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ON MAXIMAL ANTICHAINS CONTAINING NO SET AND ITS COMPLEMENT

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Let $1 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ be integers and let S denote the set of all vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ with integral coordinates satisfying $0 \leqslant x_i \leqslant k_i$, $i = 1, 2, \ldots, n$; equivalently, S is the set of all subsets of a multiset consisting of k_i elements of type i, $i = 1, 2, \ldots, n$. A subset X of S is an *antichain* if and only if for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in X the inequalities $x_i \leqslant y_i$, $i = 1, 2, \ldots, n$, do not all hold. For an arbitrary subset H of S, $(i)H$ denotes the subset of H consisting of vectors with component sum i, $i = 0, 1, 2, \ldots, K$, where $K = k_1 + k_2 + \cdots + k_n$. $|H|$ denotes the number of vectors in H, and the *complement* of a vector $\boldsymbol{x} \in S$ is $(k_1 - x_1, k_2 - x_2, \ldots, k_n - x_n)$. What is the maximal cardinality of an antichain containing no vector and its complement? The answer is obtained as a corollary of the following theorem: *if X is an antichain, K is even and $|(\frac{1}{2}K)X|$ does not exceed the number of vectors in $(\frac{1}{2}K)S$ with first coordinate different from k_1, then*

$$\sum_{\substack{i=0 \\ i \neq \frac{1}{2}K}}^{K} \frac{|(i)X|}{|(i)S|} + \frac{|(\frac{1}{2}K)X|}{|(\frac{1}{2}K - 1)S|} \leqslant 1.$$

1. Introduction and statement of results

Let I denote a set consisting of $k_i \geqslant 1$ elements of type i, $i = 1, 2, \ldots, n$. We assume $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$, a requirement of certain theorems which are applied below. For example I might be a set of $K = k_1 + k_2 + \cdots + k_n$ billiard balls, k_i of color i, $i = 1, 2, \ldots, n$. We identify the subset of I consisting of x_i elements of type i with the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and use $S = S(k_1, k_2, \ldots, k_n)$ to denote the $\theta = (k_1 + 1)(k_2 + 1) \cdots (k_n + 1)$ subsets of I. S may also be regarded as the set of divisors of $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ where the p_i are distinct primes and $\boldsymbol{x}$ is identified with the divisor $p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}$. The special case $k_n = 1$ corresponds to I being an ordinary finite set with n elements and S being its 2^n subsets or the 2^n divisors of the square free number $p_1 p_2 \cdots p_n$.

A subset X of S—in other words a set of subsets of I—is called an antichain (or Sperner family or clutter) if and only if the inequalities $x_i \leqslant y_i$, $i = 1, 2, \ldots, n$, do not all hold for any two distinct vectors $\boldsymbol{x}$, $\boldsymbol{y}$ in X. In terms of sets, this is the same as saying that no two sets $\boldsymbol{x}$, $\boldsymbol{y}$ in X are related by setwise inclusion. The complement $\boldsymbol{x}^c$ of $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is $(k_1 - x_1, k_2 - x_2, \ldots, k_n - x_n)$, and the rank of $\boldsymbol{x}$ is $|\boldsymbol{x}|=x_1+x_2+\cdots+x_n$, the number of elements of I in $\boldsymbol{x}$. For an arbitrary subset H of S, $|H|$ is the number of vectors in H, and $(i)H$ denotes the subset of H consisting of vectors which satisfy $|\boldsymbol{x}|=i$.

Now let $g(k_1, k_2, \ldots, k_n)$ denote the maximal cardinality of a complement-free (c–f) subset X of S (that is $\boldsymbol{x}\in X$ implies $\boldsymbol{x}^c\notin X$) and let $G(k_1, k_2, \ldots, k_n)$ denote the corresponding maximum when X is also required to be an antichain.

It is easy to see that $g(k_1, k_2, \ldots, k_n)=[\frac{1}{2}\theta]$, where $[\frac{1}{2}\theta]$ is the largest integer not exceeding $\frac{1}{2}\theta$, since S, with $(\frac{1}{2}k_1, \frac{1}{2}k_2, \ldots, \frac{1}{2}k_n)$ removed if each k_i is even, can be partitioned into pairs of distinct vectors $(\boldsymbol{x}_1, \boldsymbol{x}_1^c), (\boldsymbol{x}_2, \boldsymbol{x}_2^c), \ldots, (\boldsymbol{x}_{[\frac{1}{2}\theta]}, \boldsymbol{x}^c_{[\frac{1}{2}\theta]})$ and X is a maximal c–f subset of S if and only if X contains exactly one vector from each pair.

As for $G(k_1, k_2, \ldots, k_n)$, since $([\frac{1}{2}(K-1)])S$ is a c–f antichain (if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $([\frac{1}{2}(K-1)])S$, $|\boldsymbol{x}|+|\boldsymbol{y}|<K$), we immediately have $G(k_1, k_2, \ldots, k_n)\geq |([\frac{1}{2}(K-1)])S|$. If K is odd, the reverse inequality also holds since Theorem 1 of DeBruijn *et al.* [8, p. 191] states that no antichain in S has more than $|([\frac{1}{2}(K-1)])S|$ vectors. That the reverse inequality also holds if K is even follows from Theorem 1.

Theorem 1. *If X is an antichain in $S=S(k_1, k_2, \ldots, k_n)$ where $K=k_1+k_2+\cdots+k_n$ is even and X contains no more than α vectors of rank $\frac{1}{2}K$, where α is the number of vectors of rank $\frac{1}{2}K$ with first coordinate different from k_1, then*

$$|X|\leq|([\tfrac{1}{2}(K-1)])S|.$$

In particular, if X is c–f, then, since complements of vectors of middle rank are also of middle rank, X contains at most $\frac{1}{2}|(\frac{1}{2}K)S|\leq\alpha$ vectors of rank $\frac{1}{2}K$.

We thus have the following corollary.

Corollary 1. $G(k_1, k_2, \ldots, k_n)=|([\frac{1}{2}(K-1)])S|$.

Calculation of $|([\frac{1}{2}(K-1)])S|$ is explained in [5, p. 1239]. The $k_n=1$ case of this result is due to Purdy [16] and similar problems are considered by Milner in [15].

Besides the maximal cardinality of antichains, general structural assertions are of interest. Hence, the following theorem, which presents a so-called LYM inequality for $S(k_1, k_2, \ldots, k_m)$, is important, It is a corollary of a result of Anderson [1, Theorem 3]. In the $k_n=1$ case this inequality was discovered by Lubell [13], Yamamoto [18] and Meshalkin [14]. For related results see also [2], [9] and [12].

The numbers $p_i=|(i)X|$, $i=0, 1, \ldots, K$ are called the parameters of X.

Theorem 2. *If X is an antichain in $S=S(k_1, k_2, \ldots, k_n)$ with parameters $p_0, p_1, \ldots, p_K$ where $K=k_1+k_2+\cdots+k_n$, then*

$$\sum_{i=0}^{K}\frac{p_i}{|(i)S|}\leq 1.$$

The inequality $|X| \leq |([\frac{1}{2}K])S|$ of DeBruijn *et al.* [9, p. 191] now follows immediately from $|(i)S| \leq |([\frac{1}{2}K])S|$, $i = 0, 1, \ldots, K$ [5, p. 128], just as Sperner's theorem [17] follows from the LYM inequality in the $k_n = 1$ case.

Theorem 3. *If X is an antichain in $S = S(k_1, k_2, \ldots, k_n)$ with parameters $p_0, p_1, \ldots, p_K$ where $K = k_1 + k_2 + \cdots + k_n$ is even and X contains no more than α vectors of rank $\frac{1}{2}K$, where α is the number of vectors of rank $K/2$ with first coordinate different from k_1, then*

$$\sum_{\substack{i=0 \\ i \neq \frac{1}{2}K}}^{K} \frac{p_i}{|(i)S|} + \frac{p_{\frac{1}{2}K}}{|(\frac{1}{2}K-1)S|} \leq 1.$$

This theorem implies Theorem 1. We obtain the following corollary by Theorem 2, if K is odd, and Theorem 3, if K is even (see Theorem 1).

Corollary 2. *If X is a c-f antichain in $S(k_1, k_2, \ldots, k_n)$ with parameters $p_0, p_1, \ldots, p_K$, where $K = k_1 + k_2 + \cdots + k_n$, then*

$$\sum_{\substack{i=0 \\ i \neq \frac{1}{2}K}}^{K} \frac{p_i}{|(i)S|} + \frac{p_{\frac{1}{2}K}}{|(\frac{1}{2}K-1)S|} \leq 1.$$

In the $k_n = 1$ case we abbreviate $S(1, 1, \ldots, 1)$ by $S(n)$. Corollary 2 then reads as Corollary 3.

Corollary 3. *If X is a c-f antichain in $S(n)$ with parameters $p_0, p_1, \ldots, p_n$, then*

$$\sum_{\substack{i=0 \\ i \neq \frac{1}{2}n}}^{n} p_i \Big/ \binom{n}{i} + p_{\frac{1}{2}n} \Big/ \binom{n}{\frac{1}{2}n-1}.$$

Consideration of examples shows that our inequalities are best possible in the sense that none of the denominators can be replaced by smaller numbers. For instance, in the case of Corollary 3, consideration of the anitchains $X = (i)S$ if $i \neq \frac{1}{2}n$ and the antichain described in Theorem 4(ii)(b) suffices. We have been able to characterize the extremal sets only in case $k_n = 1$.

Theorem 4. *If $k_n = 1$ and X is a maximal c-f antichain in $S(n)$, then X is one of the following configurations or its complement:*

(i) *if n is odd, X consists of all subsets of I of cardinality $\frac{1}{2}(n-1)$.*

(ii) *if n is even, X consists of*

(a) *all subsets of I of cardinality $\frac{1}{2}(n-2)$, or*

(b) *all subsets of $I \setminus \{v\}$ of cardinality $\frac{1}{2}n$, where v denotes an arbitrary element of I, and all subsets of I of cardinality $\frac{1}{2}n - 1$ which contain v.*

Many of the techniques and results we use are discussed in the review article [10].

2. Proof of Theorem 3

Let X satisfying the hypothesis of Theorem 3 be given and let $l=\frac{1}{2}K$. It is known [3, p. 368] that corresponding to X there is an antichain X' with parameters

$$\begin{aligned} p_l' &= p_l \\ p_i' &= p_i + p_{K-i}, && \text{if } i>l, \\ p_i' &= 0, && \text{if } i<l. \end{aligned}$$

Then $X''=\{\boldsymbol{x} : \boldsymbol{x}^c \in X'\}$ has parameters $p_i''=p_{K-i}'$, $i=0,1,\dots,K$. Next we *compress* X''. For any subset H of S, let $F(m,H)$ denote the first m elements of H, where the ordering is lexicographic: $\boldsymbol{x}<\boldsymbol{y}$ if and only if $x_i<y_i$ for the smallest integer i such that $x_i \neq y_i$. Furthermore, for $\boldsymbol{x}\in S$ define

$$\Gamma\boldsymbol{x}=\{(x_1-1,x_2,\dots,x_n),(x_1,x_2-1,x_3,\dots,x_n),\dots,(x_1,x_2,\dots,x_n-1)\}\cap S$$

and for any $H\subseteq S$ let $\Gamma H=\bigcup_{\boldsymbol{x}\in H}\Gamma\boldsymbol{x}$. It is shown in [4] that if X is an antichain with parameters $p_0,p_1,\dots,p_K$, then the *compression* of X, that is the set Y defined by

$$\begin{aligned} (K)Y &= F(p_K,(K)S), \\ (K-1)Y &= F(p_{K-1},(K-1)S-\Gamma(K)Y), \\ (K-2)Y &= F(p_{K-2},(K-2)S-\Gamma\Gamma(K)Y-\Gamma(K-1)Y), \text{ etc.,} \end{aligned}$$

is also an antichain. Let X''' be the compression of X''. Thus X''' contains the first p_l vectors of $(l)S$. Since X''' has no more than α vectors of middle rank, the first component x_1 of the last vector in $(l)X'''$ satisfies $x_1\leq k_1-1$ and therefore

$$((l)X'''\cup\Gamma(l)X''')\subset S(k_1-1,k_2,\dots,k_n)=S'.$$

It follows from K being even that $|(l-1)S'|=|(l)S'|$ [5, p. 1287]. Thus by the inequality

$$\frac{|\Gamma(j)H|}{|(j-1)S'|}\geq\frac{|(j)H|}{|(j)S'|}$$

of Anderson [1, Theorem 3], which has come to be known as the normalized matching property and which holds for any $H\subseteq S'$ and $j=1,2,\dots,K$, we have $p_l=|(l)X'''|\leq|\Gamma(l)X'''|$. $X''''=(X'''-(l)X''')\cup\Gamma(l)X'''$ is an antichain again and has parameters

$$\begin{aligned} p_i'''' &= p_i+p_{K-i}, && \text{if } i\leq l-2, \\ p_{l-1}'''' &\geq p_{l-1}+p_l+p_{l+1} \\ p_i'''' &= 0, && \text{if } i\geq l. \end{aligned}$$

To X'''' we apply Theorem 2 remarking that $|(i)S|=|(K-i)S|$, $i=0,1,\dots,\frac{1}{2}K-1$

[6, p. 1287], and obtain

$$\sum_{\substack{i=0\\ i\neq l}}^{K} \frac{p_1}{|(i)S|} + \frac{p_l}{|(l-1)S|} \leq \sum_{i=0}^{l-1} \frac{p_i''''}{|(i)S|} \leq 1.$$

3. Proof of Theorem 4

In this section we will use several times a theorem of Katona concerning systems of subsets of a finite set [11, Theorem 2 and p. 334, Remark 2]. Reformulated slightly, it reads as follows.

Katona's Theorem. *Let* $1 \leq g < m \leq n$, $m-g \leq k \leq m$, $H \subseteq (m)(S(n))$ *be such that for every pair* $\mathbf{x}, \mathbf{y} \in H$ *there are at least* k *indices* $i_1, i_2, \ldots, i_k$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, *with* $x_{i_j} = y_{i_j} = 1$ *for* $j = 1, 2, \ldots, k$ *(that is, the intersection of the subsets of* I *corresponding to* $\mathbf{x}$ *and* $\mathbf{y}$ *has at least* k *elements). Furthermore, if* $\Gamma^1 = \Gamma$ *and* $\Gamma^{i+1} = \Gamma^i\Gamma$, $1 = 1, 2, \ldots$, *then*

$$|\Gamma^{m-g}H| \geq \left(\binom{2m-k}{g} \Big/ \binom{2m-k}{m}\right)|H|. \tag{1}$$

There is strict inequality in (1) *unless*

(a) $|H| = 0$, *or*

(b) *there is a* $(2m-k)$*-element subset* C *of* I *such that* H *is the set of* $\binom{2m-k}{m}$ *vectors corresponding to the* m*-element subsets of* C.

Let X be a maximal c-f antichain in $S = S(n)$. If n is odd, our theorem follows from Sperner's theorem [17], so we henceforth assume n is even. It follows from Corollary 3 that $X \subseteq (l+1)S \cup (l)S \cup (l-1)S$ since, if $p_i \neq 0$ for some $i \neq l-1$, l, $l+1$,

$$1 \geq \sum_{\substack{i=0\\ i\neq l}}^{n} p_i \Big/ \binom{n}{i} + p_l \Big/ \binom{n}{l-1} > \sum_{i=0}^{n} p_i \Big/ \binom{n}{l-1} = |X| \Big/ \binom{n}{l-1}$$

contradicts our assumption that X is maximal. Thus at most the parameters p_{l-1}, p_l, p_{l+1} are nonzero. Let us return to the proof of Theorem 3. X''' is a compressed antichain with parameters $p_{l-1}''' = p_{l-1} + p_{l+1}$, $p_l''' = p_l$ and all other parameters zero. Since X is c-f, $p_l''' \leq \frac{1}{2}\binom{n}{l}$ and $(l)X''' = F(p''', (l)S)$ is a subset of S_0, the part of $S(n)$ consisting of vectors with 0 first coordinate. It follows that for every pair of vectors in $(l)X'''$ there is at least one coordinate in which both vectors have a "1". Applying Katona's Theorem we find

$$|\Gamma(l)X'''| \geq \left(\binom{2l-1}{l-1} \Big/ \binom{2l-1}{l}\right) |(l)X'''| = p_l,$$

where equality holds if and only if $p_l = 0$ or $p_l = \binom{2l-1}{l} = \binom{n-1}{l-1}$. But equality must hold since, because $\Gamma(l)X'''$ and $(l-1)X'''$ are disjoint subsets of $(l-1)S$, we otherwise get the contradiction

$$\binom{n}{l-1} \geq |\Gamma(l)X'''| + |(l-1)X'''| > p_l + p_{l-1} = \binom{n}{l-1}.$$

Hence we have only to consider the possibilities

(a) $p_l = 0$, $p_{l-1} + p_{l+1} = \binom{n}{l-1} = \binom{n}{l+1}$ and

(b) $p_l = \binom{n-1}{l-1}$, $p_{l-1} + p_{l+1} = \binom{n-1}{l-2} = \binom{n-1}{l+1}$.

It suffices to prove in both cases that p_{l-1} or p_{l+1} vanishes, for then if (a) holds, Theorem 4(ii)(a) follows immediately. If (b) holds, we may assume $p_{l+1} = 0$ (if $p_{l-1} = 0$, consider $\{x : x^c \in X\}$]. For each pair of vectors in $(l)X$ there is a component in which both vectors have a "1"; otherwise these vectors would be complementary, contradicting that X is a c-f antichain. Hence Katona's Theorem yields

$$\begin{aligned} \binom{n}{l-1} &\geq |\Gamma(l)X| + |(l-1)X| \\ &\geq \left(\binom{2l-1}{l-1} \Big/ \binom{2l-1}{l}\right) |(l)X| + |(l-1)X| \\ &= \binom{n-1}{l-1} + \binom{n-1}{l-2} = \binom{n}{l-1}. \end{aligned}$$

Thus equality actually holds everywhere and from the equality case of Katona's Theorem it follows that X has the form described in Theorem 4(ii)(b).

We now discuss the two possibilities separately.

(a) For every pair of vectors in $(l+1)X$ there are at least two components in which both vectors have 1's, i.e., the corresponding subsets of I have two elements in common. Applying Katona's Theorem we get

$$|\Gamma^2(l+1)X| \geq \left(\binom{n}{l-1} \Big/ \binom{n}{l+1}\right) |(l+1)X| = p_{l+1}, \tag{1}$$

where equality holds if and only if $p_{l+1} = 0$ or $p_{l+1} = \binom{n}{l+1}$. Since $\Gamma^2(l+1)X$ and $(l-1)X$ are disjoint subsets of $(l-1)S$, we have

$$\binom{n}{l-1} \geq |\Gamma^2(l+1)X| + |(l-1)X| \geq p_{l+1} + p_{l-1} = \binom{n}{l-1};$$

thus equality actually does hold in (2) and we have the situation described in Theorem 4(ii)(a).

(b) Without loss of generality we may assume that X is compressed. To complete the proof, we show that the assumption $0 < p_{l-1}, p_{l+1} < \binom{n-1}{l+1}$ leads to a contradiction. To this end, consider $A = \Gamma(B \cup (l)X)$ where $B = \Gamma(l+1)X$. If S_0 denotes the vectors in $S(n)$ with 0 first component, then

$$p_l = \binom{n-1}{l} = |(l)S_0|$$

and

$$B \cup (l)X = F(|B| + p_l, (l)S) = (l)S_0 \cup F(|B|, (l)(S \setminus S_0)).$$

The first equality here follows from X being compressed. Since the result of changing the first component of a vector in $F(|B|, (l)(S \setminus S_0))$ from 1 to 0 is a vector in $\Gamma(l)S_0$, it follows that

$$\begin{aligned} |A| &= |\Gamma((l)S_0 \cup F(|B|, (l)(S \setminus S_0)))| \\ &= |\Gamma(l)S_0| + |\Gamma F(|B|, (l-1)S(n-1))| \\ &= \binom{n-1}{l-1} + |\Gamma F(|B|, (l-1)S(n-1))|. \end{aligned} \tag{3}$$

By the normalized matching property (see Section 2) we have

$$|\Gamma(|B|, (l-1)S(n-1))| \geq \left(\binom{n-1}{l-2} \Big/ \binom{n-1}{l-1}\right) |B|. \tag{4}$$

Next, observe that since $p_{l+1} < \binom{n-1}{l+1}$, $(l+1)X \subset (l+1)S_0$ and therefore for each pair of vectors in $(l+1)X$, there are at least three components in which both vectors are 1. It then follows from Katona's Theorem that

$$|B| = |\Gamma(l+1)X| > \left(\binom{n-1}{l} \Big/ \binom{n-1}{l+1}\right) p_{l+1}. \tag{5}$$

The strictness of the inequality here follows from the fact that p_{l+1} is neither 0 nor $\binom{n-1}{l+1}$. In view of (3), (4), (5) and the fact that A and $(l-1)X$ are mutually disjoint subsets of $(l-1)(S(n))$ we now have the contradiction

$$\begin{aligned} \binom{n}{l-1} &\geq |A| + p_{l-1} \\ &> \binom{n-1}{l-1} + \left(\binom{n-1}{l-2}\binom{n-1}{l} \Big/ \binom{n-1}{l-1}\binom{n-1}{l+1}\right) p_{l+1} + p_{l-1} \\ &= \binom{n-1}{l-1} + p_{l+1} + p_{l-1} = \binom{n}{l-1}, \end{aligned}$$

and the proof is complete.

4. Remarks

(1) We mention that the discussion of the cases (a) and (b) in the proof of Theorem 4 can also be done as follows.

Let $D = \{x : x^c \in (l-1)X\}$. Then the set of vectors of $(l)S$ which are related to vectors of $(l-1)X$ is $E = \{x : x^c \in \Gamma D\}$. The generalized Macaulay theorem [7] yields

$$|\Gamma(l+1)X| \geq |\Gamma F(p_{l+1}, (l+1)S)|,$$
$$|E| = |D| \geq |\Gamma F(p_{l-1}, (l+1)S)|,$$
$$\left|\Gamma F\left(\binom{n}{l+1}, (l+1)S\right)\right| = \binom{n}{l}$$

as well as

$$\left|\Gamma F\left(\binom{n-1}{l+1}, (l+1)S\right)\right| = \binom{n-1}{l}.$$

Furthermore we use the inequality

$$|\Gamma F(p_{l+1}, (l+1)S)| + |\Gamma F(p_{l-1}, (l+1)S)| \geq |\Gamma F(p_{l+1} + p_{l-1}, (l+1)S)|. \tag{6}$$

Since $(l)X$, $\Gamma(l+1)X$, and E are mutually disjoint sets of $(l)S$ we obtain

$$\begin{aligned}\binom{n}{l} &\geq |\Gamma(l+1)X| + p_l + |E| \\ &\geq |\Gamma F(p_{l+1}, (l+1)S)| + |\Gamma F(p_{l-1}, (l+1)S)| + p_l \\ &\geq |\Gamma F(p_{l+1} + p_{l-1}, (l+1)S)| + p_l = \binom{n}{l}.\end{aligned} \tag{7}$$

The inequality (6) appears, among other places, in [6]; scrutiny of the tedious proof given there reveals that in the $k_n = 1$ case (and only in the $k_n = 1$ case) equality holds only if one of p_{l-1}, p_{l+1} is zero. In view of (7), equality does indeed hold, so one of p_{l+1}, p_{l-1} is actually zero.

(2) Professor D.J. Kleitman has pointed out to us that the proof of (6) given [6] is unnecessarily complicated since (6) follows directly from the $S' = S(1, 1, k_1, k_2, \ldots, k_n)$ case of the generalized Macaulay theorem [7]: with $S = S(k_1, k_2, \ldots, k_n)$ let H, U and T denote respectively the subsets of S' obtained by preceding each vector of $(l+2)S$ by 00, each vector of $F(p_{l-1}, (l+1)S)$ by 01 and each vector of $F(p_{l+1}, (l+1)S)$ by 10. Then

$$\begin{aligned}|\Gamma(H \cup U \cup T)| &= |\Gamma(l+2)S| + |\Gamma F(p_{l-1}, (l+1)S)| + |\Gamma F(p_{l+1}, (l+1)S)| \\ &\geq |\Gamma F(|l+2)S| + p_{l-1} + p_{l+1}, (l+2)S')| \\ &= |\Gamma(l+2)S)| + |\Gamma F(p_{l-1} + p_{l+1}, (l+1)S)|.\end{aligned}$$

But we were not able to determine the cases of equality using this elegant argument.

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