Expositiones Mathematicae 29 (2011) 142–149

Contents lists available at ScienceDirect

Expositiones Mathematicae

journal homepage: www.elsevier.de/exmath

Rigid Calabi–Yau threefolds over $\mathbb{Q}$ are modular[☆]

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ARTICLE INFO

Article history:
Received 13 July 2010
Received in revised form
19 July 2010

2000 Mathematics Subject Classification:
14J32
11F30
11F03
11F11

ABSTRACT

The proof of Serre's conjecture on Galois representations over finite fields allows us to show, using a method due to Serre himself, that all rigid Calabi–Yau threefolds defined over $\mathbb{Q}$ are modular.

In the mid-1980s, J.-P. Serre conjectured in [11] that all absolutely irreducible odd two-dimensional representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite field come from modular forms of prescribed weight, level, and character. This has now been proved by Khare and Wintenberger; see [6,7]. Because this result can be seen as a generalization of Artin Reciprocity to the GL_2 case (over $\mathbb{Q}$), we will refer to it as "Serre Reciprocity".

Already in [11], Serre showed how, given a compatible system of ℓ-adic Galois representations and bounds on the weight and level of the predicted modular forms in characteristic ℓ, one can use Serre Reciprocity to obtain results in characteristic zero. We refer to this as "Serre's method" and state and prove a generalized form of it in Section 1.

Serre's method allows us to show that certain geometric Galois representations are modular. Specifically, we show that the representation obtained from the third étale cohomology of a rigid Calabi–Yau threefold defined over $\mathbb{Q}$ comes from a modular form of weight 4 on $\Gamma_0(N)$. The proof is an application of Serre's method; it can, in fact, be read off directly from [11, Section 4.8], which is why one might describe this short paper as a "footnote to Serre". Recent results allow a slightly simpler version of the proof.

The observation that the proof of Serre's reciprocity allows us to establish the modularity of odd irreducible motives of rank two has also been made independently, in more general terms, by Mark Kisin in [8] (see his Corollary 0.5).

☆ This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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0723-0869/$ – see front matter
doi:10.1016/j.exmath.2010.09.001

The question of the modularity of the Galois representations obtained from Calabi–Yau threefolds over $\mathbb{Q}$ has been much studied, and a large number of examples are now available; see [10] for a survey. Since current methods restrict us to low-dimensional Galois representations, many of the examples involve rigid Calabi–Yau threefolds, defined below, simply because in that case the representation is automatically of dimension two.

Dieulefait and Manoharmayum have shown in [2] that if X has good reduction at small primes then it is modular. Richard Taylor showed in [12] that rigid Calabi–Yau manifolds over $\mathbb{Q}$ are *potentially modular*, i.e., that there exists a totally real field F such that the restrictions to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of the representations ρ_ℓ are attached to automorphic representations over F. These results were based on the same family of modular lifting theorems that was used to finally prove Serre reciprocity.

As we will indicate below, the same methods also apply to the non-rigid case if one can isolate an irreducible two-dimensional "piece" of the cohomology. In this case, we obtain modular forms of weight 4 and of weight 2, in agreement with many examples found by Meyer and others. In general, however, the middle cohomology groups of non-rigid Calabi–Yau threefolds do not decompose into products of two-dimensional pieces.

1. Serre reciprocity

Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, let $\mathbb{F}$ be a finite field of characteristic ℓ, and let $\rho : G \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible representation. We will assume throughout that ρ is *odd*, that is, we will assume that if $c \in G$ is (any) complex conjugation, we have $\det \rho(c) = -1$. We will let S be the finite set of primes such that ρ is unramified at all primes not in S.

In [11], Serre associated to any such ρ a triple (N, k, ε), where N and k are positive integers, $k \geq 2$, and ε is a Dirichlet character modulo N. We will briefly recall below how this triple is obtained, but we refer the reader to [11] and [4] for details.

In order to avoid technical problems related to finite fields of small characteristic, we assume $\ell \neq 2, 3$. See [4] for how to modify the statement below so that it remains true in those cases.

The result conjectured in [11] and proved in [6,7] is:

Theorem 1 (*Serre–Khare–Wintenberger*)**.** *Let G, ℓ, and $\mathbb{F}$ be as above. Suppose $\rho : G \to \mathrm{GL}_2(\mathbb{F})$ is an odd absolutely irreducible representation, and let (N, k, ε) be the Serre parameters attached to ρ. Then there exist:*

- *a cuspidal modular eigenform f on $\Gamma_0(N)$, of weight k and character ε and defined over a number field K, and*
- *a prime λ of K with residue field $\mathbb{F}$*

such that the reduction modulo λ of the λ-adic representation attached to f is isomorphic to ρ.

Part of the power of this result comes from the fact that the triple (N, k, ε) is specified in advance in terms of ρ, which restricts us to a finite number of possibilities for the eigenform f. It will be helpful to recall how these parameters are obtained.

The level N is fairly easy to describe: it is the prime-to-ℓ part of the Artin conductor of the representation ρ. As such, it is divisible only by primes $p \in S$, $p \neq \ell$. If we set

$$N = \prod_{p \in S} p^{e(p)},$$

the exponent $e(p)$ is entirely determined by the image of the inertia group at p. In particular, it is useful to note that if ρ is *tamely* ramified at p then $e(p) = 1$.

This choice of the level parameter has as a useful side effect that the modular form f will necessarily be a *newform*, i.e., it will not come from a level lower that N. (If f did come from a form of lower level, the p-adic representation attached to f would have smaller conductor, and therefore so would the mod p representation.)

The weight k is the most delicate of the three parameters. It depends only on the image of the inertia group at ℓ, but the recipe for computing k is complicated; see [11] and [4] for details. We will use Serre's normalization of the weight, so that $k \geq 2$.

Finally, the character ε is determined by the formula

$$\det \rho = \varepsilon \chi_\ell^{k-1},$$

where χ_ℓ is the (reduction mod ℓ of the) ℓ-adic cyclotomic character. Notice that this formula determines k modulo $\ell - 1$.

While Serre Reciprocity is a statement about Galois representations over finite fields, it can often be used to show the modularity of representations in characteristic zero as well. The idea is due to Serre himself; it will perhaps be useful to have a formalized version of it.

Theorem 2 (*Serre's Method*)**.** *Fix a number field K, and let λ run over primes of K. For each λ, let K_λ denote the completion of K at λ and let $\kappa(\lambda)$ be the residue field. Let ℓ be the characteristic of $\kappa(\lambda)$.*

Fix a finite set S of primes in $\mathbb{Q}$. For each $p \notin S$, let Frob_p be a choice of arithmetic Frobenius element in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Suppose we have, for each λ, a two-dimensional K_λ vector space V_λ with a continuous action of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This gives a continuous representation ρ_λ. Assume ρ_λ is odd and unramified outside $S \cup \{\ell\}$.

For each λ we can find a G-stable lattice, reduce modulo λ, and semisimplify if necessary to obtain a semisimple odd two-dimensional Galois representation

$$\overline{\rho}_\lambda : G \longrightarrow \mathrm{GL}_2(\kappa(\lambda))$$

unramified outside $S \cup \{\ell\}$.

Fix an infinite set I of primes in K. Suppose we can show that:

(1) *For all $\lambda \in I$, the representation $\overline{\rho}_\lambda$ is absolutely irreducible.*
(2) *There exists a family $Q_p(X) = X^2 - A_pX + D_p \in K[X]$ of polynomials of degree 2, indexed by primes $p \in \mathbb{Q}$, $p \notin S$, such that for all $\lambda \in I$ and (given λ) all $p \notin S \cup \{\ell\}$, the characteristic polynomial of Frob_p acting on V_λ is equal to $Q_p(X)$.*
(3) *There exists an integer k_0 such that for all $\lambda \in I$ the Serre weight k_λ attached to $\overline{\rho}_\lambda$ satisfies $1 < k_\lambda \leq k_0$.*
(4) *There exists an integer N_0 such that for all $\lambda \in I$ the Serre level N_λ attached to $\overline{\rho}_\lambda$ is a divisor of N_0. We choose N_0 to be minimal with this property.*

Then there exists a cuspidal Hecke eigenform form f (new of level N dividing N_0, weight k less than or equal to k_0, defined over K) such that for all λ the λ-adic representation $\rho_{f,\lambda}$ attached to f is isomorphic to ρ_λ.

Proof. We may, and will, assume that the set S has been chosen to be as small as possible, so that for every $p \in S$ there is at least one λ that does not divide p and such that ρ_λ is ramified at p. Note also that since $\overline{\rho}_\lambda$ is absolutely irreducible for some λ, so is ρ_λ. (Because characters can be inserted into compatible systems, it follows from the compatibility condition (2) that ρ_λ is absolutely irreducible for *all* λ. We will not actually use this, and once we have shown that the representation comes from a cuspform, it will follow that all the ρ_λ are absolutely irreducible.)

Choose $\lambda \in I$ and apply Serre Reciprocity to $\overline{\rho}_\lambda$. We get an eigenform f_λ of weight k_λ less than or equal to k_0, character ε, and level dividing N_0. A priori, f may be defined over an extension of K whose residue field at a prime λ' over λ is still $\kappa(\lambda)$. The fact that f_λ corresponds to ρ_λ tells us that

$$A_p \equiv a_p(f_\lambda) \pmod{\lambda'}$$

and

$$D_p \equiv \varepsilon(p)p^{k_\lambda - 1} \pmod{\lambda'}$$

for all $p \notin S \cup \{\ell\}$.

Since the set of all eigenforms of weight bounded by k_0 and level dividing N_0 is *finite* and there are infinitely many $\lambda \in I$, there must exist a modular form f such that $f_\lambda = f$ for infinitely many λ. Let k be the weight of f. But then, for each $p \notin S$ we will have

$$A_p \equiv a_p(f) \pmod{\lambda}$$

and

$$D_p \equiv \varepsilon(p)p^{k-1} \pmod{\lambda}$$

for infinitely many λ.

This implies that in fact $A_p = a_p$ and $D_p = \varepsilon(p)p^k$ for all $p \notin S$. Since we know f is a newform, this is enough to show that f is the eigenform we wanted to find and (together with the minimality of S) implies, in particular, that it has coefficients in K. □

In the case of representations coming from geometry, the representations will typically be obtained from the (dual of the) étale cohomology of an algebraic variety X defined over $\mathbb{Q}$. The field K is then just $\mathbb{Q}$. The set S is then contained in the set of primes of bad reduction for X and the existence of the $Q_p(X)$ follows from the Weil Conjectures as proved by Deligne.

For an example with K a totally real field, consider the case of abelian varieties with real multiplication; see [11, Section 4.7].

2. Modularity of rigid Calabi–Yau threefolds over $\mathbb{Q}$

We want to apply Serre's method to the representation obtained from the middle étale cohomology of a rigid Calabi–Yau threefold defined over $\mathbb{Q}$. We recall the definitions.

Definition 1. Let X be a smooth projective threefold defined over $\mathbb{C}$. We call X a *Calabi–Yau threefold* if

(1) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, and
(2) $K_X := \wedge^3 \Omega^1_X \simeq \mathcal{O}_X$, that is, the canonical bundle is trivial.

As usual, we define the Hodge numbers

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X).$$

By complex conjugation, $h^{i,j}(X) = h^{j,i}(X)$, and by Serre duality, $h^{i,j}(X) = h^{3-j,3-i}(X)$ for $0 \leq i, j \leq 3$. The Hodge decomposition gives

$$h^k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C}) = \sum_{i+j=k} h^{i,j}(X).$$

The number $h^k(X)$ is called the kth Betti number of X and often denoted $B_k(X)$.

If X is Calabi–Yau, then the first condition implies that

$$h^{1,0}(X) = h^{2,0}(X) = 0,$$

and the second condition, together with Serre duality, yields

$$h^{3,0} = h^{0,3} = 1.$$

We can summarize all this by drawing the "Hodge diamond" of X:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{1,2}(X) & & 1 \\ & 0 & & h^{2,2}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{l} h^0(X) = 1 \\ h^1(X) = 0 \\ h^2(X) = h^{1,1}(X) \\ h^3(X) = 2(1 + h^{2,1}(X)) \\ h^4(X) = h^{2,2}(X) = h^{1,1}(X) \\ h^5(X) = 0 \\ h^6(X) = 1. \end{array}$$

Calabi–Yau threefolds are Kähler manifolds, so $h^{1,1}(X) > 0$. All 2-cycles on Calabi–Yau threefolds are algebraic, as follows from the Lefschetz (1, 1) theorem that $H^2(X, \mathbb{Z}) \cong \mathrm{Pic}(\overline{X})$. In particular, $h^{1,1}(X) = h^2(X) = \mathrm{rk}\,\mathrm{Pic}(\overline{X})$.

Definition 2. Let X be a Calabi–Yau threefold defined over $\mathbb{C}$. We say that X is *rigid* if $h^{2,1}(X) = h^{1,2}(X) = 0$, so that $h^3(X) = 2$.

The name "rigid" comes from the fact that the space of deformations of a Calabi–Yau manifold has dimension $h^{1,2}(X)$. There are more than 50 known examples of rigid Calabi–Yau threefolds (up to birational equivalence over $\mathbb{C}$). It is still an open problem to decide whether the number of such examples is finite up to birational transformation over $\mathbb{C}$.

Since we are interested in Galois representations, we focus on Calabi–Yau threefolds defined over $\mathbb{Q}$. Of course, a given Calabi–Yau threefold over $\mathbb{C}$ may well have many different realizations over $\mathbb{Q}$. Notice that the fact that X is a rigid Calabi–Yau manifold is independent of the choice of model over $\mathbb{Q}$, but the Galois representation (and therefore the modular forms we will find) depend strongly on that choice. We will comment further on this below.

Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$. Then X always has a model defined over $\mathbb{Z}$; we assume one has been chosen and fixed. We will apply Theorem 2 with $K = \mathbb{Q}$ and S the (finite) set of primes at which X has bad reduction. Let $\overline{X} = X \otimes \overline{\mathbb{Q}}$. For each prime ℓ in $\mathbb{Q}$, let

$$V_\ell = H^3(\overline{X}, \mathbb{Q}_\ell)^\vee.$$

(We need to dualize because we want to work with the arithmetic Frobenius.) We know that if $p \notin S \cup \ell$, this representation will be unramified at p.

The assumption that X is rigid means that V_ℓ is two dimensional and that its Hodge decomposition is of the form $(3, 0) + (0, 3)$. By Pontryagin duality, we have

$$\det \rho_\ell = \chi_\ell^3$$

so that ρ_ℓ is odd. Let $\overline{\rho}_\ell$ be the representation obtained by reducing modulo ℓ.

In [11, Section 4.8], Serre checked that conditions (1) and (2) above hold for sufficiently large ℓ. A theorem of Fontaine (see also [4]) shows that for all large enough ℓ the Serre weight parameter will be $k = 4$.

In order to verify the condition on the level, Serre used a bound for the Artin conductor proved in [11, Section 4.9]: under certain congruence conditions on ℓ, the conductor N is a divisor of

$$N_0 = \prod_{p \in S} p^{e(p)},$$

where $e(2) = 8$, $e(3) = 5$, and $e(p) = 2$ for all other primes $p \in S$. We can therefore let I be the (infinite) set of primes ℓ that satisfy Serre's congruence conditions.

This can now be simplified by using the results in [12]. Since the Hodge numbers are 0 and 3 and we know that the representation is crystalline at all $p \notin S$ (because X has good reduction at all such primes), the ρ_ℓ form what Taylor calls a *weakly compatible* system of representations; by Theorem A in [12], the system must in fact be *strongly* compatible, which implies that the conductor N is *independent* of $\ell \notin S$. (We thank Luis Dieulefait for pointing this out to us.) Hence we can take our infinite set to be all primes not in S.

Theorem 2 then gives our result:

Theorem 3. *Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$, and use the notations above. Then there exists a Hecke eigenform f of weight 4, level dividing N, and trivial character such that ρ_ℓ is equivalent to $\rho_{f,\ell}$ for all ℓ.*

In other words, all rigid Calabi–Yau threefolds defined over $\mathbb{Q}$ are modular. In particular, this implies that the L-function corresponding to the third étale cohomology of such a threefold is the same as that of a modular form of weight 4, and hence is holomorphic and satisfies a functional equation relating values at s to values at $4 - s$.

Notice that while we do not need to use Serre's bound on the level for the argument, a posteriori the bound will apply to the level of the form f. This is in fact the bound obtained by Dieulefait in [1].

Serre's method is applicable, as he shows in [11], to all odd-dimensional smooth algebraic varieties whose middle-dimensional cohomology is of dimension two and of Hodge type $(*, 0) + (0, *)$.

3. The non-rigid case

The reason to focus on the rigid case is, of course, that we get a Galois representation of dimension two, which should then come from a modular form. Higher-dimensional representations should be automorphic, but the type of corresponding automorphic representation we expect to find will depend on that dimension.

If we drop the assumption that the Calabi–Yau manifold X is rigid, then h^3 will not be equal to two. It is still possible, nevertheless, that the Galois representation attached to the third cohomology contains an irreducible subrepresentation of dimension two. If such a subrepresentation occurs in $H^3(\overline{X}, \mathbb{Q}_\ell)$ for every ℓ and the resulting Galois representations are (weakly) compatible, the same argument will apply. Such a system of compatible representations is usually described as a submotive of rank two.

If the submotive happens to be the $(3, 0) + (0, 3)$ part, exactly the same argument will show that it is modular, i.e., the subrepresentation of dimension two over $\mathbb{Q}_\ell$ will be isomorphic to the ℓ-adic representation attached to a modular form f of weight 4.

If the submotive M_ℓ occurs instead in the $(2, 1) + (1, 2)$ part, the method described above is not directly applicable, because the Serre weight k_ℓ will in general be $\ell + 3$, and hence not bounded. This can be easily fixed, however, by twisting: $M_\ell \otimes \chi_\ell^{-1}$ has Hodge numbers $(1, 0) + (0, 1)$, and the argument above will show that it corresponds to a modular form of weight 2. Hence $M_\ell = V_\ell(f) \otimes \chi_\ell$ is a Tate twist of the representation coming from such a form of weight two.

There are several (proved and conjectural) examples of this in [10]. Many of them are of the type studied in [5], namely, Calabi–Yau threefolds containing a large number of elliptic ruled surfaces. To be specific, let

$$V_\ell = H^3(\overline{X}, \mathbb{Q}_\ell)^\vee.$$

The examples in [5] and [10] look like

$$V_\ell \cong V_\ell(f) \oplus [V_\ell(g_1) \otimes \chi_\ell] \oplus [V_\ell(g_2) \otimes \chi_\ell] \oplus \cdots \oplus [V_\ell(g_k) \otimes \chi_\ell],$$

where $h^3(X) = 2 + 2k$, f is a modular form of weight 4, the g_i are all modular forms of weight 2, and $V_\ell(h)$ is the ℓ-adic representation attached to a modular eigenform h. In many cases, Meyer finds examples where the g_i are in fact all the same; see, for example, pages 23–24 of [10]. Another example can be found in [9], where we have $h^3 = 4$ and the representation splits into two two-dimensional components.

Finally, if the decomposition of the cohomology representation takes place only after extending scalars, but we still obtain weakly compatible systems, Serre's method applies to deduce modularity (after twisting) of all of them, the only difference being that the relevant modular forms are no longer defined over $\mathbb{Q}$. We thank the referee for pointing that out to us.

4. Some speculations

Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$, and let f be the associated modular form of weight 4.

(1) The level N of the form f is going to be a delicate arithmetic invariant of X (over $\mathbb{Q}$, rather than over the algebraic closure). The primes dividing N should be primes at which X has bad reduction in every model of X over $\mathbb{Z}$, but it is unclear whether N *must* be divisible by all such primes. In addition, the precise power of such primes that occurs in N presumably depends on the type of singularities, but we do not know how that should work.

(2) Suppose we have X and its modular form f of level N. Then if we twist f by a quadratic character of level d, we get another eigenform of weight 4 and level dividing Nd^2. Will this form be attached to another rigid Calabi–Yau threefold over $\mathbb{Q}$? If so, then there must exist an algebraic correspondence between X and X_d that is defined over $\mathbb{Q}(\sqrt{d})$. We might even hope that X_d is a Galois twist of X, so that $X \otimes \mathbb{Q}(\sqrt{d}) \cong X_d \otimes \mathbb{Q}(\sqrt{d})$. (Note that the existence of X_d which are Galois twists is compatible with the conjecture that there are only finitely many rigid Calabi–Yau threefolds over $\mathbb{C}$ up to birational equivalence.)

Helena Verrill has found an example where such a Galois twist X_d can be constructed (see [13]), M. Schütt pointed out that twists exist for all fiber products; does such a "twist" always exist? In [10], Meyer conjectured that the answer is yes. An interesting test case is the Schoen quintic

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 5X_0X_1X_2X_3X_4.$$

This is a singular threefold, and resolving those singularities produces a rigid Calabi–Yau threefold X that is known (see [10] and the references therein) to be associated to a modular form of weight 4 and level 25. Can one construct the requisite Galois twists?

Bert van Geemen has informed us that the answer is "yes". Since the Schoen quintic X has an automorphism ϕ of order 2 defined over $\mathbb{Q}$, which acts on $H^3(X)$ by -1, and we can use this to twist the quintic X. Let $K = \mathbb{Q}(\sqrt{d})$ and let X_K be the quintic defined over K. Descend X_K back to $\mathbb{Q}$ by taking the quotient by the automorphism which is ϕ on the coordinates and which is the non-trivial automorphism of K on the scalars. Then we get a quintic X_d on whose $H^3(\overline{X}_d, \mathbb{Q}_\ell)$ the Galois representation is the twist of the one on $H^3(\overline{X}, \mathbb{Q}_\ell)$.

In fact, the automorphism ϕ of order 2 is given, for instance, explicitly by

$$\phi(X_0) = X_1, \qquad \phi(X_1) = X_0, \qquad \phi(X_i) = X_i \quad \text{for } i = 2, 3, 4.$$

Put $U = X_0 + X_1$ and $V = X_0 - X_1$. Then the equation for the quintic equation can be written as a polynomial in U and V^2 as follows:

$$U^5 + 10U^3V^2 + 5UV^4 + 16(X_2^5 + X_3^5 + X_4^5) - 20(U^2 - V^2)X_2X_3X_4 = 0.$$

Now replace V by $\sqrt{d}V$, then we obtain the quintic equation for X_d:

$$U^5 + 10dU^3V^4 + 16(X_2^5 + X_3^5 + X_4^5) - 20(U^2 - dV^2)X_2X_3X_4 = 0.$$

By counting points on X_d, we can see explicitly that the Galois representation has been twisted.

(3) Can we reverse this process? In other words, given an eigenform f of weight 4 on $\Gamma_0(N)$ and defined over $\mathbb{Q}$, does there exist a rigid Calabi–Yau threefold X corresponding to f? Since Barry Mazur first called attention to this question, it is known as *Mazur's problem*.

Of course, Mazur's problem is also connected to the issue of whether there are infinitely many different birational equivalence classes of rigid Calabi–Yau threefolds. If the answer to Mazur's question is "yes", then we can translate the question to the setting of modular forms, where it becomes the question of understanding whether there are, up to twists, infinitely many modular forms of weight 4 (of any level) that are defined over $\mathbb{Q}$.

Acknowledgements

The authors would like to thank J.-P. Serre, Richard Taylor, and J.-M. Fontaine for their willingness to answer questions over email. They also thank Matthew Emerton, Luis Dieulefait, and Matthias Schütt for their very helpful comments on a preliminary version of this article, and Bert van Geemen for pointing out the existence of a Galois twist of the Schoen quintic.

The first author thanks Queen's University for its hospitality when this work was done and Colby College for research funding. The second author thanks the Natural Sciences and Engineering Research Council of Canada (NSERC) for their support.

Added in proof: L. Dieulefait [3] has an alternative argument to arrive at the same result as ours.

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