Ideals and Quotients in Crossed Products of Hopf Algebras

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INTRODUCTION

Let $H$ be a Hopf algebra over a field $k$, acting on the $k$-algebra $R$ with action twisted by a cocycle $\sigma$ such that the crossed product algebra $R \#_{\sigma} H$ can be constructed. This paper is concerned with the relationship between the ideals of $R \#_{\sigma} H$ and those of $R$, and with computing the extended center and symmetric quotient ring of $R \#_{\sigma} H$ in terms of the extended center $C$ and symmetric quotient ring $Q$ of $R$.

Our best results are obtained when $H$ is an irreducible Hopf algebra, or more generally when $H$ is of the form $K \# kG$, where $K$ is irreducible and $kG$ is the group algebra of a group $G$. Some examples of irreducible Hopf algebras are enveloping algebras of Lie algebras $U(L)$ along with their restricted counterparts $u(L)$ in characteristic $p > 0$, divided power Hopf algebras, and the algebra $O(G)$ of regular functions on a unipotent algebraic group $G$. Hopf algebras of the form $K \# kG$ include all pointed cocommutative Hopf algebras.

More specifically, in Section 2 we consider the question of when ideals of $R \#_{\sigma} H$ intersect $R$ non-trivially. We obtain a complete answer for the case of $(R, R)$-subbimodules when $R$ is prime. That is (Theorem 2.2), we prove that every non-zero $(R, R)$-subbimodule intersects $R$ non-trivially if and only if $H$ is irreducible and the Lie algebra $L$ of primitive elements of $H$.
acts as "outer" derivations on the quotient $Q$ of $R$, in a sense which will be made precise. More generally if $H = K \# kG$ as above, then every non-zero ideal of $R \#_\sigma H$ intersects $R$ non-trivially if both $G$ and the primitive elements $L$ of $K$ are outer on $Q$ (Theorem 2.5). As a consequence of this result, we obtain a criterion for $R \#_\sigma H$ to be a simple ring.

In Section 3 we prove that if $R$ is prime and $H = K \# kG$ as above with the actions of $G$ and $L$ outer on $Q$, then the extended center $C(R \#_\sigma H) = C(R)^H$ (Corollary 3.5). If we also assume that every non-zero $H$-stable ideal of $R$ contains a left regular element, then the symmetric quotient ring $Q(R \#_\sigma H) = Q_H(R) \#_\sigma H$, where $Q_H(R)$ is the quotient with respect to the filter of $H$-stable ideals of $R$ (Theorem 3.10).

Finally in Section 4 we specialize to the situation when $H$ is irreducible and commutative and $\sigma$ is trivial. By restricting $H$ to be commutative, we can weaken our hypothesis on $R$ and only require that $R$ be $H$-prime. However, some of our results require using the left (or right) quotient rings $Q'$ of $R$ and $R \# H$, as well as certain "infinite sums" $Q'_H(R) \#_\infty H$ which determine elements of $Q'(R \# H)$. Then if $R \neq H$ is prime, we prove that

$Q'(R \# H) = (Q'_H(R) \#_\infty H) Z^{-1}$,

the localization at the non-zero elements of the center $Z$ of $Q_H(R) \# H$ (Theorem 4.3). If every non-zero $H$-stable ideal of $R$ contains a left regular element, then we can eliminate both the infinite sums and the one-sided quotients, to obtain $Q(R \# H) = (Q_H(R) \# H) Z^{-1}$ (Theorem 4.7). However, as shown by Example 4.10, the infinite sums are necessary in general. For extended centers, the situation is nicer: whenever $R \neq H$ is prime, $C(R \# H)$ is just the quotient field of $Z$.

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1. Preliminaries

Throughout this paper, $H$ denotes a Hopf algebra over a field $k$, with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. We write $H^+ = \ker \varepsilon$ and $K^+ = K \cap \ker \varepsilon$, for any subset $K$ of $H$. We will be particularly interested in irreducible Hopf algebras; in such Hopf algebras every non-zero subcoalgebra contains $k \cdot 1$.

Any $H$ has a coradical filtration $H = \bigcup_{n \geq 0} H_n$, where $H_0$ is the coradical of $H$ and $\Delta(H_n) \subseteq \sum_{i=0}^n H_i \otimes H_{n-i} [S]$. When $H$ is irreducible, $H_0 = k \cdot 1$, and we also have the following well-known lemma:

**Lemma 1.1.** Let $H$ be an irreducible Hopf algebra. Then

1. if $x \in H_n$, then $\Delta x = x \otimes 1 + 1 \otimes x + y$, where $y \in \sum_{i=0}^{n-1} H_i \otimes H_{n-i}$.

Moreover if $x \in H^+$ then $y \in \sum_{i=0}^{n-1} H_i^+ \otimes H_{n-i}^+$. 

(2) for all \( n \geq 1 \), \( H_n = \{ x \in H \mid \Delta x \in H_0 \otimes H + H \otimes H_{n-1} \} \)

(3) the antipode \( S \) of \( H \) is bijective.

**Proof.** (1) is [S, 10.0.2].

(2) By definition, \( H_n = \bigwedge^{n+1} H_0 = (\bigwedge^n H_0) \wedge H_0 \) [S, p. 179]. By [S, 9.0.0.c] also \( H_n = H_0 \wedge (\bigwedge^n H_0) \), and thus by [S, 9.0.0.a], \( H_n = \Delta^{-1}(H_0 \otimes H + H \otimes H_{n-1}) \), using \( C = H \), \( X = H_0 \), and \( Y = H_{n-1} \).

(3) follows from a theorem of Takeuchi [T] since \( H_0 \) is cocommutative.

Set \( X_0 = H_0 \), and for each \( n \geq 1 \) let \( X_n \) be a subspace of \( H_n \) such that \( H_n = H_{n-1} \oplus X_n \). Thus by choosing a basis \( B_n \) for each \( X_n \), we see that \( B = \bigcup_{n \geq 0} B_n \) is a basis for \( H \), and that \( \bigcup_{n = 0}^\infty B_n \) is a basis for \( H_n \). The elements in \( B_n \) will be called the **basis elements of degree** \( n \). When \( H \) is irreducible we assume \( B_0 = \{1\} \).

For any \( H \), the **group-like elements** are the set \( G(H) = \{ 0 \neq x \in H \mid \Delta x = x \otimes x \} \) and the **primitive elements** are the set \( P(H) = \{ x \in H \mid \Delta x = x \otimes 1 + 1 \otimes x \} \); furthermore \( P(H) \) is a Lie algebra under \( [x, y] = xy - yx \). Thus Lemma 1.1 says that when \( H \) is irreducible, elements in \( H_n \) are "primitive mod \( H_{n-1} \)." This will be extremely useful when \( H \) acts on an algebra \( R \).

We will be concerned here with **crossed products** \( R \#_\sigma H \). A crossed product can be formed whenever there is a weak action of \( H \) on the \( k \)-algebra \( R \) and \( \sigma : H \otimes H \rightarrow R \) is a convolution invertible \( k \)-bilinear map. By a weak action we mean that \( H \) measures \( R \) (\( h \cdot (rs) = \sum_{(h)} (h_{(1)} \cdot r)(h_{(2)} \cdot s) \), for all \( h \in H \) and \( r, s \in R \), and \( h \cdot 1 = \varepsilon(h)1 \), for all \( h \in H \)) and that \( 1 \cdot r = r \) for all \( r \in R \) [BCM, 1.1]. Then \( R \#_\sigma H \) is the algebra whose underlying vector space is \( R \otimes_k H \) and whose multiplication is given by

\[
(a \otimes h)(b \otimes l) = \sum_{(h)(l)} a(h_{(1)} \cdot b) \sigma(h_{(2)}, l_{(1)}) \otimes h_{(3)} l_{(2)},
\]

for all \( a, b \in R \) and \( h, l \in H \). \( R \#_\sigma H \) is associative with identity element \( 1 \neq 1 \) if and only if \( \sigma(1, h) = \sigma(h, 1) = \varepsilon(h)1 \), for all \( h \in H \), and the following conditions hold [BCM, 4.5; DT, Lemma 10]::

(1.2) (cocycle condition) For all \( h, l, m \in H \),

\[
\sum_{(h)(l)(m)} [h_{(1)} \cdot \sigma(l_{(1)}, m_{(1)}) \sigma(h_{(2)}, l_{(2)} m_{(2)}) = \sum_{(h)(l)} \sigma(h_{(1)}, l_{(1)}) \sigma(h_{(2)} l_{(2)}, m)\]

and

(1.3) (twisted module condition) For all \( h, l \in H \) and \( a \in R \),

\[
h \cdot (l \cdot a) = \sum_{(h)(l)} \sigma(h_{(1)}, l_{(1)}) (h_{(2)} l_{(2)} \cdot a) \sigma^{-1}(h_{(3)}, l_{(3)}).\]
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The element $a \otimes h$ of $R \#_\sigma H$ will be written as $a \# h$. Since $R \cong R \# 1$, we will also sometimes write $r$ for $r \# 1 \in R \#_\sigma H$.

If $\sigma$ is trivial, that is $\sigma(h, l) = \varepsilon(h) \varepsilon(l)1$ for all $h, l \in H$, then $R$ is an $H$-module algebra and $R \#_\sigma H$ is the usual smash product $R \# H$ [S]. In this case $H \cong 1 \# H$ and we will sometimes write $rh$ for $r \# h \in R \# H$.

If $B$ is a basis for $H$ and $w = \sum a_x \# u_x \in R \#_\sigma H$, where the sum is taken over those $u_x \in B$, then we define $\text{supp}(w) = \{u_x \in B \mid a_x \neq 0\}$.

Now consider $R \#_\sigma H$ when $H$ is irreducible with coradical filtration $\{H_n\}$; choose $h \in H_n$ and write $\Delta h = h \otimes 1 + 1 \otimes h + \sum y_i \otimes z_i$, $y_i, z_i \in H_{n-1}$. Then for any $r, s \in R$,

$$h \cdot (rs) = (h \cdot r)s + r(h \cdot s) + \sum_l (y_l \cdot r)(z_l \cdot s). \quad (1.4)$$

Thus $h$ acts as a derivation modulo the action of $H_{n-1}$. Also, for $r \in R$, we have

$$h \cdot r = (1 \# h)r - r(1 \# h) - \sum_l (y_l \cdot r) \# z_l. \quad (1.5)$$

Thus the action of $h$ on $R$ looks like an inner derivation via $1 \# h$, modulo lower degree terms. We will need the following result on ideals in crossed products:

**Lemma 1.6.** Let $R \#_\sigma H$ be a crossed product. If $I$ is an ideal of $R \#_\sigma H$, then $I \cap R$ is an $H$-stable ideal of $R$.

**Proof.** This fact follows [BIM, Prop. 1.8] which says that since $\sigma$ is invertible, the map $\gamma: H \rightarrow R \#_\sigma H$ given by $\gamma(h) = 1 \# h$ is also invertible. This implies that for all $h \in H$, $r \in R$, $h \cdot r = \sum \gamma(h_{(1)}) r \gamma^{-1}(h_{(2)})$; see [BIM, 1.19].

We will also use various quotient rings of the $k$-algebra $R$. If $\mathcal{F}$ is the filter of ideals of $R$ with zero annihilator, let $Q^l$ (respectively $Q^r$, $Q$) be the left (respectively right, symmetric) Martindale quotient ring with respect to $\mathcal{F}$, see [P, Chap. 3]. $R$ embeds into $Q^l$ (resp. $Q^r$) as right (left) multiplications, and any $q \in Q^l$ (resp. $Q^r$) has the property that there exists $I \in \mathcal{F}$ such that $Iq \subseteq R \ (qI \subseteq R)$. The symmetric quotient ring can then be described as

$$Q = \{q \in Q^l \mid qI \subseteq R \text{ for some } I \in \mathcal{F}\} = \{q \in Q^r \mid Iq \subseteq R \text{ for some } I \in \mathcal{F}\}.$$

The extended center of $R$ is $C = C(R)$, the center of $Q$ (and also of $Q^l$ and $Q^r$); it is also the centralizer of $R$ in each of these quotient rings.

When $H$ acts on $R$, a smaller quotient ring is useful, and we may repeat the above constructions replacing $\mathcal{F}$ by $\mathcal{F}_H$, the filter of $H$-stable ideals of
$R$ with zero annihilator. Thus one obtains $Q^I_H$, $Q^{'I}_H$, and finally $Q_H$, the 
$H$-symmetric ring of quotients of $R$. The center of $Q_H$ is $C_H = C \cap Q_H$, 
since we can view $R \subseteq Q_H \subseteq Q$ and $C$ is the centralizer of $R$ in $Q$.

Some difficulties arise in trying to extend the $H$-action to $Q_H$ (or to $Q^I_H$ 
and $Q^{'I}_H$). If $R$ is an $H$-module algebra, this is done by Cohen in \[C\]; if $R$ 
is an $H$-prime algebra with a weak action arising from a crossed product, 
then Chin \[Ch\] defines an $H$-action on $Q^I_H$ and then (indirectly) on $Q_H$ 
under the assumption that the action is "fully anti-invertible." More 
recently, it is shown in \[M3\] that the action can be extended to $Q_H$ 
directly if it is also "invertible." These assumptions are satisfied when $H$ 
is irreducible. A more precise discussion of these results will appear in 
Section 3.

Finally, we need the notion of "$X$-inner" automorphisms and derivations. 
When $R$ is prime, an automorphism $\tau$ or derivation $\delta$ of $R$ is called $X$-inner 
if it becomes inner when extended to $Q(R)$, or equivalently to $Q^I(R)$ or to 
$Q^{'I}(R)$. The following internal characterizations of $X$-inner are known:

**Lemma 1.7.** Let $R$ be a prime ring;

1. If $\tau \in \operatorname{Aut} R$, then $\tau$ is $X$-inner $\iff$ there exist non-zero elements $a, b, c, d \in R$ such that $arb = crd$, for all $r \in R$.

2. If $\delta : R \rightarrow Q$ is a derivation such that $\delta(J) \subseteq R$, for some non-zero ideal $J$ of $R$, then $\delta$ is $X$-inner $\iff$ there exist $a, b \in R$, $a \neq 0$ such that $a\delta(ra) = bra - arb$, for all $r \in R$.

**Proof.** Part (1) is \[P, 12.1\] although it really goes back to Kharchenko. Part (2) is \[BeM, Proposition 1.1\].

In fact both parts of Lemma 1.7 extend an old lemma of Martindale, 
which says that for non-zero $a, b \in R$ there exists $\lambda \in C$ such that $a = \lambda c$ if 
and only if $arb = crd$ for all $r \in R$ and some non-zero $b, d \in R$.

We denote the $k$-linear derivations from $R$ to $Q$ by $\operatorname{Der}_k(R, Q)$ and those 
which become inner on $Q$ are denoted by $X$-inn $\operatorname{Der}_k(R, Q)$.

2. **The Bimodule Property and Ideals in Crossed Products**

Let $R \#_H H$ be a crossed product; we say that $R \#_H H$ has the **bimodule property** if every non-zero $(R, R)$-subbimodule of $R \#_H H$ intersects $R$ non-
trivially. In this section we show that $R \#_H H$ can have this property only 
if $H$ is irreducible and then find necessary and sufficient conditions for 
$R \#_H H$ to have this property in terms of whether the primitive elements of 
$H$ act as $X$-inner derivations. This result is then applied to study ideals in 
smash products for more general Hopf algebras.
The bimodule property was considered previously in [BeM] for the special case of $H = U(L)$, and our first main result generalizes [BeM, Theorem 1.2] and provides a converse. We note that even for $H = U(L)$, assuming no $0 \neq x \in L$ acts as an $X$-inner derivation is not sufficient to guarantee that $R \# H$ has the bimodule property [M2, Example 1.2].

Now let $R$ be prime with extended center $C$, and consider $\delta_1, \delta_2, \ldots, \delta_n \in \text{Der}_k(R)$. If $a_1, a_2, \ldots, a_n \in C$, then $\delta = \sum_{i=1}^n a_i \delta_i \in \text{Der}_k(R, Q)$ and has the property that $\delta(J) \subset R$ for some ideal $J \neq 0$ of $R$ (let $J = I^2$, where $a, I \subset R$ for all $i$). The composition of the maps $L \to \text{Der}_k(R), C \otimes_k \text{Der}_k(R) \to \text{Der}_k(R, Q)$, and $\text{Der}_k(R, Q) \to \text{Der}_k(R, Q)/X$-inn $\text{Der}_k(R, Q)$ give a map

$$\theta : C \otimes_k L \to \text{Der}_k(R, Q)/X\text{-inn Der}_k(R, Q) \quad (2.1)$$

**Theorem 2.2.** Let $R$ be a prime $k$-algebra and $H$ a Hopf algebra with a (weak) action on $R$ such that $R \#_a H$ is a crossed product. Then $R \#_a H$ has the bimodule property if and only if $H$ is irreducible and the map $\theta$ in (2.1) is injective on $C \otimes L$, where $L = P(H)$ is the Lie algebra of primitive elements of $H$.

**Proof.** First, suppose that $H$ is not irreducible; then $H$ contains a simple subcoalgebra $L$ which does not contain $k \cdot 1$. Thus $(R \#_a 1)(1 \#_a K)$ is an $(R, R)$-subbimodule of $R \#_a H$ which does not intersect $R$ non-trivially. Hence if $R \#_a H$ does have the bimodule property, then $H$ must be irreducible. Now assume that $H$ is irreducible and suppose that $\theta$ is not injective on $C \otimes L$. Thus for some $0 \neq w = \sum a_i \otimes x_i \in C \otimes L$, $\theta(w)$ is an $X$-inner derivation; that is, there exists $0 \neq s \in Q$ such that $\sum a_i(x_i \cdot r) = [s, r]$, for all $r \in R$. Let $I \neq 0$ be an ideal of $R$ such that $I_s \subset R$ and $I_a \subset R$ for all $a$, and let $B = \{ \sum_i a_{i} x_{i} - a_{i} \neq 1 | a \in I \} \subset R \#_a H$. $B$ is clearly a left $R$-submodule of $R \#_a H$. To see that it is also a right $R$-submodule, choose $r \in R$. Using $(1 \#_a x_i)(r \#_a 1) = r \cdot x_i + (x_i \cdot r) \#_a 1$, we see that $(\sum_i a_{i} x_{i} - a_{i} \neq 1)(r \#_a 1) = \sum_i a_{i} x_{i} \#_a 1 - ar \in B$, since $ar \in I$. Thus $B$ is an $(R, R)$-subbimodule of $R \#_a H$ which does not intersect $R$ non-trivially since the $\{x_i\}$ and $\{1\}$ are independent and the $a_i$ are not zero divisors. Thus $R \#_a H$ does not have the bimodule property.

Conversely, assume that $H$ is irreducible and $\theta$ is injective on $C \otimes L$. Let $B \neq 0$ be an $(R, R)$-subbimodule of $R \#_a H$ and let $n$ be minimal such that $B \cap (R \#_a H_n) \neq 0$, where $\{H_n\}$ is the coradical filtration of $H$. We may assume that $n > 0$. There exist subspaces $X$ and $Y$ of $H$ such that, as vector spaces, we can write $H_n = H_{n - 1} \oplus X$ and $H_{n - 1} = H_{n - 2} \oplus Y$ as noted in Section 1. Let $\{w_k\}$ be a basis of $Y$ and $\{s_l\}$ a basis of $H_{n - 2}$. Among all $0 \neq z \in B \cap (R \#_a H_n)$, written as

$$z = \sum_{i=1}^m b_i \# u_i + \sum_{k} c_k \# w_k + \sum_{l} d_l \# s_l,
where \( u, c, d \in R \), choose one for which \( m \) is minimal. By the minimality of \( m \), both the \( \{b_i\} \) and the \( \{u_i\} \) are linearly independent. Now extend the \( \{u_i\} \) to a basis for \( X \).

We can also write \( Au_i = u_i \otimes 1 + 1 \otimes u_i + \sum_k w'_{i,k} \otimes w_k + \sum_i s'_{i,l} \otimes s_i \) for each \( i \). By Lemma 1.1(2), \( w'_{i,k} \neq 0 \) for some \( k \), since \( u_i \in H_{n-1} \). Moreover, we may assume \( u_i \), \( w_k \), and \( w'_{i,k} \) are all in \( H^+ \). Thus, since \( Au_i \in \sum_j^\infty H_j \otimes H_{n-j} \), it follows that \( w'_{i,k} \in H_1^+ = L = P(H) \).

Choose any \( a \in R \) and consider \( b_1 az - zab_1 \), using \( (1 \neq u_i)(ab_1 \neq 1) = ab_1 \neq u_i + (u_i \cdot (ab_1)) \neq 1 + \sum_k w'_{i,k} \cdot (ab_1) \neq w_k + \sum_i s'_{i,l} \cdot (ab_1) \neq s_i \), it follows that the coefficient of \( u_1 \) in \( b_1 az - zab_1 \) is \( b_1ab_1 - b_1ab_1 = 0 \).

Since \( b_1 az - zab_1 \in B \) and has fewer non-zero coefficients of the \( u_i \) than does \( z \), we conclude that \( b_1 az - zab_1 = 0 \). Therefore, by examining the coefficient of each \( u_i \) in \( b_1 az - zab_1 = 0 \), we have \( b_1ab_1 - b_1ab_1 = 0 \), for each \( i \) and every \( a \in R \). Thus, by Martindale's lemma, this implies \( b_1 = a_1b_1 \), where \( a_1 \in C \).

Now consider the coefficients of \( w_k \) in \( b_1 az - zab_1 = 0 \):

\[
- \sum_i b_i \sum_k w'_{i,k} \cdot (ab_1) \neq w_k + (b_1ac_k - c_kab_1) \neq w_k = 0,
\]

for all \( k \).

Choose \( k \) so that \( w'_{i,k} \neq 0 \) for some \( i \), and use \( b_i = a_i b_i \) to obtain

\[
b_i \sum_j -a_j(w'_{i,k} \cdot (ab_1)) = c_kab_1 - b_1ac_k,
\]

for all \( a \in R \). (*)

Let \( \delta = -\sum j \alpha_j d_j \), where \( d_j \in \text{Der}_k(R) \) is given by \( d_i(r) = w'_{i,k} \cdot r \). Then (*) becomes

\[
b_i \delta(ab_1) = c_kab_1 - b_1ac_k,
\]

for all \( a \in R \).

As noted before the theorem, \( \delta(J) \subseteq R \) for some non-zero ideal \( J \) of \( R \). Thus, by Lemma 1.7(2), \( \delta \) is \( X \)-inner. But \( \delta \) is the image in \( \text{Der}_k(R, \mathbb{Q}) \) of \( w = \sum j -a_j \otimes w'_{i,k} \) and \( w \neq 0 \) since, by construction, the \( \{a_i\} \) are linearly independent. But \( \theta(w) = 0 \), a contradiction. Thus \( n = 0 \) and \( B \cap R \neq 0 \).

It is surprising in Theorem 2.3 that the bimodule property depends only on the behavior of the primitive elements, since \( L = P(H) \) does not generate \( H \) in general. In particular, \( L \) does not generate if \( H \) is a divided power Hopf algebra in characteristic \( p \neq 0 \), or if \( H = \mathcal{O}(G) \), for \( G \) a non-abelian unipotent group.

**Lemma 2.3.** Assume that \( R \) is prime and that \( S = R \#_a H \) satisfies the bimodule property. If \( g \in \text{Aut} S \) such that \( R^g = R \) and \( g \) is \( X \)-outer on \( R \), then \( g \) is \( X \)-outer on \( S \).

**Proof.** Since \( R \) is prime and the bimodule property holds, we immediately note that \( S \) is also prime. If \( g \) is \( X \)-inner on \( S \), there exists
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a unit \( w \in Q(S) \) such that \( s^w = w^{-1} s w \), for all \( s \in S \). Thus \( rw = wr^e \), for all \( r \in R \). Choose \( I \neq 0 \) an ideal of \( S \) with \( wI \subseteq S \); now \( I \cap R \neq 0 \) by the bimodule property, but then \( B = w(I \cap R) \) is also a non-zero \((R, R)\)-subbimodule and so, \( B \cap R \neq 0 \). Let \( J = \{ a \in I \cap R \mid wa \in R \} \); \( J \neq 0 \) is an ideal of \( R \). For any \( a \in J \), write \( b = wa \in R \). Thus \( asb = b^s s^a \), for all \( s \in S \). By Lemma 1.7(1), \( g \) is \( X \)-inner on \( R \), a contradiction.

We continue with

**Lemma 2.4.** Assume that \( R \#_\tau H \) is a crossed product and that \( H \) is of the form \( K \# kG \), for \( G \subseteq G(H) \). Then there exists a cocycle \( \tau: G \times G \to R \#_\sigma H \) such that \( R \#_\sigma H = (R \#_\tau K) \#_\tau kG \).

**Proof.** We use the criterion for crossed products as given in [BLM, 1.19]. Let \( A = R \#_\tau K \) and \( B = R \#_\sigma H \); now \( B \) is a right \( kG \)-comodule algebra via \( \rho: B \to B \otimes kG \) given by \( \rho(r \# (h \# g)) = r \# ((h \# g) \otimes g) \), for all \( r \in R \), \( h \in K \), \( g \in G \), and it is easy to see that \( B^{\text{co} \ H} = A \), the subring of coinvariants. Thus \( A \subseteq B \) is a "\( kG \)-extension." It is clear, since the map \( \gamma: kG \to B \) given by \( \gamma(g) = 1 \#(1 \# g) \) is a right \( kG \)-comodule morphism which has a (convolution) inverse, namely \( \gamma^{-1}(g) = \sigma^{-1}(g^{-1}) \# (1 \# g^{-1}) \). Thus [BLM, 1.18] gives that \( B \cong A \#_\sigma kG \). Moreover, it is possible to show using [BLM, 1.20] that \( \tau \) is simply \( \sigma \) restricted to \( G \times G \).

It should be pointed out that Lemma 2.4 is not obvious and requires proof: Schneider [Sch] gives an example of three Hopf algebras \( H, K, M \) such that \( H \#_\sigma (K \#_\tau M) \) cannot be written as an \( M \)-crossed product over \( H \#_\sigma K \). Also, it is not easy to show directly that \( \sigma \mid_{G \times G} \) satisfies the conditions for a crossed product over \( R \#_\sigma K \).

We are now able to extend Theorem 2.2 to more general Hopf algebras.

**Theorem 2.5.** Let \( H \) be a Hopf algebra of the form \( H = K \# kG \), where \( K \) is irreducible, and let \( R \) be a prime \( k \)-algebra with extended center \( C \) such that \( R \#_\sigma H \) is a crossed product. Let \( G = G(H) \), \( L = P(H) \), and assume

1. \( G \) is \( X \)-outer on \( R \)
2. the map \( \theta: C \otimes L \to \text{Der}_k(R, Q)/X-\text{inn Der}_k(R, Q) \) is injective.

Then every non-zero ideal of \( R \#_\tau H \) intersects \( R \) non-trivially.

**Proof.** By Theorem 2.2, \( R \#_\sigma K \) has the bimodule property. Since \( G \) stabilizes \( K \), we may apply Lemma 2.3 to \( R \#_\sigma K \) to see that \( G \) is \( X \)-outer on \( R \#_\sigma K \). By Lemma 2.4, \( R \#_\sigma H = (R \#_\tau K) \#_\tau kG \) for some \( \tau \). Applying [M1, 3.16], any non-zero ideal of \( R \#_\sigma H \) intersects \( R \#_\sigma K \) non-trivially, and thus intersects \( R \) by the bimodule property.
Examples of Hopf algebras of the type considered in Theorem 2.5 are any pointed cocommutative Hopf algebra; more generally, any Hopf algebra which is the sum of its irreducible components.

The next corollary overlaps [McS, 1.5]; they consider the case in which $R$ is any commutative ring.

**Corollary 2.6.** Let $R$ be a simple $k$-algebra with center $C$, such that $R \#_o H$ is a crossed product with $H = K \# kG$, where $K$ is irreducible. Let $G = G(H)$, $L = P(H)$, and assume

1. $G$ is outer on $R$
2. $\theta: C \otimes L \to \text{Der}_k(R)/\text{Inn Der}_k(R)$ is injective.

Then $R \#_o H$ is simple.

*Proof.* When $R$ is simple, $Q(R) = R$ and $C(R)$ is the center of $R$. Thus $X$-inner means inner in the usual sense and we may apply Theorem 2.5

Specializing to smash products of finite dimensional Hopf algebras, we may apply the results of [BeM] to obtain a number of ring theoretic consequences:

**Corollary 2.7.** Let $H$ be a finite dimensional Hopf algebra of the form $K \# kG$, where $K$ is irreducible, and let $R$ be a prime $H$-module algebra satisfying conditions (1) and (2) of Theorem 2.5. Then

1. $R^H$ is (right) Goldie if and only if $R$ is (right) Goldie;
2. if $R^H$ is (right) Noetherian or Artinian, then so is $R$;
3. if $R^H$ satisfies a polynomial identity of degree $d$, then $R$ satisfies a polynomial identity of degree $\leq (\dim_k H)d$. Moreover, $R$ is Goldie with $Q_{el}(R) = RT^{-1}$, where $T$ is the set of non-zero central elements of $R^H$, and $Q_{cl}(R^H) = Q_{cl}(R)^H$.

*Proof.* Since $R$ is an $H$-module algebra, we may form the smash product $R \# H$, a crossed product with trivial cocycle. By Theorem 2.5, every non-zero ideal of $R \neq H$ intersects $R$ non-trivially. However this is the "ideal intersection property" of [BeM]. Thus (1) and (2) follow from [BeM, 3.8] and (3) follows from [BeM, 3.9].

We close this section with two examples. In the first example, we show that if $H$ is not the sum of its irreducible components then the conclusions of Theorem 2.5 and Corollary 2.6 need not be true. In the second example, we show that although the bimodule property implies that all non-zero ideals of $R \#_o H$ intersect $R$ non-trivially, the converse does not hold.
EXAMPLE 2.8. In [M2, Ex. 3.2], an action of the (unique) 4-dimensional non-commutative, non-cocommutative Hopf algebra \( H \) over \( \mathbb{C} \) such that \( \mathbb{C} \neq H \) is not simple. However, \( P(H) = 0 \) and the unique \( x \in G(H) \), \( x \neq 1 \) acts as complex conjugation, so hypotheses (1) and (2) are satisfied.

The coradical of \( H \) is \( H_0 = \mathbb{R} \langle 1, x \rangle \) and each irreducible component is one-dimensional. Thus the sum of the irreducible components is just \( H_0 \).

EXAMPLE 2.9. Let \( R \) be the 2 \times 2 matrices over a field \( F \) of characteristic 2 and let \( L \) be the 2-dimensional restricted Lie algebra of derivations with basis elements \( x \) and \( y \) where \( x \) and \( y \) act, respectively, as commutation by the matrix units \( e_{12} \) and \( e_{21} \). We now consider the smash product \( R \# u(L) \). The bimodule property is not satisfied since, as in the proof of Theorem 2.2, both \( R(x - e_{12}) \) and \( R(y - e_{21}) \) are \((R, R)\)-subbimodules of \( R \# u(L) \) which do not intersect \( R \) non-trivially.

A direct calculation shows that \( R \) is an irreducible \( R \# u(L) \)-module with commuting ring \( R^L \cong F \). Now let \( w = a_0 + a_1 x + a_2 y + a_3 xy \in R \# u(L) \) act on \( R \); thus we obtain \( w \cdot 1 = a_0, w \cdot e_{21} = a_0 e_{21} + a_1, w \cdot e_{12} = a_0 e_{12} + a_2, \) and \( w \cdot e_{11} = a_0 e_{11} + a_1 e_{12} + a_2 e_{21} + a_3 \). Therefore it follows that the action of \( R \# u(L) \) on \( R \) is also faithful, hence \( R \# u(L) \) is isomorphic to the 4 \times 4 matrices over \( F \). Since \( R \# u(L) \) is simple, its unique non-zero ideal intersects \( R \) non-trivially.

3. THE BIMODULE PROPERTY, EXTENDED CENTERS, AND QUOTIENT RINGS

In order to study quotients and extended centers for crossed products, we will need some additional assumptions on the (weak) action of \( H \) on \( R \). Consider the action as an element \( \phi \in \text{Hom}(H, \text{End}(R)) \). Then we say that \( \phi \) is invertible if, under convolution, it has an inverse \( \theta \in \text{Hom}(H, \text{End}(R)) \), that is \( \sum_{(h)} \theta_{h_{(1)}} \circ \phi_{h_{(2)}} \circ e(h) \text{id}_R = \sum_{(h)} \phi_{h_{(1)}} \circ \theta_{h_{(2)}} \circ e(h) \text{id}_R = \theta \). \( \phi \) is anti-invertible if it has an anti-inverse \( \psi \in \text{Hom}(H, \text{End}(R)) \), that is \( \sum_{(h)} \psi_{h_{(2)}} \circ \phi_{h_{(1)}} = \text{id}_R \) if \( \phi \) is biinvertible if it is both invertible and anti-invertible.

Fully anti-invertible actions were studied in [Ch] and fully biinvertible actions in [M3]. They are a natural notion, since if \( R \) is an \( H \)-module algebra and \( H \) has a bijective antipode, then the action is always fully biinvertible: let \( \theta_h = Sh \) and \( \psi_h = \overline{Sh} \), where \( \overline{S} \) is the inverse of \( S \). For crossed products \( R \#_{\phi} H \) with a weak action \( \phi \), \( \phi \) will be biinvertible whenever the coradical of \( H \) is cocommutative [M3]; in particular this includes the case where \( H \) is irreducible.

We summarize some known results.
PROPOSITION 3.1. Let $R \#_\sigma H$ be a crossed product with fully biinvertible action.

(1) Let $\mathcal{F}_H$ be the filter of $H$-stable ideals of $R$ with zero annihilator. Then one may construct the left, right, and symmetric quotient rings, $Q'_H(R)$, $Q''_H(R)$, and $Q_H(R)$ respectively, with respect to $\mathcal{F}_H$.

(2) If $J$ is an $H$-stable ideal of $R$, then the annihilator of $J$ is also $H$-stable when the action is biinvertible. If $R$ is $H$-prime, $\mathcal{F}_H$ is the set of non-zero $H$-stable ideals of $R$.

(3) The $H$-action on $R$ extends uniquely to a (weak) $H$-action on $Q'_H(R)$, $Q''_H(R)$, and $Q_H(R)$, such that the cocycle condition and twisted module condition hold for $\sigma$. Thus one may form the crossed products $Q'_H(R) \#_\sigma H$, $Q''_H(R) \#_\sigma H$, and $Q_H(R) \#_\sigma H$.

(4) Each of the crossed products in (3) embeds into the appropriate quotient of $R \#_\sigma H$, for example $Q_H(R) \#_\sigma H \subseteq Q(R \#_\sigma H)$.

(5) $Q(\sigma)(R)^H \subseteq Q_H(R)$, and so $Q(\sigma)(R)^H \subseteq \sigma Q(R \#_\sigma H)$. Under this embedding, $C(\sigma)(R)^H \subseteq C(R \#_\sigma H)$, where $C(\sigma)(R)^H$ means $C(R) \cap Q(\sigma)(R)^H$.

(6) If $J$ is an $H$-stable ideal of $R$, then $J \#_\sigma H = (1 \# H)(J \# 1) = HJ$.

Proof: For (1), $Q'_H(R)$ and $Q''_H(R)$ are done in [Ch] and $Q_H(R)$ in [M3]. Part (2) is done in [C] for smash products and in [Ch] for crossed products.

Parts (3) and (4) are done for $Q'_H(R)$ and $Q''_H(R)$ in [Ch], although the action on $Q_H(R)$ and the embeddings are done by an indirect method. These facts are reproved directly in [M3] together with the result on $Q'_H(R)$. Finally, (5) is in [M3] and (6) is in [Ch].

The following technical proposition will be useful in this section for computing extended centers and quotient rings.

PROPOSITION 3.2. Let $R$ be prime and $R \#_\sigma H$ a crossed product such that the action is biinvertible. Suppose $w \in Q'(R \#_\sigma H)$ such that $Jw \subseteq R \#_\sigma H$ for some $H$-stable ideal $J \neq 0$ of $R$ and conjugation by $w$ induces an automorphism $g$ of $R$. Then $w \in Q'_H(R) \#_\sigma H$. Furthermore, if $R \#_\sigma H$ has the bimodule property then $w \in Q'_H(R)$.

Proof: Let $B$ be a basis for $H$, as in Section 1, and for each $u_a \in B$ define an element of $Q'_H(R)$, $t_a : J \rightarrow R$, by letting $rt_a$ be the coefficient of $u_a$ in $rw$, for all $r \in J$. Now if $0 \neq a \in J$, we will let $K$ be the subcoalgebra of $H$ generated by all those $u_a$ belonging to $\text{supp}(aw)$. Since $\text{supp}(aw)$ is a finite set, $K$ is finite dimensional by the fundamental theorem of coalgebras. Let $b \in J$, $r \in R$ and consider $awrb^\sigma$; by the definition of multiplication in $R \#_\sigma H$, $\text{supp}(awrb^\sigma) \subseteq K$. However, $awrb^\sigma = arbw$ and therefore if $u_a \notin K$
then \(arb_{t} = 0\). As a result, \((aR)(Jt_{x}) = 0\) and so, \(Jt_{x} = 0\) since \(R\) is prime. Thus \(t_{x} = 0\), for all \(u_{x} \notin K\).

Let \(v = \sum t_{x}u_{x}\), where the sum is a finite sum since it is taken only over those \(u_{x} \in K\). Therefore \(v \in Q_{H}^{l}(R) #_{\sigma} H\) and, by Proposition 3.1(4), \(v \in Q_{H}^{l}(R #_{\sigma} H)\). However, by the definition of the \(t_{x}\), \(J(w - v) = 0\). Since the action is biinvertible, \(HJ = J #_{\sigma} H\) by Proposition 3.1(6), and \((J #_{\sigma} H)(w - v) = 0\). \(J #_{\sigma} H\) is an ideal of \(R #_{\sigma} H\) with zero annihilator, therefore as elements of \(Q_{H}^{l}(R #_{\sigma} H)\), \(w = v\). Thus \(w \in Q_{H}^{l}(R) #_{\sigma} H\).

Now suppose \(R #_{\sigma} H\) satisfies the bimodule property. Since \(wR = Rw\), \(Jw\) is an \((R, R)\)-subbimodule of \(R #_{\sigma} H\), hence it intersects \(R\) non-trivially. Therefore there exists some \(a \in J\) such that \(0 \neq aw \subset R\). However, as before \(aRJw = awR^{g}b^{g} \subset R\). Since \(w = \sum t_{x}u_{x}\), we have \(\sum (aRJt_{x})u_{x} = aRJw \subset R\) thus, as in the first paragraph \(t_{x} = 0\), for all \(u_{x} \neq 1\). Hence, \(w \in Q_{H}^{l}(R) #_{\sigma} H\).

We can now prove the main result of this section.

**Theorem 3.3.** Let \(R\) be a prime \(k\)-algebra and \(R #_{\sigma} H\) a crossed product which has the bimodule property. Then \(C(R #_{\sigma} H) = C(R)^{H} #_{\sigma} 1 \cong C(R)^{H}\).

**Proof.** By Proposition 3.1(5), it is always true that \(C(R)^{H} #_{\sigma} 1 \subset C(R #_{\sigma} H)\). Conversely, choose \(w \in C(R #_{\sigma} H)\) and an ideal \(I \neq 0\) of \(R #_{\sigma} H\) such that \(wI = Iw \subset R #_{\sigma} H\). Let \(J = I \cap R\); \(J\) is \(H\)-stable by Lemma 1.6 and \(J \neq 0\) by the bimodule property. Since \(Jw \subset R #_{\sigma} H\) and \(w\) commutes with \(R\), we can apply Proposition 3.2 to conclude that \(w \in Q_{H}^{l}(R)\). However, since \(w\) commutes with \(R\), \(w \in C(R)\).

Since \(w\) also commutes with \(H = 1 # H\), for all \(h \in H\), we have \(w \neq h = (1 # h)w = \sum_{h} h_{(1)}/h_{(2)} # w \neq h_{(2)}\). Applying \(id \otimes \varepsilon\) to this equation yields \(\varepsilon(h)w = h \cdot w\), hence \(w \in C(R)^{H} #_{\sigma} 1 \cong C(R)^{H}\).

We continue looking for cases where \(C(R #_{\sigma} H) = C(R)^{H} #_{\sigma} 1 \cong C(R)^{H}\). The next result may be well known.

**Lemma 3.4.** If \(R\) is prime and the action of \(G\) is \(X\)-outer on \(R\), then \(C(R #_{\sigma} kG) = C(R)^{G} #_{\sigma} 1 \cong C(R)^{G}\).

**Proof.** By Proposition 3.1(5), \(C(R)^{G} \subset C(R #_{\sigma} kG)\); note that since \(kG\) is cocommutative, the action is fully biinvertible. Conversely, as in Theorem 3.3, choose \(w \in C(R #_{\sigma} kG)\) and \(I \neq 0\) such that \(wI = Iw \subset R #_{\sigma} kG\). It is known that \(R #_{\sigma} kG\) has the ideal intersection property since \(G\) is \(X\)-outer [M1, p. 52]. Thus \(J = I \cap R \neq 0\), \(J\) is a \(G\)-stable ideal of \(R\), and \(wJ \subset R #_{\sigma} kG\).

We claim \(wJ \subset R\). For if \(a \in J\), write \(wa = b = \sum_{g} r_{g} g \# g\). Since, for any \(r \in R\), \(arb = bra\), we see \(arr_{g} = r_{g} g^{g} a^{g}\), for all \(g \in G\). If \(r_{g} \neq 0\), this implies, by Lemma 1.7, that \(g\) is \(X\)-inner, a contradiction unless \(g = 1\). Thus
We can now prove

**Corollary 3.5.** Let $R \#_\sigma H$ be a crossed product such that $R$ is prime, $H = K \# kG$ where $K$ is irreducible, and $G = G(H)$ and $L = P(H)$ satisfy conditions (1) and (2) of Theorem 2.5. Then $C(R \#_\sigma H) = C(R)^H \neq 1 \cong C(R)^G$.

**Proof:** By Lemma 2.4, $R \#_\sigma H = (R \#_\sigma K) \#_\sigma kG$. Then, by Lemma 2.3, the action of $G$ on $R \#_\sigma K$ is $X$-outer since by Theorem 2.2, $R \#_\sigma K$ satisfies the bimodule property. Thus, by Lemma 3.4, $C(R \#_\sigma H) = C(R \#_\sigma K)^G \neq 1 \cong C(R \#_\sigma K)^G$. Since $R \#_\sigma K$ has the bimodule property, Theorem 3.3 applies to give $C(R \#_\sigma K) = C(R)^K$. Thus $C(R \#_\sigma H) = (C(R)^K)^G = C(R)^H$.

We can also apply Proposition 3.2 to obtain some information about the $X$-inner automorphisms of $R \#_\sigma H$. The following corollary is similar to work of Osterburg and Passman [OP] on the $X$-inner automorphisms of $R \#_\sigma U(I)$.

**Corollary 3.6.** Let $R$ be prime and $R \#_\sigma H$ a crossed product with the bimodule property. Then every $X$-inner automorphism of $R \#_\sigma H$ which stabilizes $R$ is induced by an element of $QH(R)$.

**Proof:** Suppose $w \in Q(R \#_\sigma H)$ induces an $X$-inner automorphism $g$ of $R \#_\sigma H$ which stabilizes $R$. Let $I \neq 0$ be an ideal of $R \#_\sigma H$ such that $wI, Iw \subseteq R \#_\sigma H$ and let $J = I \cap R$. $J$ is $H$-stable and is non-zero by the bimodule property. Since $Jw \subseteq R \#_\sigma H$, $Rw = wR$, and $R \#_\sigma H$ has the bimodule property, it follows by Proposition 3.2 that $w \in Q^I_H(R)$. Thus $wJ \subseteq (R \#_\sigma H) \cap Q^I_H(R)J \subseteq R$ and so, $w \in Q_H(R)$.

The assumption in Corollary 3.6 that the $X$-inner automorphism stabilize $R$ is necessary, for in Example 4.10 we will see an $X$-inner automorphism of $R \#_\sigma H$ which is not induced by any element of $Q_H(R)$.

If we consider the case where every non-zero $H$-stable ideal of $R$ contains a left regular element, then we can describe $Q(R \#_\sigma H)$. In order to do so, we need the following important proposition which will also be used in the next section.

**Proposition 3.7.** Let $R \#_\sigma H$ be a crossed product with $H = K \# kG$, where $K$ is irreducible. Suppose $w \in Q^I(R \#_\sigma K)$ and $J \neq 0$ is an $H$-stable
ideal of \( R \) such that \( Jw, wJ \subseteq R \# K \). If \( J \) contains a left regular element, then \( w \in Q_H(R) \#_\sigma K \).

**Proof.** Let \( b \in J \) be left regular. Choose a basis \( B = \{ u_x \} \) of \( K \) as in Section 1, so that it includes a basis for every \( K_n \). Let \( D \) be the subcoalgebra of \( K \) generated by \( \text{supp}(wb) \); \( D \) is finite dimensional by the fundamental theorem of coalgebras. First note that for any \( a \in J \), \( \text{supp}(awb) \subseteq \text{supp}(wb) \subseteq D \). We claim that for any \( a \in J \), in fact \( \text{supp}(aw) \subseteq D \).

If not, there exists a largest integer \( n \) such that there exists some \( u_x \in B_n \) and \( u_y \in \text{supp}(aw) - D \). Therefore for some \( 0 \neq r \in R \),

\[
aw = \sum c_i u_i + ru_y + \sum d_\beta u_\beta,
\]

where \( c_i, d_\beta \in R \), \( u_i \in B_n \) are distinct from \( u_x \), and \( u_\beta \in D \cap (B - B_n) \). Now

\[
awb = \sum c'_i u'_i + rbu_y + \sum d'_\beta u'_\beta,
\]

where \( c'_i, d'_\beta \in R \), \( u'_i \in K_n \) are distinct from \( u_x \), and \( u'_\beta \in D \). Since \( \text{supp}(awb) \subseteq D \), and \( rb \neq 0 \) since \( b \) is left regular, it follows that \( u_y \in D \), a contradiction. Thus \( \text{supp}(aw) \subseteq D \).

If \( \{ d_x \} \) is a basis for \( D \), then for each \( d_x \) we define an element of \( Q_H'(R) \), \( t_x : J \rightarrow R \), by letting \( ut_x \) be the coefficient of \( d_x \) in \( aw \), any \( a \in J \). Now let \( v = \sum t_x d_x, \ v \in Q_H'(R) \#_K \subseteq Q'(R) \#_K \) by Proposition 3.1. Also by Proposition 3.1, \( KJ = J \#_\sigma K \), thus \( 0 = KJ(w - v) = (J \#_\sigma K)(w - v) \). Therefore \( w = v \in Q_H'(R) \#_K \).

We now claim that \( t_x J \subseteq R \), for all \( d_x \in \text{supp}(w) \). If not, let \( n \) be the largest integer such that there exists some \( d_y \in K_n \cap \text{supp}(w) \) with \( t_x J \not\subseteq R \). Therefore we have

\[
w = \sum t_x d_x + t_y d_y + \sum t_\beta d_\beta,
\]

where \( d_x \in K_n, d_\beta \in K - K_n, t_\beta J \subseteq R \), and the \( d_x \) are all different from \( d_y \).

If \( r \in J \) then by (1.5), the coefficient of \( d_y \) in \( wr \) is \( t_y r + \sum_\beta t_\beta (y_i \cdot r) \), for appropriate \( y_i \in K \). However, \( J \) is \( K \)-stable and each \( t_\beta J \subseteq R \), by induction, thus \( \sum_\beta t_\beta (y_i \cdot r) \in R \). Since the coefficient of \( d_y \) in \( wr \) belongs to \( R \), it follows that \( t_y r \in R \), hence \( t_y J \subseteq R \), a contradiction. As a result, all the \( t_x \) belong to \( Q_H'(R) \) and \( w \in Q_H'(R) \#_K \).

The next result now follows directly from Proposition 3.7; note we do not require that \( R \) is prime.

**Theorem 3.8.** Suppose that \( R \#_\sigma H \) is a crossed product with the bimodule property. If every non-zero \( H \)-stable ideal of \( R \) contains a left regular element then \( Q(R \#_\sigma H) = Q_H(R) \#_\sigma H \).
Proof. By Proposition 3.1(4), it is always the case that \( Q_H(R) \#_\sigma H \subseteq Q(R) \#_\sigma H \). Conversely, let \( w \in Q(R) \#_\sigma H \) and let \( I \neq 0 \) be an ideal of \( R \#_\sigma H \) such that \( Iw, wI \subseteq R \#_\sigma H \). By the bimodule property, \( H \) is irreducible, and if \( J = I \cap R \) then \( J \) is both non-zero and \( H \)-stable. Since \( Jw, wJ \subseteq R \#_\sigma H \), we may use Proposition 3.7 with \( G = \langle 1 \rangle \) to see \( w \in Q_H(R) \#_\sigma H \). Thus \( Q(R) \#_\sigma H = Q_H(R) \#_\sigma H \).

When \( R \) is prime we can consider Hopf algebras of the form \( K \# kG \), where \( K \) is irreducible. We first prove the quotient analog of Lemma 3.4.

**Lemma 3.9.** If \( R \) is prime such that the action of \( G \) is \( X \)-outer of \( R \) and every non-zero \( G \)-stable ideal of \( R \) contains a left regular element, then \( Q(R) \#_\sigma kG = Q_R(R) \#_\sigma kG \).

**Proof.** By Proposition 3.1, \( Q_G(R) \#_\sigma kG \subseteq Q(R) \#_\sigma kG \). Conversely choose \( w \in Q(R) \#_\sigma kG \) and \( I \neq 0 \) an ideal of \( R \#_\sigma kG \) such that \( Iw, wI \subseteq R \#_\sigma kG \). As in Lemma 3.4, \( J = I \cap R \neq 0 \), \( J \) is \( G \)-stable, and \( Jw, wJ \subseteq R \#_\sigma kG \).

Choose \( b \) left regular in \( J \) and let \( D \) be the finite subset of \( G \) consisting of the support of \( wb \) in \( R \#_\sigma kG \). Then for any \( a \in J \), \( \text{supp}(awb) = \text{supp}(awb) \subseteq D \).

For each \( a \in J \), we may write \( aw = \sum_{g \in D} rg \# g \). Define \( z_g : J \to R \) by \( az_g = r_g \), for each \( g \in D \); clearly \( z_g \in Q_G^G(R) \). Now let \( v = \sum_{g \in D} z_g \# g \in Q^G_G(R) \#_\sigma kG \subseteq Q^G_R(R) \#_\sigma kG \). Then \( 0 = (kG)J(w - v) = (J \#_\sigma kG)(w - v) \) and so, \( w = v \in Q_G^G(R) \#_\sigma kG \).

Finally we claim \( z_g \in Q_G(R) \), for each \( g \in D \). For, if \( a \in J \), \( wa = (\sum_{g \in D} z_g \# g)(a \# 1) = \sum_{g \in D} z_g a^g \# g \in R \#_\sigma kG \), and thus \( z_g a^g \in R \). Since \( J^g = J \), we have \( z_g J \subseteq R \) and thus \( z_g \in Q_G(R) \).

We can now prove the main result of this section.

**Theorem 3.10.** Let \( R \#_\sigma H \) be a crossed product such that \( R \) is prime, \( H = K \# kG \), where \( K \) is irreducible, and \( G = G(H) \) and \( L = P(H) \) are \( X \)-outer on \( R \) as in Theorem 2.5. Assume also that every \( J \in \mathbb{F}_H \) contains a left regular element. Then \( Q(R) \#_\sigma H = Q_H(R) \#_\sigma H \).

**Proof.** By Proposition 3.1, \( Q_H(R) \#_\sigma H \subseteq Q(R) \#_\sigma H \). For the converse, we need several observations. First, by Lemma 2.4, there exists a cocycle \( \tau \) such that \( R \#_\sigma H = (R \#_\sigma K) \#_\sigma kG \) and \( Q_H(R) \#_\sigma H = (Q_H(R) \#_\sigma K) \#_\sigma kG \). In addition, \( R \#_\sigma K \) is prime since the bimodule property holds by Theorem 2.2. Also, \( G \) is \( X \)-outer on \( R \#_\sigma K \) by Lemma 2.3. Now consider any \( G \)-stable ideal \( I \neq 0 \) of \( R \#_\sigma K \); by the bimodule property, \( J = I \cap R \neq 0 \). \( J \) is \( K \)-stable as \( I \) is \( K \)-stable, and \( J \) is also \( G \)-stable since \( G \) acts on \( R \). Thus \( J \) is \( H \)-stable, so it contains a left regular element \( r \in R \). Then \( r \neq 1 \) is left regular in \( R \#_\sigma K \), and \( r \neq 1 \in I \). We may
therefore apply Lemma 3.9 with $R \#_K$ as the base ring to conclude that $Q(R \#_K H) = Q((R \#_K K) \# kG) = Q_G(R \#_K K) \# kG$.

In order to prove the theorem it now suffices to show that $Q_G(R \#_K K) \subseteq Q_H(R) \#_K$. Choose $w \in Q_G(R \#_K K)$; then there exists a $G$-stable ideal $I \neq 0$ of $R \#_K K$ with $Iw$, $wI \subseteq R \#_K K$. By the bimodule property $J = I \cap R \neq 0$; $J$ is $K$-stable and $G$-stable and so, $H$-stable. Thus $J$ contains a left regular element and we conclude, by Proposition 3.7, that $w \in Q_H(R) \#_K$.

It is reasonable to wonder if the hypothesis about left regular elements in Theorem 3.8, Lemma 3.9, and Theorem 3.10 can be removed. However, Examples 4.10 and 4.11 show that the remaining hypotheses do not suffice.

4. Quotients and Extended Centers for $H$ Commutative Irreducible

In this last section we specialize to the case $H$ is commutative and irreducible. We will fix a basis $\{u_i\}$ for $H$ as in Section 1; in particular, it is a union of bases for all the $H_n$. We also specialize to the situation in which $R$ is an $H$-module algebra, and thus our crossed product becomes the usual smash product $R \# H$.

Recall that $R$ is $H$-prime if the product of two non-zero $H$-stable ideals is always non-zero; in such an algebra, $\mathcal{F}_H = \{\text{all non-zero } H\text{-stable ideals}\}$ since $R$ is an $H$-module algebra [C, Cor. 3]. In this situation, (1) and (3) of Proposition 3.1 are also due to Cohen.

Throughout this section, unless otherwise stated, we will always be assuming that $R$ is an $H$-prime $H$-module algebra, where $H$ is commutative irreducible. We require two new definitions.

**Definition 4.1.** (1) $Q^I_H(R) \#_\infty H = \{\sum_i t_i \# u_i \mid t_i \in Q^I_H(R) \text{ for all } i \text{ and there exists } J \in \mathcal{F}_H \text{ such that } Jt_i \subseteq R, \text{ all } i, \text{ and for each } a \in J, at_i = 0 \text{ for all but finitely many } i\}$.

(2) $Q^H_H(R) \#_\infty H = \{\sum_i t_i \# u_i \in Q^H_H(R) \#_\infty H \mid t_i \in Q^H_H(R) \text{ for all } i\}$.

The elements in $Q^I_H(R) \#_\infty H$ are infinite sums; however, for each element $w = \sum t_i \# u_i$ as above, $Jw \subseteq R \# H$ for some $J \in \mathcal{F}_H$. But now since $J$ is $H$-stable and $S$ is bijective, $J \# H = HJ$, by Proposition 3.1, and thus $(J \# H)w \subseteq R \# H$. Since $J \# H$ has zero annihilator, $w$ determines an element $\tilde{w}$ in $Q^I(R \# H)$ and the map $w \mapsto \tilde{w}$ is injective. Thus we have proved part of

**Lemma 4.2.** $Q^I_H(R) \#_\infty H = \{q \in Q^I(R \# H) \mid \text{there exists some } J \in \mathcal{F}_H \text{ such that } Jq \subseteq R \# H\}$. 
One containment is proved in the discussion above. Now suppose \( q \in Q^I(R \# H) \) and \( J \in \mathscr{F}_H \) such that \( Jq \subseteq R \# H \) and let \( \{u_i\} \) be a basis for \( H \). Therefore for any \( a \in J \), \( aq = \sum_i r_i u_i \in R \# H \), so for each \( i \) we may define \( t_i : J \to R \) via \( a \mapsto r_i \).

Each \( t_i \) is a left \( R \)-module map, hence \( \{t_i\} \subseteq Q^I_H(R) \). Furthermore the \( \{t_i\} \) satisfy the conditions \( Jt_i \subseteq R \) for all \( i \) and for each \( a \in J \), \( at_i = r_i = 0 \) for all but a finite number of \( i \). Thus \( \sum_i t_i u_i \in Q^I_H(R) \#_\infty H \). Also \( a(q - \sum_i t_i u_i) = 0 \), for all \( a \in J \). Since \( S \) is bijective \( J \# H = HJ \), hence \( (J \# H)(q - \sum_i t_i u_i) = 0 \). Thus \( q = \sum_i t_i u_i \in Q^I_H(R) \#_\infty H \) and the second containment is proved.

The main result of this section is the following.

**Theorem 4.3.** Let \( R \) be an \( H \)-prime \( H \)-module algebra, where \( H \) is commutative irreducible. Let \( Z \) denote the center of \( Q_H(R) \# H \); then if \( 0 \neq q \in Q^I(R \# H) \) there exists \( f \in Z \) such that \( 0 \neq fq \in Q^I_H(R) \#_\infty H \). Furthermore if \( R \# H \) is prime then \( Q^I(R \# H) = (Q^I_H(R) \#_\infty H) Z^{-1} \), the localization at the non-zero elements of \( Z \).

Before proving Theorem 4.3, we need several preliminary results.

**Lemma 4.4.** Let \( R \) be an \( H \)-module algebra with \( H \) commutative irreducible, let \( I \) be an ideal of \( R \# H \), and assume \( r_1 u_1 + \cdots + r_m u_m \in I \), where the basis \( \{u_i\} \) is as above. Then for all \( h \in H \), \( (h \cdot r_1) u_1 + \cdots + (h \cdot r_m) u_m \in I \).

**Proof.** We proceed by induction on \( n \). If \( h \in H_0 = k \cdot 1 \), it is surely true. Now assume that it is true for \( h \in H^+_n \), and choose \( h \in H^+_n \). By (1.5), if \( Ah = h \otimes 1 + 1 \otimes h + \sum_i y_i \otimes z_i \), for \( y_i, z_i \in H^+_n \), then \( h \cdot r = hr - rh - \sum_i (y_i \cdot r) z_i \). Thus

\[
(h_1 \cdot r_1) u_1 + \cdots + (h_m \cdot r_m) u_m = (hr_1 - r_1 h) u_1 + \cdots + (hr_m - r_m h) u_m - \sum_i (y_i \cdot r_i) z_i u_i,
\]

using that \( H \) is commutative. Since \( y_i \in H^+_n \), the double sum is in \( I \) by induction. Thus the entire expression is in \( I \).

The proof of Theorem 4.3 will follow from the following proposition. If \( f = 1 \) in the proposition then \( J \subseteq I \), thus the proposition is essentially a substitute for the "ideal intersection property" in [BeM].
PROPOSITION 4.5. Assume $R$ is an $H$-prime $H$-module algebra, where $H$ is commutative irreducible. Choose a non-zero ideal $I$ of $R \# H$. Then there exists $J \in \mathcal{P}_H$ and $0 \neq f \in Z = Z(Q_H^l(R) \# H)$ such that $Jf \subseteq I$.

Proof. Let $n$ be such that $I \cap (R \# H_n) \neq 0$, but $I \cap (R \# H_{n-1}) = 0$. Consider the ordered basis of $H$ chosen above, thus it is a union of bases for all the $H_n$. Choose $0 \neq a \in I \cap (R \# H_n)$ such that $a$ has the smallest number of basis elements of “degree $n$” in its support, where by “degree $n$” we mean elements of $X_n$. So

$$a = r_1u_1 + \sum_{i=2}^{k} r_iu_i + \sum_{u_a \in H_{n-1}} r_a u_a,$$

$u_i \in X_n$, and $X_n \cap \text{supp}(a) = \{u_1, \ldots, u_k\}$. Then any other non-zero $b \in I \cap (R \# H_n)$ has at least as many elements of $X_n$ in its support. Fix some $u_1 \in \text{supp}(a)$ and define

$$J = \left\{ r \in R \mid ru_1 + \sum_{i=2}^{k} s_iu_i + \sum_{u_s \in H_{n-1}} s_su_s \in I \right\},$$

where some of the $s_a$ may equal 0. $J$ is clearly a left ideal of $R$, and it is $H$-stable by Lemma 4.4. It is also a right ideal, since by (1.5) for any $s \in R$, 

$$ru_1 + \sum_{i=2}^{k} s_iu_i + \sum_{u_s \in H_{n-1}} s_su_s \in (ru_1 + \sum_{i=2}^{k} s_iu_i + \sum_{u_s \in H_{n-1}} s_su_s) \subseteq I,$$

and so $rs \in J$.

Fix $r \in J$, and say $a = ru_1 + \sum_{i=2}^{k} s_iu_i + \sum_{u_s \in H_{n-1}} s_su_s$, then the $s_i$ and $s_a$ in this expression are unique, for if $a' = ru_1 + \sum_{i=2}^{k} s_i'u_i + \sum_{u_s \in H_{n-1}} s_s'u_s \in I$, then $a - a' \in I$ and has fewer degree $n$ elements in its support. Therefore, by our choice of $a$, $a - a' = 0$ and so $s_i = s_i'$ and $s_a = s_a'$ for every $i, a$. Thus for all $i \geq 2$ and $a$ there are well-defined maps $t_i : J \rightarrow R$ and $t_a : J \rightarrow R$ given by $r \mapsto s_i$ and $r \mapsto s_a$. Clearly each $t_i$ and $t_a$ is a left $R$-module map, and thus $t_i$ and $t_a$ determine elements of $Q_H^l(R)$ since $J$ is $H$-stable and $R$ is $H$-prime. Although there may be an infinite number of $t_a$, for any $a \in J$, $at_a = 0$ for all but finitely many $a$. By Lemma 4.2, $Q_H^l(R) \# \infty H$ exists and can be embedded into $Q_H^l(R \# H)$. Thus

$$f = u_1 + \sum_{i=2}^{k} t_iu_i + \sum_{u_a \in H_{n-1}} t_a u_a \in Q_H^l(R \# \infty H).$$

By construction, $Jf \subseteq I$.

We next need to show that $f$ is central in $Q_H^l(R \# \infty H)$. To do this it suffices to show that $[f, R] = 0$ and $[f, H] = 0$, for then we will have $[f, R \# H] = 0$, hence $[f, Q_H^l(R) \# \infty H] \subseteq [f, Q_H^l(R \# H)] \subseteq 0$. Choose $r \in R$, then $J( rf - fr ) = ( Jr ) f - ( Jf ) r \subseteq I$. But using (1.5), $rf - fr \in \sum_{i=2}^{k} Q_H^l(R) \# u_i + \sum_{u_a \in H_{n-1}} Q_H^l(R) \# u_a$, thus $J( rf - fr )$ would contain
elements of $I$ with fewer degree $n$ elements than $a$ in its support, a contradiction unless $J(rf - fr) = 0$. Since $J$ is $H$-stable, $J \neq H$ is an ideal of $R \neq H$ with zero annihilator, and $J \neq H = (1 \neq H)(J \neq 1)$ since $S$ is bijective. Thus $(J \neq H)(rf - fr) = 0$. Considering $rf - fr \in Q'(R \neq H)$ implies that $rf - fr = 0$. Thus $[J, R] = 0$.

To show that $[f, H] = 0$, we proceed by induction on $m$ for $h \in H_m$; in fact we will prove that $[f, H_m] = 0$ and $t_i, t_x \in Q'_H(R)^H_m$. If $m = 0$, then $H_0 = k \cdot 1$ gives the result trivially. So assume, by induction that for all $v \in H_{m-1}$, $[f, v] = 0$ and $v \cdot t = \varepsilon(v)t$, where $t$ is any $t_i$ or $t_x$. Now choose $h \in H_m^+$ and $s \in J$. By (1.5), $sh = hs - h \cdot s - \sum (y_j, s) z_t$, for $y_j, z_t \in H_{m-1}^+$. Since $[f, H_{m-1}] = 0$, we have $(sh)f = hs - (h \cdot s)f - \sum (y_j, s)f_j, \in hI + Jf + Jfz_t \in I$. Thus $f(t_j)f = I$, but clearly $Jh = 0$, and so $J(hf - fh) \in J$.

Now $hf - fh = \sum_{i=2}^k (ht_i - t_ih) u_i + \sum_{u \in H_{n-1}} (ht_a - t_ah) u_x$; using (1.5) again where $t$ is any $t_i$ or $t_x$, $ht - ih = h \cdot t + \sum (y_j, t)z_t = h \cdot t$, since $y_j, t = \varepsilon(y_j)t = 0$ by induction. Thus $hf - fh = \sum_{i=2}^k (h \cdot t_i) u_i + \sum_{u \in H_{n-1}} (h \cdot t_ah) u_x$. Since $J(hf - fh) \subseteq I$, $J\sum_{i=2}^k (h \cdot t_i) u_i + \sum_{u \in H_{n-1}} (h \cdot t_ah) u_x \subseteq I$. But this gives elements in $I$ with fewer degree $n$ elements in its support than $a$; thus $\sum_{i=2}^k J(h \cdot t_i) u_i + \sum_{u \in H_{n-1}} J(h \cdot t_ah) u_x = 0$. It follows that whenever $t = t_i$ or $t_x$, $J(h \cdot t) = 0$, and so, $h \cdot t = 0 = \varepsilon(h)t$. Thus $t \in Q'_H(R)^H_m$. We also obtain $J(hf - fh) = 0$, thus $0 = HJ(hf - fh) = (J \neq H)(hf - fh)$ and so $hf - fh = 0$. Thus $[f, H_m] = 0$, thereby proving the inductive step.

Since $t_i, t_x$ belong to $Q'_H(R)^H$, they commute with $H$, so in particular $f = u_1 + \sum_{i=2}^k u_i t_i + \sum_{u \in H_{n-1}} u_x t_x$. But then, since $fJ = Jf \subseteq I \subseteq R \neq H = HR$, we see that $t_iJ, t_xJ \subseteq R$, for all $i, x$. Thus $t_i, t_x \in Q_H(R)$ and $f \in Z(Q_H(R) \neq \infty H) \cap (Q_H(R)^H \neq \infty H)$.

Finally we need to show that $f$ is a finite sum and we use that each $t_x \in Q_H(R)^H$ to adapt an argument used in the proof of Proposition 3.2. Let $0 \neq s_1 \in J$ and let $K$ be the subcoalgebra of $H$ generated by $\text{supp}(s_1 f)$; we claim that $\text{supp}(f) \subseteq K$. Therefore we must show that if $u_a \notin K$ then $t_x = 0$. If $s \in J$ then since $u_a \notin K$, it follows by the multiplication in $R \neq H$ that $u_a \notin \text{supp}(s_1 f)$. However, $s_1 f s = s_1 s f$, hence $u_a \notin \text{supp}(s_1 s f)$. Therefore $s_1 s t_x = 0$, and so $s_1 Jt_x = 0$. Since $J$ is $H$-stable and $t_x \in Q_H(R)^H$, $Jt_x$ is an $H$-stable left ideal of $R$, thus either $Jt_x = 0$ or $Jt_x$ has zero left annihilator. But since $s_1 \neq 0$, we have $Jt_x = 0$ and so, $t_x = 0$. By the fundamental theorem of coalgebras, $K$ is finite dimensional, thus all but finitely many of the $t_x$ are zero and $f$ is therefore a finite sum. Thus $f \in Z(Q_H(R) \neq \infty H)$.

We can now prove Theorem 4.3.

Proof of Theorem 4.3. We write $Q = Q'_H(R)$ to simplify the notation. By Lemma 4.2, we know $Q \neq \infty H \subseteq Q'(R \neq H)$. Since $Z$ commutes with $R \neq H$, it lies in $C(R \neq H)$. Thus if $R \neq H$ is prime then $C(R \neq H)$ is a field, hence $Z^{-1} \subseteq C(R \neq H)$ and $(Q \neq \infty H) Z^{-1} \subseteq Q'(R \neq H)$.

For the reverse containment, choose $q \in Q'(R \neq H)$. Let $I$ be a non-zero
ideal of $R \# H$ such that $Iq \subseteq R \# H$. By Proposition 4.5, there exists $J \in \mathcal{F}_H$ and $0 \neq f \in Z$ such that $Jf \subseteq I$. Thus $J(fq) = (Jf)q \subseteq Iq \subseteq R \# H$. Therefore, by Lemma 4.2, $fq \in Q_H \# \infty H$ and the first part of the theorem is proved. In addition, if $R \# H$ is prime then $q = (fq)f^{-1} \in (Q_H \# \infty H)Z^{-1}$ and the second part of the theorem is proved.

We can now obtain some information on $C(R \# H)$.

**Corollary 4.6.** Let $R$ be an $H$-module algebra, where $H$ is commutative irreducible, such that $R \# H$ is prime. Then $C(R \# H) = Q_{\mathfrak{q}}(Z)$, where $Z$ is the center of $Q_H(R) \# H$.

**Proof.** As noted in the proof of Theorem 4.3, $Q_{\mathfrak{q}}(Z) \subseteq C(R \# H)$. For the other inclusion, choose $q \in C(R \# H)$ and let $I \neq 0$ be an ideal of $R \# H$ such that $Iq \subseteq R \# H$. By Proposition 4.5, there exists $0 \neq f \in Z$ and $J \in \mathcal{F}_H$ such that $Jf \subseteq I$. We now consider $fq$; clearly $fq \in C(R \# H)$ and $J(fq) \subseteq R \# H$. Therefore, by Lemma 4.2, $fq \in Q'_H(R) \# \infty H$. As a result, we can write $fq = \sum t_i u_i$. Since $H$ is commutative and $fq$ commutes with $H$, for every $h \in H$ and $i$, we claim that $t_i \in Q'_H(R)$. As in the proof of Proposition 4.5, we will show by induction on $m$ that $s t_i = \epsilon(h) t_i$, for all $i$ and $h \in H_m$. Trivially the result holds when $m = 0$ since $H_0 = k \cdot 1$, so we may assume that $v \cdot t_i = \epsilon(v) t_i$, for all $i$ and $v \in H_{m-1}$. Now choose $h \in H_m^1$. Thus by (1.5), $h \cdot t_i = [h, t_i] + \sum (y_{1} \cdot t_i) z_i = 0$, since $y_{1} \cdot t_i = \epsilon(y_i) t_i = 0$ by induction.

Since $fq$ commutes with $R$ and its coefficients all belong to $Q'_H(R)$, the arguments applied to $f$ in the last two paragraphs of Proposition 4.5 now apply directly to $fq$. Thus we can conclude that $fq \in Q_H(R) \# H$. However, since $fq$ is central in $Q_H(R) \# H$, we now have $fq \in Z$. Since $q = (fq)f^{-1}$ we have $q \in Q_{\mathfrak{q}}(Z)$.

In the final main result of this paper we show that if every non-zero $H$-stable ideal of $R$ contains a left regular element, then we can compute $Q(R \# H)$ without using infinite sums.

**Theorem 4.7.** If $R \# H$ is prime with $H$ commutative irreducible such that every non-zero $H$-stable ideal of $R$ contains a left regular element, then $Q(R \# H) = (Q_H(R) \# H)Z^{-1}$, where $Z$ is the center of $Q_H(R) \# H$.

**Proof.** Clearly $(Q_H(R) \# H)Z^{-1} \subseteq Q(R \# H)$, so it suffices to show the other inclusion. By Proposition 4.5, if $q \in Q(R \# H)$ then there exists an ideal $I \neq 0$ of $R \# H$, $J \in \mathcal{F}_H$, and $0 \neq f \in Z$ such that $Iq$, $qI \subseteq R \# H$ and $Jf \subseteq I$. Since $fq \in Q(R \# H)$, $J(fq) \subseteq R \# H$, and $(fq)J = q(Jf) \subseteq qI \subseteq R \# H$, it follows by Proposition 3.7 that $fq \in Q_H(R) \# H$. Since $q = (fq)f^{-1}$, we have $q \in (Q_H(R) \# Z)Z^{-1}$. 

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We now give an example where \( Q'(R \# H) \) can be computed without using infinite sums. In light of Theorem 4.3, it suffices to show that \( Q'(R) \neq \infty H \) can replaced by \( Q'_H(R) \neq H \).

**Example 4.8.** If \( R \) is prime Goldie then \( Q'_H(R) \neq \infty H = Q'_H(R) \neq H \). To see this, suppose \( w = \sum t_iu_i \in Q'_H(R) \neq \infty H \) and let \( J \in \mathcal{F}_H \) such that \( Jw \subseteq R \neq H \). If\( a \in J \) is regular in \( R \), then \( a \) remains regular in \( Q'(R) \) since \( Q'(R) \) is contained in the classical quotient ring of \( R \). But since \( aw \in R \neq H \) and \( at \neq 0 \) whenever \( t_i \neq 0 \), it follows that \( w \) is a finite sum and \( w \in Q'_H(R) \neq H \).

Unfortunately \( Q'(R \# H) \) and \( Q(R \# H) \) cannot, in general, be computed without using "infinite sums." We conclude this paper with three examples which show that infinite sums are often necessary.

In Example 4.9 we show that even if \( R \) is a domain then infinite sums are necessary in computing \( Q'(R \# H) \).

**Example 4.9.** Let \( R = k \langle x_1, x_2, \ldots, x_n, \ldots \rangle \) be the free algebra in an infinite number of variables over a field of characteristic 0. Let \( L = \langle y \rangle \) be the one-dimensional Lie algebra acting on \( R \) via the outer derivation \( y; x_i \mapsto x_{i+1} \), for all \( i \). Since \( C(R) = k \) (in fact \( Q(R) = R \), by a theorem of Kharchenko [P, 13.111]), we have \( C(R \# U(L)) = C(R)^L = k \). Thus by Theorem 4.3, \( Q'(R \# U(L)) = Q'_H(R) \neq \infty U(L) \). Now let \( J \) be the ideal of \( R \) of polynomials with zero constant term, and define \( t_i; J \to R \) by

\[
(r_1 x_1 + \cdots + r_i x_i + \cdots) t_i = r_{i+1} x_{i+1} + r_{i+2} x_{i+2} + \cdots.
\]

Thus for any \( a \in J \), there exists \( N \) such that \( at \neq 0 \) for all \( n \geq N \). Hence \( \sum_{i=1}^{\infty} t_iy \in Q'_H(R) \neq \infty U(L) \) and infinite sums are necessary.

In light of Theorem 4.7, this example cannot be extended to \( Q(R \# H) \). However, the next examples show that for prime rings in which not all non-zero \( H \)-stable ideals contain regular elements, infinite sums can be necessary in \( Q(R \# H) \). In fact, these examples show many things: in Corollary 3.6 the assumption that the \( X \)-inner automorphism stabilizes \( R \) is necessary; and in Proposition 3.7, Theorem 3.8, Lemma 3.9, Theorem 3.10, and Theorem 4.7 it is necessary to assume that every non-zero \( H \)-stable ideal of \( R \) contains a left regular element.

**Example 4.10.** Let \( k \) be a field of characteristic 0 and \( F \) a field extension of \( k \) with a \( k \)-linear derivation \( d \neq 0 \). Let \( R \) be the ring of infinite matrices over \( F \) generated by those matrices with a finite number of non-zero entries and \( k \cdot 1 \). Extend \( d \) to \( R \) entrywise. Then \( R \) is prime and \( d \) is outer.
Let \( L = \langle x \rangle \) be the one-dimensional Lie algebra over \( k \) acting on \( R \) via \( d \), and let \( H = U(L) \), noting that \( P(H) = L \). Now \( C(R) = F \) and \( C \otimes_k L \) is the 1-dimensional Lie algebra over \( F \). \( C \otimes_k L \) cannot become inner on \( Q(R) \), so by Theorem 2.2, \( R \# H \) has the bimodule property. It is easy to see that

\[
w = e_{12}x + e_{34}x^2 + e_{56}x^3 + \cdots + e_{2n-1,2n}x^n + \cdots \in Q(R \# H),
\]

by using \( J \) to be the ideal of \( R \) with a finite number of non-zero entries: \( Jw \subseteq R \# H \). However, \( w \in Q_H(R) \# H \), thus infinite sums are necessary in \( Q(R \# H) \). This shows that Theorems 3.8 and 4.7 cannot be improved. Furthermore, since \( w^2 = 0 \), \( 1 + w \) is invertible in \( Q(R \# H) \) and conjugation by \( 1 + w \) is an \( X \)-inner automorphism of \( R \# H \), hence Corollary 3.6 cannot be improved. Finally, since \( (1 + w)J = J(1 + w) \), it follows that \( wJ \subseteq R \# H \), thus Proposition 3.7 also cannot be improved.

Example 4.11. Let \( F \) be a field extension of \( k \) which has a \( k \)-automorphism \( g \) of infinite order and let \( R \) be as in Example 4.10. Then \( G = \langle g \rangle \) is \( X \)-outer on \( R \), but \( w = e_{12}g + e_{34}g^2 + e_{56}g^3 + \cdots + e_{2n-1,2n}g^n + \cdots \in Q(R \# kG) \), but \( w \notin Q_G(R) \# kG \). Thus Lemma 3.9 cannot be improved.

References


