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Journal of Symbolic Computation 40 (2005) 999–1012
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Operations with regular holonomic $\mathcal{D}$-modules with support a normal crossing

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Received 4 October 2004; accepted 7 March 2005
Available online 31 May 2005

Abstract

The aim of this work is to describe some operations in the category of regular holonomic $\mathcal{D}$-modules with support a normal crossing and variation zero introduced in [Àlvarez Montaner, J., García López, R., Zarzuela, S., 2003. Local cohomology, arrangements of subspaces and monomial ideals. Adv. Math. 174 (1), 35–56]. These operations will allow us to compute the characteristic cycle of the local cohomology supported on homogeneous prime ideals of these modules. In particular, we will be able to describe their Bass and dual Bass numbers.
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MSC: primary 32C38; secondary 16D40

Keywords: $\mathcal{D}$-modules; Bass numbers

1. Introduction

Let $X = \mathbb{C}^n, \mathcal{O}_X$ the sheaf of holomorphic functions in $\mathbb{C}^n$, and $\mathcal{D}_X$ the sheaf of differential operators in $\mathbb{C}^n$ with holomorphic coefficients. Galligo et al. (1985) described in terms of linear algebra the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ of regular holonomic $\mathcal{D}_X$-modules such that their solution complex $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ are perverse sheaves relatively to the stratification given by the union T of the coordinate hyperplanes in $\mathbb{C}^n$.

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doi:10.1016/j.jsc.2005.03.001

In Section 2 we recall the definition and the basic properties of the category $\mathcal{D}^T_{v=0}$ of modules with variation zero introduced in Àlvarez Montaner et al. (2003) (see also Àlvarez Montaner and Zarzuela, 2003). Moreover, we define the category of modules with unipotent monodromy that is also a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$. This category is closed under extensions and includes $\mathcal{D}^T_{v=0}$.

In Section 3 we describe some operations in the category $\mathcal{D}^T_{v=0}$. We have to point out that we will consider the case of $R = k[x_1, \ldots, x_n]$ being the polynomial ring in n independent variables over any field k of characteristic zero and $\mathcal{D}$ being the ring of differential operators over R. We can consider this case due to the good behavior of this category with respect to flat base change (see Àlvarez Montaner and Zarzuela, 2003).

First, we describe the restriction to a homogeneous prime ideal of a module with variation zero. However, the main result of this section is a description of the kernel, the cokernel and the image of the homomorphism $\lambda_i : M \longrightarrow M[\frac{1}{x_i}]$ of localization of a module with variation zero M by the variable x_i.

In Section 4, by using the results of the previous section and Brodmann's exact sequence, we give an algorithm that allows us to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\alpha}(M)$, where $\mathfrak{p}_\alpha \subseteq R$ is an homogeneous prime ideal. In particular, we give a different approach to the computation of the Bass numbers $\mu_p(\mathfrak{p}_\alpha, M) := \dim_{k(\mathfrak{p}_\alpha)} \mathrm{Ext}^p_{R_{\mathfrak{p}_\alpha}}(k(\mathfrak{p}_\alpha), M_{\mathfrak{p}_\alpha})$ given in Yanagawa (2001).

Finally, in Section 5 we define Matlis duality in the category $\mathcal{D}^T_{v=0}$. This duality theory is nothing but a duality in the lattice $\{0,1\}^n$. The Matlis dual of an injective $\mathcal{D}^T_{v=0}$-module is projective so, by using the results of the previous section, we describe projective resolutions in $\mathcal{D}^T_{v=0}$.

In the sequel we will denote $\mathbf{1} = \varepsilon_1 + \cdots + \varepsilon_n$ where $\varepsilon_1, \ldots, \varepsilon_n$ is the natural basis of $\mathbb{Z}^n$. For all $\alpha \in \{0,1\}^n$, X_α will be the linear subvariety of X defined by the homogeneous prime ideal $\mathfrak{p}_\alpha =< x_i \mid \alpha_i = 1 >$. For unexplained terminology on the theory of algebraic $\mathcal{D}$-modules we refer to Björk (1979) and Coutinho (1995).

2. Preliminaries

Let $\mathcal{C}_n$ be the category whose objects are families $\{\mathcal{M}_\alpha\}_{\alpha\in\{0,1\}^n}$ of finitely dimensional complex vector spaces, endowed with linear maps

$$\mathcal{M}_\alpha \xrightarrow{u_i} \mathcal{M}_{\alpha+\varepsilon_i}, \qquad \mathcal{M}_\alpha \xleftarrow{v_i} \mathcal{M}_{\alpha+\varepsilon_i}$$

for each $\alpha \in \{0,1\}^n$ such that $\alpha_i = 0$. These maps are called canonical (resp., variation) maps, and they are required to satisfy the conditions:

$$u_i u_j = u_j u_i, \quad v_i v_j = v_j v_i, \quad u_i v_j = v_j u_i \ \text{ and } \ v_i u_i + id \ \text{ is invertible.}$$

Such an object will be called an n-hypercube. A morphism between two n-hypercubes $\{\mathcal{M}_\alpha\}_\alpha$ and $\{\mathcal{N}_\alpha\}_\alpha$ is a set of linear maps $\{f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha\}_\alpha$, commuting with the canonical and variation maps. In Galligo et al. (1985), an equivalence of categories between $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ and $\mathcal{C}_n$ is described.

The functor $\mathrm{Mod}(\mathcal{D}_X)^T_{hr} \longrightarrow \mathcal{C}_n$ is a contravariant exact functor. The construction of the n-hypercube corresponding to an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ is given in Galligo et al. (1985). From the construction one can describe the composition $v_i \circ u_i$ in terms of the partial monodromy around the hyperplane $x_i = 0$. We also want to point out that the n-hypercube describes the characteristic cycle of the corresponding $\mathcal{D}_X$-module $\mathcal{M}$. Namely, let $CC(\mathcal{M}) = \sum m_\alpha \, T^*_{X_\alpha}\mathbb{C}^n$ be the characteristic cycle of $\mathcal{M}$. Then, one has the equality $\dim_{\mathbb{C}}\mathcal{M}_\alpha = m_\alpha$.

The papers Àlvarez Montaner et al. (2003) and Àlvarez Montaner and Zarzuela (2003) study objects in the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ having the following property:

Definition 2.1. We say that an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has variation zero if the morphisms v_i are zero for all $1 \leq i \leq n$ and all $\alpha \in \{0,1\}^n$ with $\alpha_i = 0$.

Modules with variation zero form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ that will be denoted $\mathcal{D}^T_{v=0}$.

The simple objects of $\mathcal{D}^T_{v=0}$ are of the form:

$$\frac{\mathcal{D}_X}{\mathcal{D}_X(\{x_i \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}.$$

This module is isomorphic to the local cohomology module $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$.

Every holonomic module has finite length, so if $\mathcal{M} \in \mathcal{D}^T_{v=0}$, then there exists a finite increasing filtration $\{\mathcal{F}_j\}_{j\geq 0}$ of $\mathcal{M}$ by objects of $\mathcal{D}^T_{v=0}$ such that for all $j \geq 1$ one has $\mathcal{D}_X$-module isomorphisms

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X), \quad \alpha \in \{0,1\}^n.$$

The category $\mathcal{D}^T_{v=0}$, regarded as a subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$, is not closed under extensions. However, its objects can be characterized by the following particular filtration:

Proposition 2.2 (*Àlvarez Montaner et al., 2003*). *An object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has variation zero if and only if there is a increasing filtration $\{\mathcal{F}_j\}_{0\leq j\leq n}$ of $\mathcal{M}$ by objects of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ and there are integers $m_\alpha \geq 0$ for $\alpha \in \{0,1\}^n$ such that for all $1 \leq j \leq n$ one has $\mathcal{D}$-module isomorphisms*

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \bigoplus_{|\alpha|=j} (\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X))^{\oplus m_\alpha}.$$

2.1. Closing the category $\mathcal{D}^T_{v=0}$ for extensions

In this section we will find the minimal full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ containing $\mathcal{D}^T_{v=0}$ that is closed under extensions.

Definition 2.3. We say that an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has m-trivial monodromy if the composition of morphisms $(v_i \circ u_i)^m$ is zero for all $1 \leq i \leq n$ and all $\alpha \in \{0,1\}^n$ with $\alpha_i = 0$.

Modules with m-trivial monodromy form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ that will be denoted $\mathcal{D}^T_{(vu)^m=0}$. Notice that we have

$$\mathcal{D}^T_{v=0} \subseteq \mathcal{D}^T_{vu=0} \subseteq \mathcal{D}^T_{(vu)^2=0} \subseteq \cdots \subseteq \mathcal{D}^T_{(vu)^m=0} \subseteq \cdots .$$

In particular, we get an increasing filtration of the following category:

Definition 2.4. We say that an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has unipotent monodromy if the composition of morphisms $v_i \circ u_i$ is nilpotent for all $1 \leq i \leq n$ and all $\alpha \in \{0,1\}^n$ with $\alpha_i = 0$.

Modules with unipotent monodromy form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ that will be denoted $\mathcal{D}^T_{uni}$.

It is easy to see that the simple objects of the category $\mathcal{D}^T_{uni}$ are isomorphic to the local cohomology modules $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$, for $\alpha \in \{0,1\}^n$. The categories $\mathcal{D}^T_{(vu)^m=0}$, regarded as subcategories of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$, are not closed under extensions for all m, but $\mathcal{D}^T_{uni}$ is.

Proposition 2.5. *The category $\mathcal{D}^T_{uni}$, regarded as a subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$, is closed under extensions.*

Proof. Let $0 \longrightarrow \mathcal{M}' \xrightarrow{i} \mathcal{M} \xrightarrow{\pi} \mathcal{M}'' \longrightarrow 0$ be an exact sequence in $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ such that $\mathcal{M}', \mathcal{M}'' \in \mathcal{D}^T_{uni}$. Consider, for all $\alpha \in \{0,1\}^n$ such that $\alpha_i = 0$, the commutative diagram in the category $\mathcal{C}_n$ of n-hypercubes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & u''_i \uparrow & & u_i \uparrow & & u'_i \uparrow & & \\
0 & \longrightarrow & \mathcal{M}''_\alpha & \xrightarrow{\pi_\alpha} & \mathcal{M}_\alpha & \xrightarrow{i_\alpha} & \mathcal{M}'_\alpha & \longrightarrow & 0 \\
 & & v''_i \uparrow & & v_i \uparrow & & v'_i \uparrow & & \\
0 & \longrightarrow & \mathcal{M}''_{\alpha+\varepsilon_i} & \xrightarrow{\pi_{\alpha+\varepsilon_i}} & \mathcal{M}_{\alpha+\varepsilon_i} & \xrightarrow{i_{\alpha+\varepsilon_i}} & \mathcal{M}'_{\alpha+\varepsilon_i} & \longrightarrow & 0 \\
 & & u''_i \uparrow & & u_i \uparrow & & u'_i \uparrow & & \\
0 & \longrightarrow & \mathcal{M}''_\alpha & \xrightarrow{\pi_\alpha} & \mathcal{M}_\alpha & \xrightarrow{i_\alpha} & \mathcal{M}'_\alpha & \longrightarrow & 0
\end{array}$$

where $(v'_i \circ u'_i)^{m'} = 0$ and $(v''_i \circ u''_i)^{m''} = 0$. It is easy to check that for $m \gg \max\{m'', m'\}$, $(v_i \circ u_i)^m = 0$ so $\mathcal{M} \in \mathcal{D}^T_{uni}$. $\square$

Since every holonomic module has finite length, it follows that the objects of $\mathcal{D}^T_{uni}$ are characterized as follows:

Proposition 2.6. *An object $\mathcal{M}$ of $\mathcal{D}^T_{hr}$ has unipotent monodromy if and only if there is a finite increasing filtration $\{\mathcal{F}_j\}_{j\geq 0}$ of $\mathcal{M}$ by objects of $\mathcal{D}^T_{hr}$ such that for all $j \geq 1$ one has $\mathcal{D}$-module isomorphisms*

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X), \quad \alpha \in \{0,1\}^n.$$

However, notice that the modules $\mathcal{M} \in \mathcal{D}^T_{uni}$ are not characterized by a filtration given by the height (as in Proposition 2.2), unless they are modules with variation zero, i.e., we cannot give a filtration $\{\mathcal{F}_j\}_{j\geq 0}$ of $\mathcal{M}$ where the submodules $\mathcal{F}_j$ correspond to the n-hypercubes:

$$(\mathcal{F}_j)_\beta = \begin{cases} \mathcal{M}_\beta & \text{if } \mid \beta \mid \leq j \\ 0 & \text{otherwise,} \end{cases}$$

the canonical and variation maps being either zero or equal to those in $\mathcal{M}$.

3. Operations in the category $\mathcal{D}^T_{v=0}$

By using flat base change, we can define the category $\mathcal{D}^T_{v=0}$ of modules with variation zero, as well the corresponding category of n-hypercubes, for the case of $\mathcal{D}$ being the ring of differential operators over R, where R is the polynomial or the formal power series ring in n independent variables, $x_1, \ldots, x_n$, over any field k of characteristic zero (see Àlvarez Montaner and Zarzuela, 2003, Remark 4.1). From now on, we will only consider the case $R = k[x_1, \ldots, x_n]$ in order to take advantage of the $\mathbb{Z}^n$-graded structure of these modules given by the equivalence of categories (see Àlvarez Montaner et al., 2003), between the category of modules with variation zero and the category of straight modules introduced by K. Yanagawa (Yanagawa, 2001).

3.1. Restriction to a face ideal

Let $\mathbb{Z}^\alpha \subseteq \mathbb{Z}^n$ be the coordinate space spanned by $\{\varepsilon_i \mid \alpha_i = 1\}$, $\alpha \in \{0,1\}^n$. The restriction of R to the homogeneous prime ideal $\mathfrak{p}_\alpha \subseteq R$ is the $\mathbb{Z}^\alpha$-graded k-subalgebra of R

$$R_{[\mathfrak{p}_\alpha]} := k[x_i \mid \alpha_i = 1].$$

The restriction to $\mathfrak{p}_\alpha$ of a $\mathbb{Z}^n$-graded module M is the $R_{[\mathfrak{p}_\alpha]}$-module

$$M_{[\mathfrak{p}_\alpha]} := \bigoplus_{\beta \in \mathbb{Z}^\alpha} M_\beta.$$

Restriction gives us a functor that plays in some cases the role of the localization functor. For details of the description of the morphisms and further considerations we refer to Miller (2000).

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. Then, the restriction of I to the face ideal $\mathfrak{p}_\alpha$ is the squarefree monomial ideal $I_{[\mathfrak{p}_\alpha]} \subseteq R_{[\mathfrak{p}_\alpha]}$ whose face ideals in the minimal primary decomposition are those face ideals I_{α_j} contained in $\mathfrak{p}_\alpha$. Namely

$$I_{[\mathfrak{p}_\alpha]} = \bigcap_{\alpha_j \leq \alpha} I_{\alpha_j}.$$

Notice that the restriction to $\mathfrak{p}_\alpha$ of the local cohomology module $H^{|\beta|}_{\mathfrak{p}_\beta}(R)$ supported on a face ideal $\mathfrak{p}_\beta \subseteq R$ vanishes if and only if $\mathfrak{p}_\beta \nsubseteq \mathfrak{p}_\alpha$.

The restriction to $\mathfrak{p}_\alpha$ of a module $M \in \mathcal{D}^T_{v=0}$ is again a module with variation zero. This can be seen considering the restriction of a fixed increasing filtration of M, $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ given by Proposition 2.2.

Proposition 3.1. (i) *The vertices of the $|\alpha|$-hypercube corresponding to $M_{[\mathfrak{p}_\alpha]}$ are the vector spaces $(\mathcal{M}_{[\mathfrak{p}_\alpha]})_\gamma = \mathcal{M}_\gamma$ for $\gamma \leq \alpha$.*
(ii) *The map $u_j : (\mathcal{M}_{[\mathfrak{p}_\alpha]})_\gamma \to (\mathcal{M}_{[\mathfrak{p}_\alpha]})_{\gamma+\varepsilon_j}$ is the same map as $u_j : \mathcal{M}_\gamma \to \mathcal{M}_{\gamma+\varepsilon_j}$.*

Proof. Let $CC(M) = \sum m_\gamma \, T^*_{X_\gamma} X$ be the characteristic cycle of M. Then, the characteristic cycle of $M_{[\mathfrak{p}_\alpha]}$ is

$$CC(M_{[\mathfrak{p}_\alpha]}) = \sum_{\gamma \leq \alpha} m_\gamma \, T^*_{X_\gamma} X$$

so we get the vertices of the n-hypercube.

In order to get the linear maps u_i's we only have to point out that the filtration of the module $M_{[\mathfrak{p}_\alpha]}$ is determined by the filtration of M; in particular, they have the same extension problems. Then we are done by Àlvarez Montaner et al. (2003, Section 3) and Àlvarez Montaner and Zarzuela (2003, Theorem 4.1). □

3.2. Localization by a variable

Let M be a module in $\mathcal{D}^T_{v=0}$ and $x_i \in R$ a variable. We will describe the n-hypercube of the localization $M[\frac{1}{x_i}]$. First of all, it is worthwhile pointing out that localization by a variable is an exact functor in the category of modules with variation zero due to the fact that $R[\frac{1}{x_i}]$ is a flat module in $\mathcal{D}^T_{v=0}$. It also follows that localization commutes with restriction to homogeneous prime ideals.

Proposition 3.2. (i) *The vertices of the n-hypercube corresponding to $M[\frac{1}{x_i}]$ are the vector spaces $\mathcal{M}[\frac{1}{x_i}]_\gamma = \mathcal{M}_\gamma$ if $\gamma_i = 0$. In this case we also have $\mathcal{M}[\frac{1}{x_i}]_{\gamma+\varepsilon_i} = \mathcal{M}_\gamma$.*
(ii) *The map $u_j : \mathcal{M}[\frac{1}{x_i}]_\gamma \to \mathcal{M}[\frac{1}{x_i}]_{\gamma+\varepsilon_j}$, where $\gamma_i = 0$, is:*
- *Id if $j = i$*
- *$u_j : \mathcal{M}_\gamma \to \mathcal{M}_{\gamma+\varepsilon_j}$ if $j \neq i$. In this case, this map is also equal to*

$$u_j : \mathcal{M}\left[\frac{1}{x_i}\right]_{\gamma+\varepsilon_i} \to \mathcal{M}\left[\frac{1}{x_i}\right]_{\gamma+\varepsilon_i+\varepsilon_j}.$$

Proof. The vertices of the n-hypercube corresponding to $M[\frac{1}{x_i}]$ can be easily described by means of a formula given in Briançon et al. (1994). Namely, let $CC(M) = \sum m_\alpha \, T^*_{X_\alpha} X$ be the characteristic cycle of M. Then, the characteristic cycle of $M[\frac{1}{x_i}]$ is

$$CC\left(M\left[\frac{1}{x_i}\right]\right) = \sum_{\alpha_i = 0} m_\alpha \, (T^*_{X_\alpha} X + T^*_{X_{\alpha+\varepsilon_i}} X).$$

In order to get the linear maps u_i's we will use induction on the length l of M. Let $l = 1$, i.e., $M = H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)$ is a local cohomology module. If $\alpha_i \neq 0$ then $M[\frac{1}{x_i}] = 0$. If

$\alpha_i = 0$, we just have to prove that the map $u_i : \mathcal{M}[\frac{1}{x_i}]_\alpha \to \mathcal{M}[\frac{1}{x_i}]_{\alpha+\varepsilon_i}$ is the identity. For simplicity we will use the restriction to $\mathfrak{p}_{\alpha+\varepsilon_i}$. Then localizing by x_i the minimal graded injective resolution:

$$0 \longrightarrow H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \longrightarrow {}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1}) \longrightarrow {}^*E(R/\mathfrak{p}_{\alpha+\varepsilon_i})(\mathbf{1}) \longrightarrow 0$$

we get an isomorphism $H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)[\frac{1}{x_i}] \cong {}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1})$, so we get the desired result by the description of injective modules given in the proof of Àlvarez Montaner et al. (2003, Theorem 4.3). Notice that the same argument can be used to describe the n-hypercube of the localization $H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)[\frac{1}{x^\beta}]$ of a local cohomology module by any squarefree monomial $x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$, $\beta \in \{0,1\}^n$.

The case $l = 2$ is proved as well since a module of length two has to be, for convenient $\alpha, \beta \in \{0,1\}^n$ and $j \in \{1, \ldots, n\}$, isomorphic to one of the following modules

$$H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \oplus H^{|\alpha|}_{\mathfrak{p}_\alpha}(R), \quad H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \oplus H^{|\beta|}_{\mathfrak{p}_\beta}(R), \quad H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)\left[\frac{1}{x_j}\right]$$

that correspond to the n-hypercubes with non-vanishing part

$$\begin{array}{ccccc} k^2 & , & k & , & k \\ & & & & \downarrow 1 \\ & & k & & k \end{array}$$

If $l > 2$ we consider the submodule $M' \subseteq M$ whose corresponding n-hypercube only has the vertices and linear map we want to study. Namely, $u_j : \mathcal{M}_\gamma \longrightarrow \mathcal{M}_{\gamma+\varepsilon_j}$, where $\gamma_i = 0$. If the n-hypercube of M has another vertex different from zero then length $M' <$ length M so we are done by induction. If the n-hypercube of M only has the vertices $\mathcal{M}_\gamma$ and $\mathcal{M}_{\gamma+\varepsilon_j}$, i.e., $M' = M$, we can give a precise description of this module in terms of the rank of u_j. Namely, let $m = \mathrm{rk}(u_j)$, $m_\gamma = \dim_k \mathcal{M}_\gamma$ and $m_{\gamma+\varepsilon_j} = \dim_k \mathcal{M}_{\gamma+\varepsilon_j}$. Then, M is the direct sum of

- m copies of $H^{|\gamma|}_{\mathfrak{p}_\gamma}(R)[\frac{1}{x_j}]$,
- $m_\gamma - m$ copies of $H^{|\gamma|}_{\mathfrak{p}_\gamma}(R)$,
- $m_{\gamma+\varepsilon_j} - m$ copies of $H^{|\gamma+\varepsilon_j|}_{\mathfrak{p}_{\gamma+\varepsilon_j}}(R)$.

So we are done since we can reduce to the cases of length $l = 1$ and $l = 2$. □

3.3. *Image, kernel and cokernel of the localization by a variable*

Let M be a module in $\mathcal{D}^T_{v=0}$ and $x_i \in R$ a variable. We will describe the n-hypercubes of the image, kernel and cokernel of the morphism $\lambda_i : M \longrightarrow M[\frac{1}{x_i}]$ of localization of M by the variable x_i.

Proposition 3.3. (i) *The vertices of the n-hypercube corresponding to* $\operatorname{Im}\lambda_i$ *are computed from the characteristic cycle*

$$CC(\operatorname{Im}\lambda_i) = \sum_{\alpha_i=0} (m_\alpha \;\; T^*_{X_\alpha} X + \operatorname{rk}(u_i) \;\; T^*_{X_{\alpha+\varepsilon_i}} X).$$

(ii) *The maps* $u_j : (\mathcal{I}m\,\lambda_i)_\gamma \to (\mathcal{I}m\,\lambda_i)_{\gamma+\varepsilon_j}$ *are the same as the corresponding for the module M.*

Proof. We will use induction on the length l of M. Let $l = 1$, i.e., $M = H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)$ is a local cohomology module. If $\alpha_i \neq 0$ then $\operatorname{Im}\lambda_i = 0$. If $\alpha_i = 0$ we have the exact sequence:

$$0 \longrightarrow H^0_{(x_i)}(M) \longrightarrow M \longrightarrow M[\tfrac{1}{x_i}] \longrightarrow H^1_{(x_i)}(M) \longrightarrow 0\ .$$

Notice that $H^0_{(x_i)}(M) = 0$ and $H^1_{(x_i)}(M) = H^{|\alpha+\varepsilon_i|}_{\mathfrak{p}_{\alpha+\varepsilon_i}}(R)$ so we are done. The case $l = 2$ is easy to compute using the description of a module of length two.

If $l > 2$, we consider the submodule $M' \subseteq M$ whose corresponding n-hypercube only has the vertices and linear map we want to study. Namely, $u_j : \mathcal{M}_\gamma \longrightarrow \mathcal{M}_{\gamma+\varepsilon_j}$, where $\gamma_i = 0$. It follows as in the proof of Proposition 3.2. □

Once the n-hypercube for $\operatorname{Im}\lambda_i$ is determined we can easily compute the n-hypercube for $\operatorname{Ker}\lambda_i$ and $\operatorname{Coker}\lambda_i$.

Proposition 3.4. (i) *The vertices of the n-hypercube corresponding to* $\operatorname{Ker}\lambda_i$ *are computed from the characteristic cycle*

$$CC(\operatorname{Ker}\lambda_i) = CC(M) - CC(\operatorname{Im}\lambda_i).$$

(ii) *The maps* $u_j : (\mathcal{K}er\,\lambda_i)_\gamma \to (\mathcal{K}er\,\lambda_i)_{\gamma+\varepsilon_j}$ *are the same as the corresponding for the module M.*

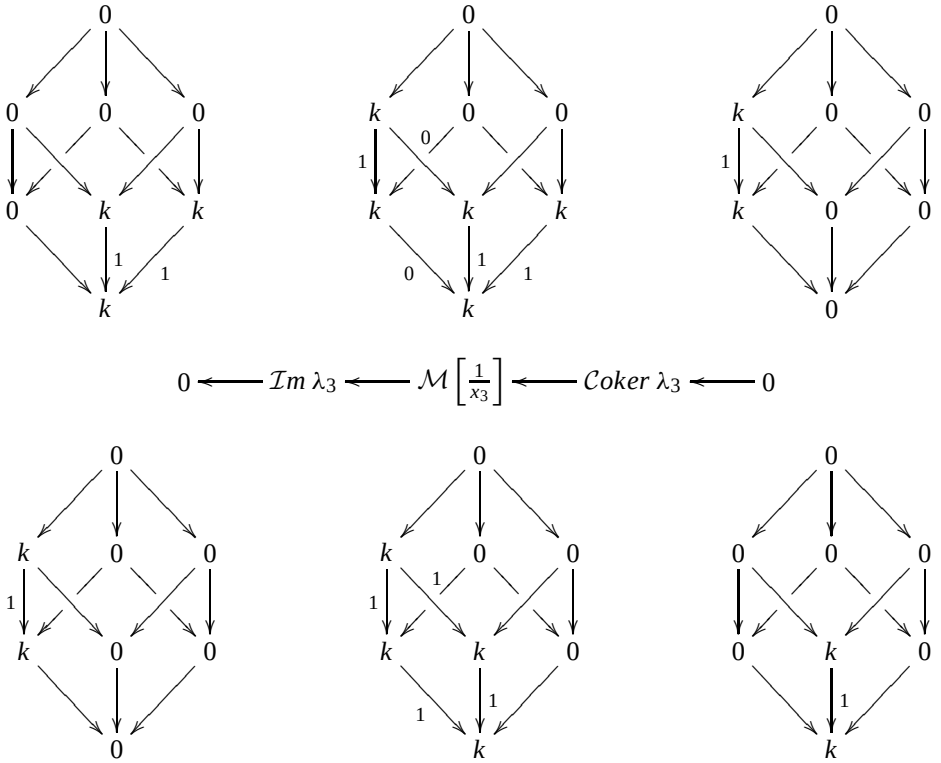
Proposition 3.5. (i) *The vertices of the n-hypercube corresponding to* $\operatorname{Coker}\lambda_i$ *are computed from the characteristic cycle*

$$CC(\operatorname{Coker}\lambda_i) = CC(M[\frac{1}{x_i}]) - CC(\operatorname{Im}\lambda_i).$$

(ii) *The maps* $u_j : (\mathcal{C}oker\,\lambda_i)_\gamma \to (\mathcal{C}oker\,\lambda_i)_{\gamma+\varepsilon_j}$ *are the same as the corresponding for the module* $M[\frac{1}{x_i}]$.

Example. Let $R = k[x_1, x_2, x_3]$. Given the 3-hypercube of a module $M \in \mathcal{D}^T_{v=0}$ (see below), consider the morphism $\lambda_3 : M \longrightarrow M[\frac{1}{x_3}]$. Then we have the following exact sequences of 3-hypercubes:

$$0 \longleftarrow \mathcal{K}er\,\lambda_3 \longleftarrow \mathcal{M} \longleftarrow \mathcal{I}m\,\lambda_3 \longleftarrow 0$$



It is not difficult to check out that:

- $M \cong H^1_{(x_1)}(R)[\frac{1}{x_2}] \oplus H^2_{(x_1x_2,x_3)}(R)$
- $M[\frac{1}{x_3}] \cong H^1_{(x_1)}(R)[\frac{1}{x_2x_3}] \cong {}^*E(R/(x_1))(\mathbf{1})$
- $\mathrm{Ker}\lambda_3 \cong H^2_{(x_1x_2,x_3)}(R)$
- $\mathrm{Im}\,\lambda_3 \cong H^1_{(x_1)}(R)[\frac{1}{x_2}]$
- $\mathrm{Coker}\lambda_3 \cong H^2_{(x_1,x_3)}(R)[\frac{1}{x_2}] \cong {}^*E(R/(x_1,x_3))(\mathbf{1})$

4. Bass numbers of modules with variation zero

It is easy to prove that the injective objects of $\mathcal{D}^T_{v=0}$ are of the form:

$$\frac{\mathcal{D}}{\mathcal{D}(\{x_i \mid \alpha_i = 1\}, \{x_j\partial_j + 1 \mid \alpha_j = 0\})}, \quad \alpha \in \{0,1\}^n.$$

These modules are isomorphic to the shifted graded injective hulls ${}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1})$ of the quotients $R/\mathfrak{p}_\alpha$. In particular, the minimal injective resolution of a module with variation zero M is in the form:

$$\mathbb{I}^\bullet(M): \quad 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} I^m \xrightarrow{d^m} \cdots,$$

where the j-th term is

$$I^j = \bigoplus_{\alpha\in\{0,1\}^n} {}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1})^{\mu_{\mathbf{j}}(\mathfrak{p}_\alpha,\mathbf{M})}.$$

The Bass numbers of M with respect to the face ideal $\mathfrak{p}_\alpha \subseteq R$ are the invariants defined by $\mu_j(\mathfrak{p}_\alpha, M)$.

In general, the Bass numbers $\mu_p(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})}\mathrm{Ext}^p_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$ with respect to any prime ideal $\mathfrak{p} \subseteq R$ can be described as the multiplicities of the characteristic cycle of $H^p_\mathfrak{p}(M)$. Namely, by using the same arguments as in Àlvarez Montaner (2004, Proposition 2.1) we have:

Proposition 4.1. *Let $\mathfrak{p} \subseteq R$ be a prime ideal and*

$$CC(H^p_\mathfrak{p}(M)) = \sum \lambda_{\mathfrak{p},p,\alpha} \quad T^*_{X_\alpha}X$$

be the characteristic cycle of the local cohomology module $H^p_\mathfrak{p}(M)$. Then, the Bass numbers with respect to $\mathfrak{p}$ of M are

$$\mu_p(\mathfrak{p}, M) = \lambda_{\mathfrak{p},p,\alpha_\mathfrak{p}},$$

where $X_{\alpha_\mathfrak{p}}$ is the subvariety of $X = \mathrm{Spec}\,(R)$ defined by $\mathfrak{p}$.

Our aim in this section is to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\alpha}(M)$, where $\mathfrak{p}_\alpha \in R$ is a face ideal. In particular, we give a different approach to the computation of the Bass numbers of these modules given in Yanagawa (2001). We have to point out that, by means of Goto and Watanabe (1978, Theorem 1.2.3), one may compute the Bass numbers of M with respect to any prime ideal.

To this purpose we will use the Brodmann sequence

$$\cdots \longrightarrow H^p_{\mathfrak{p}_{\alpha+\varepsilon_i}}(M) \longrightarrow H^p_{\mathfrak{p}_\alpha}(M) \longrightarrow H^p_{\mathfrak{p}_\alpha}(M)\left[\frac{1}{x_i}\right] \longrightarrow H^{p+1}_{\mathfrak{p}_{\alpha+\varepsilon_i}}(M) \longrightarrow \cdots$$

in an iterated way. By using the additivity of the characteristic cycle with respect to short exact sequences it will be enough to compute the characteristic cycle of the kernel and cokernel of the localization morphism. This will be done by using the description given in Propositions 3.4 and 3.5.

We present the following:

Algorithm.

INPUT: A module with variation zero $M \in \mathcal{D}^T_{v=0}$ and the face ideal $\mathfrak{p}_\alpha \subseteq R$.
Denote $\mathfrak{p}_{\alpha_k} := (x_i \mid \alpha_i = 1, \ i \leq k)$.

OUTPUT: The characteristic cycle of $H^p_{\mathfrak{p}_\alpha}(M)\ \forall p$.

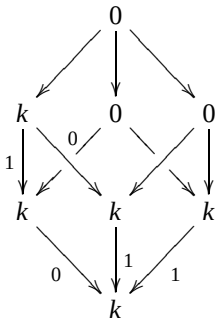
- For $i = 1, \ldots, n$, while $\alpha_i = 1$:
  - Consider the Brodmann exact sequence:

$$\cdots \longrightarrow H^p_{\mathfrak{p}_{\alpha_i}}(M) \longrightarrow H^p_{\mathfrak{p}_{\alpha_{i-1}}}(M) \xrightarrow{\lambda_{i,p}} H^p_{\mathfrak{p}_{\alpha_{i-1}}}(M)[\tfrac{1}{x_i}] \longrightarrow H^{p+1}_{\mathfrak{p}_{\alpha_i}}(M) \longrightarrow \cdots$$

• Compute the characteristic cycle of $\operatorname{Ker}\lambda_{i,p}$ and $\operatorname{Coker}\lambda_{i,p}$, $\forall p$ by using Propositions 3.4 and 3.5.

• $CC(H^p_{\mathfrak{p}_{\alpha_i}}(M)) = CC(\operatorname{Ker}\lambda_{i,p}) + CC(\operatorname{Coker}\lambda_{i,p-1})$, $\forall p$.

Example. Let $R = k[x_1, x_2, x_3]$. Consider the 3-hypercube



of the module $M \in \mathcal{D}^T_{v=0}$ studied in the previous example. Then:

• For $i = 1, 2, 3$, consider the Brodmann sequences

$$0 \longrightarrow H^0_{(x_i)}(M) \longrightarrow M \xrightarrow{\lambda_{i,0}} M[\tfrac{1}{x_i}] \longrightarrow H^1_{(x_i)}(M) \longrightarrow 0\ .$$

By using Propositions 3.4 and 3.5 we get:

$$CC(H^0_{(x_1)}(M)) = T^*_{X_{(1,0,0)}}X + T^*_{X_{(1,1,0)}}X + T^*_{X_{(1,0,1)}}X.$$

$$CC(H^0_{(x_2)}(M)) = T^*_{X_{(0,1,1)}}X.$$

$$CC(H^0_{(x_3)}(M)) = T^*_{X_{(1,0,1)}}X + T^*_{X_{(0,1,1)}}X + T^*_{X_{(1,1,1)}}X.$$

$$CC(H^1_{(x_3)}(M)) = T^*_{X_{(1,0,1)}}X + T^*_{X_{(1,1,1)}}X.$$

• For $1 \le i < j \le 3$, consider the Brodmann sequences

$$0 \longrightarrow H^0_{(x_i,x_j)}(M) \longrightarrow H^0_{(x_i)}(M) \xrightarrow{\lambda_{j,0}} H^0_{(x_i)}(M)[\tfrac{1}{x_j}] \longrightarrow H^1_{(x_i,x_j)}(M) \longrightarrow 0\ .$$

By using Propositions 3.4 and 3.5 we get:

$$CC(H^0_{(x_1,x_2)}(M)) = T^*_{X_{(1,1,1)}}X.$$

$$CC(H^0_{(x_1,x_3)}(M)) = T^*_{X_{(1,0,1)}}X.$$

$$CC(H^1_{(x_1,x_3)}(M)) = T^*_{X_{(1,0,1)}}X + T^*_{X_{(1,1,1)}}X.$$

$$CC(H^0_{(x_2,x_3)}(M)) = T^*_{X_{(0,1,1)}}X.$$

• Consider the Brodmann sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{(x_1,x_2)}(M) \xrightarrow{\lambda_{3,0}} H^0_{(x_1,x_2)}(M)[\tfrac{1}{x_3}] \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0\ .$$

By using Propositions 3.4 and 3.5 we get:

$$CC(H^1_{(x_1,x_2,x_3)}(M)) = T^*_{X_{(1,1,1)}}X.$$

In particular, the Bass numbers of M are:

$\mathfrak{p}_\gamma$	μ_0	μ_1	μ_2
(x_1)	1	–	–
(x_2)	–	–	–
(x_3)	–	–	–
(x_1, x_2)	–	–	–
(x_1, x_3)	1	1	–
(x_2, x_3)	1	–	–
(x_1, x_2, x_3)	–	1	–

5. Dual Bass numbers of modules with variation zero

It is easy to prove that the projective objects of $\mathcal{D}^T_{v=0}$ are of the form:

$$\frac{\mathcal{D}}{\mathcal{D}(\{x_i\partial_i + 1 \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}, \quad \alpha \in \{0,1\}^n.$$

This modules are isomorphic to the localizations $R[\frac{1}{\mathbf{x}^\alpha}]$. In particular, the minimal projective resolution of a module with variation zero M is in the form:

$$\mathbb{P}^\bullet(\mathcal{M}): \quad \cdots \xrightarrow{d^m} P^m \xrightarrow{d^{m-1}} \cdots \xrightarrow{d^1} P^1 \xrightarrow{d^0} P^0 \longrightarrow 0\ ,$$

where the j-th term is

$$P^j = \bigoplus_{\alpha\in\{0,1\}^n} R\left[\frac{1}{\mathbf{x}^\alpha}\right]^{\pi_j(\mathfrak{p}_\alpha, M)}.$$

The dual Bass numbers of M with respect to the face ideal $\mathfrak{p}_\alpha \subseteq R$ are the invariants defined by $\pi_j(\mathfrak{p}_\alpha, M)$.

5.1. Matlis duality

Recall that for $\mathbb{Z}^n$-graded modules, the Matlis dual of a $\mathbb{Z}^n$-graded R-module M is defined as:

$$M^* := {}^*\mathrm{Hom}_R(M, {}^*E(R/\mathfrak{m})).$$

Notice that the Matlis dual defines a duality of the type $\alpha \to -\alpha$ among its graded pieces. In order to find a duality theory for the n-hypercubes we should look for a duality in the lattice $\{0,1\}^n$. More precisely, a duality of the type $\alpha \to \mathbf{1} - \alpha$.

In the category of module with variation zero we can mimic the definition of Matlis duality because of the equivalence of categories given in Àlvarez Montaner et al. (2003, Theorem 4.3).

Definition 5.1. Let $M \in \mathcal{D}^T_{v=0}$ be a module with variation zero. We define the Matlis dual of M as:

$$M^* := \mathrm{Hom}_{\mathcal{D}}\left(M, \frac{\mathcal{D}}{\mathcal{D}(x_1, \ldots, x_n)}\right).$$

Nevertheless we can prove that this definition gives in fact the notion of duality in the lattice $\{0,1\}^n$.

Proposition 5.2. (i) *The vertices of the n-hypercube corresponding to the module with variation zero M^* are the vector spaces*

$$\mathcal{M}^*_\alpha = \mathcal{M}_{\mathbf{1}-\alpha}$$

(ii) *The maps $u_i : \mathcal{M}^*_\gamma \longrightarrow \mathcal{M}^*_{\gamma+\varepsilon_i}$ is the dual of $u_i : \mathcal{M}_{\mathbf{1}-\gamma-\varepsilon_i} \longrightarrow \mathcal{M}_{\mathbf{1}-\gamma}$.*

Proof. We only have to notice that, by using Àlvarez Montaner et al. (2003, Lemma 4.4), we have:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}\left(M, \frac{\mathcal{D}}{\mathcal{D}(x_1, \ldots, x_n)}\right) &= \mathrm{Hom}_{\mathcal{D}}(M, E_{\mathbf{1}}) = {}^*\mathrm{Hom}_R(M, {}^*E(R/\mathfrak{m})(\mathbf{1})) \\ &= {}^*\mathrm{Hom}_R(M, {}^*E(R/\mathfrak{m}))(\mathbf{1}) = M^*(\mathbf{1}),\end{aligned}$$

where the last module is the Matlis dual of the $\mathbb{Z}^n$-graded module M shifted by $\mathbf{1}$. □

The Matlis dual functor is exact contravariant and it is easy to prove that the Matlis dual of an injective $\mathcal{D}^T_{v=0}$-module is projective; more precisely we have

$$({}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1}))^* = R\left[\frac{1}{\mathbf{x}^{\mathbf{1}-\alpha}}\right]$$

and the Matlis dual of a simple $\mathcal{D}^T_{v=0}$-module is simple. Namely we have

$$(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^* = H^{|\mathbf{1}-\alpha|}_{\mathfrak{p}_{\mathbf{1}-\alpha}}(R).$$

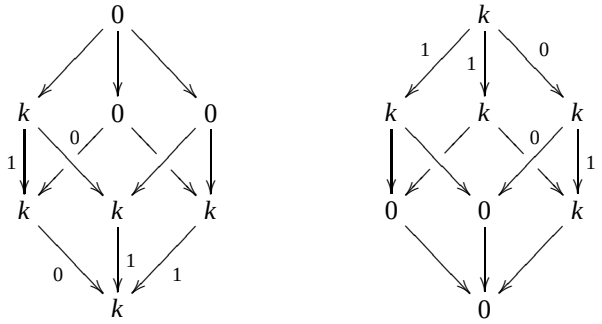
5.2. Dual Bass numbers

By using Matlis duality we can compute the dual Bass numbers of a projective resolution of a module with variation zero $M \in \mathcal{D}^T_{v=0}$. Namely we have:

Proposition 5.3. *Let M^* be the Matlis dual of a module with variation zero $M \in \mathcal{D}^T_{v=0}$. Then, we have:*

$$\pi_j(\mathfrak{p}_\alpha, M) := \mu_j(\mathfrak{p}_{\mathbf{1}-\alpha}, M^*).$$

Example. Let $R = k[x_1, x_2, x_3]$. Consider the 3-hypercubes of the module with variation zero $M \in \mathcal{D}^T_{v=0}$ studied in the previous examples and its Matlis dual M^*:



Then, the dual Bass numbers of M^* are:

$\mathfrak{p}_\gamma$	π_0	π_1	π_2
(0)	–	1	–
(x_1)	1	–	–
(x_2)	1	1	–
(x_3)	–	–	–
(x_1, x_2)	–	–	–
(x_1, x_3)	–	–	–
(x_2, x_3)	1	–	–
(x_1, x_2, x_3)	–	–	–

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