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Best Approximation of a Nonnormal Operator in the Trace Norm

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1. Introduction and Preliminaries

In [11] P. R. Halmos suggested the problem of classifying the operators that have a unique best approximation among the nonnegative operators (a unique positive approximant) in one norm or another. For the operator norm this problem was solved in [2, 7]. Those results were generalized in [8, 12, 4], and other papers. The problem of approximation in trace norm was specifically excluded in [4], and it was noted how the methods given there failed in the case of the trace norm. Normal operators with a unique positive approximant in trace norm are characterized by [5] and the result is a striking contrast to the results in [4, 7]. This paper uses a theorem of T. Ando and other new techniques to determine when certain nonnormal operators have a unique positive approximant in trace norm. The author is grateful to Professor Ando for his suggestions concerning an earlier version of this paper.

We are concerned throughout this paper with (bounded linear) operators on a separable complex Hilbert space H. For any operator T we use the associated operator $|T| = (T^*T)^{1/2}$ and the Cartesian decomposition $T = B + iC$ with $B = \frac{1}{2}(T + T^*)$ and $C = (1/2i)(T - T^*)$. We refer to B as re T and to C as im T. For a compact operator T we let $s_1(T)$, $s_2(T)$, ... denote the eigenvalues of $|T|$ in nonincreasing order repeated according to multiplicity. If we have

$$\sum_{j=1}^{\infty} s_j(T) < \infty$$

then we say that T is trace class and the preceding sum is the trace norm, denoted $\|T\|_1$. If T is not trace class then $\|T\|_1$ is defined to be infinity.

For a self-adjoint operator B we define B^+ to be $\frac{1}{2}(|B| + B)$ and B^- to be $\frac{1}{2}(|B| - B)$; we note that $B = B^+ - B^-$ and $|B| = B^+ + B^-$. If $E(\cdot)$ is the

0022-247X/88 $3.00

spectral measure for B then it follows from the usual operational calculus that $B^+ = BE([0, \infty])$ and $B^- = BE((-\infty, 0])$. If T is a given operator and P is a nonnegative operator such that

$$\infty > \|T-P\|_1 \quad \text{and} \quad \|T-R\|_1 \geqslant \|T-P\|_1$$

for every nonnegative operator R then we say that P is a *trace class positive approximant* of T.

We shall use the following inequality for the trace class operator T, where $\{e_j\}$ is some orthonormal set;

$$\|T\|_1 \geqslant \sum_j |\langle Te_j, e_j \rangle|.$$

This follows for the corollary on p. 40 of [13].

In [5] the next theorem was proved using the theory of the Weyl spectrum as stated in [3, 6]. This theorem and the next theorem provide a proper context for considering positive approximation in trace norm.

1.1. THEOREM. *For a given operator $T = B + iC$, $B^* = B$, $C^* = C$, the following conditions are equivalent*:

(i) *There exists a nonnegative operator P such that $T - P$ is trace class.*

(ii) *The operator C is trace class and the spectrum of B, denoted $\sigma(B)$, not in the interval $[0, \infty)$ consists of isolated eigenvalues, say $\{\lambda_j\}$ repeated according to multiplicity, such that $\sum_j |\lambda_j| < \infty$.*

(iii) *The operator $(T - B^+)$ is trace class.*

The next theorem, which was proved in [5], shows that trace class positive approximants exist. The proof regards the trace class operators on H as the Banach space dual of the compact operators on H. (See p. 48 of [13], for example.) Then the proof exploits various facts from functional analysis and elementary topology.

1.2. THEOREM. *If the operator T satisfies one of the conditions in Theorem 1.1 then T has a trace class positive approximant.*

2. ANDO'S THEOREM

The purpose of this section is to extend the theorem proved by T. Ando in [1] for a finite dimensional Hilbert space to a separable Hilbert space. Because the proof given in [1] does not readily generalize, we supply a complete proof for the theorem and the relevant corollaries.

Given various results in [10], the next lemma is not surprising. However, we could not find a suitable reference, so we supply a proof.

2.1. LEMMA. *Let B be a trace class operator and V be a contraction. The equation* $\operatorname{tr} VB = \operatorname{tr} |B|$ *is true if and only if* $VB = |B|$.

Proof. Let $U|B|$ be the usual polar factorization of B, and let $\{e_j\}$ be an orthonormal basis consisting of eigenvectors of $|B|$ with λ_j denoting the eigenvalue corresponding to e_j. Note that

$$\operatorname{tr} VB = \sum_j \langle VBe_j, e_j\rangle = \sum_j \langle VU|B|\, e_j, e_j\rangle = \sum_j \lambda_j \langle VUe_j, e_j\rangle.$$

Assume that $\operatorname{tr} VB = \operatorname{tr} |B|$; since $\operatorname{tr} VB$ is real, we have

$$\operatorname{tr} VB = \sum \lambda_j \operatorname{re}\langle VUe_j, e_j\rangle$$

and since VU is a contraction, it must be that

$$\lambda_j \operatorname{re}\langle VUe_j, e_j\rangle \leqslant \lambda_j \qquad \text{every } j. \tag{$*$}$$

The inequalities

$$\operatorname{tr} VB = \sum_j \lambda_j \operatorname{re}\langle VUe_j, e_j\rangle \leqslant \sum_j \lambda_j = \operatorname{tr} |B| = \operatorname{tr} VB$$

imply that $\operatorname{re}\langle VUe_j, e_j\rangle \neq 0$ whenever $\lambda_j \neq 0$. Provided $\lambda_j \neq 0$ equality must hold in $(*)$; thus,

$$1 = \operatorname{re}\langle VUe_j, e_j\rangle \leqslant |\langle VUe_j, e_j\rangle| \leqslant 1$$

which implies that $VUe_j = e_j$.

If $\lambda_j = 0$ then $|B|\, e_j = 0$ and $Be_j = 0$, so $VB = VU|B|$ coincides with $|B|$ on the basis $\{e_j\}$. This proves that $VB = |B|$ and the converse is trivial.

The preceding lemma is a key step in the proof of Ando's theorem.

2.2. ANDO'S THEOREM. *Let T be an operator satisfying one of the conditions in Theorem* 1.1. *If there is a contraction V and a nonnegative operator R such that*

$$\operatorname{re} V \leqslant 0, \qquad (\operatorname{re} V)\, R = 0, \qquad \text{and} \qquad V(T-R) = |T-R|.$$

then R is a trace class positive approximant of T. Provided that T is trace class there is such a contraction V and it can be chosen common to all trace class positive approximants.

Proof. First we show that the conditions on V and R suffice to conclude that R is a trace class positive approximant. For any nonnegative operator P it is easy to see that

$$\operatorname{tr}((\operatorname{re} V)\, P) = \operatorname{tr}(P^{1/2}(\operatorname{re} V)\, P^{1/2}) \leqslant 0$$

and

$$\operatorname{re} \operatorname{tr} VP = \operatorname{re} \operatorname{tr} P^{1/2} V P^{1/2} = \operatorname{tr} P^{1/2}(\operatorname{re} V)\, P^{1/2} = \operatorname{tr}((\operatorname{re} V)\, P).$$

Since

$$\begin{aligned} &\operatorname{re} \operatorname{tr}(VT) - \operatorname{tr}((\operatorname{re} V)\, P) = \operatorname{re} \operatorname{tr} V(T-P) \\ &\leqslant \|V\|\; \|T-P\|_1 \\ &\leqslant \|T-P\|_1, \end{aligned}$$

we have

$$\begin{aligned} \|T-P\|_1 \geqslant \operatorname{re} \operatorname{tr}(VT) &= \operatorname{re} \operatorname{tr} V(T-R) \\ &= \operatorname{tr} |T-R| = \|T-R\|_1. \end{aligned}$$

It follows that R is a trace class positive approximant for T.

Let R be a trace class positive approximant for T. We shall now show the existence of V. Recall that the Banach space conjugate of the trace class operators with the trace norm is the space of all operators with the operator norm. (See p. 47 of [13], for example.) Since the open ball with center T and radius $\delta = \|T-R\|_1$ does not intersect the nonnegative operators, there is a continuous linear functional that separates the two. (See p. 417 of [9], for example.) Thus, there is an operator V such that

$$\operatorname{re} \operatorname{tr} V(T+A) > \operatorname{re} \operatorname{tr} VP$$

for every operator A with $\|A\|_1 < \delta$ and every nonnegative trace class operator P. Since the quantity

$$\operatorname{re} \operatorname{tr} VP = \operatorname{re} \operatorname{tr} P^{1/2} V P^{1/2} = \operatorname{tr} P^{1/2}(\operatorname{re} V)\, P^{1/2} = \operatorname{tr}(\operatorname{re} V)\, P$$

is bounded uniformly in P, it must be that

$$\operatorname{re} V \leqslant 0$$

and the inequality implies that $V \neq 0$.

It is straightforward to see that

$$\inf\{\operatorname{re} \operatorname{tr} VA \colon \|A\|_1 < \delta\} = -\delta\, \|V\|.$$

Thus, we have

$$\text{re tr } VT - \delta \, \|V\| \geqslant 0$$

and dividing by $\|V\|$, if necessary, we may assume that $\|V\| = 1$. Note that

$$\text{re tr } VT \geqslant \delta = \|T - R\|_1$$

and

$$\|T - R\|_1 \leqslant \text{re tr } V(T-R) \leqslant |\text{tr } V(T-R)|$$
$$\leqslant \|V\| \, \|T-R\|_1 = \|T - R\|_1.$$

Thus,

$$\text{re tr } VT \geqslant \text{re tr } VT - \text{re tr } VR$$

or

$$0 \geqslant \text{tr } R^{1/2}(-\text{re } V)\, R^{1/2}.$$

This implies that $(\text{re } V)\, R = 0$.

The final property of V follows from Lemma 2.1 and the equations

$$\text{re tr } V(T-R) = |\text{tr } V(T-R)| = \|T-R\|_1 = \text{tr } |T-R|.$$

For positive approximants in the operator norm the order structure for nonnegative operators is very important, as demonstrated in [7]. The next corollary suggests that the order structure is not very useful for trace class positive approximants.

2.3. COROLLARY. *If R_1 and R_2 are trace class positive approximants for the trace class operator T then $R_1 - R_2$ is unitarily equivalent to $R_2 - R_1$. Consequently $R_1 \geqslant R_2$ implies $R_1 = R_2$ and if 0 is a trace class positive approximant then it is the unique trace class positive approximant.*

Proof. Let V be the contraction given by Ando's theorem. Since

$$V(T - R_j) = |T - R_j| \qquad \text{for} \quad j = 1, 2$$

and for any vector f

$$\|(T - R_j) f\| = \| \, |T - R_j| \, f \, \|,$$

we see that V is isometric on the range of $T - R_j$, denoted $(T - R_j)\, H$, for $j = 1, 2$. It follows that each $(T - R_j)\, H$ and $(R_1 - R_2)\, H$ are contained in the kernel of $(I - V^* V)$. Thus, V is isometric on $(R_1 - R_2)\, H$ and $(R_2 - R_1)\, H$.

Since $(\text{re } V) R_j = 0$ for $j = 1, 2$, it follows that V coincides with $i \text{ im } V$ on $(R_1 - R_2) H$. Note that

$$\begin{aligned}(i \text{ im } V)(R_1 - R_2) &= V(R_1 - R_2)\\ &= V(T - R_2) - V(T - R_1)\\ &= |T - R_2| - |T - R_1|.\end{aligned}$$

Thus, $(i \text{ im } V)(R_1 - R_2)$ is self adjoint and $\text{im } V(R_1 - R_2) = (R_2 - R_1) \text{ im } V$. It follows that $(R_1 - R_2)$ is unitarily equivalent to $(R_2 - R_1)$.

2.4. COROLLARY. *Let $A = B + iC$, $B \leqslant 0$, $C^* = C$ be an operator satisfying one of the conditions in Theorem* 1.1. *Then* 0 *is a trace class positive approximant for A.*

Proof. If f belongs to the kernel of A then

$$0 = \langle Af, f \rangle = \langle Bf, f \rangle + i \langle Cf, f \rangle$$

and

$$0 = \langle -Bf, f \rangle = \langle (-B)^{1/2} f, (-B)^{1/2} f \rangle,$$

which proves that f belongs to the kernel of B. Thus,

$$0 = Af = Bf + iCf = iCf$$

and so f belongs to the kernels of C and $A^* = B - iC$. By the symmetry of the argument we conclude that

$$\ker A = \ker A^*$$

and

$$(AH)^- = (\ker A)^\perp = (\ker A^*)^\perp = (A^* H)^-.$$

If $U |A|$ is any polar factorization of A then it follows from the preceding paragraph that U maps $(|A| H)^- = (A^* H)^- = (AH)^-$ isometrically onto $(AH)^-$. Let U equal $-I$ on $(AH)^\perp$. It follows from Theorem 1.1 that A is trace class and, consequently, $|A|$ is diagonalizable. Consideration of an orthonormal basis of eigenvectors of $|A|$ shows that $\text{re } U \leqslant 0$. It follows from the equation

$$U^*(A - 0) = |A - 0|$$

and Theorem 2.2 that 0 is a trace class positive approximant of A.

3. Operators with Nonnegative or Nonpositive Real Part

It is difficult to construct trace class positive approximate for nonnormal operators; no general formula is known for a canonical trace class positive approximant. Indeed, evidence suggests that there is no canonical trace class positive approximant. The next lemma gives the most general result that is known; the generality cannot be improved easily as we indicate with a counterexample after the theorem. We shall pursue the method of proof subsequently.

3.1. Lemma. *Let $A = B + iC$, $B^* = B$, $C^* = C$, be an operator satisfying one of the conditions in Theorem 1.1. If B is nonnegative or nonpositive then B^+ is a trace class positive approximant for A.*

Proof. Let $\{e_j\}$ be an orthonormal basis consisting of eigenvectors for C with λ_j the eigenvalue corresponding to e_j. If R is any nonnegative operator then

$$\|A-R\|_1 \geqslant \sum_j |\langle (A-R))\, e_j, e_j\rangle| = \sum_j [(\langle (B-R)\, e_j, e_j\rangle)^2 + \lambda_j^2]^{1/2}.$$

If $B \geqslant 0$ then we note that

$$\|A-R\|_1 \geqslant \sum_j |\lambda_j| = \|C\|_1 = \|A-B\|_1,$$

which proves that $B^+ = B$ is a trace class positive approximant.

If $B \leqslant 0$ then it follows from Corollary 2.4 that $B^+ = 0$ is a trace class positive approximant for A.

Lemma 3.1 might lead the reader to conjecture that B^+ is always a trace class positive approximant for $A = B + iC$, $B^* = B$, $C^* = C$. Such a conjecture is false, as we demonstrate. Define A by

$$A = B + iC \qquad \text{with } B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix}.$$

It follows that $B^+ = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and it is routine to determine that the spectrum of $|A - B^+|$ is

$$\{\sqrt{6+2\sqrt{5}}, \sqrt{6-2\sqrt{5}}\}$$

consequently,

$$\|A+B^+\|_1 = \operatorname{tr} |A-B^+| = \sqrt{6+2\sqrt{5}} + \sqrt{6-2\sqrt{5}} = 2\sqrt{5}.$$

Since $\|A\|_1 = 4$, we see that the zero operator is closer to A than B^+ is.

The next theorem gives a condition that is sufficient for B to be the unique trace class positive approximant of $A = B + iC$, $B \geqslant 0$, $C^* = C$. The next theorem is an essential step in the proof of our main result.

3.2. THEOREM. *Let $A = B + iC$, $B \geqslant 0$, $C^* = C$, be an operator satisfying one of the conditions in Theorem 1.1. If C is nonnegative or nonpositive then B is the unique trace class approximant for A.*

Proof. As in the proof of Lemma 3.1, we obtain the following for R any nonnegative operator:

$$\begin{aligned}\|A - R\|_1 &\geqslant \sum_j |\langle (A - R)\, e_j, e_j \rangle| \\ &\geqslant \left| \sum_j \langle (A - R)\, e_j, e_j \rangle \right| \\ &= \left| \sum_j \langle (B - R)\, e_j, e_j \rangle + i \sum_j \lambda_j \right| \\ &\geqslant \left| i \sum_j \lambda_j \right| \\ &= \|C\|_1 = \|A - B\|_1.\end{aligned}$$

If R is a trace class positive approximant of A then equality holds throughout the preceding inequalities, and we have

$$\|A - R\|_1 = |\mathrm{tr}(A - R)|.$$

By the last part of Theorem 8.6 of pp. 104–105 of [10], we conclude that $e^{-i\theta}(A - R)$ is a nonnegative operator for $\theta = \arg \mathrm{tr}(A - R)$. The equality in the preceding inequalities implies that

$$\mathrm{tr}(A - R) = \pm i \sum_j \lambda_j.$$

Thus, we know that either

$$-i(A - R) = -i(B - R) + C$$

or

$$i(A - R) = i(B - R) - C$$

is a nonnegative operator. In either case this implies that $B - R = 0$, as desired.

We shall prove a very strong partial converse to Theorem 3.2 in the next lemma. The counterexample following the lemma shows that it cannot easily be improved.

3.3. LEMMA. *Assume that $A = B + iC$, $C^* = C$, satisfies one of the conditions of Theorem* 1.1 *and that B is an invertible nonnegative operator. If C is neither nonnegative nor nonpositive, then A does not have a unique trace class positive approximant.*

Proof. Decompose H into $H_0 \oplus H_1 \oplus H_2$ where orthonormal bases for the three summands are $\{e_j\}$, $\{f_j\}$, and $\{g_j\}$, respectively, and

$$Ce_j = 0, \qquad Cf_j = \lambda_j f_j \qquad \text{with} \quad \lambda_j > 0$$

and

$$Cg_j = \mu_j g_j \qquad \text{with } \mu_j < 0 \text{ for } j = 1, 2, \ldots .$$

Note that if the summand H_1 is trivial then it follows that C is nonpositive. A similar observation applies to H_2. Thus, the bases $\{f_j\}$ and $\{g_j\}$ exist.

Define V^* by

$$V^* e_j = ie_j, \qquad V^* f_j = if_j \qquad \text{and} \qquad V^* g_j = -ig_j \qquad \text{for every } j.$$

Note that V^* is unitary and

$$A - B = iC = V^* \, |C|.$$

For positive ε define P_ε by

$$P_\varepsilon = \langle \cdot, g_1 \rangle \, i\varepsilon f_1 - \langle \cdot, f_1 \rangle \, i\varepsilon g_1$$

and define R_ε by

$$R_\varepsilon = B + P_\varepsilon.$$

Since B is invertible there exists a positive δ such that

$$\langle Bf, f \rangle \geqslant \delta \, \|f\|^2$$

for every $f \in H$. It is routine to verify that P_ε is self adjoint and $\|P_\varepsilon\| = \varepsilon$. Thus, for positive ε not exceeding δ we have

$$R_\varepsilon \geqslant 0.$$

We shall show that R_ε is a trace class positive approximant of A for all positive ε sufficiently small. Note that

$$\begin{aligned} A - R_\varepsilon &= iC - \langle \cdot, g_1 \rangle i\varepsilon f_1 + \langle \cdot, f_1 \rangle i\varepsilon g_1 \\ &= V^*(|C| - \langle \cdot, g_1 \rangle \varepsilon f_1 - \langle \cdot, f_1 \rangle \varepsilon g_1) \end{aligned}$$

and so

$$V(A - R_\varepsilon) = |C| - \langle \cdot, g_1 \rangle \varepsilon f_1 - \langle \cdot, f_1 \rangle \varepsilon g_1.$$

For all $v \in \operatorname{span}\{f_1, g_1\}$, we have

$$\langle |C| v, v \rangle \geqslant \min\{\lambda_1, |\mu_1|\} > 0.$$

Thus, for all positive ε not exceeding this minimum

$$|C| - \langle \cdot, g_1 \rangle \varepsilon f_1 - \langle \cdot, f_1 \rangle \varepsilon g_1$$

is a nonnegative operator.

For any operator T if U is a unitary operator and UT is a nonnegative operator, say P, then $U^* P$ is a polar factorization for T because

$$T^* T = PUU^* P = P^2$$

and

$$|T| = P.$$

It follows from the preceding sentence and the preceding paragraph that

$$V(A - R_\varepsilon) = |A - R_\varepsilon|.$$

Since $\operatorname{re} V = 0$ the hypotheses of Theorem 2.2 are satisfied and we have shown that R_ε is a trace class positive approximant of A for all positive ε sufficiently small.

According to Lemma 3.1, B is a trace class positive approximant for $A = B + iC$, $C^* = C$, whenever $B \geqslant 0$. This might lead the reader to conjecture that it is not necessary to require B to be invertible in Lemma 3.3. That conjecture is false.

3.4. Counterexample. If B is not required to be invertible in Lemma 3.3 then the conclusion is false.

Proof. Let $A = B + iC$, *where*

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Define v_1 and v_2 by

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and note that

$$Cv_1 = v_1 \quad \text{and} \quad Cv_2 = -v_2.$$

If follows from Lemma 3.1 that B is a trace class positive approximant of A and so

$$\begin{aligned} \inf\{\|A - R\|_1 : R \geqslant 0\} &= \|A - B\|_1 = \|C\|_1 \\ &= \operatorname{tr} |C| = \operatorname{tr} I = 2. \end{aligned}$$

Let R be a trace class positive approximant of A; for brevity, let $D = B - R$ so that $A - R = D + iC$. Note that

$$\begin{aligned} \|D + iC\|_1 &\geqslant |\langle (D + iC)\, v_1, v_1 \rangle| + |\langle (D + iC)\, v_2, v_2 \rangle| \\ &= |\langle Dv_1, v_1 \rangle + i| + |\langle Dv_2, v_2 \rangle - i| \\ &\geqslant |i| + |-i| = 2 \end{aligned}$$

and the equation $\|D + iC\|_1 = 2$ implies

$$\langle Dv_1, v_1 \rangle = 0 = \langle Dv_2, v_2 \rangle.$$

Thus, we have

$$R = \begin{bmatrix} 1 & b \\ b & 0 \end{bmatrix}.$$

It is easily seen that b must be 0 in order for R to be nonnegative. Thus, $R = B$ and A has a unique trace class positive approximant despite the fact that C is neither nonnegative nor nonpositive.

Now we can state our main theorem.

3.5. THEOREM. *Assume that $A = B + iC$, $C^* = C$, satisfies one of the conditions of Theorem* 1.1 *and that B is an invertible nonnegative operator. The operator A has a unique trace class positive approximant if and only if C is either nonnegative or nonpositive.*

Proof. The "if" part of the theorem was proved in Theorem 3.2 and the "only if" part was proved in Lemma 3.3.

4. Remarks

In [5] (and in [1] for dim $H < \infty$) it is proved that B^+ is a trace class positive approximant for $A = B + iC$, $B^* = B$, $C^* = C$, provided A is normal. In [5] it is proved that B^+ is the unique trace class positive approximant for the normal operator A if and only if the spectrum of A does not intersect both of the sets $\{z: \operatorname{re} z > 0, \operatorname{im} z > 0\}$ and $\{z: \operatorname{re} z > 0, \operatorname{im} z < 0\}$. For a finite dimensional Hilbert space T. Ando discovered that Theorem 3.2 is true without requiring that $B \geqslant 0$; his proof, which is based on Theorem 2.2, is completely different from the proof given here.

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