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Full Minimal Steiner Trees on Lattice Sets[1]

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1. INTRODUCTION

Given a finite set of points P in the Euclidean plane, the *Steiner problem* asks us to constuct a shortest possible network interconnecting P. Such a network is known as a *minimal Steiner tree*. The Steiner problem is an intrinsically difficult one, having been shown to be NP-hard [7]; however, it often proves far more tractable if we restrict our attention to points in special geometric configurations. One such restriction which has generated considerable interest is that of finding minimal Steiner trees for nice sets of integer lattice points. The first significant result in this direction was that of Chung and Graham [4], which, in effect, precisely characterized the minimal Steiner trees for any horizontal $2 \times n$ array of integer lattice points. In 1989, Chung *et al.* [3] examined a related problem, which they described as the Checkerboard Problem. They asked how to find a minimal Steiner tree for an $n \times n$ square lattice, that is, a collection of $n \times n$ points arranged in a regular lattice of unit squares like the corners of the cells of

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a checkerboard. Although their paper gave a series of conjectured solutions to this problem, not all of which turn out to be correct, they were unable to suggest a method for proving their claims. The case $n = 2^k$ was recently solved in [1].

In this paper, we examine a more general situation, namely the nature of minimal Steiner trees for *Steiner-closed lattice sets*, which we define to be sets of integer lattice points satisfying two conditions, the first of which says that they have a spanning tree all of whose edges have length 1, and the second of which is a technical condition which we believe to be redundant. These conditions, which ensure that the points are not sparsely scattered, are given in Section 2. Our analysis converts the largely geometric problem of constructing these trees to a somewhat simpler combinatorial one, which we study in the sequel to this paper [2].

Let T^* denote a minimal Steiner tree for a Steiner-closed lattice set. The key feature of the conjectured solutions of Chung *et al.* [3] for the cases where the Steiner-closed lattice set is an $n \times n$ square lattice is that they use as their principal building block for T^* the minimal Steiner tree for the corners of a unit square (shown in Fig. 1), which we will denote by X. A Steiner tree, such as X, is *full* if each of its terminals have degree 1. The *full components* of T^* can be thought of as being the smallest irreducible 'blocks' from which the T^* is composed (by union at the terminals). When $n = 2^k$, all the full components of T^* are Xs. This is proved in [1] by showing that, per terminal, X is in some sense the most efficient possible component forming part of T^*. If $n \neq 2^k$ then T^* cannot be built up solely from Xs, hence it becomes necessary to examine what other full trees can occur in checkerboards.

In Theorem 6.6 we completely classify all such full components. In particular, we show that all possible full components of T^* belong to a small number of easily understood classes. This classification greatly simplifies the problem of constructing minimal Steiner trees for specific Steiner-closed lattice sets. Our next paper [2] will demonstrate how the classification allows us to find a minimal Steiner tree for any rectangular array of integer lattice points using the concept of *excess* established in [1].

The strategy for achieving the classification is as follows. We first establish some preliminary definitions and general techniques in Section 2. The results in this section are not specific to Steiner trees on lattice sets, and

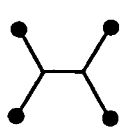

FIG. 1. The Steiner tree X

can be thought of as comprising a basic 'toolchest' of techniques for constructing minimal Steiner trees in a wide range of different contexts. We then consider a full subtree, T, of T^* and define $G(S_T)$ to be the graph on S_T, the collection of square cells and triangular half cells containing parts of T, with the obvious adjacency. In Section 3 it is shown that $G(S_T)$ is a tree and that there are precisely two Steiner points in each square of S_T and one Steiner point in each triangle of S_T. Section 4 introduces the concept of quasi-leaves of $G(S_T)$ which allow us to further investigate the structure of T, in Section 5, as we move inwards from leaves of $G(S_T)$. This results in a structure theorem for S_T which states that $G(S_T)$ has only two leaves and S_T has a restricted internal structure. In this case S_T is said to be a strip. Finally, in Section 6, we closely examine minimal steiner trees corresponding to strips to determine which ones can possibly occur as subtrees of T^*. This final classification is complete in the sense that it lists all full components that can occur in T^*, and every full component listed does occur for some choice of Steiner-closed lattice set.

2. PRELIMINARIES

A tree, T, in the Euclidean plane, consisting of vertices and straight-line edges connecting the points of P is called a *Steiner tree* if the angle between any two edges meeting at a vertex is greater than or equal to 120° and all vertices of T not in P have degree 3. Such vertices are called *Steiner points*, and it is clear that the edges meeting at a Steiner point make angles of precisely 120° with each other. Any minimal length network interconnecting P is a Steiner tree. A tree connecting the points of P without the addition of any new vertices is called a *spanning tree*, and the shortest such tree a *minimal spanning tree*.

The points in P are referred to as *terminals*. Throughout this paper we will denote the terminals by $a, b, c, d, \ldots$ and indicate the terminals of a unit square by listing them counterclockwise from the top left-hand corner. Steiner points are usually denoted by s with subscripts.

After Cockayne [5], (ab) denotes the third vertex of the equilateral triangle $ab(ab)$ where the vertices are listed in counterclockwise order. To differentiate between open and closed line segments, we will denote the line segment between points a and b by $[ab]$ if it is closed (that is, includes a and b) or simply by ab if it is open.

Consider an infinite square unit lattice on the Euclidean plane. A finite subset, P, of vertices of this lattice will be said to form a *Steiner-closed lattice set* if it satisfies the following conditions:

(i) there exists a spanning tree for P all of whose edges have length 1; and

(ii) given lattice points a and b such that $|ab| = 1$, if a minimal Steiner tree for P intersects the interior of ab then a and b are elements of P.

Note that if a set of lattice points P has the property that for any unit lattice edge meeting a lattice point not in P the interior of that edges lies entirely outside the convex hull of P, then P is Steiner-closed. It follows, for example, that an $n \times n$ square lattice forms a Steiner-closed lattice set. Indeed, we believe the following to be true:

Conjecture. Condition (ii) is redundant in the above definition of a Steiner-closed lattice set; that is, P is Steiner-closed if and only if there exists a spanning tree for P all of whose edges have length 1.

We will use the word *square* to refer exclusively to a unit square of a Steiner-closed lattice set, and the word *triangle* to refer exclusively to an isoceles right triangle whose vertices belong to a Steiner-closed lattice set and whose orthogonal edges have length 1. Hence, a triangle is half a square.

First we establish some simple facts.

LEMMA 2.1. *Suppose T is a minimal Steiner tree for a Steiner-closed lattice set.*

(i) *If p, q are two points (not necessarily vertices) in T and s and t are vertices of T, adjacent to each other and lying in the path between p and q, then $|pq| \geqslant |st|$, and the inequality must be strict if s or t are Steiner points. Moreover, no edge of T has length greater than* 1.

(ii) *No edge of T intersects the interior of two orthogonal sides of a triangle or two opposite sides of a square.*

(iii) *No convex path in a full component of T intersects two parallel lines of distance two. Hence, the terminals at the ends of a convex path in a full component of T are the endpoints of either a side or a diagonal of a square.*

(iv) *If a path in T is convex with respect to a vertex of a right angle and crosses each of its two legs at exactly one point, then the path has only one Steiner point in the right angle and meets the legs at no more than* $30°$.

(v) *No two parallel edges of T intersect the interior of a single side of a square.*

Proof. Statement (i) follows immediately from the minimality of T and the fact that a Steiner-closed lattice set can be spanned by edges of length 1.

In order to see (ii) we argue by contradiction. Let abc be a triangle with right angle at a, and assume a single edge of T intersects ab and ac at p

and q respectively. By definition, a is an element of the Steiner-closed lattice set, hence we can assume, without loss of generality, that there is a path in T from p to a not passing through q (otherwise swap the roles of p and q). But this implies T is not a minimal as we can replace the line segment pq by the shorter line segment qa to create a shorter tree. The second part of (ii) follows directly from (i).

The remaining statements have easy proofs. In particular, (iii) follows from (i); (iv) is a consequence of angle considerations and is independent of the minimality of T; and (v) is a corollary of (ii). ∎

We use four well-known techniques, outlined in the following propositions, to help eliminate non-optimal Steiner trees.

PROPOSITION 2.2 (The Simpson–Heinen Construction) (See [9]). *Let abc be a triangle, all of whose angles are less than* 120°. *Let S be the minimal Steiner tree on a, b, c with Steiner points s (as in Fig.* 2). *Then* (ac) *lies on the extended line bs, and* $|S| = |b(ac)|$.

This proposition provides a convenient way of refering to the topology of a given Steiner tree. For example, the Steiner tree T on terminals d, u, p, q and r has topology $(p(ud))(rq)$ only if it immediately follows, by repeated application of Proposition 2.2, that $|T| = |(p(ud))(rq)|$. This topology is illustrated by the tree in solid lines in Fig. 3. The repeated use of Proposition 2.2 to calculate the length of a Steiner tree with a given topology forms the basis of Melzak's algorithm [10]. The repeated use of this proposition also gives a practical method for constructing the Simpson line of a Steiner tree from any point on the tree.

Suppose $p_1p_2p_3p_4$ is a convex quadrilateral. Let $\phi(p_1p_2, p_3p_4)$ denote the angle at the intersection of the diagonals which faces p_1p_2.

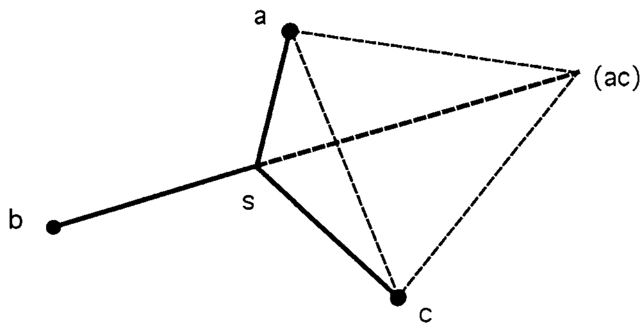

FIG. 2. The Simpson–Heinen Construction.

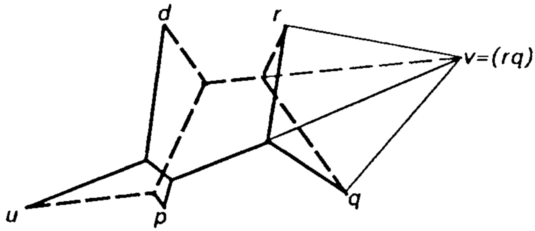

FIG. 3. The tree in solid lines has topology $(p(ud))(rq)$ and is not minimal.

PROPOSITION 2.3 (Pollak's Theorem) [11]. *Suppose both full Steiner trees* $(p_1p_2)(p_3p_4)$ *and* $(p_4p_1)(p_2p_3)$ *exist, then* $(p_2p_1)(p_4p_3)$ *is minimal if* $\phi(p_1p_2, p_3p_4) \leqslant 90°$.

Note that Proposition 2.3 can be applied to more than four points. For example, if $v=(rq)$, $|ud| \geqslant |up|$, $|pv| > |dv|$ in Fig. 3, and if both trees $(p(ud))(rq)$ (in solid lines) and $(d(rq))(pu)$ (in broken lines) exist, then the former is longer than the latter, since $\phi(up, vd) < 90°$.

PROPOSITION 2.4 (The Variational Argument) [12]. *Let* T_1 *and* T_2 *be two Steiner trees on the same set of terminals. We will consider* $|T_1|$ *and* $|T_2|$ *to be functions of* x *in the range* $[x_1, x_2]$ *measuring the lengths of the perturbed Steiner trees as we move the terminals from one position to another. Then* $|T_2(x_1)| \leqslant |T_1(x_1)|$ *if*

(1) $|T_2(x_2)| \leqslant |T_1(x_2)|$ *and*

(2) $\dfrac{d\,|T_2|}{dx} \geqslant \dfrac{d\,|T_1|}{dx} \geqslant 0$ *or* $\dfrac{d\,|T_1|}{dx} \leqslant \dfrac{d\,|T_2|}{dx} \leqslant 0.$

The basic principle, from [12], for computing the relative size of $d\,|T_1|/dx$ and $d\,|T_2|/dx$ is as follows. If each of the terminals, a_i, being moved is perturbed at a particular instant in the direction of a unit vector v_i then the contribution of an edge incident with a_i to the derivative is minus the cosine of the angle between v_i and the edge. The derivative is the sum of all such contributions.

The following lemma, which will prove useful in Sections 4 and 5, represents a typical example of an application of Proposition 2.4.

LEMMA 2.5. *The tree* T_1 *in solid lines in Figure* 4(a) *cannot be part of a Steiner minimal tree for a Steiner-closed lattice set.*

Proof. We will show that the tree T_2, drawn in broken lines in Fig. 4a, is shorter than T_1. Let p move to a along da, and q to b along cb, and perturb the two trees appropriately. Clearly $-\cos(\angle s_2pa) > -\cos(\angle s_1p_a) > 0$

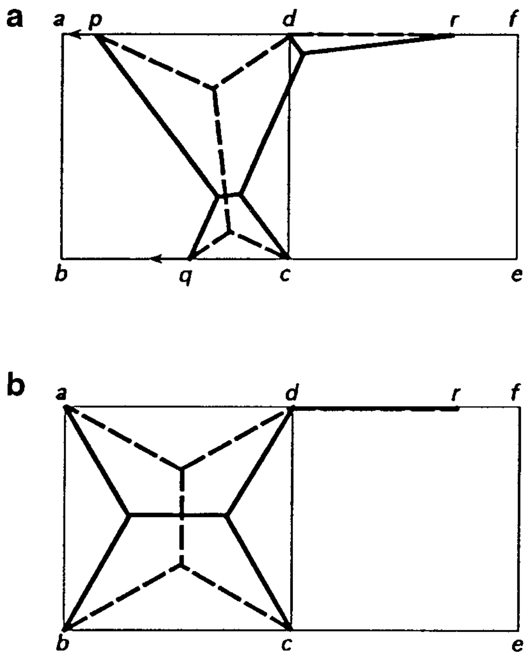

FIG. 4. The tree in solid lines in (a) is not minimal.

and $-\cos(\angle s_3 qb) > -\cos(\angle s_1 qb) > 0$. Hence, $d|T_2|/dx > d|T_1|/dx > 0$, and in the end, the Steiner point of T_1 adjacent to d degenerates into d and $|T_1| = |T_2|$ (Fig. 4b). Thus, the lemma holds by Proposition 2.4. ∎

In applying the proposition in this section, it is useful to have the concept of a left-turn path. Let s be a terminal or Steiner point of the Steiner tree T and let s_1 be an adjacent Steiner point. Consider a walk starting at s in the direction towards s_1, turning left at each Steiner point, and finishing at the first terminal reached, say t. We refer to the path traced by this walk as the left-turn path $ss_1 \cdots$ (terminating at t). A right-turn path is defined similarly.

PROPOSITION 2.6 (Non-minimal Paths) [13]. *Let $p \cdots rq$ be a path in a Steiner tree T such that $p \cdots rq$ is a simple polygon and $\angle prq \leqslant 60°$. Let m be the point on the line through rq such that $\angle mpr = \angle prq$ (Fig. 5). Suppose that every terminal that exists inside or on the boundary of the polygon $p \cdots rm$ is connected to q via r (and in particular that q is not a terminal if q lies on $[rm]$). Then T is not a minimal Steiner tree.*

The following useful lemma is a corollary of this result.

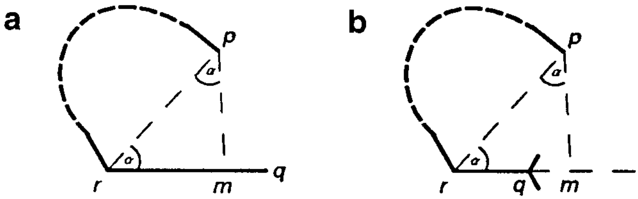

FIG. 5. The paths $p \cdots rq$ make T non-minimal if $\alpha \leqslant 60°$.

LEMMA 2.7. *Suppose $p \cdots uvrq$ is a path in a minimal tree T so that $p \cdots uvrq$ is a simple polygon, v, r and q are Steiner points, and every terminal that exists inside the polygon $p \cdots uvrq$ is connected to q via v. If $|pu| \geqslant |uv|$ and $|pu| \geqslant |vr|$, then $\angle puv > 120°$ (Fig. 6).*

Proof. Assume, on the contrary, that $\angle puv \leqslant 120°$, and that, consequently, $\angle pvr \leqslant 120°$. If $\angle pvr \leqslant 60°$, then T is not minimal by applying Proposition 2.6 to $p \cdots vr$. Hence, $\angle uvp < 60°$. Since $|pu| \geqslant |uv|$, it follows by the geometry of the triangle puv that $\angle puv \geqslant 60°$. Furthermore, by the geometry of the quadrilateral $puvr$ it is easy to see that $\angle prv \geqslant 60°$ since $|pu| \geqslant |vr|$, $\angle uvr = 120°$ and $\angle puv \leqslant 120°$. Hence, $\angle prq \leqslant 60°$ which again contradicts the minimality of T by applying Proposition 2.6 to $p \cdots rq$. ▮

A weakness of Proposition 2.6 is that the condition that every terminal inside the polygon is connected to q via r is often difficult to check. The following theorem is, in a sense, a stronger version of the proposition which provides a method of overcoming this difficulty in many situations.

THEOREM 2.8. *Suppose s_1 is a Steiner point in a Steiner tree T and s_0, s_2 are two vertices adjacent to s_1. Let p be a point in T such that p lies on the same side of the line through $s_0 s_1$ as s_2, s_0 lies on the path connecting p and s_1, and $\angle p s_0 s_1 \leqslant 60°$. Let c be the point on $s_0 s_1$ or its extension such that $pc \parallel s_1 s_2$, and let c' be the point on $s_1 s_2$ or its extension such that $pc' \parallel s_0 s_1$. Define the trap region of $p \cdots s_0 s_1$, R, as follows:*

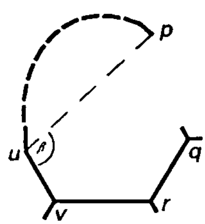

FIG. 6. The path $p \cdots uvrq$ makes T non-minimal if $\beta \leqslant 120°$.

(i) $R = p(cp)\, c(pc)$, *if* $\angle s_0 s_1 p \leqslant 120°$ (*Fig.* 7);

(ii) $R = p(c'p)\, c'(pc')$, *if* $\angle s_0 s_1 p > 120°$ *and* $\angle p s_2 s_1 \leqslant 120°$;

(iii) $R = \triangle s_1(pc)(c'p) \cup p(cp)\, c(pc) \cup p(c'p)\, c'(pc')$ *otherwise* (*Fig.* 8).

If there are no terminals in the interior of R, then T is not minimal.

Proof. Assume T is minimal. We consider three cases corresponding to the three possibilities for R.

(i) If (pc) lies on $s_0 s_1$, then $|p(pc)| \leqslant |s_0(pc)|$ which contradicts the minimality of T. Hence we assume s_1 lies on $c(pc)$. Let $s_0 s_1 s_2 \cdots s_{k+1}$ be a path in T such that: $s_i s_{i+1} \parallel s_1 s_2$ if i is odd; $s_i s_{i+1} \parallel p(pc)$ if i is even and s_i lies in $\triangle pc(pc)$; $s_i s_{i+1} \parallel p(cp)$ if i is even and s_i lies in $\triangle cp(cp)$; and $s_k s_{k+1}$ intersects $p(cp)$ or $p(pc)$ at a point u (see Fig. 7). Let j be the largest integer less than k such that $s_j s_{j+1}$ intersects $[cp]$, say at the point r. It immediately follows that $|pu| \leqslant |r s_{j+1}| \leqslant |s_j s_{j+1}|$, contradicting Lemma 2.1(i).

(ii) Clearly, this case is symmetric to Case (i).

(iii) Consider the path $s_0 s_1 s_2 \cdots$ where $s_i s_{i+1} \parallel s_0 s_1$ if i is even and $s_i s_{i+1} \parallel s_1 s_2$ if i is odd (as in Fig. 8). Clearly this path intersects $[cp]$ or $[c'p]$, say at u. Let $s_j s_{j+1}$ be the edge of T such that either $u = s_j$ or u lies in the interior of $s_j s_{j+1}$. Then $\angle p s_j s_{j+1} \leqslant 60°$ and $\angle p s_{j+1} s_j < 120°$, hence we can apply the argument in Case (i), since the region $p(up)\, u(pu)$ lies in R. This completes the proof. ▮

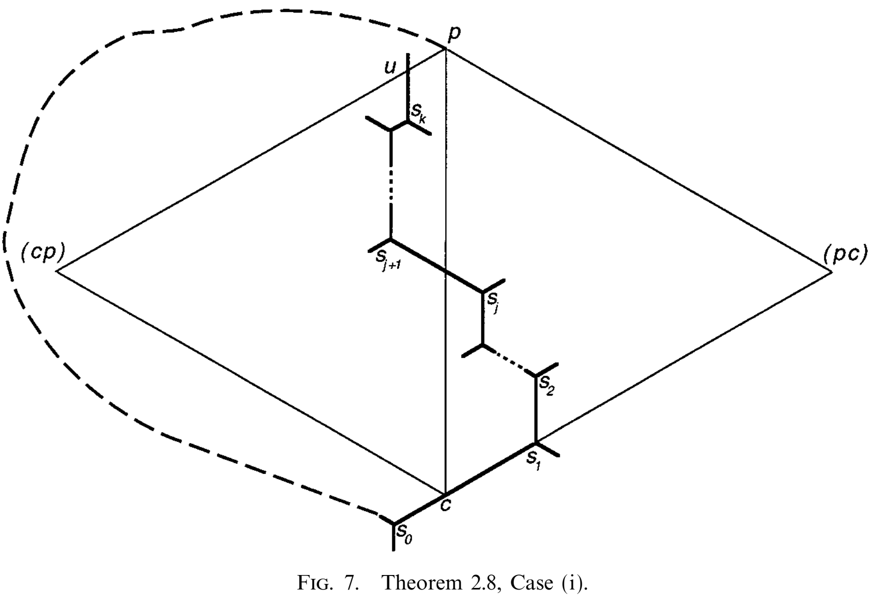

FIG. 7. Theorem 2.8, Case (i).

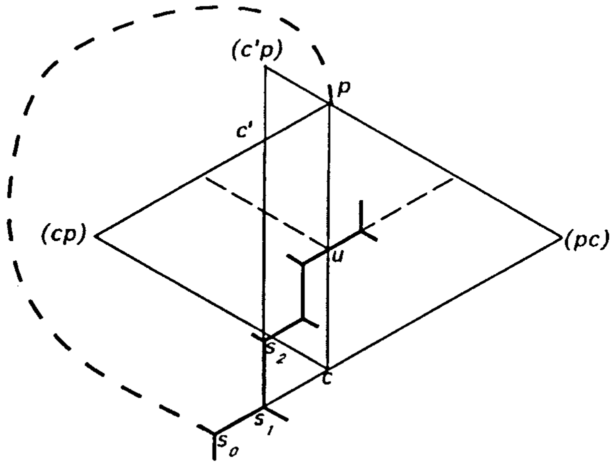

FIG. 8. Theorem 2.8, Case (iii).

Given a line segment pq, the polygon $p((qp)p)(qp)q(pq)(p(pq))$, shown in Fig. 9, is referred to as the *butterfly* of pq and q is referred to as its *head.*

COROLLARY 2.9. *Suppose $p \cdots rs_0$ is a path in a Steiner tree T and s_0 is a Steiner point. Let q be a point on rs_0 such that $\angle pqs_0 \leqslant 60°$. If there is no terminal in the interior of the butterfly of pq with head q, then T, is not minimal.*

Proof. Clearly, the trap region defined in Theorem 2.8 is completely covered by the butterfly of pq. ∎

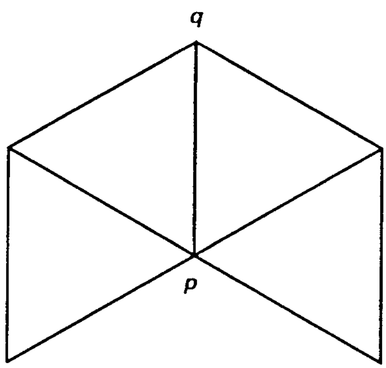

FIG. 9. The butterfly of pq.

LEMMA 2.10. *Let abcd be a square, and let T be a minimal Steiner tree for a Steiner-closed lattice set. Suppose there is an edge $s_1 s_2$ of T intersecting ad at p such that s_2 lies in abcd and $\angle s_1 pd \leqslant 60°$. Then s_1 lies on the path from s_2 to d* (*see Fig.* 12).

Proof. The butterfly of pd contains no terminals, hence the result follows by Corollary 2.9. ∎

Finally, in Lemma 2.12, we give some very general local conditions which allow us to move a terminal of a Steiner tree T along a circle whose center lies on the Simpson line at that terminal without increasing the length of T. This will be used in the next section to show that there are strong restrictions on the way an edge of a minimal Steiner tree can cross a lattice edge. Lemma 2.12 is preceded by a necessary technical lemma.

LEMMA 2.11. *Let T be a Steiner tree containing a Steiner point s adjacent to a terminal p. If T is perturbed by moving p and fixing the other terminals of T, such that the topology of T remains unchanged, then the trajectory of s makes an angle of at least* 60° *to the edge of T incident with p.*

Proof. Suppose p moves a very small distance to a point p', such that s moves to a different point s'. Let q denote the other end of the Simpson line originating at p, and let r denote the other end of the Simpson line originating at s and on the same side of pq as p'. Let the point of intersection of sr and $s'q$ be O. Then angles qsO and $rs'O$ are both 60°, so the triangles qsO and $rs'O$ are similar. Furthermore, $|sq| > |sr|$, since $|sq|$ is the total length of two subtrees of T at s whilst $|sr|$ is the length of one of those subtrees. Thus $|sO| > |s'O|$. For $|pp'|$ arbitrarily small, angle sOs' is arbitrarily close to 60°, and the lemma follows. ∎

LEMMA 2.12. *Let p be a terminal of a full Steiner tree T with terminal set $A \cup \{p\}$ and let q be the other end of the Simpson line from p. Choose points r, s and O such that s and O are on pq, and such that $|Op| = |Or|$, $\angle psr \leqslant 60°$, and no Steiner point lies in the interior of ps. Suppose further that if p' is any point on the circle through p and r centred at O, and lying on the smaller of the two arcs, C, between p and r, then for any full Steiner tree with terminals p' and a subset of A, the other end of the Simpson line from p' does not lie inside the triangle Opr. Then there exists a tree with terminals $A \cup \{r\}$ which is no longer than T.*

Proof. Perturb T by fixing all terminals in A and moving p along C towards r either to the first point where the topology of T is about to change or, if no such point exist, all the way to r. Denote the new position of p by p', denote the new tree by T', and denote the intersections of $p'q$ with Or and sr by O' and s' respectively. Then q is at the other end of the

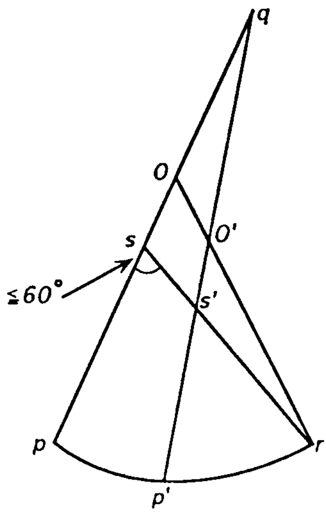

FIG. 10. Under the conditions of Lemma 2.12, p can travel along the circular arc about O to r without increasing the length of T.

Simpson line of T' at p', and using Lemma 2.11 the assumptions of the lemma are satisfied with t, p, O and s replaced by t', p', O' and s' (see Fig. 10). There are two possible reasons that the tree topology must change. One is that a Steiner vertex is about to collide with a terminal. In this case we can replace T' by the full component containing p' of the tree which results at the collison. The other possible reason is that two Steiner vertices collide. In this case the tree can be cut into two Steiner trees with two crossing edges, and p' is a terminal in one of them. In both cases the conditions of the lemma are still satisfied (since the terminals of the new tree contain p' and a subset of A) and so the process can be continued. However, if in the continuation, an edge of the tree crosses a terminal of one of the trees which have been cut off, that edge needs to be cut short at that terminal in order to ensure that the present tree, together with all the trees cut off, form a connected network. After a finite number of stages, we must have $p' = r$. At all stages of the process, the movement of p is about a circle centred on a point lying on the Simpson line from p, and so the length of the tree is not increased, as required. ▮

3. $G(S_T)$ IS A TREE

Throughout the remainder of this paper, let T be a full subtree of a minimal Steiner tree T^* for a Steiner-closed lattice set P. Let S'_T be the set of all squares in the lattice whose interiors contain parts of T. Define S_T from

S'_T as follows: for each square *abcd* of S_T, if there is a triangle in the lattice such that the part of T contained in the interior of *abcd* is completely contained in that triangle then replace *abcd* by that triangle. So, for example, if T is a unit lattice edge then S_T is empty, whereas if T is the Steiner tree in Fig. 11 then S_T contains two squares and two triangles, as shown in the figure. A square or triangle is said to be *adjacent* to another square or triangle if they share a side. Let $G(S_T)$ be the graph on S_T with the adjacency as defined above. Note that $G(S_T)$ is a connected graph (since T is full), and that all vertices of the squares and triangles of S_T are elements of P.

The word *component* will be used to refer to a connected component of the intersection of T with the interior of a given square or triangle.

LEMMA 3.1. *There is only one component in a square or a triangle of* S_T.

Proof. We prove the lemma only for squares since the proof is similar and easier for triangles. Suppose, on the contrary, there are two components in the square *abcd*. By Lemma 2.1(ii) each component has at least one Steiner point. Let P_1 and P_2 be convex paths in separate components, each reaching from one edge of *abcd* to another, such that no part of T lies between them. It is clear from Lemma 2.1 that P_1 and P_2 cannot both join the interiors of opposite sides of the square without forming a loop. Hence we can assume P_1 is part of the left-turn path $s_1 s_2 s_3 \cdots$ where $s_1 s_2$ meets $[ad]$, s_2 is in the interior of the square *abcd* and $s_2 s_3$ meets $[ab]$. It follows from Lemmas 2.1(iv) and 2.10, that s_1, s_2 and s_3 lie on the path in T^* from d to b (see Fig. 12). Similarly, assume P_2 is part of the left-turn path $s'_1 s'_2 s'_3 \cdots$ where s'_2 lies in the interior of *abcd*. By symmetry we can assume that s_1 does not lie on the path in T^* joining s_2 and s'_2. This immediately tells us that P_2 cannot meet $[ad]$. Furthermore, it is clear that P_2 does not join *ab* to *bc* by angle considerations, and does not join *bc* to *cd* by Lemma 2.10 (since otherwise s_1 lies on the path joining s_2 and s'_2). Hence P_2 must join *ab* to *cd* with $s_3 = s'_1$ and $s'_1 s'_2$ meeting *ab*, as shown in Fig. 12.

Now let *iadh* and *dcef* be squares adjacent to *abcd*. By Lemma 2.1(iii) the left-turn Steiner path $s'_1 s'_2 s'_3 \cdots$ cannot reach the line through *ef*; it must terminate at h or i. Applying Lemma 2.7 to $d \cdots s'_1 s'_2 s'_3 s'_4$ we conclude that T is not minimal. ∎

A useful consequence of this lemma is the following corollary.

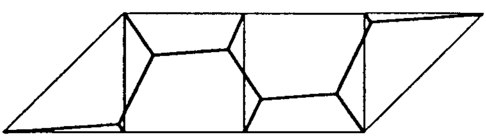

FIG. 11. Here, S_T contains two squares and two triangles.

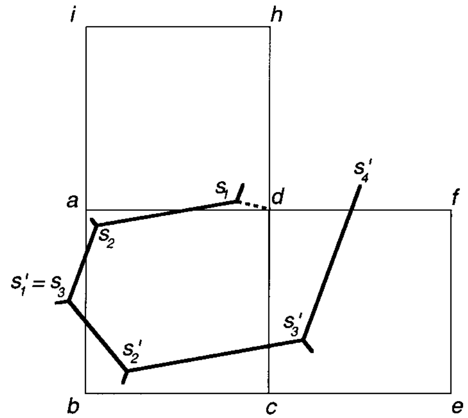

FIG. 12. There cannot be two components in *abcd*.

COROLLARY 3.2. *The interior of any edge of a square intersects at most one edge of T and no Steiner points of T.*

We now wish to show that there are at most two Steiner points in each square of S_T. The key to this lies in the following lemma.

LEMMA 3.3. *Let abcd be a square of the Steiner-closed lattice set P. Suppose that $s_1 s_2$ is an edge of T^* between Steiner vertices s_1 and s_2 crossing bc at q such that s_1 lies above bc and the extensions of other edges of T^* at s_1 do not intersect the interior of the inverval qc. Then the interval $[c(ad)]$ does not intersect the extension of qs_1.*

Proof. Arguing by contradiction, assume the extension of qs_1 intersects the interval $[c(ad)]$. First, consider the subtree T_0 of T^* containing s_1 obtained by cutting T^* at q. Suppose c is in T_0, or in other words s_1 lies on the path in T^* from c to s_2, and apply Theorem 2.8(i). Since $\angle s_1 s_2 c < 120°$ and the extension of qs_1 intersects $[c(ad)]$, it easily follows (by considering extreme cases) that the trap region of $c \cdots s_1 s_2$ contains no terminals, contradicting the minimality of T^*. Hence c is not a terminal of T_0. Let O denote the point of intersection of the Simpson line of T_0 at q and $(ad)c$. Note that $\angle(ad)\,cq = 75°$. We now consider two separate cases.

Assume, in the first case, that angle $s_1 qc < 75°$. Observe that $|Oq| > |Oc|$, and also that $|Os_1| < |Oc|$, since $\angle cs_1 O \geqslant 120°$. Hence there exists a point q_0 on the interval $s_1 q$ such that $|Oq_0| = |Oc|$. Since T_0 does not contain c, the hypotheses of Lemma 2.12 are satisfied with $s = s_1$, $r = c$, $p = q_0$ and $T = T_0 - qq_0$. Thus T_0 can be replaced by a shorter tree connecting the terminals of T^* in T_0 to c, contradicting the minimality of T^*.

So assume, on the other hand, that $\angle s_1qc \geqslant 75°$. Applying Lemma 2.12, as in the previous case, we conclude that there exists a tree, containing c and all the terminals of T^* in T_0, whose length is at most $|T_0| + |qq_0|$. We complete the contradiction by finding a tree containing all terminals of T^* not in T_0 whose length is less than $|T^* - T_0| - |qq_0|$. If q_0 lies on qs_2 then the contradiction immediately follows, so we may assume $|qq_0| \geqslant |qs_2|$.

First, note that the maximum value of $|qq_0|$ occurs when $O = (ad)$ and $|(ad)q_0| = |(ad)c|$. Thus $|qs_2| \leqslant \sqrt{2+\sqrt{3}} - (1+\sqrt{3}/2) < 0.0658$. Now let s_3 and s_4 denote the two Steiner vertices adjacent to and below s_2, and let w denote the distance between the parallel edges of T^* below them (see Fig. 13). Consider the hexagon $s_2s_3r_1r_2r_3s_4$, where all angles of the hexagon are $120°$ and $|s_3r_1| = |s_4r_3| = w$. Since the convex path in T^* containing s_3, s_2 and s_4 reaches two distinct terminals, it follows, by the minimality of T^*, that there must be a terminal on or inside this hexagon. In particular, some terminal must be within distance $w + 2w \tan 30°$ below s_2. Hence $0.0658 + w + 2w/\sqrt{3} > 1$, so $w > 0.433$.

Now consider cutting T^* apart at s_2, and let T_3 and T_4 be the two subtrees containing s_3 and s_4 respectively. Note that $|T_3| > 1$ and $|T_4| > 1$, since each subtree contains at least two terminals of T^*. Consider joining T_3 and T_4 directly to f instead of through s_2, where f is the point of intersection of the extensions of the third edges at s_3 and s_4 as shown in Fig. 13. For each $i \in \{3, 4\}$ let O_i be the point on the Simpson line for T_i at s_2 such that $|O_is_2| = 1$. It is clear that we can apply Lemma 2.12 to each subtree ensuring that r is the point f, and O is the point O_i in each case. Hence it follows from Lemma 2.12 that the total decrease in length of T_3 and T_4 when they are joined directly at f is at least $(|O_3s_2| - |O_3f|) + (|O_4s_2| - |O_4f|) = 2 - (|O_3f| + |O_4f|)$. Observe that f lies on a line segment from O_3s_2 to O_4s_2 parallel to O_3O_4 and of length $2w$. In particular,

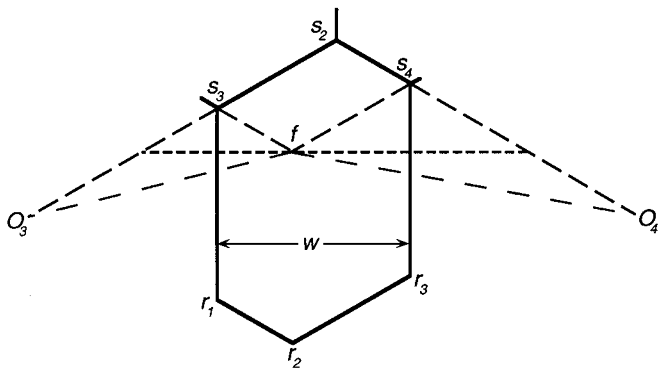

FIG. 13. The hexagon $s_2s_3r_1r_2r_3s_4$ must contain a terminal of T^*.

this line segment depends only on w and is otherwise independent of the positions of s_3 and s_4. Hence the minimum decrease in the sum of lengths of T_3 and T_4 occurs when f is, say, on the line O_4s_2, in which case $|s_2f| = w/\sin 60°$, $|Of|^2 = (\sqrt{w \tan 30° + 1})^2 + w^2$ and the decrease is at least $|s_2f| + 1 - |Of| > 0.177 > 0.0658$, as required. ▮

LEMMA 3.4. *There are at least two Steiner points in a square of S_T.*

Proof. Suppose, on the contrary, there is only one Steiner point s_1 in the square of S_T, *abcd*. Consider three rays radiating from s_1 in the opposite directions to the edges themselves. Since there are three 120° angles formed by these rays, at least two corners of *abcd*, say b and c, lie in the same 120° angle. Let the edge of T, s_1s_2, lying in this angle intersect the boundary of *abcd* at p. Without loss of generality, we may assume that either p lies on ab or $p = b$ or p lies on bc.

Firstly, suppose that p lies on ab or $p = b$. Let the other two edges incident with s_1 intersect the boundary of *abcd* at q and r (reading counterclockwise around the square from p). Then, by geometry and the fact that *abcd* is a square of S_T, it follows that q lies on cd, r lies on $[ad]$ and $r \neq d$. The extension of ps_1 must meet ad; otherwise we are done by applying Lemma 3.3 to r. Hence the point (ba) must lie above the extension of qs_1; otherwise we are done by Lemma 3.3 applied to q. But now the variation which moves r towards a and p towards b, with q fixed, rotates the edge qs_1 downwards, and so (ba) must remain above the extension of qs_1. This contradicts the fact that it lies on the extension of qs_1 when $p = b$ and $r = a$.

So finally suppose that p lies on bc. We can assume by symmetry that (ad) lies on or to the left of the extension of ps_1. Then we are done by Lemma 3.3. ▮

Suppose there are m_1 squares and m_2 triangles in S_T. By a simple induction argument the number of terminals of T is less than or equal to $2 + 2m_2 + m_1$, with equality occuring only if $G(S_T)$ is a tree. Since T is full, the number of Steiner points of T is less than or equal to $2m_2 + m_1$. But clearly each triangle must contain at least one Steiner point, and by Lemma 3.4 each square contains at least two Steiner points. Hence, the above inequalities are forced to be equalities. This immediately implies the following two lemmas:

LEMMA 3.5. *There are precisely two Steiner points in each square of S_T and one Steiner point in each triangle of S_T.*

LEMMA 3.6. *$G(S_T)$ is a tree.*

4. LEAVES AND QUASI-LEAVES OF $G(S_T)$

Before introducing the concept of a quasi-leaf we require three lemmas. The first is an elementary observation, the second is a technical result which will prove useful in this and the following section, and the third gives us valuable information about the structure of the part of T inside triangles of S_T.

LEMMA 4.1. *Let abcd be a square in the Steiner-closed lattice set. Let $s_1 s_2$ be an edge of T such that $s_1 s_2$ intersects ab, say at p, and s_2 lies in the interior of abcd. If $60° \leqslant \angle aps_1 \leqslant 120°$ then abcd is a square of S_T.*

LEMMA 4.2. *Let abcd and dcef be adjacent squares in the Steiner-closed lattice set. Let u, t, q be points on the line segments $[bc]$, $[ad]$ and $[ef]$ respectively, let p lie on ce, and let r be a point on either $[df]$ or qf (Fig. 14). Let T_1 be a full Steiner tree on d, u, t, p, q and r with topology $((ud)p)(rq)$ if $t = d$, or topology $((u(td))p)(rq)$ if $t \neq d$ (as in the figure). Let s' be the Steiner point in abcd adjacent to u, and suppose $\angle s'uc \leqslant 30°$. Then*

(i) *T_1 exists only if r lies on $[df]$, and*

(ii) *T_1 is not a subtree of T.*

Proof. Let T_1 be a subtree of T. If $t \neq d$, let t' be the point where the line through the two Steiner points in *abcd* intersects *ad*; otherwise let $t' = d$. Let T'_1 be the full Steiner tree on t', u, p, q and r with topology $((ut')p)(rq)$. We first show that T'_1 exists only if r lies on $[df]$, from which (i) immediately follows.

Suppose, on the contrary, that r lies on the interior of qf and T'_1 exists. First note, by Corollary 3.2, that $q = e$. By Proposition 2.2, $\angle(ut')\,p(rq) < 120°$. We will show this cannot occur. Observe that $\angle(bd)\,c(fe) = 120°$. It is clear that $\angle(ut')cb \leqslant \angle(bd)cb$ and $\angle(rq)ce \leqslant \angle(fe)ce$; hence $\angle(ut')\,c(qr) \geqslant 120°$. If we let p move along the line segment ce from c to e, then $\angle(ut')p(qr)$ increases at first, since $\angle(ut')cb > \angle(rq)ce$, then decreases again as p approaches e. However $\angle(ut')e(qr) > 120°$, hence it follows that $\angle(ut')p(qr) > 120°$ for any point p on ce, giving the desired contradiction.

Thus r lies on $[df]$ and either q lies on ef or $q = e$ (Fig. 14). Define s_1 to be the Steiner point in *dcef* adjacent to r and q, and s_2 to be the Steiner point adjacent to s_1 and p. Note, by Lemma 2.10, that q lies on the path from s_1 to e. Let $bb'c'c$ and $cc'e'e$ be the squares below *abcd* and *dcef* respectively, and let the next vertices in the left-turn path $s_1 s_2 \cdots$ be s_3, s_4 and s_5. Clearly s_3 lies in the intrior of $cc'e'e$. If the right-turn Steiner path $u \cdots s_2 s_3 \cdots$ intersects the interior of cc' then, by Lemma 4.1, $bb'c'c$ is a square of S_T, contradicting Lemma 3.6. Hence the right-turn path $u \cdots s_2 s_3 \cdots$ terminates at c or c'. Assume, in the first case, that the path ends at c (Fig. 14a). By angle considerations, there must be a single Steiner point, say s_6, between s_3 and c. Hence $s_3 s_4$ intersects ee' since it is parallel to cs_6 and $\angle c_6 ce < 45°$. Let L_1 be the line through e parallel to $s_1 s_2$. Let θ be the acute angle between L_1 and ce. Note that $15° < \theta \leqslant 30°$. We first show that s_3 lies above L_1. Suppose, on the contrary, that $[s_2, s_3]$ intersects L_1. Then $[cs_6]$ intersects L_1, say at x, and $|cs_6| \geqslant |cx| = 2\sin(\theta)/\sqrt{3}$. Let y be the point where T intersects cd and let y' be the point where the line through (bd) parallel to cs_6 intersects cd. By Proposition 2.2, $|cy| \leqslant |cy'|$, and a simple calculation shows that $|cy'| = 1 - \sqrt{2}\sin(45-\theta)/\sin(30+\theta)$. It can now be checked that, over the domain $15° < \theta \leqslant 30°$,

$$1 - \frac{\sqrt{2}\sin(45-\theta)}{\sin(30+\theta)} \leqslant \frac{2\sin(\theta)}{\sqrt{3}}$$

with equality when $\theta = 30°$. It follows that T is not minimal, since we can replace cs_6 by cy to form a shorter tree. Hence s_3 lies above L_1 and $\angle es_3 s_4 < 60°$. Let z be the point on $s_3 s_4$ such that $ez \parallel s_2 s_3$. Clearly there are no terminals in the region $e(ze)\, z(ez)$, so T is not minimal by Theorem 2.8.

If, on the other hand, the right-turn path $u \cdots s_2 s_3 \cdots$ ends at c' (Fig. 14b) then s_4 lies in $cc'e'e$ and it is clear, by another easy angle argument, that $s_4 s_5$ intersects ee'. Note that $\angle es_4 s_5 < 60°$. Let z be the point on $s_4 s_5$ such that $ez \parallel s_2 s_3$. Again, since there are no terminals in $e(ze)\, z(ez)$, T is not minimal by Theorem 2.8. ▮

LEMMA 4.3. *The Steiner point in a triangle of S_T is adjacent to the terminal at the right angle.*

Proof. Let bcd be a triangle of S_T with right angle at c. By Lemma 3.5, this triangle contains a unique Steiner point, s_1. By Corollary 3.2, s_1 is adjacent to at least one vertex of $\triangle bcd$. Suppose, contrary to the lemma, that s_1 is not adjacent to c, but is adjacent to d. Without loss of generality, let the second edge incident with s_1 meet $[bc]$ at u, and the third edge incident with s_1 intersect the interior of dc. If $\triangle bcd$ shares dc with another triangle $\triangle dce$ then there are two possible cases: either both right angles occur at c (Fig. 15a) or one occurs at c and the other at d (Fig. 15b). In the first case an edge must cross de, since $\triangle dce$ contains exactly one Steiner point; in the second case the part of T in the two triangles is clearly non-minimal by Pollak's Theorem (Proposition 2.3). Thus, in each case there is a contradiction. Consequently, we may assume that bcd is adjacent to a square of S_T, $dcef$. Let the right-turn Steiner path starting with us_1 be $us_1 s_2 s_3$. We consider two cases.

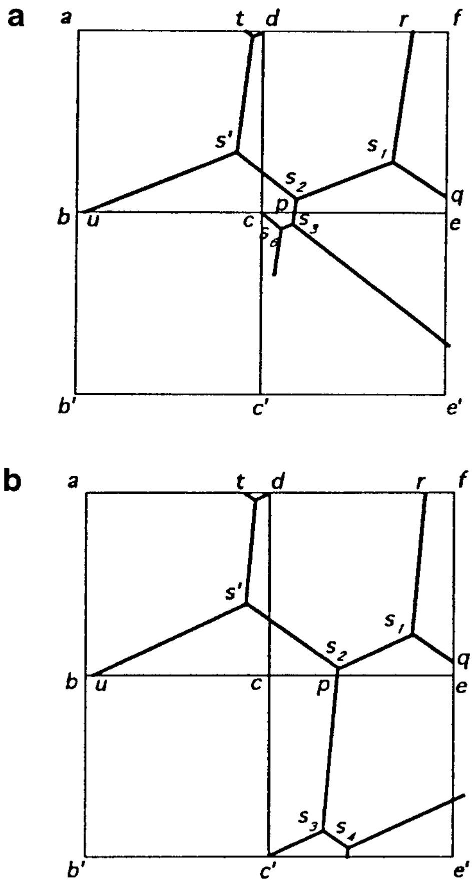

FIG. 14. The Steiner tree T_1 and two possibilities for a neighbouring square.

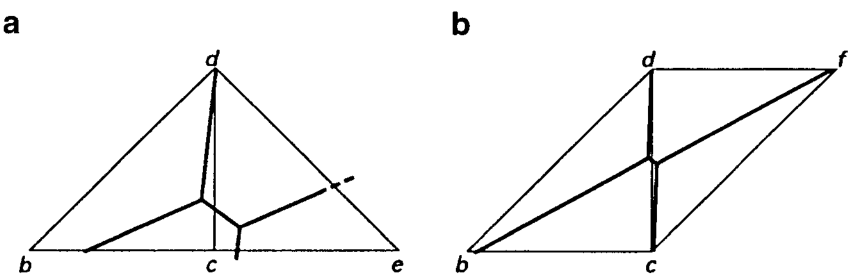

FIG. 15. Trees in two adjacent triangles.

(i) Assume $s_2 s_3$ meets $[ce]$ at p. If $p \neq c$ then T is not optimal, by Lemma 4.2. So it follows that $p = c$ (Fig. 16). Let $v = (rq)$ and note that the tree $(cu)(dv)$, shown in broken lines in the figure, does exist. Consequently, by Proposition 2.3 and the remark following it, we have a contradiction to the minimality of T.

(ii) Assume s_3 is also in $dcef$ (Fig. 17a). By Corollary 3.2 it follows that s_3 is adjacent to c and its third edge intersects ce, say at p. Another edge incident with s_2 meets df or ef, say at q. Let T_1 be this Steiner tree on u, c, p, q and d; let T_2 be uc plus the tree $(dq)(pc)$ (shown in broken lines in Fig. 17a). We now argue by variation (Proposition 2.4). Let p move along ec to c and let q move along qe to e. The resulting trees are shown in Fig. 17b. This process decreases the length of T_1 at a greater rate than the length of T_2, but the length of the tree resulting from T_2 is clearly less than or equal to the length of the tree resulting from T_1. So T_1 is not minimal, and consequently neither is T. ▮

We define the direction of the edges of T to range from $-15°$ to $165°$ from the horizontal. Of the three directions of edges in T, one must be either in the range

$(-15°, 0°]$, and called *negative horizontal*, or

$(0°, 15°]$, and called *positive horizontal*, or

$(75°, 90°]$, and called *positive vertical*, or

$(90°, 105°]$, and called *negative vertical*.

Note that these directions, referred to as *main directions*, are exclusive. That is, T cannot have two directions which are both main directions.

If a leaf of $G(S_T)$ is a triangle then, by Lemma 4.3, precisely one edge of T in triangle is incident with one of the acute angles of the triangle, and clearly lies in the main direction (Fig. 18a). If a leaf is a square, the it is as shown in Figure 18b. (The other possible Steiner tree, is not minimal by Proposition 2.3, as shown in [1].) The edge joining two Steiner points s_1 and s_2 is in the main direction.

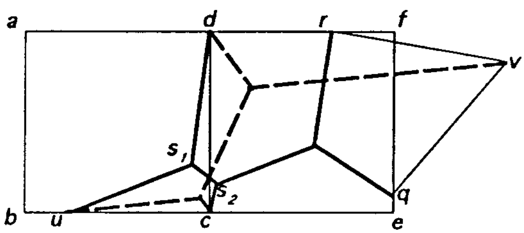

FIG. 16. The tree in solid lines is not minimal.

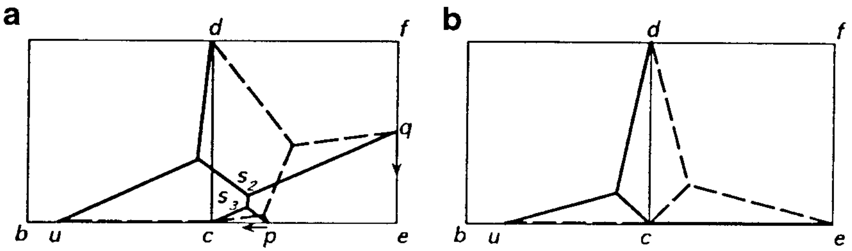

FIG. 17. By the variational argument, the tree in solid lines in (a) is not minimal.

We will call the sides of a square whose interiors intersect T *shared* sides. Suppose V_m is a vertex of degree two in $G(S_T)$. V_m is called a *quasi-leaf* if there is a sequence of adjacent vertices $VV_1V_2 \cdots V_m$ in $G(S_T)$ such that

(1) V is a leaf,

(2) V_i, $1 \leqslant i < m$, are quasi-leaves,

(3) V_m is either a triangle or square of degree 2 in $G(S_T)$, and in the latter case its shared sides are opposite.

If a quasi-leaf is a triangle, one of the edges intersecting its sides is in a main direction (Fig. 19a). It is quasi-leaf *abcd* is a square with two Steiner points s_1 and s_2, then $s_1 s_2$ is in a main direction. The square is referred to as *normal* if s_1, s_2 are adjacent to the endpoints of an unshared side (Fig. 19b). Otherwise s_1, s_2 are adjacent to the endpoints of a diagonal *ac* or *bd*, and the square is referred to as *abnormal* (Fig. 19c). Note that whether a square is normal or abnormal depends on the topology of T. In all cases we will classify a leaf or quasi-leaf by its main direction, for example as positive horizontal if that is its main direction. Similarly, we can classify T by its main direction.

We conclude this section with a few useful lemmas on quasi-leaves, beginning with a simple observation. This is a stronger version of Lemma 4.1.

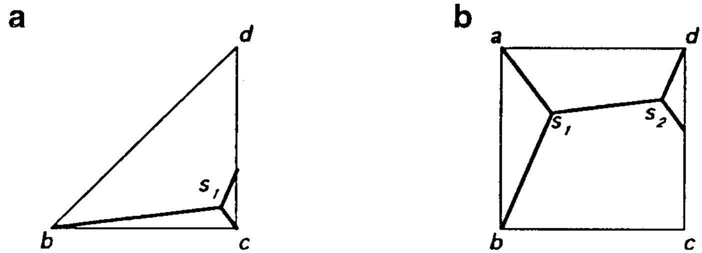

FIG. 18. The two possible topologies in leaves of $G(S_T)$.

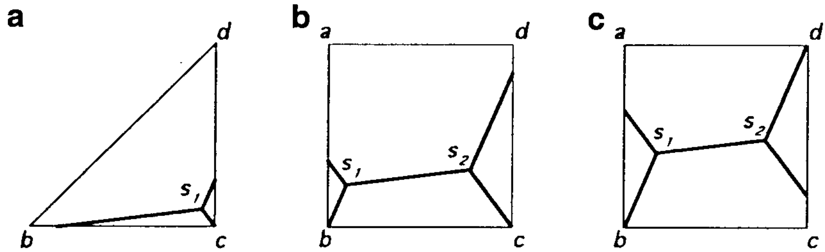

FIG. 19. The three possible topologies in quasi-leaves of $G(S_T)$.

LEMMA 4.4. (i) *If an edge of T intersects the interior of a side of a triangle of S_T, then the angle between them is in* $(0, 30°)$.

(ii) *If an edge of T intersects a shared side of a square leaf or square quasi-leaf, then the angle between them is in* $(15°, 45°)$.

Note that this result tells us that if an edge of T intersects a shared side of a leaf or quasi-leaf then the angle between them is less than 45°.

LEMMA 4.5. *In any sequence of adjacent square quasi-leaves in $G(S_T)$ at most one square is abnormal.*

Proof. Let $V_i V_{i+1} \cdots V_{i+m}$ be a sequence of adjacent square quasi-leaves in $G(S_T)$. Suppose, contrary to the lemma, that V_i and V_{i+m} are both abnormal quasi-leaves with no abnormal quasi-leaves lying between them. Note, by angle considerations, that m must be odd. Let $V_i = abcd$ and $V_{i+m} = a'b'c'd'$ and let T intersect ab at p, and $c'd'$ at q. Let T_1 be the subtree of T in these $m+1$ squares (Fig. 20). Let T_2 be the Steiner tree on p, q and the terminals of T_1 whose topology in each square is the same as that of a normal quasi-leaf (Fig. 19b), as shown in broken lines in the figure. As p moves along ab towards a and q moves along $c'd'$ towards c' it is clear that $d|T_2|/dx > d|T_1|/dx > 0$, and eventually $|T_1| = |T_2|$ when p coincides with a and q with c'. Hence, by Proposition 2.4, we have a contradiction to the minimality of T. ∎

LEMMA 4.6. *Let abcd be a square of S_T and let V_1 be an adjacent vertex of $G(S_T)$ lying to the right of abcd. Suppose V_1 is either a leaf or quasi-leaf.*

(i) *If abcd has degree two in $G(S_T)$ then T is horizontal.*

(ii) *If abcd is adjacent to another vertex of $G(S_T)$ which is a leaf or quasi-leaf, then T is horizontal.*

Proof. Suppose V_1 lies to the right of *abcd*, but is not horizontal. By symmetry, we may assume without loss of generality that V_1 is negative

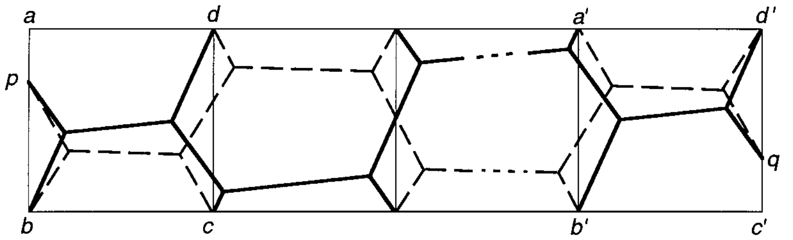

FIG. 20. The tree in solid lines is not minimal.

vertical. V_1 is not a square, by our previous descriptions of leaves and quasi-leaves; hence V_1 is a triangle, with Steiner point s_1. Let s_2 be the Steiner point in *abcd* adjacent to s_1, and let s_3 be the other Steiner point in *abcd*. Clearly, s_3 must lie on the left-turn path $cs_1s_2s_3$ (as in Fig. 21), otherwise *abcd* would not be a square of S_T. Since V_1 is negative vertical, the extension of s_2s_3 intersects *ad*. Hence, one edge incident with s_3 intersects *ad*. By Corollary 3.2, s_2 has to be adjacent to *d*. Since $\angle s_2dc > 45°$, the third edge incident with s_3 cannot end at *b*, but rather intersects the interior of *ab*. This implies that *abcd* has degree three in $G(S_T)$, proving (i).

The vertex of $G(S_T)$ above *abcd* cannot be a leaf or quasi-leaf by Lemma 4.4 since *ad* meets an edge incident with s_3 at more than 60°. Furthermore, the vertex to the left of *abcd* is not a leaf or quasi-leaf since *ab* meets an edge incident with s_3 at more than 45°. Hence, by contradiction, (ii) is also true. ▮

LEMMA 4.7. *Let $V_1V_2 \cdots V_m$ be a sequence of adjacent vertices of $G(S_T)$ so that V_1 is a leaf and the others are quasi-leaves. Let $V_{m+1} = abcd$ be a square of $G(S_T)$ adjacent to V_m along the side cd. Suppose the component of T in abcd has Steiner points s_1 and s_2 so that s_1 is adjacent to b, s_2 is adjacent to d and one of the edges incident with s_1 intersects ad at p (Fig. 22). Then T is positive horizontal.*

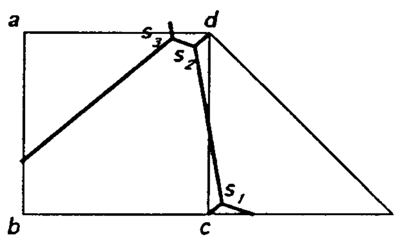

FIG. 21. The part of *T* in *abcd*, for *T* vertical.

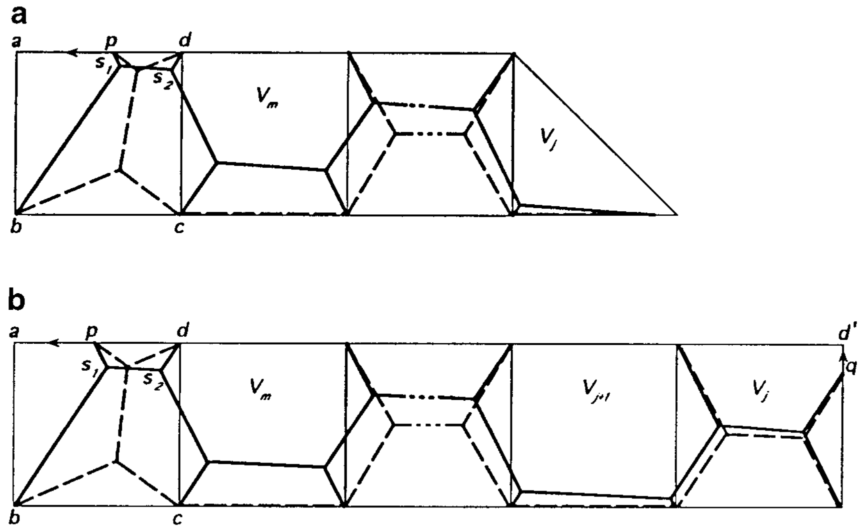

FIG. 22. In each case, the tree in solid lines is not minimal.

Proof. By Lemma 4.6, T is horizontal. Assume, contrary to the lemma, that the main direction is negative horizontal. If V_m is a triangle, then the part of T in V_m and V_{m+1} fails to be minimal by Lemma 2.5. Hence V_m must be a square and is clearly normal since the main direction is negative horizontal. Let j be the largest element of $\{1, ..., m-1\}$ such that V_j is either

(i) a triangle (Fig. 22a), or

(ii) a square leaf or abnormal quasi-leaf (Fig. 22b).

If V_j is an abnormal quasi-leaf, let q be the point where T intersects the edge shared by V_j and V_{j-1} and let d' be the terminal on V_j adjacent to a Steiner point in V_{j-1} (as in the figure). Note that in Case (i) $m-j$ is necessarily even, while in Case (ii) $m-j$ is odd. Let T_1 be the part of T in $V_j V_{j+1} \cdots V_{m+1}$.

The lemma now follows from a variational argument similar to that used in Lemma 2.5. Observe that the orientation of a leaf, triangle or abnormal quasi-leaf determines whether the main direction is positively or negatively nearly horizontal. Hence, as we move p to a and (in the case of V_j being an abnormal quasi-leaf) q to d', the main direction of T_1 cannot change from negative horizontal to positive horizontal. This forces T_1 to degenerate into an alernating series of Xs and edges when p coincides with a (and q with d'). In each case, let T_2 be the Steiner tree shown in broken lines in Fig. 22. Clearly, as p moves to a (and q to d'), T_2 is perturbed to the same alternating series of Xs and edges, except that the X in V_{m+1} is differently oriented; but T_2 increases in length faster than T_1. Hence, by Proposition 2.4, T_1 is not minimal. ∎

5. THE STRUCTURE OF $G(S_T)$

The aim of this section is to establish a structure theorem for S_T. In essence, we show that all vertices of $G(S_T)$ are leaves or quasi-leaves, and consequently that there can be no branching in $G(S_T)$. This theorem follows from Corollary 5.2, Lemma 5.4, and Lemma 5.5, which systematically demonstrate that certain vertices which are neither leaves nor quasi-leaves do not occur in $G(S_T)$. Moreover, using simple angle arguments we are able to further restrict the structure of S_T to a form we describe as a *strip*.

The first lemma follows directly from Lemma 4.6 and the fact that the main direction is exclusive.

LEMMA 5.1. *Suppose a square of S_T, abcd, is adjacent to two vertices of $G(S_T)$, V_1 and V_2, each of which is either a leaf or quasi-leaf.*

(i) *If V_1 lies to the right or left of abcd, then V_1 is horizontal; if V_1 lies above or below abcd, it is vertical.*

(ii) *V_1 and V_2 lie on opposite sides of abcd.*

An immediate consequence of this lemma is the following result.

COROLLARY 5.2. *$G(S_T)$ has no vertex of degree four adjacent to three vertices, each of which is a leaf or quasi-leaf.*

Before proving our next main result, we need a small technical lemma.

LEMMA 5.3. *None of the trees T_1, drawn in solid lines in Fig. 23, can be subtrees of T.*

Proof. We will show that in each case the tree T_2, drawn in broken lines in Fig. 23, is shorter than T_1, using Proposition 2.4. In Fig. 23a and 23b, let p, q, u, v move to the corners i, b, c, h respectively. Clearly, we always have $d|T_2|/dx > d|T_1|/dx > 0$. In the end, $|T_1| = \sqrt{11+6\sqrt{3}}$, while $|T_2| = 2+\sqrt{5+2\sqrt{3}}$ in Fig. 23a and $|T_2| = 3+\sqrt{3}$ in Fig. 23b. In both cases, $|T_1| \geqslant |T_2|$. Hence, by Proposition 2.4, T_1 is not minimal. In Fig. 23c, let q, u, v move to b, d, i respectively. It then follows that $d|T_2|/dx > d|T_1|/dx > 0$ for q, v, and $d|T_1|/dx \leqslant d|T_2|/dx < 0$ for u. In the end, $|T_1| = |T_2|$. So, again in this case T_1 is not minimal. ∎

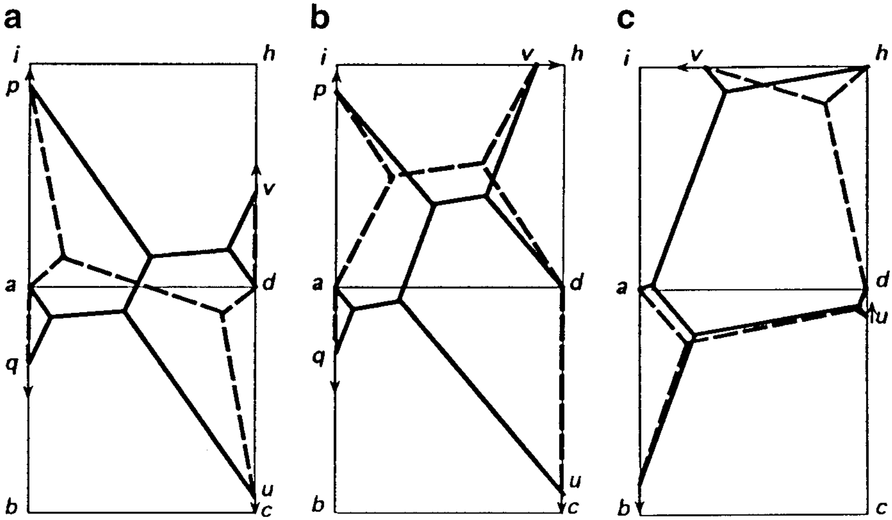

FIG. 23. The Steiner trees in solid lines are not minimal.

LEMMA 5.4. *$G(S_T)$ has no vertex of degree three which is adjacent to two vertices, V_1 and V_2, each of which is either a leaf or quasi-leaf.*

Proof. Assume *abcd* is such a vertex of $G(S_T)$, and that V_1 lies on its left side and V_2 on its right side. Also assume V_1 and V_2 are positive horizontal and the third vertex is above *abcd*. By Lemma 5.1 all these assumptions can be made without any loss of generality. Let s_2 and s_3 be the Steiner points in *abcd* such that s_2 is adjacent to a Steiner point, s_1, in V_1. Since V_1 is a leaf or quasi-leaf, the edge incident with s_2 in the main direction cannot meet $[ab]$ or c by angle considerations, or intersect *cd* by Lemma 4.4. Also, it cannot meet *ad*, since in that case s_1 being adjacent to *a* would force one of the edges adjacent to s_3 to intersect *ab* (Fig. 24) whereas s_1 being adjacent to *b* would clearly force *abd* to be a triangle of S_T. Hence, the edge in the main direction is s_2s_3. Moreover, since *abcd* is a vertex of degree three in $G(S_T)$, exactly one of the corners of *abcd* is adjacent to s_2 or s_3. Since *abcd* is positive horizontal and the third vertex in $G(S_T)$ is above *abcd*, this corner cannot be *c*. This leaves three cases to be eliminated. In each case, the nearby vertices of the checkerboard are labelled as indicated in the figures.

(i) Assume s_2 is adjacent to *b* (Fig. 25).

Both V_1 and V_2 are squares by Lemma 4.4. Let s_4 and s_5 be the next vertices on the left-turn path $s_2s_3s_4s_5$. Since $G(S_T)$ is a tree, *jkai* is not a square of S_T, which means T enters this square only if *jai* is a triangle of S_T. So, by Lemma 4.4, the path $s_2s_3s_4s_5\cdots$ cannot intersect *ai*. Hence, s_6,

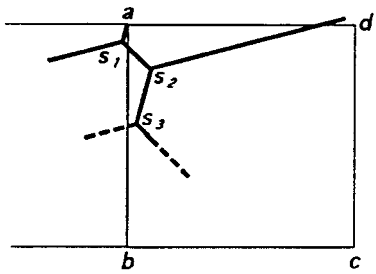

FIG. 24. T is not minimal if $s_2 s_3$ is not in the main direction.

the next Steiner point on the right-turn path $s_3 s_4 s_6$ lies in *iadh* by Lemma 3.5. Since *hdfg* is not a square of S_T, s_6 must be adjacent to d. Now if the third edge of s_6 meets $[hi]$, Proposition 2.3 is contradicted, and if it meets dh, T is not minimal by Lemma 2.5.

(ii) Assume s_2 is adjacent to a (Fig. 26).

Let s_4, s_5 and s_6 be vertices of T as defined above. If V_1 is a square or $s_4 s_5$ meets $[ih]$ then the minimality of T is again contradicted by the argument in (i). Hence, V_1 is a triangle and $s_4 s_5$ intersects ai. As before, s_6 must be adjacent to d since V_2 is clearly a square. If the third edge incident with s_6 meets dh (Fig. 26a), T is not minimal by Lemma 5.3, Fig. 23a. If the thrid edge incident with s_6 meets $[ih]$ (Fig. 26b), T is not minimal by Lemma 5.3, Fig. 23b.

(iii) Assume s_3 is adjacent to d (Fig. 27).

Again V_2 is a square by Lemma 4.4. Let s_4 and s_5 be the next Steiner points on the right-turn path $s_3 s_2 s_4 s_5$ and let the third edge incident with

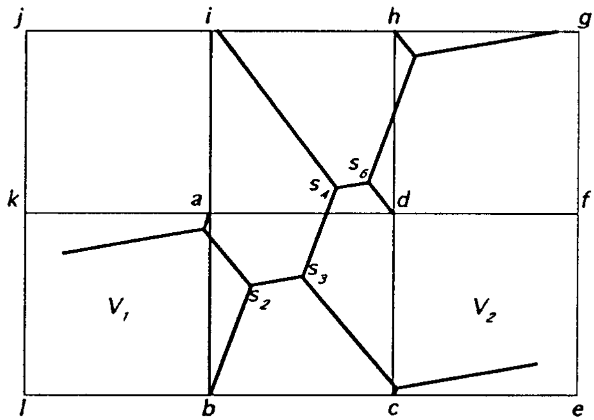

FIG. 25. The case where s_2 is adjacent to b.

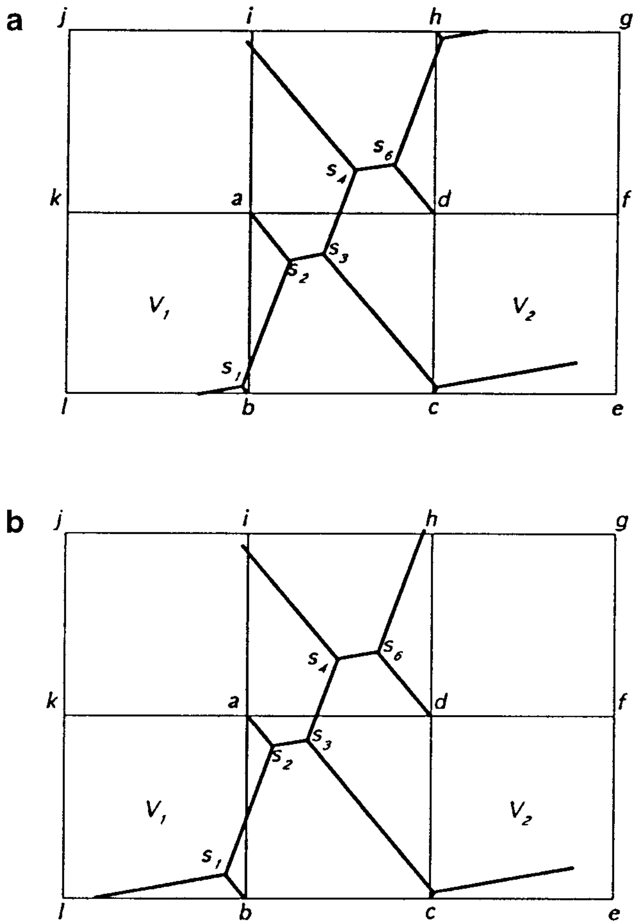

FIG. 26. The case where s_2 is adjacent to a.

s_4 end at the vertex s_6. We will assume, in the first case, that s_6 does not lie in the interior of the square *iadh* (Fig. 27a). This implies that s_5 lies in *iadh*. If s_7 is the next vertex on the right-turn path $s_3 s_2 s_4 s_5 s_7$ then $s_5 s_7$ must meet $[ih]$ since *hdfg* is not a square of S_T (by Lemma 3.6). It is now easily seen that the left-turn path $s_4 s_5 \cdots$ cannot intersect the interior of *ai*. It immediately follows from Lemma 4.2(i) that s_5 is adjacent to h (replace *abcd* in the statement of that lemma with *dabc* here). If s_6 lies in the interior of *jkai* then, by Lemma 4.2(ii), T is not minimal. If, on the other hand, $s_6 = a$ then T is not minimal by Lemma 5.3, Fig. 23c.

Thus s_6 must lie in *iadh* (Fig. 27b). In this case, s_6 lies on the path connecting i and s_4 by Lemma 2.10. Note that $s_2 s_4$ intersects *ad* at less than

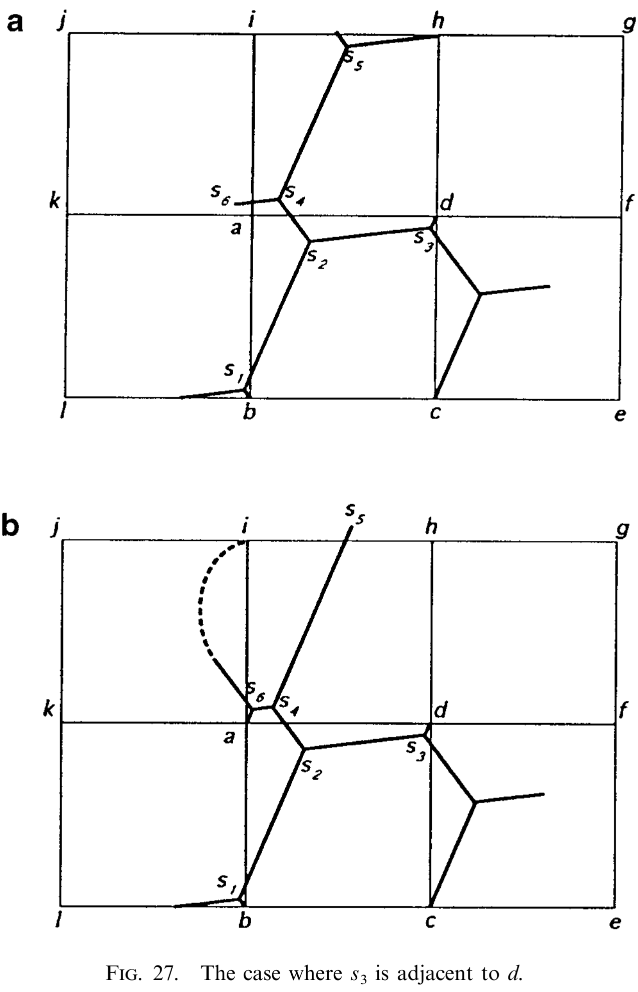

FIG. 27. The case where s_3 is adjacent to d.

60°. Hence, the line through s_2s_4 intersects dc or ai, from which it follows that either $\angle s_3s_2c > 60°$ or $\angle s_6s_4i > 60°$. If the first of these possibilities holds then the left-turn path $c \cdots s_3s_2s_1$ shows that T is not minimal by Proposition 2.6. If the second holds then the left-turn path $i \cdots s_6s_4s_5$ shows that T is not minimal, again by Proposition 2.6. This completes the proof of the lemma. ▌

LEMMA 5.5. *Every vertex of degree two in $G(S_T)$ adjacent to a leaf or quasi-leaf is itself a quasi-leaf.*

Proof. Much of this proof parallels that of the previous lemma. Assume $abcd$ is a vertex of $G(S_T)$ which is not a quasi-leaf, but is adjacent to a leaf or quasi-leaf, V. By Lemma 4.6, if V lies to the right of $abcd$ then T is not vertical. By symmetry, we can now assume, without loss of generality, that V lies to the right of $abcd$, T is horizontal and the second vertex in $G(S_T)$ adjacent to $abcd$ lies above $abcd$. Let s_2 and s_3 be the Steiner points in $abcd$, such that s_3 is adjacent to a Steiner point in V. It is clear that s_2s_3 is in the main direction. Since $abcd$ is a vertex of degree two, two corners of $abcd$ are adjacent to s_2 or s_3. This results in three cases to eliminate.

(i) Assume s_2 is adjacent to both a and b.

In this case s_2s_3 must be positive horizontal. The possibilities for T correspond to those in Figure 26, where s_1 now coincides with b. It follows that T is not minimal by the argument used in the proof of Lemma 5.4, Case (ii).

(ii) Assume s_2 is adjacent to b and s_3 is adjacent to d.

There are two subcases. If s_2s_3 is negative horizontal (as in Fig. 22), then T is not minimal by Lemma 4.7. If s_2s_3 is positive horizontal, then here the possibilities for T correspond to those in Figure 27, where s_1 now coincides with b. Again it follows that T is not minimal by the argument used in the proof of Lemma 5.4, Case (iii).

(iii) Assume s_2 is adjacent to b and s_3 is adjacent to c (Fig. 28).

Here s_2s_3 is negative horizontal. Let s_4 and s_5 be the next Steiner points on the right-turn path $d \cdots s_3s_2s_4s_5$ which intersects cd at q, and intersects ad at p. Let s_6 be the next Steiner point on the left-turn path $s_2s_4s_6$. Since

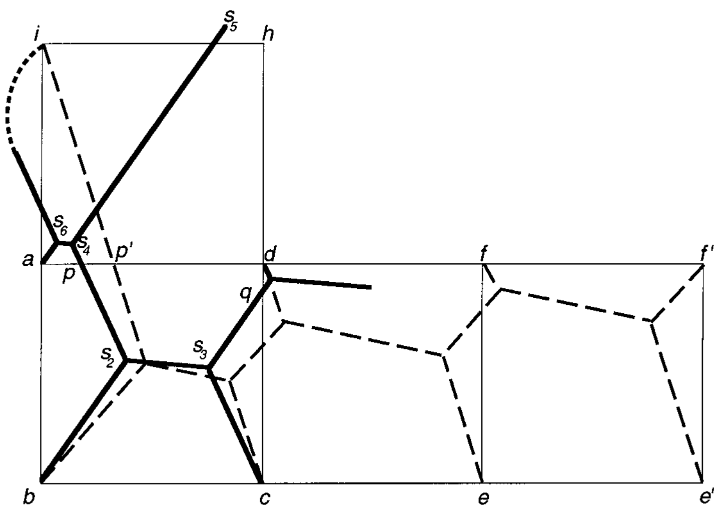

FIG. 28. Case (iii).

the right-turn path $s_3s_2s_4s_5\cdots$ cannot intersect dh, s_6 lies in $iadh$. By Lemma 2.10, s_6 is on the path connecting i and s_4. Let α be the absolute value of the slope of the main direction. If, for a fixed p, we wish to maximise α, we should choose $dcef$ to be an abnormal quasi-leaf. In this case, fef' cannot be a triangle, by Lemma 4.1, nor can $fee'f'$ be an abnormal quasi-leaf, by Lemma 4.5. It follows that α is maximised if $fee'f'$ is a square leaf. Now construct the full Steiner tree, T' on $i, a, b, c, d, e, f, e', f'$ shown in broken lines in Figure 28, where $fee'f'$ is a square leaf, $dcef$ is an abnormal quasi-leaf, and the part of T' inside $abcd$ is similar in topology to the part of T inside $abcd$, as shown. Suppose T' intersects ad at p'. It is easy to calculate that $|ap'| = 1/3$ and the main direction of T' is $-11.565°$. If p lies on $[ap']$, then it immediately follows from the construction that $\alpha \leqslant 11.565°$, and hence that $\angle is_4s_5 \leqslant 60°$. Applying Theorem 2.8 to $i\cdots s_4s_5$, we conclude that T is not minimal. Similarly, if p lies on $p'd$ and the line through s_2s_4 intersects $[ia]$ then again T is not minimal. But if, on the other hand, p lies on $p'd$ and the line through s_2s_4 intersects ih, then a simple calculation (using the fact that $|s_4s_5| < 1$) shows that the subtree $(bs_4)(qc)$ is not minimal by Proposition 2.3, again contradicting the minimality of T. ▮

Before stating the main theorem of this section we require some definitions. Given an infinite unit square lattice in the Euclidean plane, we define a *ladder* to be a finite sequence of adjacent squares all lying in the one row or column. A ladder is said to be *horizontal* if the squares all lie in the same row, and *vertical* if they all lie in the same column. We define a *staircase* to be a finite sequence of adjacent triangles in the square lattice with the property that they are adjacent along unit edges and all the hypotenuses of the triangles are parallel. A staircase is said to be *ascending* if the hypotenuses lie at an angle of 45° from the horizontal and *descending* if they lie at an angle of 135° from the horizontal.

Let S be a finite alternate sequence of adjacent ladders and staircases, with the adjacencies occurring at the ends of the ladders and staircases. A staircase in S is said to be *internal* if it is adjacent to two ladders, and *external* if it is adjacent to precisely one ladder. We say that S is a *strip* if it satisfies the following conditions:

(i) Either all ladders in S are horizontal, or all ladders in S are vertical. Likewise, all staircases in S are ascending, or all are descending.

(ii) If S contains no ladders, then S contains exactly one or an even number of triangles. If S contains one or more ladders, then all internal staircases of S contain an even number of triangles, and all external staircases of S contain an odd number of triangles.

THEOREM 5.6. *S_T is a strip.*

Proof. Let S^* be the subset of all elements of S_T which are not leaves or quasi-leaves of $G(S_T)$. If S^* is non-empty then $G(S^*)$ is clearly a tree. Let L be a leaf of $G(S^*)$. Clearly L has degree 4, 3 or 2 in $G(S_T)$. However, these possibilities contradict, respectively, Corollary 5.2, Lemma 5.4 and Lemma 5.5. It follows that two vertices in $G(S_T)$ are leaves and all other vertices are quasi-leaves. Hence, S_T consists of a sequence of adjacent ladders and staircases. Moreover, condition (i) follows easily from the fact that the main direction of T is exclusive. For example, if S_T has both kinds of ladders, horizontal and vertical, then there are two directions, resulting in a contradiction. The fact that condition (ii) is satisfied follows from condition (i). ∎

Recall that a tree is called a *caterpillar* if the subtree obtained by removing all leaves forms a path (i.e., a caterpillar is a tree which is a path in Autumn). From the description of T in leaves and quasi-leaves of $G(S_T)$ we have the following result.

COROLLARY 5.7. *T is a caterpillar.*

6. CLASSIFYING THE FULL COMPONENTS OF T^*

In the previous section we showed that S_T is a strip. The aim of this final section is to completely classify those strips whose vertices can be spanned by a full minimal Steiner tree. This will provide us with a list of all possible full components T for any T^*, and their lengths. The key to this classification is the following geometric construction for computing lengths and main directions of such trees, which is based on a more general result for caterpillars to appear in [14]. Throughout this section, let T be a positive horizontal full minimal Steiner tree for a Steiner-closed lattice set.

Assume S_T contains more than ione square or triangle. Let $V_1, V_2, ..., V_k$ be the sequence of adjacent squares and triangles in S_T ordered from left to right. Assume the set $\{V_j, V_{j+1}, ..., V_k\}$ contains no abnormal squares. Later in this section we will show that abnormal squares in fact never occur in S_T. Let s_1, s_2 be adjacent Steiner points in T such that s_2 is to the right of s_1, and s_1 lies in V_{j-1} (see Fig. 29). Since T is a caterpillar, all the Steiner points to the right of s_1 lie on a path $s_1 s_2 \cdots s_m$. Let x be the terminal in T adjacent to s_2. Let $p_2 = x$; let q be the terminal of T such that q is adjacent to s_3 and $|p_2 q| = 1$; and let p_1 be the terminal of T such that $p_1 p_2$ is a unit edge of V_{j-1} and $\angle p_1 p_2 q = 90°$ (in particular, if V_{j-1} is not an abnormal square then p_1 is adjacent to s_1). Let s_{m+1} be the terminal of T adjacent to s_m such that $s_{m-1} s_m s_{m+1}$ is a left-turn path if V_k is a square or a right-turn path if V_k is a triangle. We now construct a path

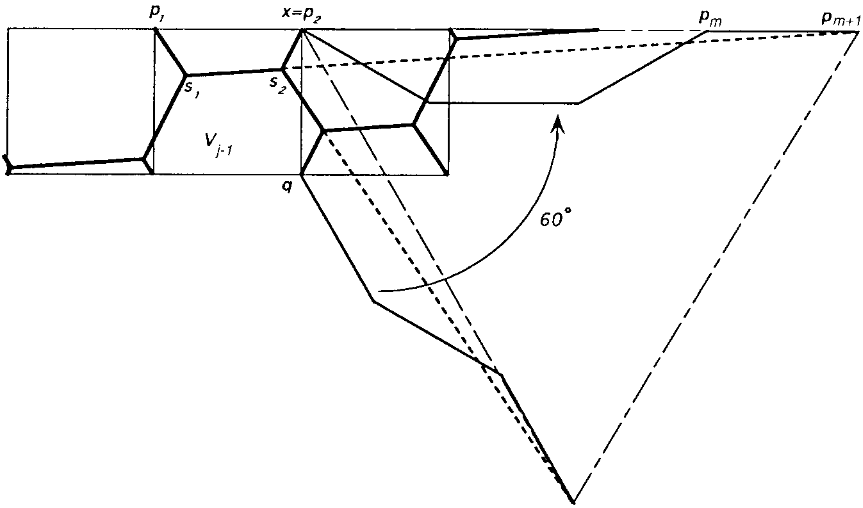

FIG. 29. Illustration of the inductive step in the proof of Lemma 6.1.

$p_1 p_2 \cdots p_{m+1}$, denoted M_x, and defined as follows: $|p_i p_{i+1}| = 1$ for all $1 \leqslant i \leqslant m$; $\angle p_i p_{i+1} p_{i+2} = 150°$ for all $1 \leqslant i \leqslant m-1$; and if we walk along the path from p_1 to p_{m+1} we turn left at p_i if $s_{i-1} s_i s_{i+1}$ is part of a left-turn path and right at p_i if $s_{i-1} s_i s_{i+1}$ is part of a right-turn path. Note that T is divided into three subtrees by s_1. Let T_2 be the subtree containing s_2. A simple inductive argument, using the methods of Melzak, shows that p_{m+1} lies on the line through $s_1 s_2$ and $|s_1 p_{m+1}| = |T_2|$. (This is illustrated in Fig. 29. By the inductive hypothesis, the end of the constructed path beginning $p_2 q$ coincides with the end of the Simpson line from s_3 shown in the figure. By the Simpson–Heinen construction, if we swing this path around p_2 by 60° then $\angle p_1 p_2 p_3 = 150°$ and p_{m+1} coincides with the end of the Simpson line from s_2.)

If S_T contains no abnormal squares, we can extend the construction of M_x to a construction for all of T as follows. Let s_2 be the left-most Steiner point of T, let s_3 be the Steiner point of T adjacent to s_2, let s_1 be the terminal of T adjacent to s_2 such that $s_1 s_2 s_3 \cdots$ is a right-turn path if V_1 is a square or a left-turn path if V_1 is a triangle, and let x be the other terminal adjacent to s_2. Then we define the path $M_x = p_1 p_2 \cdots p_{m+1}$ as in the previous paragraph, and we define M_T to be M_x orientated by rotation so that p_1 is the leftmost point of M_T and $p_i p_{i+1}$ is horizontal whenever $s_i s_{i+1}$ is in the main direction. It follows, again by the methods of Melzak, that the line through p_1 and p_{m+1} is in the main direction of T and $|T| = |p_1 p_{m+1}|$.

These results are summarized in the following lemma.

LEMMA 6.1. *Let T, s_1, s_2, T_2, M_x and M_T be defined as above.*

(i) *If $M_x = p_1 p_2 \cdots p_{m+1}$ then the line through $s_1 s_2$ passes through p_{m+1} and $|s_1 p_{m+1}| = |T_2|$.*

(ii) *If $M_T = p_1 p_2 \cdots p_{m+1}$ then the line through p_1 and p_{m+1} is in the main direction of T and $|p_1 p_{m+1}| = |T|$.*

The above definition for M_T can be extended to negative horizontal Steiner trees. Let $\tilde{T}$ be a horizontal full minimal Steiner tree for a strip containing no abnormal squares. If $\tilde{T}$ is negative horizontal, let T be the reflection of $\tilde{T}$ about a vertical line. In this case we define the path $M_{\tilde{T}} = p_1 p_2 \cdots p_{m+1}$ to be the reflection of M_T about a vertical line.

We can also define the following useful quantities on $\tilde{T}$. Define $D_H(M_{\tilde{T}})$ to be the horizontal distance between p_1 and p_{m+1}, that is, the distance between the vertical lines through p_1 and p_{m+1}. Similarly, define $D_V(M_{\tilde{T}})$ to be the vertical distance between p_1 and p_{m+1}. Note that $D_H(M_{\tilde{T}})^2 + D_V(M_{\tilde{T}})^2 = |\tilde{T}|^2$.

The next lemma shows that the condition that there are no abnormal squares in S_T holds for all T.

LEMMA 6.2. *S_T contains no abnormal squares.*

Proof. Let $V_1, ..., V_k$ be a sequence of adjacent squares forming a ladder of S_T, and assume, contrary to the lemma, that the square $abcd = V_j$ is abnormal for some $1 < j < k$. By Lemma 4.5 there are no other abnormal squares in this ladder, and it immediately follows by an easy angle argument that k is odd and j is even. Let L_{ad} be the line through ad and let L_{bc} be the line through bc. Let T' be the part of T lying between L_{ad} and L_{bc}. Furthermore, let p be the rightmost point of T' lying L_{ad}, and let e be the rightmost terminal of T' lying on L_{ad}. Similarly, let q be the leftmost point of T' lying on L_{bc}, and let f be the leftmost terminal of T' lying on L_{bc} (as in Fig. 30). Let s_1 and s_2 be the two Steiner points of T' lying in $abcd$. Applying Melzak's construction to T' we obtain the Simpson line p^*q^* for T' passing through $s_1 s_2$. By the proof of Lemma 6.1, and noting that j is even, it follows that p^* lies on L_{ad} and q^* lies on L_{bc}. We can construct an alternative Steiner tree, T'', on p, q and the terminals of T' by placing an X in each V_i for i odd, and connecting the tree with unit edges and

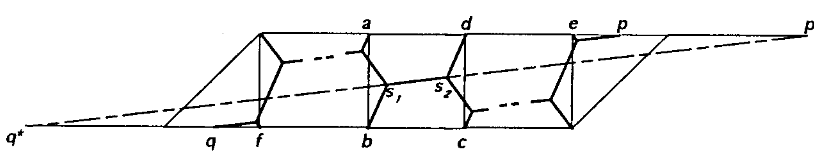

FIG. 30. The subtree T'.

the edges ep and fq. Again using the proof of Lemma 6.1, we conclude $|T''| = |q^*b| + |ap^*| < |q^*p^*| = |T'|$. Hence T' is not minimal, giving the desired contraction. ∎

Note that, since every square of $G(S_T)$ is a leaf or normal quasi-leaf, it follows that the topology of T is completely determined up to reflection or rotation by S_T.

We say that a full Steiner tree $\bar{T}$ on the vertices of a strip is *locally minimal* if its topology (up to rotation or reflection) in leaves of $G(S_{\bar{T}})$ is as in Fig. 18 and in quasi-leaves of $G(S_{\bar{T}})$ is as in Figs. 19a and 19b. In view of Lemma 6.2 it follows that every full minimal tree for a strip is locally minimal. Let A_{2k} be the locally minimal positive horizontal full Steiner tree for a $2k$-*ladder*, that is, for a ladder containing $2k$ adjacent squares (An example is illustrated in Fig. 31). Let B_{2k+1} be the locally minimal positive horizontal full Steiner tree for a $2k$-ladder with a triangle attached to one end, and let C_{2k+2} be the locally minimal positive horizontal full Steiner tree for a $2k$-ladder with a triangle attached to each end such that the two hypotenuses are parallel (as in Fig. 31). A simple argument shows that A_{2k}, B_{2k+1} and C_{2k+2} exist as full Steiner trees for all k. Define $Q(\bar{T})$ to be the main direction of $\bar{T}$. It follows from the proof of Lemma 6.1 that

$$Q(A_{2k}) > Q(B_{2k+1}) > Q(C_{2k+2}) > Q(A_{2k+2}).$$

These definitions and inequalities are used in the proof of the following lemma.

LEMMA 6.3. *Let Z be a strip which is not a square, and which contains at least one ladder. Suppose there exists a full minimal Steiner tree on the lattice points of Z. Then the following statements hold*:

(i) *every ladder in Z contains an even number of squares*;

(ii) *each external staircase of Z contains precisely one triangle and each internal staircase of Z contains precisely two triangles*;

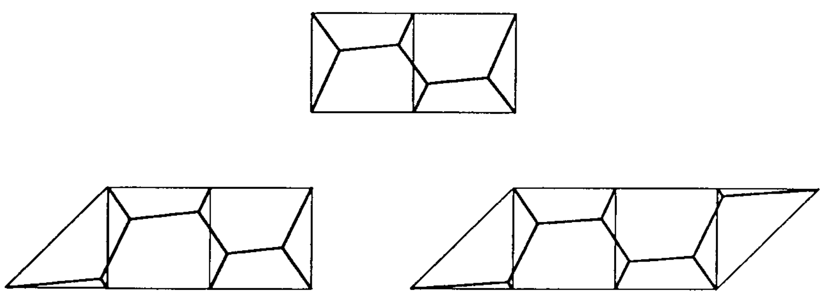

FIG. 31. The Steiner trees A_2, B_3, and C_4.

(iii) *all ladders in Z contain the same number of squares*; *and*

(iv) *if Z contains more than one ladder then Z contains either zero or two external staircases.*

Proof. We can assume the full minimal Steiner tree on the lattice points of Z is positive horizontal. Let $Z = S_T$. We prove each of the four statements in turn. Statement (i) follows directly from Lemma 6.2.

Now consider the full Steiner tree for the strip consisting of two triangles sharing a vertical edge (Fig. 32). The main direction of this strip is $\arctan(1/(4+\sqrt{3})) > 9.896°$. Hence, if Z contains two triangles sharing a vertical edge, then $Q(T) > 9.896°$. But the main direction of the full Steiner tree for the strip consisting of two squares sharing a vertical edge is $\arctan(1/(4+3\sqrt{3})) < 6.206°$. So, if Z contains a ladder then $Q(T) < 6.206°$, and consequently Z does not contain two triangles sharing a vertical edge. This immediately implies Statement (ii), noting that external staircases have an odd number of triangles.

To see Statements (iii) and (iv), we divide T into component subtrees by cutting T at each of the points where an edge of T in an internal staircase intersects the interior of a horizontal unit edge of the lattice. Note that the parts of T contained in each ladder of Z lie in separate components. Suppose, contrary to Statement (iii), that the ladders corresponding to two component subtrees T_1 and T_2 contain $2k_1$ and $2k_2$ squares respectively, where $k_1 < k_2$. Then,

$$Q(T_1) > Q(C_{2k_1+2}) > Q(A_{2k_1+2}) \geqslant Q(A_{2k_2}) > Q(T_2),$$

contradicting the uniqueness of the main direction of T (where $Q(T_1)$ and $Q(T_2)$ are defined in the obvious way).

Finally, to prove Statement (iv), assume Z contains exactly one external staircase. Let each of the ladders of Z contain $2k$ squares. Then it is clear, by the construction of M_T in Lemma 6.1, that the main direction of T is equal to $Q(B_{2k+1})$ and consequently that each of the component subtrees is a full subtree on a subset of the vertices of Z. If Z contains more than one ladder this contradicts the fact that T is full. ∎

LEMMA 6.4. *If Z is a staircase, or if Z is a strip containing at least one ladder and satisfying Statements* (i), (ii), (iii), *and* (iv) *in Lemma* 6.3, *then all minimal Steiner trees on the lattice points of Z are full.*

Proof. We can assume that Z is orientated so that its ladders are horizontal and its staircases are ascending. We will first show that if Z satisfies the hypotheses of the lemma then there exists a full locally minimal

Steiner tree $\bar{T}$ such that $S_{\bar{T}} = Z$. To complete the proof, we then prove that $\bar{T}$ is strictly shorther than any Steiner tree for Z containing more than one full component.

Consider a horizontal $2k$-ladder. Let L_1 be the horizontal line passing through the top vertices of this ladder and L_0 the horizontal line passing through its bottom vertices. Let e be the rightmost vertex of the ladder lying on L_1 and let f be the leftmost vertex of the ladder lying on L_0. Let p be a point on L_1 lying on or to the right of e and let q be a point on L_0 lying on or to the left of f. By the construction for Lemma 6.1 it follows that there exists a full Steiner tree on p, q and the vertices of the ladder which is locally minimal in the squares of the ladder and whose main direction is

$$\arctan\left(\frac{\frac{1}{2}}{k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+|fq|+|ep|}\right).$$

Now let Z be a strip satisfying the hypotheses of the lemma and containing l $2k$-ladders labelled (from left to right) $Z_1, Z_2, ..., Z_l$. If $l=1$ then the construction above clearly gives a suitable full locally minimal Steiner tree $\bar{T}$, where p and q are respectively the top rightmost and bottom leftmost vertices of Z. So suppose $l>1$ and, moreover, Z has no external staircases. For $1 \leqslant i \leqslant l-1$ let L_i be the horizontal line passing through the top vertices of Z_i, let e_i be the rightmost vertex of Z_i lying on L_i, and let p_i be the point on L_i lying to the right of e_i such that $|e_i p_i| = (l-i)/l$. Finally, let p_0 be the bottom leftmost vertex of Z and let p_l be the top rightmost vertex of Z. As above, for each i we can construct a full Steiner tree T_i on p_{i-1}, p_i and the vertices of Z_i whose main direction is

$$\arctan\left(\frac{\frac{1}{2}}{k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{i-1}{l}+\frac{l-i}{l}}\right)$$
$$=\arctan\left(\frac{\frac{1}{2}}{k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{l-1}{l}}\right).$$

Since each of the T_is has the same main direction, their union forms a full Steiner tree, $\bar{T}$, for all Z. It immediately follows from the construction that

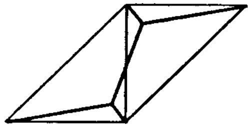

FIG. 32. The full Steiner tree for a 2-staircase.

$\bar{T}$ is locally minimal. If, on the other hand, Z contains two external staircases then we can use exactly the same argument to construct a suitable $\bar{T}$ by choosing the points p_i such that $|e_i p_i| = i/l$. Similarly, we can use this argument to construct a suitable $\bar{T}$ for the case where Z is a $2k$-staircase by viewing the staircase as a collection of k 0-ladders separated by internal 2-staircases.

Now suppose, contrary to the lemma, there exists a minimal Steiner tree T' on the vertices of Z such that T' is not full. Let $\{T'_i\}$ be the set of full components of T'. It is clear, by an easy angle argument, that T' contains no vertical unit edges. Hence, we can assume that all full components of T' are horizontal. Furthermore, there is no square *abcd* in the strip such that *ad* and *bc* are both unit edges of T' (as two such unit edges can always be replaced by a suitably oriented minimal Steiner tree for a triangle to form a shorter tree). It immediately follows, by a simple induction argument for example, that $\sum_i D_H(M_{T'_i}) \geqslant D_H(M_{\bar{T}})$.

We next show that $\sum_i D_V(M_{T'_i}) \geqslant D_V(M_{\bar{T}})$. This follows from the fact that, for almost any horizontal strip S with full minimal Steiner tree T, $D_V(M_T) = k/2$, where k is the number of ladders in S and where, as previously, a $2k$-staircase is considered to contain k 0-ladders. The sole exception to this is the case where S is a single square, $T = X$ and $D_V(M_T) = 0$. However, it is clear that for any ladder of Z there is no minimal Steiner tree on the vertices of that ladder consisting only of Xs and unit edges (since the ladder contains an even number of squares). The inequality above now easily follows.

Using a standard inequality, we deduce that

$$\begin{aligned}
|T'| &= \sum_i |T'_i| \\
&= \sum_i (D_H(M_{T'_i})^2 + D_V(M_{T'_i})^2)^{1/2} \\
&> \left(\left(\sum_i D_H(M_{T'_i})\right)^2 + \left(\sum_i D_V(M_{T'_i})\right)^2\right)^{1/2} \\
&\geqslant (D_H(M_{\bar{T}})^2 + D_V(M_{\bar{T}})^2)^{1/2} \\
&= |\bar{T}|.
\end{aligned}$$

This contradict the minimality of T'. ∎

Note that the tree $\bar{T}$ constructed in the above proof is indeed a minimal Steiner tree for Z since, up to rotation and reflection, any full locally minimal Steiner tree on Z is unique.

The next lemma tells us that any minimal Steiner tree on the vertices of a strip occurs as a subtree of a minimal Steiner tree of some Steiner-closed lattice set, since the vertices of the strip are themselves Steiner-closed.

LEMMA 6.5. *Let Z be a strip. The set of lattice points corresponding to the vertices of Z forms a Steiner-closed lattice set.*

Proof. Suppose, contrary to the proposition, there exists a minimal Steiner tree, T', on the vertices of Z, such that T' contains an edge $s_1 s_2$ crossing a lattice edge not contained in Z. Clearly s_1 and s_2 are both Steiner points at least one of which lies outside Z. If both s_1 and s_2 lie outside Z, then an easy exercise shows that the four distinct terminals of the two convex paths through $s_1 s_2$ cannot all be verties of Z, giving a contradiction. If, on the other hand, s_1 lies in Z and s_2 lies outside Z, then the four distinct terminals of the two convex paths through $s_1 s_2$ can only lie in Z if there exists another edge of T' between two Steiner points both of which lie outside Z. So again, by the previous argument, we obtain a contradiction to the existence of T'. ∎

In order to complete our classification of full components of T^* we need to introduce some new notation for some special kinds of strips. A $[2k, l]$-*strip* is defined to be a strip consisting of l $2k$-ladders separated by $l-1$ internal 2-staircases. Similarly, a $\langle 2k, l]$-*strip* is a $[2k, l]$-strip with an external 1-staircase (that is, a single triangle) on one end, while a $\langle 2k, l\rangle$-*strip* is a $[2k, l]$-strip with external 1-staircases on both ends. Since the topology of T is completely determined up to reflection or rotation by S_T, we can also use this notation to describe T. Finally, let Y denote the full Steiner tree for a triangle.

The following classification now follows from Lemma 6.3, 6.4 and 6.5.

THEOREM 6.6. *Let T be a full component of T^*, containing at least one Steiner point. Up to reflection or rotation, S_T is either*

(i) *a triangle*;

(ii) *a square*;

(iii) *a* $2k$-*staircase*;

(iv) *a* $\langle 2k, 1]$-*strip*;

(v) *a* $[2k, l]$-*strip*; *or*

(vi) *a* $\langle 2k, l\rangle$-*strip*.

In each case the main direction and length of T are as shown in Table 1.

TABLE 1

Complete Classification of All Possible Full Components T of T^*

T	Main direction of T	$\|T\|$
Unit edge	$0°$	1
Y	$15°$	$\sqrt{2+\sqrt{3}}$
X	$0°$	$1+\sqrt{3}$
$2k$-staircase	$\arctan\left(\dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}+\frac{k+1}{k}}\right)$	$k\sqrt{\frac{1}{4}+\left(\frac{\sqrt{3}}{2}+\frac{k+1}{k}\right)^2}$
$\langle 2k, 1]$-strip	$\arctan\left(\dfrac{\frac{1}{2}}{k(2k+\sqrt{3})+\frac{\sqrt{3}}{2}+1}\right)$	$\sqrt{\frac{1}{4}+\left(k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+1\right)^2}$
$[2k, l]$-strip	$\arctan\left(\dfrac{\frac{1}{2}}{k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{l-1}{l}}\right)$	$l\sqrt{\frac{1}{4}+\left(k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{l-1}{l}\right)^2}$
$\langle 2k, l\rangle$-strip	$\arctan\left(\dfrac{\frac{1}{2}}{k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{l+1}{l}}\right)$	$l\sqrt{\frac{1}{4}+\left(k(2+\sqrt{3})+\frac{\sqrt{3}}{2}+\frac{l+1}{l}\right)^2}$

Furthermore, with k and l ranging over all positive integers this gives a complete irredundant classification of possible full components of T^*.

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