in W(Sp_n)$ may be denoted by the first n entries in the permutation representing w). Now, if $\alpha = e_k - e_l$, then $m(\phi) = m(\tau)$ and $l_{SL}(\tau) = l_{SL}(\phi) - 2$. Further, we have $a_k > a_l$ (which in turn implies $a'_k < a'_l$) and for any j, $k < j < l$, a_j is either $> a_k$ or $< a_l$.

If $\alpha = e_k + e_l$, then we have $m(\phi) = m(\tau)$, $l_{SL}(\tau) = l_{SL}(\phi) - 2$; further, precisely one of $\{a_k, a_l\}$ is $> n$, $a_k > a'_l$ (and hence $a_l > a'_k$). Further, for any j, $k < j < l$, a_j is either $> a_k$ or $< a'_l$; if $l < j \leqslant n$, then a_j is either $> a_k$ or $< a'_l$ and a_j is either $> a_l$ or $< a'_k$ (these follow in view of the fact that in $W(SL_{2n})$, $\phi \to \phi(k, l')$ and $\phi(k, l') \to \phi(k, l')(l, k')$ are covers). These conditions may be restated as follows (cf. [15]). In the case $\alpha = e_k + e_l$, we have

(1)   for every j, $k < j < l$, $a_j > a_k$ or $< a'_l$, and

(2)   for every j, $l < j < n$, $|a_j|$ is either $> \max(a_k, a_l)$ or $< \max(a'_k, a'_l)$ (where for any r, $1 \leqslant r \leqslant 2n$, by $|r|$, we mean r or r' according as $r > n$ or $\leqslant n$).

Finally, if $\alpha = 2e_k$, then $m(\tau) = m(\phi) - 1$, $l_{SL}(\tau) = l_{SL}(\phi) - 1$. Further, we have, $a_k > a'_k$ (in particular, $a_k > n$) and for every j, $k < j \leqslant n$, a_j is either $> a_k$ or $< a'_k$.

We shall now apply these results to obtain the covers in $W^{(q)}$.

Covers in $W^{(q)}$.   Let $\phi \to \tau$ be a cover in $W^{(q)}$ for some q, $1 \leqslant q \leqslant n$. Further let $\phi = (a_1 \cdots a_q)$ (so that $a_1 < a_2 < \cdots < a_q$). Then, as an element in W, ϕ is given by $\phi = (a_1 \cdots a_n)$, where $\{a_{q+1}, ..., a_n\}$ = the set of first $n - q$ elements in $C\{a_1, ..., a_q, a'_1, ..., a'_q\}$ in $\{1, 2, ..., 2n\}$ arranged in ascending order (since $\phi \in W^{(q)}$). In particular $a_j \leqslant n$, $q < j \leqslant n$). Let $\tau = \phi s_\alpha$. Then the cover $\phi \to \tau$ is one of the following four types of covers.

(1)   Let $\alpha = e_k - e_l$. First, we have $k \leqslant q < l$, necessarily (since $\phi \in W^{(q)}$, $a_k \not> a_l$, if $k > q$ or $l \leqslant q$). Also, since $\tau = (a_1 \cdots a_l \cdots a_q a_{q+1} \cdots a_k \cdots a_n) \in W^{(q)}$, we have $a_k \leqslant n$, necessarily. Let then $a_k = r$ and $a_l = p$, where $p < r \leqslant n$. Also, since $\tau \in W^{(q)}$, we should have $a_{k-1} < p$ and $r < a_{l+1}$. This is equivalent to the condition that for every t, $p < t < r$, $t' \in \{a_{k+1}, ..., a_q\}$.

(2)   Let $\alpha = e_k + e_l$. Further let $l \leqslant q$. Since $\phi \to \phi s_\alpha$ is a cover, we have (from our discussion on covers in W), precisely one of $\{a_k, a_l\}$ is $> n$. This together with the fact that $a_k < a_l$ (since $\phi \in W^{(q)}$ and $l \leqslant q$) implies that $a_k = r \leqslant n$ and $a_l = p'$ for some $p \leqslant n$. Further, $a_k > a'_l$ implies that $r > p$.

Now, for any j, $n \geqslant j > q$ (note $a_j \leqslant n$), we have $a_j > r$ or $a_j < p$ (cf. the discussion above on covers in W). Hence we conclude that, for any t, $p < t < r$, neither t nor $t' \in \{a_1, ..., a_{k-1}\}$ (since $a_{k-1} < p$, as $\tau \in W^{(q)}$), nor $\in \{a_{k+1}, ..., a_{l-1}\}$ (since $a_k = r$ and $a_{l-1} < r'$, as $\tau \in W^{(q)}$) nor $\in \{a_{l+1}, ..., a_q\}$ (since $a_{l+1} > p'$), nor $\in \{a_{q+1}, ..., a_n\}$ (since for $j > q$, $a_q \leqslant n$ and a_j is either $> r$ or $< p$). But then this is impossible (unless $p = r - 1$), since given, $w = (d_1 \cdots d_n) \in W$, for any t, $1 \leqslant t \leqslant n$, either t or $t' \in \{d_1, ..., d_n\}$.

(3) Let $\alpha = e_k + e_l$. Further let $q < l \leqslant n$. Observe that $k \leqslant q$ (since $\phi \in W^{(q)}$ and $a_k > a_l$). Now $a_l \leqslant n$ and hence $a_k > n$ (since precisely one of $\{a_k, a_l\}$ is $> n$). Let $a_k = r'$, for some r, $r \leqslant n$ and let $a_l = p$. Also $a_k > a_k'$ implies that $r < p$. Now, for any j, $l < j \leqslant n$, we have $a_j > a_l = p$ (since $\phi \in W^{(q)}$) and for any j, $q < l$, $a_j < r$ (since $\tau \in W^{(q)}$) and for any j, $k < j < q$, $a_j > a_k = r'$. Further we have $a_{k-1} < p'$ (since $\tau \in W^{(q)}$). Hence we conclude that for any t, $r < t < p$, $t \in \{a_1, ..., a_{k-1}\}$.

(4) Let $\alpha = 2e_k$ (where $k \leqslant q$, for, otherwise $\phi \equiv \tau \pmod{W_q}$). Now $a_k > a_k'$ implies $a_k > n$, say $a_k = r'$ for some $r \leqslant n$. For any j, $n \geqslant j > q$ (note $a_j \leqslant n$), we have $a_j < a_k' = r$ (cf. the discussion above on covers in W). For any $k < j < q$, $a_j > a_k$ (and hence $a_j' < r$). For any $j < k$, $a_j < r$ (since $\tau \in W^{(q)}$ and the kth entry in τ is r). Thus we obtain, for all j, $1 \leqslant j \leqslant n$, $j \neq k$, min $a_j, a_j') < r$. Now this implies $r = n$ (obviously).

Let us summarize these results in the following

PROPOSITION 3.4. *Let $\phi \in W^{(q)}$, say $\phi = (a_1 \cdots a_k \cdots a_q)$. Then any cover $\phi \to \tau$ in $W^{(q)}$ is one of the following four types.*

(1) *Let $a_k = r$, for some $r \leqslant n$ (and $k \leqslant q$). Let p be the largest integer $< r$ such that neither p nor $p' \in \{a_1, ..., a_q\}$ and such that for every t, $p < t < r$, $t' \in \{a_1, ..., a_q\}$. Then $\tau = (a_1 \cdots p \cdots a_q)$ is obtained by replacing r by p in ϕ.*

(2) *Let $a_k = r$ for some $r \leqslant n$ and $a_l = (r-1)'$, where $k < l \leqslant q$. Then $\tau = (a_1 \cdots (r-1) \cdots r' \cdots a_q)$ is obtained by simultaneously replacing r and $(r-1)'$ (in ϕ) by $(r-1)$ and r', respectively.*

(3) *Let $a_k = r'$ for some $r \leqslant n$ (and $k \leqslant q$). Let p be the smallest integer $> r$ such that neither p nor $p' \in \{a_1, ..., a_q\}$ and such that for every t, $r < t < p$, $t \in \{a_1, ..., a_q\}$. Then $\tau = (a_1 \cdots p' \cdots a_q)$ is obtained by replacing r' by p' in ϕ.*

(4) *Let $a_k = n + 1$ (for some $k \leqslant q$). Then $\tau = (a_1 \cdots n \cdots a_q)$ is obtained by replacing $n + 1$ by n in ϕ.*

Covers in Z. Let $\phi \to \tau$ be a cover in Z. If ϕ and $\tau \in W^{(q)}$, then $\phi \to \tau$ is as in Proposition 3.4. If $\phi \in W^{(q)}$ and $\tau \in W^{(q+1)}$, it can be easily seen that there exists a $r \leqslant n$, such that $\phi = (12 \cdots (r-1) a_r \cdots a_q)$, where $a_r > r$ and $\tau = (12 \cdots (r-1) a_r \cdots a_q r')$ (note that there can not exist $\theta \in Z$ such that $\tau < \theta < \phi$).

Remark 3.5. The set of ϕ's as above can be identified with

$$\left\{ (a_1 \cdots a_q),\ 1 \leqslant q \leqslant n-1 \ \middle|\ \begin{array}{l} \text{the smallest } r \text{ in } \complement\,(a_1,...,a_q,a'_1,...,a'_q) \\ \text{in } (1, 2,..., 2n) \text{ has the property that} \\ a_k = k,\ 1 \leqslant k \leqslant r-1 \end{array} \right\}.$$

We shall refer to such ϕ's as *special edge points*. We also want to remark that if $(a_1 \cdots a_q)$ is a special edge point, so is $(a_1 \cdots a_{q-1})$. Also, starting with a special edge point $\phi = (a_1 \cdots a_q)$, the special edge points in $W^{(q)}$ lying below ϕ may be obtained by reducing a_t by one, $r \leqslant t \leqslant q$ (one at a time) following the rules (to obtain a cover) described in Proposition 3.4.

Next we want to prove the following

PROPOSITION 3.6. *The poset Z is graded (i.e., Z has an unique maximal and unique minimal element and all the maximal chains in Z have the same length).*

Proof. The fact that Z has an unique maximal (resp. minimal) element is obvious, the unique maximal (resp. minimal) element in Z being given by $(2n) \in W^{(1)}$ (resp. $(12 \cdots n) \in W^{(n)}$).

Next, to prove that all maximal chains in Z have the same length, we proceed as follows. Given a maximal chain $\underline{c}$, let $\theta^{(r)}(\underline{c}) \to \mu^{(r+1)}(\underline{c})$, $1 \leqslant r \leqslant n-1$ denote those edges for which $\theta^{(r)}(\underline{c}) \in W^{(r)}$ and $\mu^{(r+1)}(\underline{c}) \in W^{(r+1)}$. To $\underline{c}$, we associate a "formal weight" $m(\underline{c})$ as follows. Let $m_r(\underline{c}) = \dim X(\theta^{(r)}(\underline{c}))$ (in G/P_r), $1 \leqslant r \leqslant n-1$. Let N be a positive integer, sufficiently large (say $N > \dim G/B$). Define $m(\underline{c}) = (m_1(\underline{c}), m_2(\underline{c}),..., m_{n-1}(\underline{c}))$ in the N-adic representation, i.e., $m(\underline{c}) = \sum_{r=1}^{n-1} m_r(\underline{c})\, N^{n-r}$. If $\underline{c}$ and $\underline{c}'$ are two maximal chains in Z, with $\theta^{(r)}(\underline{c}) = \theta^{(r)}(\underline{c}')$, $1 \leqslant r \leqslant n-1$, then $\underline{c}$ and $\underline{c}'$ have the same lengths (since in any interval in $W^{(i)}$, $1 \leqslant i \leqslant n$, all maximal chains have the same length (well known)). Hence, as far as the discussion of lengths of maximal chains in Z goes, we may identify $\underline{c}$ and $\underline{c}'$ if $\theta^{(r)}(\underline{c}) = \theta^{(r)}(\underline{c}')$, $1 \leqslant r \leqslant n-1$ (observe that $\mu^{(r+1)}(\underline{c})$ is uniquely determined by $\theta^{(r)}(\underline{c})$, $1 \leqslant r \leqslant n-1$). Let $\underline{c}_0$ be the chain such that $\theta^{(r)}(\underline{c}_0) = (12 \cdots r)$, $1 \leqslant r \leqslant n-1$. It can be easily seen that $\underline{c}_0$ has length $n^2 + n$. Now we prove Proposition 3.6 by proving the following

LEMMA 3.7. *Notations as above, length of any chain $\underline{c}$ = length of $\underline{c}_0$.*

Proof (By increasing induction on $m(\underline{c})$). If $m(\underline{c}) = (0,..., 0)$ then $\underline{c} = c_0$, in which case the result follows trivially. Let $m(\underline{c}) = (0,..., 0, 1)$ (i.e., $m(\underline{c}) = N$). Now $\theta^{(r)}(\underline{c}) = \theta^{(r)}(\underline{c}_0)$, $r \neq n-1$ and $\theta^{(r)}(\underline{c}) = (12 \cdots (n-2)\, n)$, $r = n-1$ (note that $\dim X(\theta^{(n-1)}(\underline{c})) = 1$, since $\theta^{(r)}(\underline{c}) = (12 \cdots (n-2)\, n\, n-1)$ as an element in W). Hence we may assume, $\underline{c}$ and $\underline{c}_0$

have the same edges till $\theta^{(n-2)}(\underline{c})\ (= (12 \cdots n - 2))$. Now $\mu^{n-1}(\underline{c}) = (12 \cdots n - 2\,n + 2)$. We shall now write down the remaining edges for $\underline{c}$ and $\underline{c}_0$.

Remaining edges for $\underline{c}$.

$$(12 \cdots n - 2\,n + 2) \rightarrow (12 \cdots n - 2\,n + 1) \rightarrow (12 \cdots n - 2\,n)$$

$$\mu^{(n-1)}(\underline{c}) \qquad\qquad\qquad\qquad\qquad \theta^{(n-1)}(\underline{c})$$

$$\rightarrow (12 \cdots n - 2\,n\;n + 2)$$

$$\mu^{(n)}(\underline{c})$$

$$\rightarrow (12 \cdots n - 2\,n - 1\;n + 1) \rightarrow (12 \cdots n - 1\;n).$$

Remaining edges for $\underline{c}_0$.

$$(12 \cdots n - 2\,n + 2) \rightarrow (12 \cdots n - 2\,n + 1) \rightarrow (12 \cdots n - 2\,n)$$

$$\mu^{(n-1)}(\underline{c}_0)$$

$$\rightarrow (12 \cdots n - 2\,n - 1)$$

$$\theta^{(n-1)}(\underline{c}_0)$$

$$\rightarrow (12 \cdots n - 2\,n - 1\;n + 1) \rightarrow (12 \cdots n - 1\;n)$$

$$\mu^{(n)}(c_0).$$

Hence, length of $\underline{c} =$ length of $\underline{c}_0$ (since the number of remaining edges in $\underline{c}$ and $\underline{c}_0$ are the same) and thus the result is true in this case. Now let $m(\underline{c}) > N$. Fix an r, $1 \leqslant r \leqslant n - 1$, such that $\theta^{(r)}(\underline{c}) >$ projection of $\theta^{(r+1)}(\underline{c})$ on W/W_r (under $W \rightarrow W/W_r$) (such an r clearly exists since $m(\underline{c}) > (0,\ldots, 0, 1)$). Let $\theta^{(r)}(\underline{c}) = (c_1,\ldots, c_r)$ and $\theta^{(r+1)}(\underline{c}) = (k_1,\ldots, k_{r+1})$, so that $(c_1 \cdots c_r) > (k_1 \cdots k_r)$. Now $(k_1 \cdots k_r)$ is a special edge point (cf. Remark 3.5). Define the chain $\underline{c}'$ as follows.

$$\theta^{(t)}(\underline{c}') = \theta^{(t)}(\underline{c}), \qquad t \neq r$$

$$= (k_1 \cdots k_r), \qquad t = r.$$

We may assume that $\underline{c}'$ passes through $\theta^{(r)}(\underline{c})$ and $\underline{c}$ passes through $\mu^{(r+1)}(\underline{c}')$ (observe that $\theta^{(r)}(\underline{c}') < \theta^{(r)}(\underline{c}) < \mu^{(r)}(\underline{c}') \, (= \mu^{(r)}(\underline{c}))$ and $\theta^{(r+1)}(\underline{c}) < \mu^{(r+1)}(\underline{c}') < \mu^{(r+1)}(\underline{c})$; (note that, since $(k_1 \cdots k_r) < (c_1 \cdots c_r)$ and both of them being special edge points, we have $\mu^{(r+1)}(\underline{c}') < \mu^{(r+1)}(\underline{c})$)). In other words, the part where $\underline{c}$ and $\underline{c}'$ differ is the part from $\theta^{(r)}(\underline{c})$ to $\mu^{(r+1)}(\underline{c}')$. The following diagram will help in visualizing the situation.

Now from Remark 3.5, it is clear that the part of $\underline{c}$ from $\mu^{(r+1)}(\underline{c})$ to $\mu^{(r+1)}(\underline{c}')$ has the same length as the part of $\underline{c}'$ going from $\theta^r(\underline{c})$ to $\theta^r(\underline{c}')$. As already remarked, the chains $\underline{c}$ and $\underline{c}'$ differ only in this part. Thus $\underline{c}$ and $\underline{c}'$ have the same length. On the other hand, $m(\underline{c}') < m(\underline{c})$ (the first place where they differ, is the rth place, where we have

$$m_r(\underline{c}') \, (= \dim X(\theta^{(r)}(\underline{c}'))) < m_r(\underline{c}) \, (= \dim X(\theta^{(r)}(\underline{c})))$$

and we are through by the induction hypothesis on $m(\underline{c})$.

This completes the proof of Lemma 3.7 and hence that of Proposition 3.6.

Lexicographic shellability for Z. Embed $i: Z \to W(SL_{2n+1})$ by sending $(a_1 \cdots a_d) \quad (1 \leqslant d \leqslant n)$ to $(a_1 \cdots a_d(2n+1)a_{d+2} \cdots a_{2n+1})$, where $\{a_{d+2}, ..., a_{2n+1}\} = C \, (a_1, ..., a_d)$ in $(1, 2, ..., 2n)$ arranged in descending order. Then $i(Z)$ may be identified with

$$\left\{ (a_1 \cdots a_{2n+1}) \in S_{2n+1} \middle| \begin{array}{l} \text{there exists a } d, \, 1 \leqslant d \leqslant n, \text{ such that} \\ (1) \ a_1 < a_2 < \cdots < a_d \leqslant 2n \\ (2) \ a_i + a_j \neq 2n+1, \, 1 \leqslant i < j \leqslant d \\ (3) \ a_{d+1} = 2n+1 \\ (4) \ (a_{d+2}, ..., a_{2n+1}) = C \, (a_1, ..., a_d) \text{ in } (1, 2, ..., 2n) \\ \quad \text{arranged in descending order.} \end{array} \right\}$$

In terms of reduced expressions (for elements in $W(SL_{2n+1})$), $i(Z)$ may be identified with

$$
\left\{ w = w_1 \cdots w_{2n} \in W(SL_{2n+1}) \;\middle|\;
\begin{array}{l}
\text{there exists } d \ (1 \leqslant d \leqslant n) \text{ such that} \\[4pt]
(1) \ w_j = u_j, \ j \geqslant d+1 \ (\text{cf. Section 2 for} \\
\quad \text{definition of } u_j) \\[4pt]
(2) \ l(w_j) \leqslant l(w_{j+1}), \ 1 \leqslant j \leqslant d-1 \\[4pt]
(3) \ \text{There do not exist } i, j, \ 1 \leqslant i < j \leqslant d, \\
\quad \text{such that } w_i = s_{t-1} \cdots s_{i+1} s_i \text{ and} \\
\quad w_j = s_{2n-t} \cdots s_{n+1} s_n \cdots s_i \\
\quad (\text{where } t \leqslant n)
\end{array}
\right\}
$$

Corresponding to the five types of covers in Z, we obtain five types of covers in $i(Z)$ and we proceed to describe them (in terms of reduced expressions). Let $\phi \to \tau$ be a cover in $i(Z)$. Further, let $\phi = i(\phi_0)$, $\tau = i(\tau_0)$.

(1) Let $\phi_0 = (a_1 \cdots a_k \cdots a_q)$, where $a_k = r$ (for some $r \leqslant n$). Let p be the largest integer $< r$ such that p or $p' \notin \{a_1, ..., a_q\}$ and such that for every t, $p < t < r$, $t' \in \{a_{k+1}, ..., a_q\}$. Let $\tau_0 = (a_1 \cdots p \cdots a_q)$. If $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$, then $\tau = \tau_1 \tau_2 \cdots \tau_{2n}$, where

$$
\begin{aligned}
\tau_j &= \phi_j, & j \neq k \\
&= s_{p-1} \cdots s_k, & j = k
\end{aligned}
$$

(observe that $\phi_k = s_{r-1} \cdots s_k$).

(2) Let $\phi_0 = (a_1 \cdots a_k \cdots a_l \cdots a_q)$, where $a_k = r$ and $a_l = (r-1)'$, for some $r \leqslant n$ and $\tau_0 = (a_1 \cdots (r-1) \cdots r' \cdots a_q)$. If $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$, then $\tau = \tau_1 \tau_2 \cdots \tau_{2n}$, where

$$
\begin{aligned}
\tau_j &= \phi_j, & j \neq k, l \\
&= s_{r-2} \cdots s_k, & j = k \\
&= s_{2n-r} \cdots s_n \cdots s_l, & j = l
\end{aligned}
$$

(observe that $\phi_k = s_{r-1} \cdots s_k$ and $\phi_l = s_{2n+1-r} \cdots s_n \cdots s_l$).

(3) Let $\phi_0 = (a_1 \cdots a_k \cdots a_q)$, where $a_k = r'$, for some $r \leqslant n$. Let p be the smallest integer $> r$ such that neither p nor $p' \in \{a_1 \cdots a_q\}$ and such that for every t, $r < t < p$, $t \in \{a_1, ..., a_{k-1}\}$. Let $\tau_0 = (a_1 \cdots p' \cdots a_q)$. If $\phi = \phi_1 \cdots \phi_{2n}$, then $\tau = \tau_1 \tau_2 \cdots \tau_{2n}$, where

$$
\begin{aligned}
\tau_j &= \phi_j, & j \neq k \\
&= s_{2n-p} \cdots s_n \cdots s_k, & j = k
\end{aligned}
$$

(observe that $\phi_k = s_{2n-r} \cdots s_n \cdots s_k$).

(4) Let $\phi_0 = (a_1 \cdots a_k \cdots a_q)$ where $a_k = n+1$ and $\tau_0 = (a_1 \cdots n \cdots a_q)$. If $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$ and $\tau = \tau_1 \tau_2 \cdots \tau_{2n}$, then

$$\tau_j = \phi_j, \qquad\qquad j \neq k$$
$$= s_{n-1} \cdots s_k, \qquad j = k$$

(observe that $\phi_j = s_n \cdots s_k$).

(5) Let $\phi_0 = (12 \cdots r - 1\, a_r \cdots a_q)$, where $a_r > r$ (for some $r \leqslant n$) (i.e., ϕ_0 is a special edge point) and $\tau_0 = (12 \cdots r - 1\, a_r \cdots a_q r')$ (note that $\phi_0 \in W^{(q)}$ and $\tau_0 \in W^{(q+1)}$). If $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$, then $\tau = \tau_1 \cdots \tau_{2n}$, where

$$\tau_j = \phi_j, \qquad\qquad j \neq q+1$$
$$= s_{2n-r} \cdots s_n \cdots s_{i+1}, \qquad j = q+1$$

(observe that $\phi_{q+1} = u_{q+1} (= s_{2n} \cdots s_n \cdots s_{q+1})$).

Labelling of the maximal chains in $i(Z)$. Let $v_0 = i(2n) (= (2n\, 2n + 1\, 2n - 1 \cdots 1))$. Let us fix the reduced expression for v_0 as given by $v_0 = (s_{2n-1} \cdots s_1 s_1) u_2 u_3 \cdots u_{2n}$. Let $\phi \to \tau$ be a cover in $i(Z)$. Let $\phi = i(\phi_0)$, where $\phi_0 = (a_1 \cdots a_k \cdots a_q)$ for some $q \leqslant n$. Then $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$, where $\phi_m = u_m$, $m > q$ and $\phi_m = s_{b_m} \cdots s_{m+1} s_m$ if $a_m > m$ and $\phi_m = \mathrm{Id}$, if $a_m = m$ (where $b_m = a_m - 1$, $1 \leqslant m \leqslant q$). Now coming down from v_0 to ϕ through any path consisting of elements of $i(Z)$, it is clear that starting with the above reduced expression for v_0, we end up with the reduced expression $\phi_1 \phi_2 \cdots \phi_{2n}$ for ϕ. In particular, for any chain $\underline{c}$ of which $\phi \to \tau$ is an edge, we are going to give a label (that is independent of $\underline{c}$) as follows. (In other words, we are going to give a global labelling for covers in $i(Z)$). We shall label the covers by n-tuples. Now, $\phi \to \tau$ is one of the five types of cover described before. For each type, we shall describe the corresponding n-tuple. Let $\phi = \phi_1 \cdots \phi_{2n}$. Further let $\phi = i(\phi_0)$, where $\phi_0 = (a_1 \cdots a_q)$ for some $q \leqslant n$.

(1) Let $\phi_k = s_{r-1} \cdots s_k$ (for some $k \leqslant q$ and $r \leqslant n$) and $\tau_k = s_{p-1} \cdots s_k$ (where p is as in (1) of Proposition 3.4), so that τ is obtained from ϕ by dropping the reflections s_{r-1}, $s_{r-2}, \ldots, s_p$ in ϕ_k simultaneously. Let $x_t^{(j)}$, $1 \leqslant j \leqslant 2n$, $j \leqslant t \leqslant 2n$, denote the position of s_t in u_j (appearing in v_0). The corresponding n-tuple is given by $(x_{r-1}^{(k)}, x_{r-2}^{(k)}, \ldots, x_p^{(k)}, x_p^{(k)}, \ldots, x_p^{(k)})$ (of course, $x_t^{(k)} + 1 = x_{t-1}^{(k)}$, $k + 1 \leqslant t \leqslant 2n$).

(2) Let $\phi_k = s_{r-1} \cdots s_k$, $\phi_l = s_{2n+1-r} \cdots s_n \cdots s_l$, for some $k < l \leqslant q$ and $r \leqslant n$ and $\tau_k = s_{r-2} \cdots s_k$, $\tau_l = s_{2n-r} \cdots s_n \cdots s_l$, so that τ is obtained from ϕ by simultaneously dropping out the reflections s_{r-1} in ϕ_k and s_{2n+1-r} in ϕ_l. The corresponding n-tuple is given by $(x_{r-1}^{(k)}, x_{2n+1-r}^{(l)}, x_{2n+1-r}^{(l)}, \ldots, x_{2n+1-r}^{(l)})$ (the notation $x_t^{(j)}$, $1 \leqslant j \leqslant 2n$, $j \leqslant t \leqslant 2n$, being as in (1) above).

(3) Let $\phi_k = s_{2n-r} \cdots s_n \cdots s_k$ (for some $k \leqslant q$ and $r \leqslant n$) and $\tau_k = s_{2n-p} \cdots s_n \cdots s_k$, so that τ is obtained from ϕ by simultaneously dropping the reflections $s_{2n-r}, s_{2n-r-1},..., s_{2n+1-p}$ in ϕ_k. The corresponding n-tuple is given by $(x^{(k)}_{2n-r}, x^{(k)}_{2n-r-1},..., x^{(k)}_{2n+1-p}, x^{(k)}_{2n+1-p},..., x^{(k)}_{2n+1-p})$.

(4) Let $\phi_k = s_n \cdots s_k$ (for some $k \leqslant q$) and $\tau_k = s_{n-1} \cdots s_k$, so that τ is obtained from ϕ by dropping the reflection s_n in ϕ_k. The corresponding n-tuple is given by $(x^{(k)}_n, x^{(k)}_n,..., x^{(k)}_n)$.

(5) Let $\phi_j = \mathrm{Id}$, $j \leqslant r-1$, $\phi_j > \mathrm{Id}$ for $j \geqslant r$ for some $r \leqslant n$. (in other words, $\phi = i(\phi_0)$, where $\phi_0 = (12 \cdots r-1 \, a_r \cdots a_q)$, where $a_r > r$ for some $r \leqslant n$. Let $\tau_{q+1} = s_{2n-r} \cdots s_{q+1}$, so that τ is obtained from ϕ by dropping out the reflections $s_{2n}, s_{2n-1},..., s_{2n+1-r}$ in ϕ_{q+1} simultaneously (note that ϕ is a special edge point). The corresponding n-tuple is given by $(x^{(q+1)}_{2n}, x^{(q+1)}_{2n-1},..., x^{(q+1)}_{2n+1-r}, x^{(q+1)}_{2n+1-r},..., x^{(q+1)}_{2n+1-r})$.

THEOREM 3.8. *Let G be of type B_n or C_n. Then $Z = \bigcup_{j=1}^{n} W^{(j)}$ is lexicographically shellable.*

Proof. Labelling the covers in Z (or $i(Z)$) by n-tuples as described above, condition (L1) is immediately verified.

Verification of (L2). Let $[\tau, \phi]$ be any interval in $i(Z)$ (since the covers have been labelled globally, enough to consider $[\tau, \phi]$ rather than a rooted interval $([\tau, \phi], \underline{c})$, where $\underline{c}$ is some chain going down from v_0 to ϕ). Let $\tau = \tau_1 \tau_2 \cdots \tau_{2n}$ and $\phi = \phi_1 \phi_2 \cdots \phi_{2n}$. Also let $\phi = (b_1 \cdots b_q)$ and $\tau = (a_1 \cdots a_j)$, $q \leqslant j \leqslant n$. Further, let k be the smallest integer such that $\tau_k < \phi_k$ (note that $k \leqslant q + 1$ and that $\tau_m \leqslant \phi_m$, $1 \leqslant m \leqslant 2n$). A maximal chain in $[\tau, \phi]$ with increasing label may be obtained by starting with ϕ_k and dropping the reflections in ϕ_k one after the other from left to right until we end up with τ_k, bearing in mind the fact that if at any step the element under consideration looks like $\theta = \theta_1 \cdots \theta_k \cdots \theta_{2n}$ $(= i(\theta_0)$, where $\theta_0 = (c_1 c_2 \cdots c_q))$, where $\theta_t = \tau_t$, $1 \leqslant t \leqslant k-1$ and $\theta_k > \tau_k$:

(1) if $\theta_k = s_{2n-r} \cdots s_n \cdots s_k$, for some $r \leqslant n$ and p is the smallest integer $> r$ such that neither p nor $p' \in \{c_1,..., c_q\}$ and such that for every t, $r < t < p$, $t \in \{c_1,..., c_{k-1}\}$, then the reflections $s_{2n-r}, s_{2n-r-1},..., s_{2n+1-p}$ in θ_k will be simultaneously dropped out;

(2) $\theta_k = s_{r-1} \cdots s_k$, for some $r \leqslant n$ and $c_l = (r-1)'$ for some l $(k < l \leqslant q)$, then the reflections s_{r-1} in θ_k and s_{2n+1-r} in θ_l will be simultaneously dropped out, if $a_l < (r-1)'$. If $a_l = (r-1)'$ and if p is the largest integer $< r$ such that neither p nor $p' \in \{c_1,..., c_q\}$ and such that for every t, $p < t < r$, $t' \in \{c_{k+1},..., c_q\}$, then the reflections $s_{r-1}, s_{r-2},..., s_p$ in θ_k will be simultaneously dropped out. (Here we want to observe that $a_k \leqslant p$. This will follow if we show that for every t, $p < t < r$, if s is the

integer such that $c_s = t'$, then $a_s = t'$. This we shall prove by induction on $r - t$, the starting point of induction being $r - t = 1$, in which case the result is true, since $c_t = (r - 1)' = a_t$. Now, let $r - t > 1$ and assume the induction hypothesis. Let $c_s = t'$ for some s, $s \leqslant q$ (by the assumption on p, for every t, $p < t < r$, $t' \in \{c_{k+1},..., c_q\}$). Also, we have $s > l$ (since $c_l = (r - 1)' < t'$ and $\theta = (c_1 \cdots c_q) \in W^{(q)}$). Hence $a_s > n$ (since $a_l = (r - 1)'$ and $\tau = (a_1 \cdots a_l) \in W^{(l)}$). Also $a_s \leqslant t'$ (since $\tau \leqslant \theta$) and $a_s > (r - 1)'$ (since $a_s > a_l = (r - 1)'$). If a_s is not $= t'$, then $a_s = m'$ for some m, $t < m < r - 1$. If h is the integer such that $c_h = m'$ (observe that $t < m < r < 1$ implies $m' \in \{c_{k+1},..., c_q\}$) then $a_h = m'$ (by the induction hypothesis). Then, $h \neq s$, since $c_h (= m') \neq c_s (= t')$. Thus a_s cannot be m' for any m, $t < m < r - 1$. This together with the fact that $(r - 1)' < a_s \leqslant t'$ implies that $a_s = t'$. Thus, for every t, $p < t < r$, $t' \in \{a_{k+1},..., a_l\}$. This, together with the fact that $a_k < c_k (= r)$ implies that $a_k \leqslant p$, as required.

Proceeding thus, we first reduce ϕ_k to τ_k, then ϕ_{k-1} to $\tau_{k-1},..., \phi_q$ to τ_q. In terms of permutations, this corresponds to reducing $(b_1,..., b_q)$ to $(a_1,..., a_q)$ by first reducing b_1 to a_1, then b_2 to $a_2,..., b_q$ to a_q (keeping in mind the two facts mentioned above). To be very precise, as before, let k be the smallest integer such that $\phi_k > \tau_k$ (note that $k \leqslant q + 1$). At any step let $\theta = (c_1 \cdots c_q)$. Then if

(a) $c_k = r'$ for some $r \leqslant n$ and p is as in (1) above, then r' will be replaced by p' (and we will take the corresponding cover).

(b) $c_k = r$ for some $r \leqslant n$ and $c_l = (r - 1)'$ for some l $(k < l \leqslant q)$ then c_k and c_l will be simultaneously replaced by $r - 1$ and r' respectively, if $a_l < (r - 1)'$. If $a_l = (r - 1)'$ and if p is the integer as in (2) above, then r will be replaced by p (and we will take the corresponding cover).

After $(b_1 \cdots b_q)$ has been reduced to $(a_1 \cdots a_q)$ (as above), the remaining path from $(a_1 \cdots a_q)$ to $(a_1 \cdots a_q \cdots a_j)$ (in case $q < j$) is obtained as follows. Now let $(a_1 \cdots a_q) = (12 \cdots r - 1 \, a_r \cdots a_q)$ for some $r \leqslant n$ (where $a_r > r$). Then since $(a_1 \cdots a_q \cdots a_j) \in W^{(j)}$, we have $a_j \leqslant r'$. Hence $a_t < r'$ for $t < j$. This in particular implies that $(a_1 \cdots a_q)$ is a special edge point. Hence $i(a_1 \cdots a_q)$-has the reduced expression $\tau_r \tau_{r-1} \cdots \tau_q u_{q+1} \cdots u_{2n}$ (note that under the assumption $(a_1 \cdots a_q) = (12 \cdots r - 1 \, a_r \cdots a_q)$, we have $\tau_m = \mathrm{Id}$, $m < r - 1$). Now the reflections $s_{2n}, s_{2n-1},..., s_{2n+1-r}$ in u_{q+1} shall be dropped out simultaneously to obtain the cover $(1 \cdots r - 1 \, a_r \cdots a_q) \rightarrow (1 \cdots r - 1 \, a_r \cdots a_q r')$. Then we reduce $(a_1 \cdots a_q r')$ to $(a_1 \cdots a_q a_{q+1})$ by reducing r' one by one, keeping in mind the fact that if at any step, the element θ under consideration looks like $(a_1 \cdots a_q l')$, for some $l \leqslant n$, and p is the smallest integer $> l$ such that neither p nor $p' \in \{a_1,..., a_q\}$ and such that for every t, $l < t < p$, $t \in \{a_1,..., a_q\}$: then the corresponding cover to be taken is $(a_1 \cdots a_q l') \rightarrow (a_1 \cdots a_q p')$. (This amounts to dropping out the reflections $s_{2n-l}, s_{2n-l-1},..., s_{2n+1-p}$ in $\theta_{q-1} (= s_{2n-l} \cdots s_{q+1})$

simultaneously.) One proceeds thus, finally ending up with τ. Obviously this chain has an increasing label.

Any other chain is obtained by starting with ϕ_m for some $m \geqslant k$ (recall, k is the smallest integer such that $\tau_k \nleqslant \phi_k$). Suppose $m > k$, then at some point, one has to work with dropping out the reflections in ϕ_k and thus the corresponding label is not increasing and the corresponding label is (clearly) lexicographically $>$ the above increasing label. Suppose $m = k$; one may assume the corresponding chain has the first edge to be different from the first edge of the above chain with increasing label by using induction on $l[(\tau, \phi)]$ ($= l_{sp}(\phi_0) - l_{sp}(\tau_0)$, where ϕ_0 and τ_0 are given by $i(\phi_0) = \phi$, $i(\tau_0) = \tau$, the proof for the starting point of induction, namely $l_{sp}(\tau_0) = l_{sp}(\phi_0) - 1$, being trivial); but then (as happened in the case of SL_n (cf. Theorem 3.1)), the element θ ($= i(\theta_0)$) corresponding to the first edge is such that θ_0 is not covered by ϕ_0 in Z. Thus the possibility $m = k$ does not exist.

This completes the verification of (L2) and hence the proof of Theorem 3.8.

Next we want to prove the lexicographic shellability for Z_w and also for $Z_w^{(k)}$ (cf. Section 2), $w \in W$. We shall treat the two cases, namely $G = SL_n$ or Sp_{2n}, separately.

First, let $G = SL_n$. For any $w \in W$, recall (cf. Section 2)

$$Z_w = \{\tau \in W^{(q)}, 1 \leqslant q \leqslant n - 1/\tau \leqslant \text{projection of } w \text{ on } W/W_q$$

$$\text{under } W \to W/W_q\}.$$

Covers in Z_w.   A cover $\phi \to \tau$ in Z_w is either of the form

  (1) $\phi \to \tau$, where $\phi, \tau \in W^{(q)}$, for some $q \leqslant n - 1$, or

  (2) $\phi \to \tau$, where $\phi \in W^{(q)}$, $\tau \in W^{(q+1)}$, for some $q \leqslant n - 2$.

To be very precise, in (2), let $w = (b_1 \cdots b_n)$ (as a permutation). For $1 \leqslant q \leqslant n - 1$, let $\{x_1^{(q)}, x_2^{(q)},..., x_q^{(q)}\} = \{b_1,..., b_q,$ arranged in ascending order$\}$. Then it can be easily seen that ϕ and τ look as follows:

$$\tau = (c_1,..., c_q, x_{q+1}^{(q+1)}) \text{ and } \phi = (c_1,..., c_q), \text{ where } (c_1,..., c_q)$$

$$\leqslant (b_1,..., b_q, \text{ arranged in ascending order})$$

PROPOSITION 3.9.   *The poset Z_w is graded.*

Proof.   The fact that Z_w has an unique maximal and minimal element is obvious, namely, they are given by $(b_1) \in W^{(1)}$ and $(12 \cdots n - 1) \in W^{(n-1)}$.

Next, we want to prove that all maximal chains in Z_w have the same length. We prove this in the same spirit as the proof of Proposition 3.6.

Given a maximal chain $\underline{c}$, let $\theta^{(r)}(\underline{c}) \to \mu^{(r+1)}(\underline{c})$, $1 \leqslant r \leqslant n-2$, denote those edges for which $\theta^{(r)}(\underline{c}) \in W^{(r)}$ and $\mu^{(r+1)}(\underline{c}) \in W^{r+1}$. To $\underline{c}$, we associate a "formal weight" $m(\underline{c})$ as follows. Let $m_r(\underline{c}) = \dim X(\theta^{(r)}(\underline{c}))$ (in G/P_r), $1 \leqslant r \leqslant n-2$. Let N be a positive integer sufficiently large (say $N > \dim G/B$). Define $m(\underline{c}) = (m_1(\underline{c}), m_2(\underline{c}),..., m_{n-2}(\underline{c}))$, in the N-adic representation, i.e., $m(\underline{c}) = \sum_{r=1}^{n-2} m_r(\underline{c}) N^{n-r-1}$. If $\underline{c}$ and c' are two maximal chains in Z_w, with $\theta^{(r)}(\underline{c}) = \theta^{(r)}(c')$, $1 \leqslant r \leqslant n-2$, then $\underline{c}$ and $\underline{c}'$ have the same lengths (since in any interval in $W^{(i)}$, $1 \leqslant i \leqslant n-2$, all maximal chains have the same length). Hence for the discussion of lengths of maximal chains in Z_w, we may identify $\underline{c}$ and $\underline{c}'$ if $\theta^{(r)}(\underline{c}) = \theta^{(r)}(\underline{c}')$, $1 \leqslant r \leqslant n-2$ (observe that $\mu^{(r+1)}(\underline{c})$ is uniquely determined by $\theta^{(r)}(\underline{c})$. In fact, if $\theta^{(r)}(\underline{c}) = (c_1 \cdots c_r)$, then $\mu^{(r+1)}(\underline{c}) = (c_1 \cdots c_r x_{r+1}^{(r+1)})$, from our discussion above on covers in Z_w). Let $\underline{c}_0$ be the chain (in Z_w) such that $\theta^{(r)}(\underline{c}_0) = (12 \cdots r)$, $1 \leqslant r \leqslant n-2$. Now we prove Proposition 3.9 by proving the following

LEMMA 3.10. *Notations as above, length of any chain $\underline{c}$ in Z_w = length of $\underline{c}_0$.*

Proof (By increasing induction on $m(\underline{c})$). If $m(c) = (0,..., 0)$, then $\underline{c} = \underline{c}_0$, in which case the result follows trivially. If $m(\underline{c}) = (0, 0,..., 0, 1)$ (i.e., $m(\underline{c}) = N$), then $\theta^{(r)}(\underline{c}) = \theta^{(r)}(\underline{c}_0)$, $r \neq n-2$, and $\theta^{(n-2)}(\underline{c}) = (12 \cdots n-3 \; n-1)$ (since by our assumption on $m(\underline{c})$, $\dim X(\theta^{(n-2)}(\underline{c})) = 1$). This in particular implies $x_{n-2}^{(n-2)} \geqslant n-1$ (where (recall) for $1 \leqslant q \leqslant n-1$, $x_q^{(q)} = \max\{b_1,..., b_q\}$, $(b_1 \cdots b_n)$ being the permutation representing w). Also, by our assumption on $m(\underline{c})$, $\underline{c}$ and $\underline{c}_0$ have the same edges until $\theta^{(n-2)}(c)$ (and hence till $\mu^{(n-2)}(c)$). We shall now describe the remaining edges of $\underline{c}$ and $\underline{c}_0$. For this, we need to distinguish the two cases, $x_{n-2}^{(n-2)} = n$ or $n-1$ (respectively).

Let $x_{n-2}^{(n-2)} = n$.

Remaining edges of c.

$$(12 \cdots n-3 \; n) \to (12 \cdots n-3 \; n-1) \to (12 \cdots n-3 \; n-1 \; n) \to (12 \cdots n-2 \; n)$$

$$\mu^{(n-2)}(\underline{c}) \qquad\qquad \theta^{(n-2)}(\underline{c}) \qquad\qquad \mu^{(n-1)}(\underline{c})$$

$$\to (12 \cdots n-2 \; n-1).$$

Remaining edges of $\underline{c}_0$.

$$(12 \cdots n-3\, n) \to (12 \cdots n-3\, n-1) \to (12 \cdots n-3\, n-2)$$

$$\mu^{(n-2)}(\underline{c}_0) \qquad\qquad\qquad\qquad \theta^{(n-2)}(\underline{c}_0)$$

$$\to (12 \cdots n-3\, n-2\, n) \to (12 \cdots n-2\, n-1).$$

$$\mu^{n-1}(\underline{c}_0)$$

Let $x_{n-2}^{(n-2)} = n-1$.

Remaining edges of $\underline{c}$.

$$(12 \cdots n-3\, n-1) \to (12 \cdots n-3\, n-1\, n) \to (12 \cdots n-3\, n-2\, n)$$

$$\mu^{(n-2)}(\underline{c}) \qquad\qquad\qquad \mu^{n-1}(\underline{c})$$

$$\theta^{(n-2)}(\underline{c}).$$

$$\to (12 \cdots n-2\, n-1)$$

(It may be assumed that $x_{n-1}^{(n-1)} = n$. For, if $x_{n-1}^{(n-1)} \neq n$, then $x_{n-1}^{(n-1)} = n-1$, in which case $b_n = n$ and $w \in W(SL_{n-1})$ and we may use induction on n.)

Remaining edges of $\underline{c}_0$.

$$(12 \cdots n-3\, n-1) \to (12 \cdots n-3\, n-2) \to (12 \cdots n-2\, n)$$

$$\mu^{(n-2)}(\underline{c}_0) \qquad\qquad \theta^{(n-2)}(\underline{c}_0) \qquad\qquad \mu^{(n-1)}(\underline{c}_0).$$
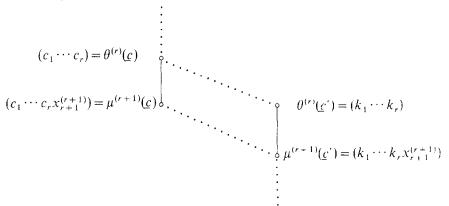
$$\to (12 \cdots n-2\, n-1).$$

In either case, we find that the number of remaining edges is the same for both $\underline{c}$ and $\underline{c}_0$. Thus the result is true in this case.

Now let $m(\underline{c}) > N$. Fix a r, $1 \leqslant r \leqslant n-2$, such that $\theta^{(r)}(\underline{c}) >$ projection of $\theta^{(r+1)}(\underline{c})$ on W/W_r, under $W \to W/W_r$. (Such an r clearly exists since $m(\underline{c}) > (0, 0,..., 0, 1)$). Let $\theta^{(r)}(\underline{c}) = (c_1 \cdots c_r)$ and $\theta^{(r+1)}(\underline{c}) = (k_1 \cdots k_{r+1})$, so that $(c_1 \cdots c_r) > (k_1 \cdots k_r)$. Define the chain $\underline{c}'$ as follows

$$\theta^{(t)}(\underline{c}') = \theta^{(t)}(\underline{c}), \qquad t \neq r$$

$$= (k_1 \cdots k_r), \qquad t = r.$$

We may assume that $\underline{c}'$ passes through $\theta^{(r)}(\underline{c})$ and $\underline{c}$ passes through $\mu^{(r+1)}(\underline{c}')$ (observe that $\theta^{(r)}(\underline{c}') < \theta^{(r)}(\underline{c}) < \mu^{(r)}(\underline{c}')$ $(= \mu^{(r)}(\underline{c}))$ and $\theta^{(r+1)}(\underline{c}) < \mu^{(r+1)}(\underline{c}') < \mu^{(r+1)}(\underline{c})$ (note that $\mu^{(r+1)}(\underline{c}') = k_1 \cdots k_r x^{(r+1)}_{r+1} <$ $(c_1 \cdots c_r x^{(r+1)}_{r+1}) = \mu^{(r+1)}(\underline{c}))$). In other words, the part where $\underline{c}$ and $\underline{c}'$ differ is the part from $\theta^{(r)}(\underline{c})$ to $\mu^{(r+1)}(\underline{c}')$. The following diagram will help in visualizing the situation

$$(c_1 \cdots c_r) = \theta^{(r)}(\underline{c})$$

$$(c_1 \cdots c_r x^{(r+1)}_{r+1}) = \mu^{(r+1)}(\underline{c})$$

$$\theta^{(r)}(\underline{c}') = (k_1 \cdots k_r)$$

$$\mu^{(r+1)}(\underline{c}') = (k_1 \cdots k_r x^{(r+1)}_{r+1})$$

Now, it is clear that the part of $\underline{c}$ going from $\mu^{(r+1)}(\underline{c})$ to $\mu^{(r+1)}(\underline{c}')$ has the same length as the part of $\underline{c}'$ going from $\theta^{(r)}(\underline{c})$ to $\theta^{(r)}(\underline{c}')$. As already remarked $\underline{c}$ and $\underline{c}'$ differ only in this part. Thus $\underline{c}$ and $\underline{c}'$ have the same length. On the other hand, $m(\underline{c}') < m(\underline{c})$ (the first place where they differ is the rth place, where we have $m_r(\underline{c}') = (\dim X(\theta^{(r)}(\underline{c}'))) < m_r(\underline{c})$ $(= \dim X(\theta^{(r)}(\underline{c})))$ and we are through by the induction hypothesis on $m(\underline{c})$.

This completes the proof of Lemma 3.10 and hence that of Proposition 3.9.

THEOREM 3.11. *The poset Z_w is lexicographically shellable.*

Labelling of maximal chains in Z_w. Now $Z_w \subset Z$ and a cover $\phi \to \tau$ in Z_w continues to be a cover in Z, if both ϕ and $\tau \in W^{(j)}$ for some $j \leqslant n-1$. In the alternate case, a cover $\phi \to \tau$ in Z_w need not be a cover in Z (for example, in SL_4, consider $w = (3124)$; then, $(1) \longrightarrow (13)$ is a cover in Z_w, but is not a cover in Z). Let N be a positive integer sufficiently large (say $N > rk(Z)$ $(=$ length of any maximal chain in Z)). We shall now label the covers in Z_w by N-tuples as follows.

Given a cover $\phi \to \tau$ in Z_w, consider the unique chain in Z going from ϕ to τ with increasing label (cf. Theorem 3.1) (note that this unique chain does not depend on the path chosen to come down from v_0, the unique maximal element in Z to ϕ, because of the global nature of the labellings of covers in Z (cf. Remark 3.3)). Let us denote this chain by $\phi = \phi_0 \to^{n_1}$ $\phi_1 \to^{n_2} \phi_2 \to \cdots \to^{n_r} \phi_r = \tau$ (where $r \leqslant N$). Now label the cover $\phi \to \tau$ by the

N-tuple $(n_1, n_2,..., n_r, n_r,..., n_r)$. (observe that because the labellings of covers in Z are global (cf. Remark 3.3), the labellings of covers in Z_w are also global, so (L1) follows immediately. To make the labelling very precise, if ϕ, $\tau \in W^{(q)}$, for some $q \leqslant n-1$, then $\phi \to \tau$ is a cover in Z also and if m is the label for the cover $\phi \to \tau$ in Z, then $n_1 = n_2 = \cdots = n_r = m$; in the alternate case, $\phi \in W^{(q)}$ and $\tau \in W^{(q+1)}$, say, $\phi = (a_1 \cdots a_q)$, $\tau = (a_1 \cdots a_q x_{q+1}^{(q+1)})$ (where $x_{q+1}^{(q+1)} = \max\{b_1,..., b_{q+1}\}$, b_j's being given by $w = (b_1 \cdots b_n)$). If $i(\phi) = \phi_1 \cdots \phi_q u_{q+1} \cdots u_n$, $i(\tau) = \tau_1 \cdots \tau_{q+1} u_{q+2} \cdots u_n$ (recall $i: Z \to W(SL_{n+1})$ (cf. Proof of Theorem 3.1)), then $\tau_k = \phi_k$, $k \neq q+1$ and $\tau_{q+1} = \mathrm{Id}$ or $s_t \cdots s_{q+1}$ according as $x_{q+1}^{(q+1)} = q+1$ or $> q+1$ (and $t = x_{q+1}^{(q+1)} - 1$). Then $n_1 = y_n^{(q+1)}$, $n_2 = y_{n-1}^{(q+1)}$ $(= n_1 + 1)$, etc., where $y_k^{(j)}$, $1 \leqslant j \leqslant n$, $j \leqslant k \leqslant n$, denotes the position of s_k appearing in u_j (in $v_0 = i((n)) = (s_{n-1} \cdots s_1) u_2 u_3 \cdots u_n)$.

Verification of (L2). Now let $[\tau_0, \phi_0]$ be any interval in Z_w. The fact that there exists a chain in Z_w going from ϕ_0 to τ_0 whose label is increasing is immediate. In fact, if $\phi_0 = (c_1 \cdots c_q)$, $\tau_0 = (a_1 \cdots a_j)$, $q < j \leqslant n-1$ (it may be assumed $q < j$, since for $q = j$, the result follows from [2]. Observe that in this case, the above labelling is the same as the labelling described in [2]), one first considers the unique chain in Z with increasing label going from $\phi_0 = \theta_0 = (c_1 \cdots c_q)$ to $(a_1 \cdots a_q) = \theta_r$ (observe that since $(c_1 \cdots c_q)$ and $(a_1 \cdots a_q)$ both belong to $W^{(q)}$ and $(a_1 \cdots a_q) \leqslant (c_1 \cdots c_q) \leqslant$ projection of w on W/W_q, the elements of the above unique chain (in Z) in fact, belong to Z_w). Now this chain is followed by $(a_1 \cdots a_q) \to (a_1 \cdots a_q x_{q+1}^{(q+1)})$. The label for this cover is $(n_1,..., n_l, n_l,..., n_l)$ say; then it is $>$ the label for $\theta_{r-1} \to \theta_r$ (the latter label looks like $(m, m,..., m)$, where $m = y_{a_k}^{(k)}, k$ being the largest integer $\leqslant q$ such that $a_k < c_k$ (if $a_k = c_k$ for all $1 \leqslant k \leqslant q$, then $\phi_0 = (a_1,..., a_q) \to (a_1,..., a_q, x_{q+1}^{(q+1)})$ is the first edge in the required chain) (here the notation $y_t^{(j)}$ is as above)). Now from our description of $(n_1, n_2,..., n_l, n_l,..., n_l)$ it is clear that $m < n$, so that the label for $\theta_{r-1} \to \theta_r$ $(= (a_1,..., a_q))$ is $<$ that for $(a_1,..., a_q) \to (a_1,..., a_q, x_{q+1}^{(q+1)})$. Now $\phi_0 = \theta_0 \to \theta_1 \to \cdots \to \theta_r \to (a_1,..., a_q, x_{q+1}^{(q+1)})$ is followed by the unique chain in Z going from $(a_1,..., a_q, x_{q+1}^{(q+1)})$ to $(a_1,..., a_q, a_{q+1})$ whose label is increasing and so on.

Let $i(\tau_0) = \tau = \tau_1 \cdots \tau_j u_{j+1} \cdots u_n$, $i(\phi_0) = \phi = \phi_1 \cdots \phi_q u_{q+1} \cdots u_n$, where for every t, $1 \leqslant t \leqslant n$, τ_t is a right-end segment of ϕ_t. Let k be the smallest integer such that $\tau_k < \phi_k$ (note that $k \leqslant q+1$). Now, in any other maximal chain, the first reflection that is dropped out occurs in ϕ_r for some $r \geqslant k$. If $r > k$, then, at some point one has to work with dropping out the reflections in ϕ_k and thus the corresponding label is not increasing and the corresponding label is (clearly) lexicographically $>$ the above increasing label. Suppose $r = k$ (may assume the corresponding chain has the first edge to be different from the first edge in the above chain with increasing

label, by using induction on $l([\tau_0, \phi_0])$ (in Z_w), the proof for the starting point of induction being trivial), then the element ρ_0 corresponding to the first edge is not covered by ϕ in Z_w. (this is obvious if $k \leq q$. If $k = q + 1$. then ρ_0 looks like $(c_1 \cdots c_q y)$ where $y \leqq x_{q+1}^{(q+1)}$ (recall $\phi_0 = (c_1 \cdots c_q)$ and $x_{q+1}^{(q-1)} = \max\{b_1, ..., b_{q-1}\}$, where $(b_1 \cdots b_n) = w$)). Thus the possibility $r = k$ does not exist.

This completes the verification of (L2) and hence the proof of Theorem 3.11.

THEOREM 3.12. *Let* $G = SL_n$ *and* $(j) = (j_1, ..., j_r)$ *be an* r-tuple $(r \leq n - 1)$, $1 \leq j_1 < \cdots < j_r \leq n - 1$ *and for any* $w \in W$. *let*

$$Z_w^{(j)} = \bigcup_{k=1}^{r} \{\tau \in W^{(j_k)} / \tau \leq \text{projection of } w \text{ on } W/W_{j_k} \text{ under } W \to W/W_{j_k}\}.$$

Then $Z_w^{(j)}$ *is lexicographically shellable.*

Proof. First we shall prove this for w_0, the unique element of maximal length in W. Now a cover $\phi \to \tau$ in $Z^{(j)}$ $(= Z_{w_0}^{(j)})$ continues to be a cover in Z, if $\phi, \tau \in W^{(j_k)}$ for some k, $1 \leq k \leq r$. In the alternate case, we have, $\phi \in W^{(j_k)}$, $\tau \in W^{(j_{k+1})}$; and if $\phi = (a_1, ..., a_{j_k})$, then $\tau = (a_1, ..., a_{j_k}, n + 1 - t..., n - 1, n)$, where $t = j_{k+1} - j_k$. In this case one considers the unique chain in Z going from ϕ to τ whose label is increasing, say,

$$\phi = \phi_0 \xrightarrow{n_1} \phi_1 \xrightarrow{n_2} \cdots \xrightarrow{n_m} \phi_m = \tau.$$

Now one labels the cover $\phi \to \tau$ in $Z^{(j)}$ by the N-tuple $(n_1, n_2, ..., n_m, n_m, ..., n_m)$ (where N is a positive integer sufficiently large, say $N > $ rank of Z). With this labelling, one can check (proceeding as in the proof of Theorem 3.11) that $Z^{(j)}$ is lexicographic shellable.

The discussion for $Z_w^{(j)}$ is completely analogous. We shall just mention what a cover $\phi \to \tau$ in $Z_w^{(j)}$ looks like. A cover $\phi \to \tau$ in $Z_w^{(j)}$ continues to be a cover in Z, if $\phi, \tau \in W^{(j_k)}$ for some k, $1 \leq k \leq r$. In the alternate case, we have $\phi \in W^{(j_k)}$, $\tau \in W^{(j_{k-1})}$. If $\phi = (a_1 \cdots a_{j_k})$, then $\tau = (a_1 \cdots a_{j_k}, c_{j_k+1}, ..., c_{j_{k-1}})$, where $\{c_{j_k+1}, ..., c_{j_{k-1}}\} = $ the first $j_{k+1} - j_k$ numbers in $\{b_1, ..., b_{j_{k-1}}\}$, arranged in descending order}.

THEOREM 3.13. *Let* $G = Sp_{2n}$ *and let* $w \in W$. *Then the set* Z_w *is lexicographically shellable.*

Proof. The proof is analogous to the proof of Theorem 3.11. We shall first describe what the covers in Z_w look like and how they are labelled. Let $w = (b_1 \cdots b_n \cdots b_{2n})$. For $1 \leq j \leq n$. let $x_j^{(j)} = \max\{b_1, ..., b_j\}$. Now a cover $\phi \to \tau$ continues to be a cover in Z, if $\phi, \tau \in W^{(q)}$ for some $q \leq n$. In the alter-

nating case, if $\phi = (a_1 \cdots a_q)$, then $\tau = (a_1 \cdots a_q a_{q+1})$, where a_{q+1} is the largest integer $< x_{q+1}^{(q+1)}$ such that neither a_{q+1} nor $a_{q+1}' \in \{a_1, ..., a_q\}$ (to be very precise, if $x_{q+1}^{(q+1)} \leq n$, then we have $a_q < x_{q+1}^{(q+1)}$ and $a_{q+1}^{(q+1)} = x_{q+1}^{(q+1)}$. If $x_{q+1}^{(q+1)} > n$, say $x_{q+1}^{(q+1)} = r'$, for some $r \leq n$, then, if k is the smallest integer $> r$ such that neither k nor $k' \in \{a_1, ..., a_q\}$ and such that for every t, $r < t < k$, $t \in \{a_1, ..., a_q\}$, then $a_{q+1} = k'$). Then as in the proof of Theorem 3.11, to get the label for the cover $\phi \to \tau$ in Z_w, one considers the unique chain in Z going from ϕ to τ with increasing label, say

$$\phi = \phi_0 \xrightarrow{m_1} \phi_1 \xrightarrow{m_2} \cdots \xrightarrow{m_r} \phi_r = \tau$$

(where each m_t is an n-tuple and $m_1 < m_2 < \cdots < m_r$). Now one labels the cover $\phi \to \tau$ by the N-tuple $(m_1, m_2, ..., m_r, m_r, ..., m_r)$ (where N is a positive integer sufficiently large, say $N = n \cdot rk(Z)$).

Now if $[\tau, \phi]$ is an interval in Z_w, say $\tau = (a_1 \cdots a_j)$, $\phi = (c_1 \cdots c_q)$, $q < j \leq n$ (it may be assumed $q < j$; for, if $q = j$, then the elements in the unique chain in Z, going from ϕ to τ, whose label is increasing, all $\in Z_w$, since they are all $\leq \phi$ and $\phi \in Z_w$). Then the required chain with increasing label is obtained by first taking the unique chain in Z going from $(c_1 \cdots c_q)$ to $(a_1 \cdots a_q)$, say $\theta_0 = (c_1 \cdots c_q) \to \theta_1 \to \cdots \to \theta_r = (a_1 \cdots a_q)$. Then this is followed by $(a_1 \cdots a_q) \to (a_1 \cdots a_q d_{q+1})$ (the cover as described above) and then, this is followed by the unique chain in Z going from $(a_1 \cdots a_q d_{q+1})$ to $(a_1 \cdots a_q a_{q+1})$ whose label is increasing and so on. Clearly this chain (in Z_w) has an increasing label.

The verification of (L2) is done in the same spirit as in the proof of Theorem 3.11.

THEOREM 3.14.   *Given an r-tuple $(j_1, ..., j_r)(r \leq n)$ $1 \leq j_1 < j_2 < \cdots < j_r \leq n$, and $w \in W$, $Z_w^{(j)}$ is lexicographically shellable.*

Proof is analogous to that of Theorem 3.12.

4. KEMPF VARIETIES

In this section our main result is that G being of type A_n, B_n, or C_n, on a given Schubert variety $X(w) \subset G/B$, the two notions of monomials being standard (cf. Section 2) coincide if and only if $X(w)$ is a Kempf variety. First we recall the definition of Kempf varieties (cf. [10]) for G being of type A_n and prove the result mentioned above and then we do it for G of type B_n or C_n.

DEFINITION 4.1.   Let G be of type A_n and let $w \in W$, say $w = w_1 w_2 \cdots w_n$, where w_i $(1 \leqslant i \leqslant n)$ is a right-end segment of u_i $(= s_n \cdots s_{i+1} s_i)$ (cf. [10]). Call w a *Kempf element* (and $X(w)$, a *Kempf variety*) if

$$l(w_i) \leqslant l(w_{i+1}) + 1 \qquad \text{whenever} \qquad w_{i+1} < u_{i+1}, \ 1 \leqslant i \leqslant n - 1.$$

We give below another characterization of Kempf elements (cf. Proposition 4.3 below). We first start with the following

PROPOSITION 4.2.   *Let w be a Kempf element. Then for any $\tau \in W$, $w \geqslant \tau$ if and only if $w_i \geqslant \tau_i$, $1 \leqslant i \leqslant d$, where $w = w_1 \cdots w_n$, $\tau = \tau_1 \cdots \tau_n$ and w_i (resp. τ_i) is a right-end segment of u_i, $1 \leqslant i \leqslant n$.*

Proof. The proof of the implication $\Leftarrow$ is obvious, since $w = w_1 \cdots w_n$ (resp. $\tau = \tau_1 \cdots \tau_n$) is a reduced expression for w (resp. τ) and the characterization of the partial order in W in terms of reduced expressions (recall (cf. [4]) that given $\theta, \rho \in W(G)$, G being of any type, $\theta \geqslant \rho$, if and only if a reduced expression of θ contains a subexpression which is a reduced expression of ρ).

Now let $w \geqslant \tau$. If possible, let $w_r \not\geqslant \tau_r$ for some r, $r \leqslant n$. This implies $w_r < \tau_r$ (since w_r and τ_r are both right-end segments of u_r). Let $w_r = s_l \cdots s_{r+1} s_r$ (resp. Id) and $\tau_r = s_k \cdots s_{r+1} s_r$, where $k > l$ (resp. $k \geqslant r$). The length condition for a Kempf element, together with Proposition 2.5 implies that the projection of w on W/W_r, as an (increasing) r-tuple has the entry m at the rth place, where $m = l + 1$ (resp. $\leqslant r$) while the projection of τ on W/W_r, as an (increasing) r-tuple has the entry at the rth place to be $\geqslant k + 1$ (identifying W with S_{n+1}). Thus projection of w on W/W_r is $\not\geqslant$ projection of τ on W/W_r. Hence $w \not\geqslant \tau$ (cf. Section 2, partial order on W), which contradicts the hypothesis that $w \geqslant \tau$. Hence our assumption that $w_r \not\geqslant \tau_r$ (for some r) is wrong.

This completes the proof of Proposition 4.2.

PROPOSITION 4.3.   *The set $\{w / \text{for any } \tau \leqslant w, \ \tau_k \leqslant w_k, \ 1 \leqslant k \leqslant n\}$ is precisely the set of Kempf elements.*

Proof. The inclusion $\supseteq$ follows from Proposition 4.2. Let now w be such that for any $\tau \leqslant w$, $\tau_k \leqslant w_k$, $1 \leqslant k \leqslant n$. We *claim*: w is a Kempf element.

If not let $w_k = s_i \cdots s_{k+1} s_k$ and $w_{k+1} = s_j \cdots s_{k+1}$, for some $k \leqslant n - 1$ (here $k + 1 \leqslant j < i$). Now, $w > s_i \cdots s_{k+1} = \tau$ say. Now $\tau = \tau_1 \cdots \tau_n$, where $\tau_l = \text{Id}$, $l \neq k + 1$ and $\tau_{k+1} = s_i \cdots s_{k+1}$. Thus $w \geqslant \tau$; but $w_{k+1} \not\geqslant \tau_{k+1}$ (since, by assumption $j < i$). (If $w_{k+1} = \text{Id}$ and $w_k = s_i \cdots s_{k-1} s_k$, where $i \geqslant k + 1$, then taking $\tau = s_{k+1}$, we obtain $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k-1}$.) This completes the proof of Proposition 4.3.

Remark 4.4.   Just for the sake of interest we would like to mention one example here. Consider $w = (s_3 s_2 s_1)(s_2)(s_3)$ and $\tau = (s_1)(s_3 s_2)$ in $W(SL_4)$. We have, $w \geqslant \tau$, but $w_2 \not\geqslant \tau_2$ (observe that w is not a Kempf element).

THEOREM 4.5.   *Let $w \in W(SL_{n+1})$. Then on w (or $X(w)$), the two notions of Young diagrams being standard coincide if and only if w is a Kempf element.*

Proof.   Let $w = w_1 w_2 \cdots w_n$ be a Kempf element. Since on any $X(\rho)$, $\rho \in W$, any Young diagram standard on $X(\rho)$ is weakly standard (cf. [13]), proving the required result is equivalent to proving the result that any Young diagram weakly standard on $X(w)$ is in fact standard. Let then $\theta = (\theta_{ij})$ be a Young diagram of type $\underline{m} = (m_1, ..., m_n)$ weakly standard on $X(w)$, so that, we have $\theta_{ij} \in W^{(i)}$, $\theta_{ij} \geqslant \theta_{ij+1}$, $\theta_{im_i} > \theta_{i+11}$ (as elements of Z) and $w > \theta_{ij}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$. To make it very precise, let $\theta_{ij} = v_1^{(j)} v_2^{(j)} \cdots v_i^{(j)}$ (reduced expression for θ_{ij}, where $v_k^{(j)}$ is a right-end segment of $u_k = s_n \cdots s_{k+1} s_k$ (cf. Proposition 2.3)). Then $\theta_{ij} \geqslant \theta_{ij+1}$ (in the usual sense, i.e., as elements of W). And $\theta_{im_i} > \theta_{i+11}$, in Z, is equivalent to the condition that $v_1^{(im_i)} v_2^{(im_i)} \cdots v_i^{(im_i)}$ is $> v_1^{(i+11)} v_2^{(i+11)} \cdots v_i^{(i+11)}$ (cf. Definition 2.7). Define $\lambda = (\lambda_{ij})$, where $\lambda_{ij} = v_1^{(j)} \cdots v_i^{(j)} v_{i+1}^{(i+11)} v_{i+2}^{(i+21)} \cdots v_n^{(n1)}$ $(= \theta_{ij}\mu_i$, say) (where $v_t^{(t1)}$ is taken to be Id, if for all $s \geqslant t$, $m_s = 0$ and $v_t^{(t1)}$ is taken to be $v_t^{(s1)}$, if $m_t = 0$ and s is the least integer $> t$ such that $m_s \neq 0$). The fact that $\lambda_{ij} < w$ follows from the fact that $w > \theta_{rs}$, $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant m_r$, and Proposition 4.2. The fact that $\lambda_{ij} > \lambda_{ij+1}$ follows from the fact $\theta_{ij} > \theta_{ij+1}$ (note that $\lambda_{ij} = \theta_{ij}\mu_i$ and $\lambda_{ij+1} = \theta_{ij+1}\mu_i$; also note that $\theta_{ij}\mu_i$ and $\theta_{ij+1}\mu_i$ are reduced (cf. Proposition 2.3)). The fact that $\lambda_{im_i} > \lambda_{i+11}$ follows from the fact that $\theta_{im_i} > v_1^{(i+11)} v_2^{(i+11)} \cdots v_i^{(i+11)}$ (note that $\lambda_{ij} = \theta_{im_i}\mu_i$ and

$$\lambda_{i+11} = v_1^{(i+11)} v_2^{(i+11)} \cdots v_{i+1}^{(i+11)} v_{i+2}^{(i+21)} \cdots v_n^{(n1)}$$
$$= v_1^{(i+11)} \cdots v_i^{(i+11)} \mu_i).$$

Thus we have $w \geqslant \lambda_{11} \geqslant \lambda_{12} \geqslant \cdots \geqslant \lambda_{1m_1} \geqslant \lambda_{21} \geqslant \cdots \geqslant \lambda_{nm_n}$ (in W), where λ_{ij} is a lift for θ_{ij} under $W \to W/W_i$. In other words λ is a defining pair for θ on $X(w)$ (cf. 12] or [13]) and thus θ is standard on $X(w)$.

Conversely, let $w = w_1 w_2 \cdots w_n (\in W)$ such that all weakly standard Young diagrams on w are in fact standard. We shall now show that w is a Kempf element. If w is not a Kempf element, let d be the smallest integer such that $l(w_d) \not\leqslant l(w_{d+1}) + 1$, say $w_d = s_i \cdots s_{d+1} s_d$, $i \geqslant d+1$ and $w_{d+1} = s_j \cdots s_{d+2} s_{d+1}$ $(j < i)$ or Id. Now, identifying W with S_{n+1}, let $\tau = w_1 w_2 \cdots w_{d-1} = (c_1 \cdots c_{n+1})$ (as a permutation). Now for $t \geqslant d$, τs_t remains reduced and hence $l(\tau s_t) (= l(\tau) + 1)$ is $> l(\tau)$. Hence $\tau(\alpha_t) > 0$, where α_t is the root $e_t - e_{t+1}$ (cf. [11, Proposition 1.4]). In particular, we have, $c_k < c_m$, $d \leqslant k < m \leqslant n+1$ (since $\tau(e_k - e_{k+1}) = e_{c_k} - e_{c_{k+1}} > 0$, so that

$c_k < c_{k+1}$ and so on). For any k, $1 \leqslant k \leqslant n$, let $w^{(k)}$ denote the projection of w on W/W_k (note that $w^{(k)}$ is just $w_1 w_2 \cdots w_k$)). Now

$$w^{(d)} = \tau w_d = (c_1 \cdots c_{n+1})(12 \cdots d - 1 \ i + 1 \ d \cdots i \cdots n + 1)$$

$$= (c_1 \cdots c_{d-1} c_{i+1})$$

and

$$w^{(d+1)} = (c_1 \cdots c_{d-1} c_{i+1} c_d) \qquad \text{if} \quad w_{d-1} = \mathrm{Id}$$

$$= (c_1 \cdots c_{d-1} c_{i+1} c_j) \qquad \text{if} \quad w_{d+1} = s_j \cdots s_{d-1}$$

(note that $d + 1 \leqslant j < i$). Now consider the weakly standard diagram on

$$\begin{array}{cc} d & d+1 \end{array}$$
$X(w)$ of type $(0, 0, ..., 1, \quad 1 \quad, 0 \cdots 0)$ given by

$$(*) \qquad \begin{pmatrix} c_1 \\ \vdots \\ c_{d-1} \\ c_i \end{pmatrix} > \begin{pmatrix} c_1 \\ \vdots \\ c_{d-1} \\ c_j \\ c_{i+1} \end{pmatrix} \qquad \text{(assume } w_{d-1} \neq \mathrm{Id})$$

(note that the d-tuples $(c_1, ..., c_{d-1}, c_i)$ and $(c_1, ..., c_{d-1}, c_j, c_{i+1})$ may not be in the ascending order; nevertheless, it is easily seen that

$(c_1, ..., c_{d-1}, c_i$, arranged in ascending order)

$> (c_1, ..., c_{d-1}, c_j, c_{i+1}$, arranged in ascending order since $c_i > c_j$).

Also, they are both $< w$. In fact $(c_1 \cdots c_{d-1} c_i)$ (resp. $(c_1 \cdots c_{d-1} c_j c_{i+1})$ has the reduced expression $w_1 \cdots w_{d-1}(s_{i-1} \cdots s_d)$ (resp. $w_1 \cdots w_{d-1} s_{j-1} \cdots s_d s_i \cdots s_{d+1})$, where $(c_1 \cdots c_{d-1} c_i)$ as an element in W $(= S_{n+1})$ is identified with the permutation $(c_1 \cdots c_{d-1} c_i x_{i+1} \cdots x_{n+1})$, where $\{x_{i+1}, ..., x_{n+1}\} = C(c_1, ..., c_{d-1}, c_i)$ in $(1, 2, ..., n+1)$ arranged in ascending order. A similar remark holds for $c_1 \cdots c_{d-1} c_j c_{i+1})$. Let $\{a_1, ..., a_d\}$ (resp. $\{b_1, ..., b_{d+1}\}$) be $\{c_1, ..., c_{d-1}, c_i\}$ (resp. $\{c_1, ..., c_{d-1}, c_j, c_j, c_{i+1}\}$) arranged in ascending order. Now the smallest λ on which $(*)$ is standard is given by the smallest element in W, which has the projection $(a_1 \cdots a_d)$ on W/W_d and which is $> (b_1 \cdots b_{d+1})$ (cf. Remark 2.16 and [13, Corollary 11.2']). Now $(*)$ is standard on w if and only if $w \geqslant \lambda$ (cf. Remark 2.16). But now $w \not\geqslant \lambda$, since projection of w on $W/W_{d+1} \not\geqslant$ projection of λ on W/W_{d+1} (note that projection of w (resp. λ) on W/W_{d+1} is $(c_1, ..., c_{d-1}, c_j, c_{i+1})$ (resp. $(c_1, ..., c_{d-1}, c_i, c_{i+1})$) arranged in ascending order and also note that $c_j < c_i < c_{i+1}$). If $w_{d+1} = \mathrm{Id}$, we carry out the same argument replacing c_j by

c_d. To be very precise, one takes (*) to be the weakly standard Young diagram

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{d-1} \\ c_i \end{pmatrix} > \begin{pmatrix} c_1 \\ \vdots \\ c_{d-1} \\ c_d \\ c_{i+1} \end{pmatrix}$$

Now the smallest λ on which this is standard has the projection on W/W_{d-1} to be $(c_1 \cdots c_{d-1} c_i c_{i+1})$ (arranged in ascending order), while the corresponding projection for w is $(c_1 \cdots c_{d-1} c_d c_{i+1})$ (arranged in ascending order). Thus $w^{(d+1)} \not\geq \lambda^{(d+1)}$ (note that $i \geq d+1$, so that $c_d < c_i$).

Thus in either case $w \not\geq \lambda$ and hence (*) is not standard on w contradicting the hypothesis that on w all weakly standard diagrams are standard. Hence such a d as above, namely $l(w_d) > l(w_{d+1}) + 1$ does not exist. Thus for all d, $1 \leq d \leq n-1$, $l(w_d) \leq l(w_{d+1}) + 1$, and hence w is a Kempf element.

This completes the proof of Theorem 4.5.

Now, Theorem 4.5 leads to the following

DEFINITION 4.6. Let Q be any parabolic subgroup of G. Then a Schubert variety $X(w)$ in G/Q will be called a *Kempf variety* if $X(w)$ has the property that all weakly standard Young diagrams on $X(w)$ are in fact standard.

We next prove the result (analogous to Theorem 4.5) for Kempf varieties in the case G is of type B_n or C_n. We first recall the definition of a Kempf variety in G/B (G being of type B_n or C_n). Let $u_i = s_i \cdots s_{n-1} s_n \cdots s_{i+1} s_i$. Then (recall cf. [10, Propositions B.1 and C.1]) any $w \in W$ has a reduced expression $w = w_1 w_2 \cdots w_n$, where w_i is a right-end segment of u_i, $1 \leq i \leq n$.

DEFINITION 4.7. Given $w = w_1 \cdots w_n$ call w a Kempf element if the following holds. Let $1 \leq i \leq n-1$.

If $w_{i+1} = u_{i+1}$, then w_i is arbitrary.
If $w_{i+1} = s_r \cdots s_{i+1}$, $i+1 \leq r \leq n-1$, then $w_i \leq s_r \cdots s_{i+1} s_i$.
If $w_{i+1} = s_r \cdots s_n \cdots s_{i+1}$, $i+1 < r \leq n$, then $w_i \leq s_{r-2} \cdots s_{i-1} s_i$.
If $w_{i+1} = \mathrm{Id}$, then $w_i \leq s_i$.

PROPOSITION 4.8. *Let w be a Kempf element. Then for any $\tau \in W$, we have $w \geq \tau$ if and only if $w_r \geq \tau_r$, $1 \leq r \leq n$, where $w = w_1 \cdots w_n$ and $\tau = \tau_1 \cdots \tau_n$.*

Proof. The proof of the implication $\Leftarrow$ is obvious. Now let $w \geq \tau$. If possible let $w_k < \tau_k$, for some k, $1 \leq k \leq n$. This implies in particular that $w_k \neq u_k$ ($= s_k \cdots s_n \cdots s_k$). Now we distinguish the following two cases.

Case 1. Let $w_k = s_r \cdots s_{k+1} s_k$ for some $r \leqslant n-1$, so that $\tau_k \geqslant s_{r+1} \cdots s_{k+1} s_k$. Now identifying W with $W(Sp_{2n})$ the projection of w on W/W_k, as an (increasing) k-tuple has the entry at the kth place to be $r+1$ (since w is a Kempf element, we have $w_t \leqslant s_r \cdots s_{t+1} s_t$, $t \leqslant k$) while the projection of τ on W/W_k, as an (increasing) k-tuple has the entry at the kth place to be $\geqslant r+2$ (since $\tau_k \geqslant s_{r+1} \cdots s_{k+1} s_k$). Hence $w^{(k)} \not\geqslant \tau^{(k)}$ (where for any $\rho \in W$ and t, $1 \leqslant t \leqslant n$, $\rho^{(t)} (\in W^{(t)})$ denotes the projection of ρ on W/W_t). Thus $w \not\geqslant \tau$, which contradicts the hypothesis that $w \geqslant \tau$. Thus, this case does not exist. (The same argument holds in the case $w_k = \mathrm{Id}$. To be very precise, if $w_k = \mathrm{Id}$, then $\tau_k \geqslant s_k$ and now, $w^{(k)}$ has the entry at the kth place to be r where $r \leqslant k$ (since w is a Kempf element, $w_t \leqslant s_{k-1} \cdots s_t$, $t \leqslant k-1$), while $\tau^{(k)}$ has the entry at the kth place to be $\geqslant k+1$ (since $\tau_k \geqslant s_k$). Thus $w^{(k)} \not\geqslant \tau^{(k)}$, which in turn implies that $w \not\geqslant \tau$ contradicting the hypothesis that $w \geqslant \tau$. Thus this possibility again does not exist.)

Case 2. Let $w_k = s_r \cdots s_n \cdots s_{k+1} s_k$, $k < r \leqslant n$, and let $\tau_k = s_t \cdots s_n \cdots s_k$, where $t < r$ (since $w_k < \tau_k$ and both w_k and τ_k are right-end segments of $s_k \cdots s_n \cdots s_k$). Now w is a Kempf element, implies that $w_i \leqslant s_{r-2} \cdots s_{i+1} s_i$, $i \leqslant k-1$. Hence we obtain $w^{(k)}$ has the entry at the kth place to be $r' (= 2n+1-r)$, while, $\tau^{(k)}$ has the entry at the kth place to be $\geqslant 2n+1-t$ (since $\tau_k = s_t \cdots s_n \cdots s_k$). Hence $w^{(k)} \not\geqslant \tau^{(k)}$ (since $r' < t'$ (as $t < r$)). Thus $w \not\geqslant \tau$, contradicting the hypothesis that $w \geqslant \tau$.

Thus both the cases lead to contradictions. Hence our assumption $w_k \not\geqslant \tau_k$ is wrong. Thus $w_k \geqslant \tau_k$ for all $1 \leqslant k \leqslant n$.

PROPOSITION 4.9. *The set* $\{ w \in W / \text{for any } \tau \leqslant w, \ \tau_k \leqslant w_k, \ 1 \leqslant k \leqslant n \}$ *is precisely the set of Kempf elements.*

Proof. The inclusion $\supseteq$ follows from Proposition 4.8. Let w be such that for any $\tau \leqslant w$, $\tau_k \leqslant w_k$, $1 \leqslant k \leqslant n$.

Claim. w is a Kempf element.
If not, let k be such that w_k and w_{k+1} do not satisfy the condition in Definition 4.7. In particular we have $w_{k+1} \neq u_{k+1} (= s_{k+1} \cdots s_n \cdots s_{k+1})$. We distinguish the following five cases.

Case 1. Let $w_k = s_i \cdots s_{k+1} s_k$ and $w_{k+1} = s_j \cdots s_{k+1}$, $k+1 \leqslant j < i \leqslant n$. Consider $\tau = \tau_1 \cdots \tau_n$, where

$$\tau_l = \mathrm{Id}, \qquad l \neq k+1$$
$$= s_i \cdots s_{k+1}, \qquad l = k+1.$$

Now $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k+1}$.

Case 2.   Let $w_k = s_t \cdots s_n \cdots s_k$ and $w_{k+1} = s_i \cdots s_{k+1}$, $i \leqslant n-1$ *and* $t \geqslant k$. Consider $\tau = \tau_1 \tau_2 \cdots \tau_n$, where

$$\begin{aligned}
\tau_l &= \mathrm{Id}, & l &\neq k+1 \\
&= s_t \cdots s_n \cdots s_{k+1}, & l &= k+1.
\end{aligned}$$

Now $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k+1}$.

Case 3.   Let $w_k = s_t \cdots s_k$, $t \leqslant n-1$ and $w_{k+1} = s_r \cdots s_n \cdots s_{k+1}$, $r-1 \leqslant t$.
   Consider $\tau = \tau_1 \tau_2 \cdots \tau_n$, where

$$\begin{aligned}
\tau_l &= \mathrm{Id}, & l &\neq k+1 \\
&= s_{r-1} s_r \cdots s_n \cdots s_{k+1}, & l &= k+1.
\end{aligned}$$

Now $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k+1}$.

Case 4.   Let $w_k = s_t \cdots s_n \cdots s_k$, $t \geqslant k$ and $w_{k+1} = s_r \cdots s_n \cdots s_{k+1}$, $r > k+1$. ($r > k+1$, since $w_{k+1} \neq u_{k+1}$). Consider $\tau = \tau_1 \tau_2 \cdots \tau_n$, where

$$\begin{aligned}
\tau_l &= \mathrm{Id}, & l &\neq k+1 \\
&= s_{r-1} s_r \cdots s_n \cdots s_{k+1}, & l &= k+1.
\end{aligned}$$

Now $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k+1}$.

Case 5.   Let $w_k \geqslant s_{k+1} s_k$ and $w_{k+1} = \mathrm{Id}$.
   Consider $\tau = \tau_1 \cdots \tau_n$, where

$$\begin{aligned}
\tau_l &= \mathrm{Id}, & l &\neq k+1 \\
&= s_{k+1}, & l &= k+1.
\end{aligned}$$

Now $w \geqslant \tau$, but $w_{k+1} \not\geqslant \tau_{k+1}$.
   Thus all possible cases contradict the hypothesis. Hence such a k does not exist, which implies that w is a Kempf element.

THEOREM 4.10.   *Let* $w \in W$ (G *being of type* B_n *or* C_n). *Then, on* $X(w)$ *the two notions of Young diagrams being standard coincide if and only if* w *is a Kempf element.*

   Proof.   The proof of the implication $\Leftarrow$ is completely analogous to the proof of the corresponding implication in Theorem 4.5 (one uses Proposition 4.8, etc.).
   Let now $X(w)$ be such that on $X(w)$ a weakly standard Young diagram is in fact standard. Let $w = w_1 w_2 \cdots w_n$. Claim: w is a Kempf element. If not, let d be the smallest integer such that the conditions relating w_d and

w_{d+1} (cf. Definition 4.7) are not satisfied (in particular this implies $w_{d+1} < u_{d+1} (= s_{d+1} \cdots s_n s_{n-1} \cdots s_{d+1})$). Then we have the following five possible cases. We shall now show that all the possible cases lead to contradictions, from which the claim will follow. (In the following proof, for a $w \in W$, we shall repeatedly use, both its reduced expression (in terms of the simple reflections, cf. Proposition 2.3) and its representation as a permutation identifying W with $W(Sp_{2n})$. To be very precise, given a Young diagram θ weakly standard on G/B, to see θ is a diagram on $X(w)$ (cf. Definition 2.9) we use the reduced expression for w and to get the smallest Schubert variety $X(\lambda)$ on which θ is standard (cf. Remark 2.16), we use the permutation forms of elements of W).

Before discussing the various cases, we want to observe the following:

(i) The element $s_r \cdots s_{d+1} s_d$ (where $r \leqslant n-1$) as a permutation

$$
\begin{array}{cccc}
d & d+1 & \cdots & r+1
\end{array}
$$
$$
= (12 \cdots d-1 \ r+1 \quad d \quad d+1 \cdots \quad r \quad r+2 \cdots n)
$$

(ii) The element $s_t \cdots s_n \cdots s_d$ (where $d \leqslant t \leqslant n$) as a permutation

$$
\begin{array}{cccc}
d \ d+1 & \cdots & & t
\end{array}
$$
$$
= (12 \cdots d-1 \ t' \quad d \quad d+1 \cdots t-1 \ t+1 \cdots n), \text{ where } t' = 2n+1-t.
$$

We also need the following lemma for our discussion.

LEMMA 4.11. *Let* $W = W(SL_n)$ *and let* $w \in W^{(d)}$, *say* $w = w_1 w_2 \cdots w_d$ *for some* d, $1 \leqslant d \leqslant n$ *(cf. Proposition 2.5). Further let* $w = (a_1 \cdots a_d a_{d+1} \cdots a_n)$, *where* $1 \leqslant a_1 < a_2 < \cdots < a_d \leqslant n$ *and* $1 \leqslant a_{d+1} < a_{d+2} < \cdots < a_n \leqslant n$. *Then for* $1 \leqslant i \leqslant n-d$,

$$
a_{d+i} = d+i, \qquad \text{if} \quad l(w_d) < i
$$
$$
= k+i-1, \qquad \text{where } k \text{ is the smallest integer such that } l(w_k) \geqslant i.
$$

Proof. (By decreasing induction on i and decreasing induction on $l(w)$.)

Starting point of Induction. $i = n-d$. Then $d+i = n$. We need to show $a_n = n$. Suppose $l(w_d) < n-d$, this implies in particular that $w_d \leqslant s_{n-2} \cdots s_{d+1} s_d$ and hence $a_d \leqslant n-1$ (cf. Remark 2.6(ii)). Hence $n \in \{a_{d+1}, ..., a_n\}$ and in fact $a_n = n$ (since $a_{d+1} < a_{d+2} < \cdots < a_n$).

$l(w) (= \dim G/P_d) = d(n-d)$. In this case we have $l(w_k) = n-d$, $1 \leqslant k \leqslant d$, and $w = (n-d+1, \ n-d+2,..., \ n, \ 1, \ 2, \ 3,..., \ n-d)$. In particular we have $a_{d+i} = i$, $1 \leqslant i \leqslant n-d$. Now, for every i, $1 \leqslant i \leqslant n-d$, $l(w_1) \geqslant i$, and hence the smallest k such that $l(w_k) \geqslant i$ is given by $k = 1$. Hence $k+i-1 = i = a_{d+i}$, as required.

Now let $i < n-d$ and w arbitrary. If $l(w_d) < i$, then $w_d \leqslant s_{d+i-2} \cdots s_{d+1} s_d$

(recall (cf. Section 2) that w_j is Id or a right-end segment of $s_{n-1} \cdots s_d$ (where $W = S_n$)) and hence $a_d \leq d + i - 1$ (cf. Remark 2.6(ii)). Hence $d + i \in \{a_{d+1}, \ldots, a_n\}$ (since $a_1 < a_2 < \cdots < a_d$). In fact $d + i$, $d + i + 1, \ldots, n \in \{a_{d+1}, \ldots, a_n\}$ and the fact that $a_{d+1} < a_{d+2} < \cdots < a_n$, implies $a_{d+j} = d + j$, $i \leq j \leq n - d$. Now let $l(w_d) \geq i$ and let k be the least integer such that $l(w_k) \geq i$. Now, $l(w_j) < i$, $j < k$, implies $w_j \leq s_{j+i-2} \cdots s_{j+1} s_j$, $j < k$ and hence $a_j = j + i - 1 < k + i - 1$. Also, $w_k \geq s_{k+i-1} \cdots s_{k+1} s_k$ (since $l(w_k) \geq i$) implies a_k (and hence a_j, $k \leq j \leq d$) is $\geq k + i$. Hence we obtain that $k + i - 1 \in \{a_{d+1}, \ldots, a_n\}$. Now consider $\tau \in W^{(d)}$, $\tau = \tau_1 \tau_2 \cdots \tau_d$, where

$$\tau_j = w_j, \qquad j \neq k - 1$$
$$= s_{k+i-2} \cdots s_k s_{k-1}.$$

Then $\tau > w$ (since $\tau_{k-1} > w_{k-1}$, by our assumption on k). Now if $\tau = (b_1 \cdots b_d \cdots b_n)$, then $b_{d+i} = k + i - 2$ (by induction hypothesis on $l(\tau)$). Hence $a_{d+i} \geq k + i - 2$ (observe that, since $\tau \geq w$, $(b_1, \ldots, b_d) \geq (a_1, \ldots, a_d)$ equivalently $(b_{d+1}, \ldots, b_n) \leq (a_{d+1}, \ldots, a_n)$). Claim. $a_{d+i} = k + i - 1$.

For, we have

$$a_{d+i+1} = d + i + 1, \qquad \text{if} \quad l(w_d) < i + 1$$
$$= k' + i, \qquad \text{where } k' \text{ is the least integer such that } l(w_{k'}) \geq i + 1$$

(by induction hypothesis on i). Now $k' \geq k$ and hence in either case a_{d+i+1} is $\geq k + i$. This together with the fact that $a_{d+i} \geq k + i - 2$ and the fact that $k + i - 1 \in \{a_{d+1}, \ldots, a_n\}$ implies that $a_{d+i} = k + i - 1$, as required.

COROLLARY 4.12. *Let* $w \in W^{(d)}$, *say* $w = (a_1 \cdots a_d \cdots a_n)$. *Let* $r > d$ *be such that* $a_r (= x)$ *is* $< r$. *Then* $w \geq s_x s_{x+1} \cdots s_{r-1}$.

Proof. Write $i = r - d$. Then, since $a_r < r$, we have $l(w_d) \geq i$ and $a_r = k + i - 1$, where k is the least integer such that $l(w_k) \geq i$ (by Lemma 4.11 above). Now $w_k \geq s_{k+i-1} \cdots s_{k+1} s_k$ (since $l(w_k) \geq i$). Hence for every j, $k \leq j \leq d$, $w_j \geq s_{j+i-1} \cdots s_{j+1} s_j$ (cf. Proposition 2.5). In particular $w_d \geq s_{d-i-1} \cdots s_{d+1} s_d = s_{r-1} \cdots s_{d+1} s_d$ (note that $r = d + i$). Thus $w (= w_1 \cdots w_k \cdots w_d)$ contains the subexpression $\prod_{j=k}^{d} s_{j+i-1} = s_x s_{x-1} \cdots s_d$, as required.

Return to the proof of Theorem 4.10.

Case A. $w_{d+1} = s_r \cdots s_n \cdots s_{d+1}$, where $r > d + 1$ (since $w_{d+1} < u_{d+1} = s_{d+1} \cdots s_n \cdots s_{d+1}$ and w_m, $1 \leq m \leq n$, is either Id or a right-end segment of u_m (cf. Proposition 2.3)).

$$w_d = s_t \cdots s_d, \qquad r - 1 \leq t \leq n - 1.$$

Let $w_1 \cdots w_{d-1} = (a_1 \cdots a_{d-1} \cdots a_n)$. Now, for k, $d \leqslant k \leqslant n$, $w_1 w_2 \cdots w_{d-1} s_k$ is reduced (cf. Proposition 2.3) and hence $a_k < a_j$, $d \leqslant k \leqslant j \leqslant n-1$, and $a_k \leqslant n$, $d \leqslant k \leqslant n$ (because, if $\tau = w_1 \cdots w_{d-1}$, then $\tau s_k > \tau$, $d \leqslant k \leqslant n$, implies $\tau(\alpha_k) > 0$, $d \leqslant k \leqslant n$, and $\tau(\alpha_k) = e_{a_k} - e_{a_{k+1}}$, if $\alpha_k = e_k - e_{k+1}$, $k \leqslant n-1$, and $\tau(\alpha_n) = 2e_{a_n}$ (cf. [14, Sect. 3])). Thus $\{a_d, a_{d+1}, ..., a_n\} = \complement (a_1, ..., a_{d-1})$ in $(1, 2, ..., n)$ arranged in ascending order (note that $a_k \leqslant n$, for $k \leqslant d-1$, since by our assumption on d, the condition in Definition 4.7, relating w_{k-1} and w_k is satisfied for $k \leqslant d$ and $w_d \leqslant s_{n-1} \cdots s_d$). In particular we have $a_m \leqslant m$, for $d \leqslant m \leqslant n$. Let $a_{r-1} = x$.

Observe that $x > 1$ (for $x = 1$ would imply $r - 1 = d$, which is not true, since $r > d + 1$). Let

$$v_x = s_x s_{x+1} \cdots s_{r-2}, \qquad \text{if} \quad x < r - 1$$
$$= \text{Id}, \qquad \text{if} \quad x = r - 1.$$

Then $w_1 \cdots w_{d-1}$ contains v_x as a subexpression (cf. Corollary 4.12). Hence $v_x s_{r-1} w_{d+1} \leqslant w$ (since $w_d = s_t \cdots s_{d+1} s_d$ where $t \geqslant r - 1$). Denote $v_x s_{r-1} w_{d+1}$ by θ. Let

$$\mu = s_{x-1} \cdots s_{d+1} s_d, \qquad \text{if} \quad x \geqslant d+1$$
$$= s_{x-1} \cdots s_{d-1} s_d, \qquad \text{if} \quad x \leqslant d$$

(note that $x > 1$, as already observed). Now $\mu \leqslant w$; for, if $x \geqslant d+1$, then $\mu \leqslant w_d$ (since $w_d = s_t \cdots s_{d+1} s_d$, where $t \geqslant r - 1 \geqslant x$); if $x \leqslant d$, (one may assume $w_{x-1} > \text{Id}$, for if $w_{x-1} = \text{Id}$, then $w \in W(Sp_{2(n-x+1)})$ and one may use induction on the rank of G; thus in fact, we may assume $w_1 > \text{Id}$, which implies that $w_k > \text{Id}$, $k \leqslant d$, since the condition in Definition 4.7, relating w_{k-1} and w_k is satisfied for $k \leqslant d$ (by our assumption)), then $w > s_{x-1} s_x \cdots s_{d-1} s_d (= \mu)$.

Now $\theta (= s_x s_{x+1} \cdots s_{r-2} s_{r-1} s_r \cdots s_n \cdots s_{d+1}) < \mu$ (as element of Z) In fact as permutations θ and μ are given by

$$\begin{array}{ccccc} & d+1 \, d+2 & \cdots & & x \\ \theta = (1 \cdots & x' & d+1 \, d+2 \cdots x-1 \, x+1 \cdots n), & & \text{if} \quad x \geqslant d+1 \end{array}$$

$$\begin{array}{ccccc} & x & x+1 & \cdots & d \quad d+1 \\ = (1 \cdots & x+1 \, x+2 & \cdots & d+1 & x' \quad d+2 \cdots n), & \text{if} \quad x \leqslant d \end{array}$$

and

$$\begin{array}{ccccc} & d \, d+1 & \cdots & & x \\ \mu = (1 \cdots x & d & \cdots & x-1 \, x+1 \cdots n), & & \text{if} \quad x \geqslant d+1 \end{array}$$

$$\begin{array}{ccccc} & x-1 & x & \cdots & d \quad d+1 \\ = (1 \cdots & x & x+1 & d+1 \, x-1 \, d+2 \cdots n), & \text{if} \quad x \leqslant d \end{array}$$

and $\theta < \mu$ (as elements of Z). Thus

(*) $0 < \mu$ is a weakly standard Young diagram on $X(w)$ (since both 0 and $\mu \leqslant w$ as already observed). Now the smallest $X(\lambda)$ on which (*) is standard is such that $\lambda^{(d+1)}$ (projection of λ on W/W_{d+1}) as an increasing $(d+1)$-tuple has the entry at the $(d+1)$th place to be $(x-1)'$ whereas $w^{(d+1)}$ has the entry at the $(d+1)$th place to be x' (for $w^{(d+1)} \equiv (a_1 \cdots a_{d-1} \cdots a_n) \, w_d w_{d+1} \pmod{W_{d+1}}$).

$$
d \quad d+1 \cdots t+1
$$

Recall, $w_d \,(\text{resp. } w_{d+1}) = (1 \cdots d-1 \, t+1 \quad d \quad \cdots \quad t \quad t+2 \cdots n)$

$$
d+1 \, d+2 \cdots \quad r
$$

$(\text{resp. } 1 \cdots d \quad r' \quad d+1 \cdots r-1 \, r+1 \cdots n))$

$= (a_1 \cdots a_{d-1} a_{t+1} a'_{r-1} \cdots) \qquad (\text{note that } d+2 \leqslant r \leqslant t+1)$

$= (a_1 \cdots a_{d-1} a_{t+1} x' \cdots)$

(where $a_k \leqslant n$, $1 \leqslant k \leqslant d-1$ and also for $k = t+1$).

Thus $w^{(d+1)} \not\geqslant \lambda^{(d+1)}$ and hence $w \not\geqslant \lambda$, which implies that (*) cannot be standard on $X(w)$. This contradicts the hypothesis that all weakly standard diagrams on $X(w)$ are in fact standard.

Case B.

$$
w_{d+1} = s_r \cdots s_n \cdots s_{d+1} \qquad (\text{where } r > d+1)
$$

$$
w_d = s_t \cdots s_n \cdots s_d, \qquad d \leqslant t \leqslant n.
$$

Let k be the smallest integer $\leqslant d$ such that $w_k \geqslant s_n \cdots s_k$ (recall (cf. Proposition 2.3) that for $1 \leqslant m \leqslant n$, w_m is either Id or a right-end segment of $u_m = s_m \cdots s_n \cdots s_m$). Then, we have, $w_m \leqslant s_{n-1} \cdots s_m$, for $m \leqslant k-1$ and $w_m = u_m$, for $k+1 \leqslant m \leqslant d$ (since by our assumption on d, the condition in Definition 4.7 which we shall henceforth refer to as the *Kempf condition* relating w_{m-1} and w_m is satisfied for $m \leqslant d$). Now for $k \leqslant m \leqslant n$, τs_m is reduced and hence $\tau(\alpha_m) > 0$, $k \leqslant m \leqslant n$. This implies that for $k \leqslant m < j \leqslant n$, $a_m < a_j \leqslant n$ (since

$$
\tau(\alpha_m) = e_{a_m} - e_{a_{m+1}}, \qquad \text{if } \alpha_m = e_m - e_{m+1} \qquad (m \leqslant n-1)
$$

$$
= 2e_{a_n}, \qquad \text{if } \alpha_m = 2e_m.
$$

Also, $a_m \leqslant n$, for $m \leqslant k-1$ (since for $m \leqslant k-1$, $w_m \leqslant s_{n-1} \cdots s_m$, etc.). Thus $(a_k, ..., a_n) = \mathbb{C} \,(a_1, ..., a_{k-1})$ in $(1, ..., n)$ arranged in ascending order. For $1 \leqslant t \leqslant n$, let $w^{(t)}$ denote the projection of w on W/W_t (observe that $w^{(t)} = w_1 w_2 \cdots w_t$ (cf. Remark 2.4)). Now writing $w^{(d+1)}$ and $w^{(d)}$ as increasing d-tuples, we have

$$w^{(d+1)} = (1) \; (c_1,..., c_{k-1}, a'_r, a'_d, a'_{d-1},..., a'_k), \qquad \text{if } w_k = u_k$$

$$= (2) \; (c_1,..., c_{k-1}, a'_r, a'_d, a'_{d-1},..., a'_k), \qquad \text{if } w_k = s_t \cdots s_n \cdots s_k$$
$$\text{where } k < t < d$$

$$= (3) \; (c_1,..., c_{k-1}, a'_r, a'_t, a'_{d-1},..., a'_k), \qquad \text{if } d \leqslant t < r \; (w_k \text{ as in } (2))$$

$$= (4) \; (c_1,..., c_{k-1}, a'_t, a'_{r-1}, a'_{d-1},..., a'_k), \quad \text{if } t \geqslant r \; (w_k \text{ as in } (2))$$

where $\{c_1,..., c_{k-1}\} = \{a_1,..., a_{k-1}\}$ arranged in ascending order) and

$$w^{(d)} = (1) \; (c_1,..., c_{k-1}, a'_d, a'_{d-1},..., a'_k)$$

$$= (2) \; (c_1,..., c_{k-1}, a'_d, a'_{d-1},..., a'_k)$$

$$= (3) \; (c_1,..., c_{k-1}, a'_t, a'_{d-1},..., a'_k)$$

$$= (4) \; (c_1,..., c_{k-1}, a'_t, a'_{d-1},..., a'_k)$$

respectively. Now consider the weakly standard Young diagram (μ, θ), on $X(w)$, where $\mu \in W^{(d)}$ and $\theta \in W^{(d+1)}$, μ and θ being defined as follows.

$$\theta = w^{(d+1)} \qquad \text{(in all the four cases)}$$

and

$$\mu = (1) \; (c_1,..., c_{k-1}, a'_{d+1}, a'_d,..., a'_{k+1})$$

$$= (2) \; (c_1,..., c_{k-1}, a'_{d+1}, a'_d,..., a'_{k+1})$$

$$= (3) \; \text{(a) } (c_1,..., c_{k-1}, a'_t, a'_d,..., a'_{k+1}), \qquad \text{if } t > d$$
$$\text{(b) } (c_1,..., c_{k-1}, a'_{t+1}, a'_d,..., a'_{k+1}), \qquad \text{if } t = d$$

$$= (4) \; (c_1,..., c_{k-1}, a'_t, a'_d, a'_{d-1},..., a'_{k+1}).$$

Observe that $\theta < \mu$ (as elements of Z (in fact in cases (1)–(3) θ and μ have the same entries in the first d places, except the kth place, where the entries are given by a'_r (resp. a'_{d+1}) in the first two cases; a'_r (resp. a'_t) in (3(a)); a'_r (resp. a'_{d+1}) in (3(b)); in (4), θ and μ have the same entries in the first d places except the $(k+1)$th place where the entries are given by a'_{r-1} and a'_d, respectively. And we have, $a'_r < a'_{d+1}$, $a'_{r-1} < a'_d$, $a'_r < a'_t$ (in (3(a)) etc., (since $a_r > a_{d+1}$, $a_r > a_t$, (when $r > t$) by our work on $w_1 \cdots w_{k-1} = (a_1 \cdots a_{k-1} a_k \cdots a_n)$, in the beginning of case B). Also, $\mu \leqslant w^{(d)}$ (obvious) and $\theta = w^{(d+1)}$. Thus (μ, θ) is weakly standard on $X(w)$. Now the smallest $X(\lambda)$ on which (μ, θ) is standard is given by

$$\lambda = (1) \; (c_1,..., c_{k-1}, a'_{d+1}, a'_d,..., a'_{k+1}, a'_k,...)$$

$$= (2) \; (c_1,..., c_{k-1}, a'_{d+1}, a'_d,..., a'_{k+1}, a'_k,...)$$

$$= (3) \; \text{(a) } (c_1,..., c_{k-1}, a'_t, a'_d,..., a'_{k+1}, a'_k,...), \qquad \text{if } t > d$$
$$\text{(b) } (c_1,..., c_{k-1}, a'_{d+1}, a'_d,..., a'_{k+1}, a'_k,...), \qquad \text{if } t = d$$

$$= (4) \; (c_1,..., c_{k-1}, a'_t, a'_d, a,..., a'_{k+1}, a'_k,...).$$

In all cases, one finds that $w^{(d+1)} \not\geqslant \lambda^{(d+1)}$ (for in (1)–(3), the entry at the kth place in $w^{(d+1)} \not\geqslant$ the entry in the kth place in $\lambda^{(d+1)}$, since $a'_r < a'_{d+1}$ (in (1) and (2)), $a'_r < a'_t$ (in (3(a))), $a'_r < a'_{d+1}$ (in (3(b)))); in (4), the entry in the $(k+1)$th place in $w^{(d+1)} \not\geqslant$ the corresponding entry in $\lambda^{(d+1)}$ (since $a'_{r-1} < a'_d$)). Thus $w \not\geqslant \lambda$ and hence (μ, θ) cannot be standard on $X(w)$ contradicting the hypothesis that all Young diagrams weakly standard on $X(w)$ are in fact standard.

It should be remarked that for the above argument to be valid, one requires $d > k$. Suppose $d = k$. The above argument holds in (1), (2), and (3b)) only. (In (3(a)) and (4), the d-tuple (corresponding to μ) as a set is $\subseteq$ the $(d+1)$-tuple corresponding to θ. Hence (μ, θ) in fact remains standard (a defining pair for (μ, θ) may be taken to be (λ, λ), where

$$\lambda = (c_1,..., c_{k-1}, a'_t, a'_r) \qquad \text{in case (3(a))}$$
$$= (c_1,..., c_{k-1}, a'_t, a'_{r-1}) \qquad \text{in case (4)}.$$

Nevertheless, in (3(a)) and (4), we consider the weakly standard Young diagram (μ', θ'), where

$$\mu' = (a_1,..., a_{d-1}, a_t), \qquad \text{if} \quad t < r$$
$$= (a_1,..., a_{d-1}, a_{r-1}), \qquad \text{if} \quad r \leqslant t$$

and

$$\theta' = (a_1,..., a_{d-1}, a_{t-1}, a'_t), \qquad \text{if} \quad t < r$$
$$= (a_1,..., a_{d-1}, a_{r-2}, a'_{r-1}), \qquad \text{if} \quad r \leqslant t$$

(observe that $\theta' < \mu'$ (as element of Z), since $a_{t-1} < a_t$, $a_{r-2} < a_{r-1}$ (recall that for $k = d \leqslant m < j \leqslant n$, $a_m < a_j$ and that $t-1 \geqslant d$ and $r-2 \geqslant d$ in (3(a)) and (4)); also, observe that, $\theta' \leqslant w^{(d+1)}$ and $\mu' \leqslant w^{(d)}$. Thus (μ', θ') is wekaly standard on $X(w)$). Now the smallest $X(\lambda')$ on which (μ', θ') is standard is such that $\lambda^{(d+1)} \not\leqslant w^{(d+1)}$ (The $(d+1)$th entry in $\lambda^{(d+1)}$ is a'_{t+1} (resp. a'_{r-2}) while the corresponding entry in $w^{(d+1)}$ is a'_t (resp. a'_{r-1})). Thus (μ', θ') can not be standard on $X(w)$ contradicting the hypothesis that on $X(w)$, all weakly standard Young diagrams are in fact standard.

Case C.

$$w_{d+1} = s_r \cdots s_{d+1}, \qquad d+1 \leqslant r < n$$
$$w_d = s_t \cdots s_d, \qquad r+1 \leqslant t < n.$$

For $k \leqslant d$, we have $w_k \leqslant s_t \cdots s_k$ (by our assumption on d). The discussion in this case is completely analogous to the proof of Theorem 4.5.

Case D.

$$w_{d+1} = s_r \cdots s_{d+1}, \qquad d+1 \leqslant r \leqslant n-1$$

$$w_d = s_t \cdots s_n \cdots s_d, \qquad d \leqslant t \leqslant n.$$

The discussion in this case is very much analogous to that of Case B. Let k be the smallest integer $\leqslant d$ such that, $w_k > s_{n-1} \cdots s_k$. Now the Kempf condition in Definition 4.7 implies that $w_m \leqslant s_{n-1} \cdots s_m$, $m \leqslant k-1$ and $w_m = u_m$, $k+1 \leqslant m \leqslant d$. Let $\tau = w_1 \cdots w_{k-1} = (a_1 \cdots a_{k-1} a_k \cdots a_n)$. Then, for $k \leqslant m \leqslant n-1$, we have $a_m < a_{m+1} \leqslant n$ (since τs_m, $m \geqslant k$ remains reduced, we have $\tau(\alpha_m) > 0$ (and

$$\left. \begin{array}{ll} \tau(\alpha_m) = e_{a_m} - e_{a_{m+1}}, & \text{if } m \leqslant n-1 \\ = 2e_{a_n}, & \text{if } m = n \end{array} \right).$$

Hence

$$w^{(d+1)} = (1) \ (c_1, ..., c_k, a'_d, ..., a'_k), \qquad \text{if } w_k = u_k$$

$$= (2) \ (c_1, ..., c_k, a'_d, ..., a'_k), \qquad \text{if } w_k = s_t \cdots s_n \cdots s_k$$
$$\text{where } k < t < d$$

$$= (3) \ (c_1, ..., c_k, a'_t, a'_{d-1}, ..., a'_k), \qquad \text{if } d \leqslant t < r+1$$
$$(w_k \text{ as in } (2))$$

$$= (4) \ (c_1, ..., c_k, a'_t, a'_{d-1}, ..., a'_k), \qquad \text{if } t \geqslant r+1$$
$$(w_k \text{ as in } (2))$$

where $\{c_1, ..., c_k\} = \{a_1, ..., a_{k-1}, a_{r+1},$ arranged in ascending order$\}$ in (1)-(3) (resp. $\{a_1, ..., a_{k-1}, a_r$ arranged in ascending order$\}$ in (4)), and

$$w^{(d)} = (1) \ (b_1, ..., b_{k-1}, a'_d, ..., a'_k)$$

$$= (2) \ (b_1, ..., b_{k-1}, a'_d, ..., a'_k)$$

$$= (3) \ (b_1, ..., b_{k-1}, a'_t, a'_{d-1}, ..., a'_k)$$

$$= (4) \ (b_1, ..., b_{k-1}, a'_t, a'_{d-1}, ..., a'_k)$$

where $\{b_1, ..., b_{k-1}\} = \{a_1, ..., a_{k-1},$ arranged in ascending order$\}$. Now consider the weakly standard Young diagram (μ, θ) on $X(w)$, where $\mu \in W^{(d)}$ and $\theta \in W^{(d+1)}$, μ and θ being defined as follows:
$\theta = w^{(d+1)}$ (in all the four cases) and

$$\mu = (1) \ (b_1, ..., b_{k-1}, a'_{d+1}, a'_d, ..., a'_{k+1})$$

$$= (2) \ (b_1, ..., b_{k-1}, a'_{d+1}, a'_d, ..., a'_{k+1})$$

$$= (3) \ (a) \ (b_1, ..., b_{k-1}, a'_t, a'_d, a'_{d-1}, ..., a'_{k+1}), \qquad \text{if } t > d$$

$$\quad \quad (b) \ (b_1, ..., b_{k-1}, a'_{t+1}, a'_d, a'_{d-1}, ..., a'_{k+1}), \qquad \text{if } t = d$$

$$= (4) \ (b_1, ..., b_{k-1}, a'_t, a'_d, a'_{d-1}, ..., a'_{k+1}).$$

Observe that $0 < \mu$ (as elements of Z) and $\mu \leqslant W^{(d)}$. Thus (μ, θ) is weakly standard on $X(w)$. Now the smallest λ on which (μ, θ) is standard has the projection $\lambda^{(d+1)}$ on W/W_{d+1} to be

$$
\begin{aligned}
\lambda^{(d+1)} &= (1) \ (b_1, ..., b_{k-1}, a'_{d+1}, a'_d, ..., a'_k) \\
&= (2) \ (b_1, ..., b_{k-1}, a'_{d+1}, a'_d, ..., a'_k) \\
&= (3) \ \text{(a)} \ (b_1, ..., b_{k-1}, a'_t, a'_d, ..., a'_k), \qquad \text{if} \quad t > d \\
&\qquad \text{(b)} \ (b_1, ..., b_{k-1}, a'_{t+1}, a'_d, ..., a'_k), \qquad \text{if} \quad t = d \\
&= (4) \ (b_1, ..., b_{k-1}, a'_t, a'_d, ..., a'_k).
\end{aligned}
$$

In all (four) cases, one finds that $w^{(d+1)} \not\geqslant \lambda^{(d+1)}$ (for the simple reason that the kth entry in $\lambda^{(d+1)} > n$, while the corresponding entry in $w^{(d+1)}$ is $\leqslant n$ (recall that $a_m \leqslant n$, for $k \leqslant m \leqslant n$).

As happened in Case B, the above argument holds in (1)–(3), without any condition on k and in (3(a)) and (4), one requires $d > k$. Let us then consider cases (3(a)) and (4) when $k = d$. In this case, one considers (μ', θ'), where

$$
\mu' = (a_1, ..., a_{d-1}, a_t)
$$

and

$$
\begin{aligned}
\theta' &= (a_1, ..., a_{d-1}, a_{t-1}, a'_t), \qquad \text{if} \quad t < r+1 \\
&= (a_1, ..., a_{d-1}, a_r, a'_t), \qquad \text{if} \quad r+1 \leqslant t.
\end{aligned}
$$

Now $a_{t-1} < a_t$ and $a_r < a_t$ (when $r+1 \leqslant t$) (since $a_m < a_j$, for $k = d \leqslant m < j \leqslant n$ and both $t-1$ and $r \geqslant d$ (since $t > d$ in (3(a)). Thus $\theta' < \mu'$ (as elements of Z). Also, it is easily seen that $\theta' \leqslant w^{(d+1)}$ (note that a_{t-1} (resp. a_r) is $\leqslant a_{r-1}$ (resp. a_r) in (3(a)) (resp. (4))) and $\mu' \leqslant w^{(d)}$. Thus (μ', θ') is weakly standard on $X(w)$. Now the smallest $X(\lambda')$ on which (μ', θ') is standard is such that $\lambda^{(d+1)} \not\leqslant w^{(d+1)}$; for, the entry at the $(d+1)$th place in $\lambda^{(d+1)}$ is a'_{t-1} while the corresponding entry in $w^{(d+1)}$ is a'_t and $a'_{t-1} > a'_t$ (since $a_{t-1} < a_t$ (recall that for $k = d \leqslant m < j \leqslant n$, $a_m < a_j$ and that $t-1 \geqslant d$, in (3(a)) and (4)). Thus (μ', θ') can not be standard on $X(w)$ contradicting the hypothesis that on $X(w)$, all weakly standard Young diagrams are in fact standard.

Case E.

$$
w_{d+1} = \text{Id} \qquad \text{and} \qquad w_d \geqslant s_{d+1} s_d.
$$

If $w_d = s_t \cdots s_d$ for some $t \leqslant n-1$, then we have $w_m \leqslant s_{n-1} \cdots s_m$ for $m \leqslant d$ and the discussion is completely analogous to the proof of Theorem 4.5.

If $w_d = s_t \cdots s_n \cdots s_d$, $d \leqslant t \leqslant n$, then the discussion in Case D (with $r+1$ replaced by $d+1$) goes through verbatim (one may observe that if $k = d$ and $w_{d+1} = \mathrm{Id}$, then (3(a)) does not exist).

Thus all possible cases contradict the hypothesis that on $X(w)$ all weakly standard Young diagrams are in fact standard. Hence our assumption on d is wrong. Thus the condition in Definition 4.7 relating w_d and w_{d+1} is satisfied, $1 \leqslant d \leqslant n - 1$. In other words, under the hypothesis that on $X(w)$ all weakly standard Young diagrams are in fact standard, we have proved that w has to be necessarily a Kempf element.

This completes the proof of Theorem 4.10.

Now, Theorem 4.10 leads to be following

DEFINITION 4.11. Let Q be any parabolic subgroup of G. Then a Schubert variety $X(w)$ in G/Q will be called a *Kempf variety* if $X(w)$ has the property that all weakly standard Young diagrams on $X(w)$ are in fact standard.

5. DEFORMATION

In this section G will denote a group of type A_n, B_n, or C_n and we shall identify $W(G)$ with $W(SL_{n+1})$ (resp. $W(Sp_{2n})$) if G is of type A_n (resp. B_n or C_n). Let Q be a parabolic subgroup of G, say, $Q = P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_d}$, $1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant n$, where P_{k_t}, $1 \leqslant t \leqslant d$, is a maximal parabolic subgroup of G. Given $\underline{m} = (m_{k_1}, m_{k_2}, \ldots, m_{k_d}) \in (\mathbf{Z}^+)^d$ and a Young diagram (θ, δ) of type $\underline{m}$ standard on $X(w)$ (cf. Definition 2.13), recall (cf. Theorem 2.14) that the standard monomials $p_{\theta, \delta} \in H^0(X(w), L)$ form a K-basis for $H^0(X(w), L)$ (K being the bases field), where $L = L_{k_1}^{m_1} \otimes L_{k_2}^{m_2} \otimes \cdots \otimes L_{k_d}^{m_d}$ and L_{k_t}, $1 \leqslant t \leqslant d$, is the ample generator of $\mathrm{Pic}(G/P_{k_d})$ (here $w \in W/W_Q$).

PROPOSITION 5.1. *Let F be a (non-zero) non-standard monomial on $X(w)$, $w \in W/W_Q$ say*

$$F = (p_{\tau_1, \phi_1} p_{\tau_2, \phi_2} \cdots p_{\tau_r, \phi_r})(p_{\lambda_1, \mu_1} p_{\lambda_2, \mu_2} \cdots p_{\lambda_s, \mu_s})(\ ) \cdots (\ )$$

(where τ's and ϕ's $\in W^{(k_1)}$; λ's and μ's $\in W^{(k_2)}$, etc.). Writing F as a sum of standard monomials, say

$$(*) \quad F = \sum (\ )(p_{\alpha_1, \beta_1} p_{\alpha_2, \beta_2} \cdots p_{\alpha_r, \beta_r})(p_{\gamma_1, \delta_1} p_{\gamma_2, \delta_2} \cdots p_{\gamma_s, \delta_s})(\ ) \cdots (\ )$$

(where the α's and β's $\in W^{(k_1)}$; γ's and δ's $\in W^{(k_2)}$, etc.), we have $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_r, \beta_r, \gamma_1, \delta_1, \gamma_2, \delta_2, \ldots, \gamma_s, \delta_s, \ldots)$ is lexicographically $\geqslant (\sigma(\tau_1), \sigma(\phi_1), \ldots, \sigma(\tau_r), \sigma(\phi_r), \theta(\lambda_1), \theta(\lambda_1), \theta(\mu_1), \ldots, \theta(\lambda_s), \theta(\mu_s), \ldots)$, where $\sigma \in S_{2r}$, $\theta \in S_{2s}$, etc.

Proof. Choose a minimal element, say α_1, among $\{\alpha_1'/p_{\alpha_i',\beta_1}$ occurs as the first factor in some term on the RHS of *$\}$. Let $\theta = \{\theta_1 \geqslant \theta_2 \geqslant \cdots\}$ be the maximal defining pair on $X(w)$ (cf. [13, Corollary 11.2]) for the standard diagram $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots \geqslant \alpha_r \geqslant \beta_r \geqslant \gamma_1 \geqslant \delta_1 \geqslant \cdots$. Now restricting (*) to $X(\theta_1)$, any term on the RHS either vanishes (namely if $\alpha_i > \alpha_1$) or remains standard on $X(\theta_1)$ (by the choice of α_1 and θ) and there is at least one term whose restriction to $X(\theta_1)$ is not zero, namely $p_{\alpha_1,\beta_1} \cdots$. Thus the RHS of (*) when restricted to $X(\theta_1)$ is a non-zero sum of standard monomials on $X(\theta_1)$. Hence the restriction of LHS to $X(\theta_1)$ should be $\neq 0$, from which we obtain $\tau_i \leqslant \alpha_1$ ($=$ the projection of θ_1 on W/W_{k_1}). Now any other α_1' is $\geqslant$ some minimal α_1 and hence is $\geqslant \tau_i$.

If τ_i (say τ_1) is such that $\tau_1 = \alpha_1$, for some α_1 (this in particular implies that α_1 is in fact *the* minimal among $\{\alpha_1'\}$), then on $X(\theta_1)$ we have

$$(**) \quad p_{\alpha_1,\phi_1} p_{\tau_2,\phi_2} \cdots p_{\tau_r,\phi_r} p_{\lambda_1,\mu_1} \cdots p_{\lambda_s,\mu_s} \cdots$$
$$= (\ ) p_{\alpha_1,\beta_1} p_{\alpha_2,\beta_2} \cdots + (\ ) p_{\alpha_1,\beta_1'} p_{\alpha_2',\beta_2'} \cdots + \cdots.$$

(note that the first factor in every term on the RHS looks like $p_{\alpha_1,(\ )}$). Now choose a minimal, say β_1 among $\{\beta_1'\}$. Let $\theta_1 \geqslant \theta_2' \geqslant \cdots$ be the maximal defining pair on $X(\theta_1)$ for the standard diagram $\alpha_1 \geqslant \beta_1 \geqslant \cdots$ on $X(\theta_1)$. Now multiplying (**) by p_{α_1,β_1}, we have

$$p_{\alpha_1} \sqrt{p_{\beta_1} \cdot p_{\phi_1}} p_{\tau_2,\phi_2} \cdots (\ ) p_{\alpha_1} \cdot p_{\beta_1} p_{\alpha_2,\beta_2} \cdots + (\ ) p_{\alpha_1} \sqrt{p_{\beta_1} p_{\beta_1'}} p_{\alpha_2',\beta_2'} \cdots + \cdots$$

up to ± 1. (recall (cf. [13, Proposition 6.1]) that on $X(\tau)$, $\tau \in W/W_P$, $p_{\tau,\phi}^2 = \pm p_\tau \cdot p_\phi$ and $p_{\tau,\phi_1} p_{\tau,\phi_2} = \pm p_\tau \sqrt{p_{\phi_1} \cdot p_{\phi_2}}$). Now cancelling p_α, and considering the maximal defining pair $\{\xi_1 \geqslant \xi_2 \geqslant \cdots\}$ on $X(\theta_1)$ for the standard diagram $\{\beta_1 \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots\}$ on $X(\theta_1)$ and restricting to $X(\xi_1)$, we obtain that the RHS is a non-trivial linear combination of standard monomials on $X(\xi_1)$. Further, each term on the RHS starts with p_{β_1} (recall (cf. [13, Proposition 6.1]) that on $X(\beta_1)$,

$$\sqrt{p_{\beta_1} p_{\beta_1'}} \begin{matrix} = 0, & \text{if } \beta_1 \not\geqslant \beta_1' \\ = p_{\beta_1,\beta_1'}, & \text{if } \beta_1 \geqslant \beta_1' \end{matrix} \Bigg).$$

Hence, restriction of the LHS to $X(\xi_1)$ should be different from zero. Thus β_1 ($=$ the projection of ξ_1 on W/W_{k_1}) is $\geqslant \phi_1, \tau_2, \phi_2,\ldots$. Now if $\phi_1 = \beta_1$, then $\sqrt{p_{\beta_1} \cdot p_{\phi_1}} = p_{\beta_1}$; if $\phi_i = \beta_1$ for some $i > 1$, then the corresponding τ_i is $= \beta_1$ (since $\beta_1 \geqslant \tau_i \geqslant \phi_i$), in which case we have $p_{\tau_i,\phi_i} = p_{\beta_1,\beta_1} = p_{\beta_1}$. Thus in either case p_{β_1} appears on the LHS and each term on the RHS (of the restriction to $X(\xi_1)$) starts with p_{β_1} (as already observed). Now cancelling p_{β_1}, the LHS becomes a monomial of degree one less than that of F. Hence using induction on degree of F, we obtain $(\alpha_2, \beta_2,\ldots, \alpha_r, \beta_r, \gamma_1, \delta_1, \gamma_2, \delta_2,\ldots)$

is lexicographically $\geq (\rho(\tau_2),\ \rho(\phi_2),...,\ \rho(\tau_r),\ \rho(\phi_r),\ \theta(\lambda_1),\ \theta(\mu_1),.......)$, where $\rho \in S_{2r\ 2}$, $\theta \in S_{2s}$, etc., from which the required result follows (since (α_1, β_1) has been proved to be lexicographically $\geq (\tau_1, \phi_1)$ and $\tau_1 \geq \phi_1$), the proof for the starting point of induction, namely deg $F = 2$, goes on the same lines as the proof above in the general case. In fact, we obtain something stronger in this case; to make it very precise, if

$$(*) \quad p_{\tau,\phi}\, p_{\theta,\sigma} = \sum (\ ) p_{\alpha,\beta}\, p_{\gamma,\delta}$$

is the expression for the non-standard binomial $p_{\tau,\phi}\, p_{\theta,\sigma}$ in terms of standard binomials, (the degree of the binomial being of the type $(0, 0,..., 0, 1, 0,..., 0, 1, 0,..., 0)$ or $(0, 0,..., 0, 2, 0,..., 0))$ then the above proof shows that for each term on the RHS, $\alpha \geq \tau$; if $\alpha = \tau$, then $\beta \geq \phi$.

This completes the proof of Proposition 5.1.

DEFINITION 5.2. Given $w \in W/W_Q$, define $J_w(R)$ (where $R = R_w = \bigoplus_{L>0} H^0(X(w), L))$ as follows:

$$\left\{ J_w(R) = \quad \alpha \in Z_w^{(k)} \left/ \begin{array}{l} \text{there exists a straightening relation in which} \\ \text{there is a term on the RHS involving } \alpha \text{ (strictly)} \\ \text{more number of times than the number of times } \alpha \\ \text{appears on the LHS} \end{array} \right. \right\}$$

(here (k) denotes $(k_1,..., k_r)$ where $Q = P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_r}$; by a straightening relation, we mean a relation in the K-algebra R, which expresses a (non-zero) non-standard monomial in R as sum of standard monomials).

Now we want to deform R (by successive flat deformations) so that for the deformed algebra R^{def}, we have $J_w(R^{\text{def}}) = \varnothing$ and then using the results of Sections 3 and 4, we shall show that R^{def} is Cohen Macaulay if w is a Kempf element, thus proving R to be Cohen Macaulay. To arrive at R^{def}, we proceed as follows. We fix a maximal element, say α in $J_\alpha(R)$ and consider I, the ideal in R generated by $p_{\alpha,\beta}$'s and $p_{\gamma,\alpha}$'s (α being fixed). Then we have the following (*In the following we shall denote* $p_{\alpha,\beta}$ *by just* (α, β)).

PROPOSITION 5.3. *For* $j \geq 1$, *I^j has a basis consisting of standard monomials involving α at least j times.*

Proof. For any monomial T, let $T_\alpha = \{\beta$, appearing in $T/\beta \geq \alpha\}$ and $T^{(\alpha)} = \{\beta$, appearing in $T/\beta \geq \alpha\}$. Now let $\tau \in I^j$. If T is not standard, let $T = \sum a_i T_i$ be the expression for T as sum of standard monomials (where each T_i is standard). Now consider any T_i appearing on the RHS and consider a β appearing in T_i, where $\beta \geq \alpha$. Now the maximality assumption on α implies that number of times β appears in T is $\geq$ the number of times

β appears in any T_k on the RHS (in particular, the number of times β appears in T_i). Now the lexicographic condition (cf. Proposition 5.1) implies that such a β appears the same number of times in T and T_i. Hence the lexicographic condition in Proposition 5.1 implies that α has to appear in T_i at least as many times as it does in T. Hence α has to appear at least j times in T_i (since it does so in T, as $T \in I^j$) which implies the required result.

Let $\mathscr{B} = \cdots Rt^2 \oplus Rt \oplus R \oplus It^{-1} \oplus I^2t^{-2} \oplus \cdots$. Observe that $\mathscr{B}$ is a $K[t]$ algebra with algebra generators given by

$$\left\{ \begin{array}{ll} (\beta, \gamma), & (1) \ \beta, \gamma \in Z_w^{(k)} \\ (\alpha, \beta)t^{-1} & (2) \ \beta \text{ and } \gamma \text{ are different from } \alpha \\ (\gamma, \alpha)t^{-1} & \quad\quad (\alpha, \text{ having been fixed as above}) \end{array} \right\}.$$

Let $R' = \mathscr{B}/(t) \, (= R/I \oplus I/I^2 \oplus \cdots)$. Then using Proposition 5.3, we obtain

PROPOSITION 5.4. *R' is a K-algebra with algebra generators given by $\{(\beta, \gamma)\}$ (where $\beta, \gamma \in Z_w^{(k)}$ and (β, γ) is an admissible pair) such that the standard monomials in these generators form a K-basis and such that $J_w(R') \subseteq J_w(R) - \{\alpha\}$. Further, $\mathscr{B}$ is a flat deformation whose special fiber $(t = 0)$ is R' and whose general fiber $(t, \text{ invertible})$ is R.*

Continuing thus we obtain

PROPOSITION 5.5. *There exists a sequence $\{R_i\}_{0 \leqslant i \leqslant m}$ of K-algebras and a sequence $\{\mathscr{B}_i\}_{0 \leqslant i \leqslant m-1}$ of flat deformations such that*

(1) *R_i, $0 \leqslant i \leqslant m$ has a set of algebra generators indexed by $\{(\beta, \gamma)$ where $\beta, \gamma \in Z_w^{(k)}$ and (β, γ) is an admissible pair$\}$ such that the standard monomials in these generators form a K-basis for R_i*

(2) *$R_0 = R$.*

(3) *$J_w(R_i) \subsetneqq J_w(R_{i-1})$.*

(4) *$J_w(R_m) = \varnothing$.*

(5) *R_i is a generic fiber and R_{i+1}, the special fiber of $\mathscr{B}_i$.*

COROLLARY 5.6. *Let us denote R_m by R^{def}. Then we have if R^{def} is Cohen Macaulay, so is R.*

Remark 5.7. Now $J_w(R^{\mathrm{def}}) = \varnothing$ implies that a monomial in R^{def} is zero, either if the α's in it are not totally ordered (as elements of $Z_w^{(k)}$) or if the α's are totally ordered but the corresponding weakly standard monomial is not standard: in the alternating case, namely, when the corresponding weakly standard monomial is in fact standard, then it is $= a$, the corresponding

unique standard monomial, where $a \in K$ (in Proposition 5.8 below, we shall show that $a = \pm 1$, if G is of type B_n or C_n and w is a Kempf element). If G is of type A_n (then all admissible pairs (τ, ϕ) being trivial, i.e., $\tau = \phi$), we obtain that in R^{def} if there is any non-zero, non-standard monomial, say $F = \sum p_{\tau_1} \cdots p_{\tau_r} p_{\phi_1} \cdots p_{\phi_r} p_{\lambda_1} \cdots p_{\lambda_s} \cdots$, then there is only one term on the RHS and that is F itself; in other words, in R^{def}, there can not be any non-zero, non-standard monomial. Thus in R^{def}, all nonstandard monomials are in fact 0.

PROPOSITION 5.8. *Let G be of type B_n or C_n and let $X(w)$ be a Kempf variety in G/Q. Let F be a non-zero, non-standard monomial on $X(w)$, say*

$$F = p_{\tau_1,\phi_1} p_{\tau_2,\phi_2} \cdots p_{\tau_r,\phi_r} p_{\lambda_1,\mu_1} \cdots p_{\lambda_s,\mu_s} \cdots$$

where τ's and ϕ's $\in W^{(k_{i_1})}$; λ's and μ's $\in W^{(k_{i_2})}$, etc. $(1 \leqslant t_1 < t_2 < \cdots \leqslant d)$ and $(\tau_1, \phi_1, \tau_2, \phi_2, ..., \tau_r, \phi_r, \lambda_1, \mu_1, ..., \lambda_s, \mu_s, ...)$ is totally ordered (after some rearrangement) in $Z_w^{(k)}$, say $(\sigma(\tau_1), \sigma(\phi_1), \sigma(\tau_2), \sigma(\phi_2), ..., \sigma(\tau_r), \theta(\lambda_1), \theta(\mu_1), ..., \theta(\lambda_s), \phi(\mu_s), ...)$ is totally ordered in $Z_w^{(k)}$ (where $\sigma \in S_{2r}$, $\theta \in S_{2s}$, etc.). Then in the expression for F as a sum of standard monomials, the monomial $p_{\sigma(\tau_1),\sigma(\phi_1)} \cdots p_{\sigma(\tau_r),\sigma(\phi_r)} p_{\theta(\lambda_1),\theta(\mu_1)} \cdots p_{\theta(\lambda_s),\theta(\mu_s)} \cdots$ occurs with coefficient ± 1 (observe that in view of Theorems 4.5 and 4.10 and Definitions 4.6 and 4.11 the weakly standard monomial $p_{\sigma(\tau_1),\sigma(\phi_1)} \cdots p_{\sigma(\tau_r),\sigma(\phi_r)} p_{\theta(\lambda_1),\theta(\mu_1)} \cdots p_{\theta(\lambda_s),\theta(\mu_s)} \cdots$ on $X(w)$ is in fact standard).

Proof. The result follows essentially from [5]. In fact, Theorem 4.1(b) and Definition 1.2 of [5] imply that

$$p_{\tau_1,\phi_1} \cdots p_{\tau_r,\phi_r} = \pm p_{\sigma(\tau_1),\sigma(\phi_1)} p_{\sigma(\tau_2),\sigma(\phi_2)} \cdots p_{\sigma(\tau_r),\sigma(\phi_r)} + F'.$$

(observe that τ's and ϕ's $\in W^{(k_t)}$ for some t, $1 \leqslant t \leqslant d$ (where $Q = P_{k_1} \cap P_{k_2} \cap \cdots \cap P_{k_d}$)). Now, if $p_{\tau_1',\phi_1'} \cdots p_{\tau_r',\phi_r'}$ is any term in F', then we obtain $(\tau_1', \phi_1', ..., \tau_r', \phi_r')$ is lexicographically $\geqslant (\sigma(\tau_1), \sigma(\phi_1), ..., \sigma(\tau_r), \sigma(\phi_r))$ (cf. [5], Definition 1.2). Similarly writing $p_{\lambda_1,\mu_1} p_{\lambda_2,\mu_2} \cdots p_{\lambda_s,\mu_s} = \pm p_{\theta(\lambda_1),\theta(\mu_1)} \cdots p_{\theta(\lambda_s),\theta(\mu_s)} +$ other terms and so on, we obtain

$$F = \pm p_{\sigma(\tau_1),\sigma(\phi_1)} \cdots p_{\sigma(\tau_r),\sigma(\phi_r)} p_{\theta(\lambda_1),\theta(\mu_1)} \cdots p_{\theta(\lambda_s),\theta(\mu_s)} \cdots + G.$$

where in G (after straightening if necessary), any term $p_{\tau_1',\phi_1'} \cdots p_{\tau_r',\phi_r'} p_{\lambda_1',\mu_1'} \cdots p_{\lambda_s',\mu_s'} \cdots$ is such that $(\tau_1', \phi_1', ..., \tau_r', \phi_r', \lambda_1', \mu_1', ..., \lambda_s', \mu_s', ...)$ is lexicographically $\geqslant (\sigma(\tau_1), \sigma(\phi_1), ..., \sigma(\tau_r), \sigma(\phi_r), \theta(\lambda_1), \theta(\mu_1), ..., \theta(\lambda_s), \theta(\mu_s), ...)$
This completes the proof of Proposition 5.8.

6. ARITHMETIC COHEN–MACAULAYNESS FOR MULTICONES OVER KEMPF VARIETIES

DEFINITION 6.1 (cf. [5]). Let $w \in W$. For j, $1 \leqslant j \leqslant n$, define

$$
H_{w,j} = \left\{ \tau = ((\alpha_1, \beta_1), (\alpha_2, \beta_2),..., (\alpha_r, \beta_r)) \left|
\begin{array}{l}
(1)\ (\alpha_i, \beta_i) \text{ is an admissible} \\
\quad \text{pair on } X(w^{(j)})(\subseteq G/P_j) \\
(2)\ \alpha_1 \geqslant \beta_1 > \alpha_2 > \beta_2 \geqslant \cdots \\
(3)\ (\beta_i, \alpha_{i+1}) \text{ is not an} \\
\quad \text{admissible pair on } X(w^{(i)})
\end{array}
\right. \right\}
$$

(refer [12] or [13] for definition of admissible pairs on $X(\theta)$, $\theta \in W/W_P$). Here $w^{(j)}$, $1 \leqslant j \leqslant n$, denotes the projection of w on W/W_j under $W \to W/W_j$.

Remark 6.2. If $G = SL_n$ then $H_{w,j}$ is nothing but chains in $\{W_w^{(j)}\}$ (where $W_w^{(j)} = \{\theta \in W^{(j)}/\theta \leqslant w^{(j)}\}$).

DEFINITION 6.3. Given τ_1, $\tau_2 \in H_{w,j}$, say

$$
\tau_1 = ((\tau_1^{(1)}, \tau_2^{(1)}), (\tau_3^{(1)}, \tau_4^{(1)}),..., (\tau_s^{(1)}, \tau_{s+1}^{(1)}))
$$

$$
\tau_2 = ((\tau_1^{(2)}, \tau_2^{(2)}), (\tau_3^{(2)}, \tau_4^{(2)}),..., (\tau_m^{(2)}, \tau_{m+1}^{(2)}))
$$

define $\tau_1 \geqslant \tau_2$, if for any $(\tau_i^{(2)}, \tau_{i+1}^{(2)})$, $i = 1, 3, 5,..., m$, there exists a $(\tau_j^{(1)}, \tau_{j+1}^{(1)})$ such that $\tau_j^{(1)} \geqslant \tau_i^{(2)} \geqslant \tau_{i+1}^{(2)} \geqslant \tau_{j+1}^{(1)}$ (note that this defines a partial order on $H_{w,j}$).

DEFINITION 6.4. Given $(k) = (k_1, k_2,..., k_d)$, where $1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant n$, define

$$
H_w^{(k)} = \left\{ (\tau_{i_1}, \tau_{i_2},..., \tau_{i_t}) \left|
\begin{array}{l}
(1)\ \{i_1, i_2,..., i_t\} \subseteq \{k_1, k_2,..., k_d\} \\
(2)\ i_1 < i_2 < \cdots < i_t \\
(3)\ \tau_{i_l} \in H_{w,i_l},\ 1 \leqslant l \leqslant t \\
(4)\ \text{If } \tau_{i_l} = ((\tau_1^{(i_l)}, \tau_2^{(i_l)}),..., (\tau_{n_l}^{(i_l)}, \tau_{n_l+1}^{(i_l)})), \\
\quad 1 \leqslant l \leqslant t, \text{ then} \\
\quad \tau_{n_l+1}^{(i_l)} > \tau_1^{(i_{l+1})},\ 1 \leqslant l \leqslant t-1, \\
\quad \text{as elements of } Z_w^{(k)}
\end{array}
\right. \right\}
$$

Remark 6.5. If $G = SL_n$, then $H_w^{(k)}$ is just the set of all chains in $Z_w^{(k)}$.

DEFINITION 6.6. Given τ, $\phi \in H_w^{(k)}$, say $\tau = (\tau_{i_1}, \tau_{i_2},..., \tau_{i_t})$ and $\phi = (\phi_{j_1}, \phi_{j_2},..., \phi_{j_s})$, call $\tau \geqslant \phi$, if $\{j_1, j_2,..., j_s\} \subseteq \{i_1, i_2,..., i_t\}$ and $\phi_{j_m} \leqslant \tau_{j_m}$, $1 \leqslant m \leqslant s$, as elements of H_{w,j_m} (note that this defines a partial order on $H_w^{(k)}$).

THEOREM 6.7. *Let $X(w)$ be a Kempf variety in G/Q, G being of type A_n, B_n, or C_n. Then*

(1) *If G is of type A_n, then R_w^{def} is an algebra with straightening law (cf. Section 2; see also* [5] *and* [7]*) on Z_w.*

(2) *If G is of type B_n or C_n, then R_w^{def} is an algebra with straightening law on $H_w^{(k)}$, where $(k) = (k_1,..., k_d)$ is given by $Q = \bigcap_{t=1}^d P_{k_t}$.*

Proof. Now (1) follows from Remark 5.7. In fact, R_w^{def} is the discrete algebra (cf. [5, Definition 1.4 and Remark 3.1]) with straightening law on Z_w.

The proof of (2) follows from Proposition 5.8 and Definitions 6.4 and 6.6. In fact, given a chain $\tau > \phi > \theta > \cdots$ in $H_w^{(k)}$, say

$$\tau = (\tau_{i_1}, \tau_{i_2},..., \tau_{i_t}), \qquad \phi = (\phi_{j_1}, \phi_{j_2},..., \phi_{j_s}),...$$

where

$$\tau_{i_r} = ((\tau_1^{(i_r)}, \tau_2^{(i_r)}), (\tau_3^{(i_r)}, \tau_4^{(i_r)}),..., (\tau_{n_r}^{(i_r)}, \tau_{n_r+1}^{(i_r)})), \qquad 1 \leqslant r \leqslant t$$

$$\phi_{j_l} = ((\phi_1^{(j_l)}, \phi_2^{(j_l)}), (\phi_3^{(j_l)}, \phi_4^{(j_l)}),..., (\phi_{m_l}^{(j_l)}, \phi_{m_l+1}^{(j_l)})), \qquad 1 \leqslant l \leqslant s$$

etc., we can associate an element $P_\tau P_\phi P_\theta \cdots \in R_w^{\mathrm{def}}$, namely $P_\tau = p_{\tau_1^{(i_1)},\tau_2^{(i_1)}} \cdots p_{\tau_{n_t}^{(i_t)},\tau_{n_t+1}^{(i_t)}}$, P_ϕ, P_θ, etc., being defined in a similar way. Now, in view of the partial order on $H_w^{(k)}$ (cf. Definition 6.6), we see that

$$\{\tau_1^{(i_1)}, \tau_2^{(i_1)},..., \tau_{n_t+1}^{(i_t)}, \phi_1^{(j_1)}, \phi_2^{(j_1)},..., \theta_{\{\ \}}^{\{\ \}},...,...\}$$

is totally ordered in $Z_w^{(k)}$ and one can associate an unique standard monomial in the $p'_{\rho^s,\sigma}$ ((ρ, σ) being an admissible pair in $W^{(k_t)}$ for some t, $1 \leqslant t \leqslant d$) and in fact this unique monomial is $= \pm P_\tau P_\phi P_\theta \cdots$ (in view of Remark 5.7 and Proposition 5.8). Conversely given a standard monomial F in the $p'_{\rho^s,\sigma}$, it is easily seen that there exists an unique chain $\tau > \phi > \theta > \cdots$ in $H_w^{(k)}$ such that $F = \pm P_\tau P_\phi P_\theta \cdots$ (in view of Remark 5.7 and Proposition 5.8). (to obtain the chain $\tau > \phi > \theta > \cdots$, one follows the rule of associating a standard monomial $P_{\tau_1} P_{\tau_2} \cdots P_{\tau_r}$ (where $\tau_1 > \tau_2 > \cdots > \tau_r$ is a chain in $H_{w,i}$ for some $i \in \{k_1,..., k_d\}$), to a standard monomial in $p'_{\rho^s,\sigma}$, $(\rho, \sigma)'s$ being admissible pairs in $W_w^{(i)}$ ($= \{\xi \in W^{(i)}/\xi \leqslant$ projection of w on $W/W_i\}$), as described in the proof of Proposition 3.4 of [5]). Thus R_w^{def} is K-algebra with a set of algebra generators indexed by the elements of $H_w^{(k)}$, such that the standard monomials $P_\tau P_\phi P_\theta \cdots$ (where $\tau > \phi > \theta \cdots$ in $H_w^{(k)}$) form a K-basis for R_w^{def}. Further, the fact that the straightening relations satisfy the required lexicographic condition (cf. Section 2 or [5] or [7]) can be seen quite easily (using Definition 6.6).

This completes the proof of Theorem 6.7.

THEOREM 6.8. *Let $X(w)$ be a Kempf variety in G/Q. Then the ring $R_w = \bigoplus_{L \geq 0} H^0(X(w), L)$ is Cohen–Macaulay.*

Proof. In view of Corollary 5.6, enough to show that R_w^{def} is Cohen–Macaulay. For the case of G being of type A_n, this is immediate; because, by Theorem 6.7, R_w^{def} is a K-algebra (K being the base field) with straightening law on $Z_w^{(k)}$ and by Theorem 3.12, $Z_w^{(k)}$ is lexicographic shellable. Hence $Z_w^{(k)}$ is shellable (cf. [1] or [2]) and hence any K-algebra with straightening law over $Z_w^{(k)}$ is Cohen–Macaulay. In particular R_w^{def} is Cohen–Macaulay (when a poset H (or the associated simplicial complex $\Delta(H)$, of chains in H) is shellable (refer [1] for definition of shellability) using Mayer–Viotoris sequence or [16] one may conclude the Cohen–Macaulayness for the *discrete algebra $K\{H\}$* ($= K[x_{\alpha, \alpha \in H}]/(x_\alpha x_\beta, \alpha, \beta$ non comparable)) and now the Cohen–Macaulayness for $K\{H\}$ implies the Cohen–Macaulayness for any K-algebra with straightening law over H (as discussed in Section 2 or [5] or [7]).

If G is of type B_n or C_n, then by Theorem 6.7, we have that R_w^{def} is a K-algebra with straightening law over $H_w^{(k)}$. It can be easily seen that the simplicial complex $\Delta(H_w^{(k)})$ of chains in $H_w^{(k)}$ is a subdivision of $\Delta(Z_w^{(k)})$ and hence the discrete algebra $K\{H_w^{(k)}\}$ over $H_w^{(k)}$ is Cohen–Macaulay if and only if the discrete algebra $K\{H\}$ (cf. [16]) is. And now the lexicographic shellability of $Z_w^{(k)}$ (cf. Theorem 3.14) implies the Cohen–Macaulayness of $K\{Z_w^{(k)}\}$ and hence that of $K\{H_w^{(k)}\}$. Hence R_w^{def} (and R_w) is Cohen–Macaulay.

This completes the proof of Theorem 6.8.

Using the result of [4] that Schubert varieties are non-singular in codimension 1 and Theorem 6.8, we obtain

THEOREM 6.9 (see also [13, Theorem 6.2]). *Let $X(w)$ be a Kempf variety in G/Q. Then the ring R_w is normal.*

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