Hardy–Littlewood–Pólya-type theorems for invex functions

Marek Niezgoda\textsuperscript{a,}\textsuperscript{*}, Josip Pečarić\textsuperscript{b}

\textsuperscript{a} Department of Applied Mathematics and Computer Science, University of Life Sciences in Lublin, Akademicka 13, 20-950 Lublin, Poland
\textsuperscript{b} University of Zagreb, Faculty of Textile Technology, Pierottijeva 6, 10000 Zagreb, Croatia

\textbf{A B S T R A C T}

In this paper, the Hardy–Littlewood–Pólya theorem on majorization is extended from convex functions to invex ones. Some variants for pseudo-invex and quasi-invex functions are also considered. The framework used is that of similarly separable vectors. The results obtained are illustrated for monotonic, monotonic in mean, and star-shaped vectors, respectively. Applications to relative invexity are given.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction and summary

A vector $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is said to be \textit{majorized} by a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, written as $y \prec_m x$, if the sum of the $k$ largest entries of $y$ does not exceed the sum of the $k$ largest entries of $x$ for all $k = 1, 2, \ldots, n$, with equality for $k = n$ (see [1, p. 7]).

The Hardy–Littlewood–Pólya (H–L–P) theorem states that, if $f : I \to \mathbb{R}$ is a convex continuous function on an interval $I \subset \mathbb{R}$, and $y \prec_m x$ with $x, y \in I^n$, then

$$
\sum_{k=1}^{n} f(y_k) \leq \sum_{k=1}^{n} f(x_k)
$$

(see [2, p. 75]; cf. [3, pp. 207–208]).

Many authors have investigated inequalities of type

$$
\sum_{k=1}^{n} p_k f(y_k) \leq \sum_{k=1}^{n} p_k f(x_k)
$$

(2)

for some convex functions $f : I \to \mathbb{R}$ and vectors $x, y \in \mathbb{R}^n$ and $p \in \mathbb{R}_+^n$ [4,3,5–9]. A continuous version of (2) can be obtained for integrals, too [10,7,11].

Assume that $I \subset \mathbb{R}$ is an interval. Let $f : I \to \mathbb{R}$ be a differentiable function, and let $\eta : I \times I \to \mathbb{R}$ be a function of two variables. The function $f$ is said to be $\eta$-\textit{invex} if, for all $x, y \in I$,

$$
f(x) - f(y) \geq f'(y) \cdot \eta(x, y)
$$

[12, p. 1]. $f$ is called \textit{invex} if $f$ is $\eta$-invex for some $\eta$.

\textsuperscript{*} Corresponding author.

E-mail addresses: marek.niezgoda@up.lublin.pl, bniezgoda@wp.pl (M. Niezgoda), pecaric@mahazu.hazu.hr (J. Pečarić).
Clearly, each differentiable convex function $f : I \to \mathbb{R}$ is an $\eta$-invex function with $\eta(x, y) = x - y$ for $x, y \in I$.

It is known that a differentiable function is invex if and only if each stationary point is a global minimum point [12,13]. This fact was the motivation to introduce invex functions in optimization theory [14].

Let $\varTheta : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function vanishing at points of the form $(0, b), b \in \mathbb{R}$. It is not hard to verify that, if a differentiable function $f$ satisfies the condition

$$
\varTheta(f'(y), \eta_0(x, y)) \geq 0 \quad \text{implies} \quad f(x) - f(y) \geq 0 \quad \text{for} \quad x, y \in I
$$

for some function $\varTheta_0$, then $f$ is invex.

In particular, each pseudo-convex function is invex [12, pp. 3–4]. In fact, it is sufficient to consider $\varTheta(a, b) = ab$ for $a, b \in \mathbb{R}$ and $\eta_0(x, y) = x - y$ for $x, y \in I$.

In the multidimensional case ($n \geq 1$), we have the following definition.

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$. Let $\eta : I^n \times I^n \to \mathbb{R}^n$ be a function of two variables.

A differentiable function $F : I^n \to \mathbb{R}$ is said to be $\eta$-invex if, for all $x, y \in I^n$,

$$
F(x) - F(y) \geq \langle \nabla F(y), \eta(x, y) \rangle,
$$

where $\nabla$ denotes the gradient [12, p. 1].

A differentiable function $F : I^n \to \mathbb{R}$ is said to be $\eta$-pseudo-invex if, for all $x, y \in I^n$,

$$
\langle \nabla F(y), \eta(x, y) \rangle \geq 0 \quad \text{implies} \quad F(x) - F(y) \geq 0
$$

(see [14]).

A differentiable function $F : I^n \to \mathbb{R}$ is said to be $\eta$-quasi-invex if, for all $x, y \in I^n$,

$$
F(x) - F(y) \leq 0 \quad \text{implies} \quad \langle \nabla F(y), \eta(x, y) \rangle \leq 0
$$

(see [14]).

A differentiable real function $F : I^n \to \mathbb{R}^n$ is said to be invex (respectively, pseudo-invex, quasi-invex), if $F$ is $\eta$-invex (respectively, $\eta$-pseudo-invex, $\eta$-quasi-invex) for some function $\eta : I^n \times I^n \to \mathbb{R}^n$.

Applications of invex functions can be found in optimization and mathematical programming [14–21].

Now, we quote some terminology on separable vectors from [22,8,23].

Let $e = (e_1, \ldots, e_n)$ be a sequence of vectors in $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Let $J_1$ and $J_2$ be two sets of indices such that $J_1 \cup J_2 = J$, where $J = \{1, 2, \ldots, n\}$. Given a vector $v \in \mathbb{R}^n$ and scalar $\mu \in \mathbb{R}$, a vector $z \in \mathbb{R}^n$ is said to be $\mu$, $v$-separable on $J_1$ and $J_2$ with respect to $e$ if

$$
\langle e_i, z - \mu v \rangle \geq 0 \quad \text{for} \quad i \in J_1, \quad \text{and} \quad \langle e_j, z - \mu v \rangle \leq 0 \quad \text{for} \quad j \in J_2.
$$

A vector $v \in \mathbb{R}^n$ is said to be e-positive if $\langle e_i, v \rangle > 0$ for all $i = 1, 2, \ldots, n$.

Condition (6) means that

$$
\max_{j \in J_2} \frac{\langle e_j, z \rangle}{\langle e_j, v \rangle} \leq \mu \leq \min_{i \in J_1} \frac{\langle e_i, z \rangle}{\langle e_i, v \rangle} \quad \text{(provided that} \ v \ \text{is e-positive)}.
$$

We say that a vector $z \in \mathbb{R}^n$ is $v$-separable on $J_1$ and $J_2$ with respect to (w.r.t.) $e$ if $z$ is $\mu$, $v$-separable on $J_1$ and $J_2$ w.r.t. $e$ for some $\mu$.

According to (6) and (7), $z$ is $v$-separable on $J_1$ and $J_2$ w.r.t. $e$ iff

$$
\max_{j \in J_2} \frac{\langle e_j, z \rangle}{\langle e_j, v \rangle} \leq \mu \leq \min_{i \in J_1} \frac{\langle e_i, z \rangle}{\langle e_i, v \rangle} \quad \text{(provided} \ v \ \text{is e-positive)}.
$$

Geometrically, the notion of $v$-separability is related to the convex cone

$$
S_e(v; J_1, J_2) = \{ z \in V : z \text{ is } v\text{-separable on } J_1 \text{ and } J_2 \text{ w.r.t. } e \} = C_v(J_1, J_2) + \text{span } v,
$$

where $C_v(J_1, J_2)$ is the dual cone of the convex cone spanned by the set $\{ e_i : i \in J_1 \} \cup \{ -e_j : j \in J_2 \}$ (see [22, pp. 234–235]).

Two sequences $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ of vectors in $\mathbb{R}^n$ are said to be dual if $\langle e_i, d_j \rangle = \delta_{ij}$ (Kronecker delta) for $i, j = 1, 2, \ldots, n$.

It is not hard to check that, if $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ are dual bases in $\mathbb{R}^n$, then the inner product of any two vectors $a, b \in \mathbb{R}^n$ can be expressed as follows:

$$
\langle a, b \rangle = \sum_{i=1}^{n} \langle e_i, a \rangle \langle d_i, b \rangle.
$$

In consequence, for $a, b, w, v \in \mathbb{R}^n$ and $\mu, \lambda \in \mathbb{R}$, one obtains the identity

$$
\langle a - \mu v, b - \lambda w \rangle = \sum_{i=1}^{n} \langle e_i, a - \mu v \rangle \langle d_i, b - \lambda w \rangle.
$$
If in addition $a$ is $\mu$, $v$-separable on $J_1$ and $J_2$ w.r.t. $e$ and $b$ is $\lambda$, $w$-separable on $J_1$ and $J_2$ w.r.t. $d$, then identity (9) leads to an interesting inequality (10) of Chebyshev type (see [22] for details; see also [24]). Moreover, by substituting some concrete dual bases of $\mathbb{R}^n$ in place of $e$ and $d$, one can get (9) and (10) for various classes of vectors, e.g., for monotonic, monotonic in mean, or star-shaped vectors in $\mathbb{R}^n$.

In what follows, the following result will be used (see [8, Lemma 2.1], [22, Theorem 3.5]).

**Lemma 1.1** ([8, Lemma 2.1]). Assume that $e = (e_1, \ldots, e_n)$ is a basis in $\mathbb{R}^n$, and that $d = (d_1, \ldots, d_n)$ is the dual basis of $e$.

Let $w$, $v$, $a$ and $b$ be vectors in $\mathbb{R}^n$ with $\langle w, v \rangle > 0$. Denote $\lambda = (b, v)/(w, v)$. If there exist index sets $J_1$ and $J_2$ with $J_1 \cup J_2 = J$, where $J = \{1, 2, \ldots, n\}$, such that

(i) $a$ is $v$-separable on $J_1$ and $J_2$ w.r.t. $e$, and
(ii) $b$ is $\lambda$, $w$-separable on $J_1$ and $J_2$ w.r.t. $d$,

then the inequality

$$\langle a, w \rangle \langle b, v \rangle \leq \langle a, b \rangle \langle w, v \rangle$$

holds.

In light of the above lemma, we introduce the following definition.

Let $e$ and $d$ be dual bases in $\mathbb{R}^n$, $v, w \in \mathbb{R}^n$ and $\mu, \lambda \in \mathbb{R}$. Two vectors $a, b \in \mathbb{R}^n$ are said to be **similarly separable** w.r.t. $(\mu, e; \lambda, w, d)$ if there exist index sets $J_1$ and $J_2$ with $J_1 \cup J_2 = \{1, 2, \ldots, n\}$ such that

(i) $a$ is $\mu$, $v$-separable on $J_1$ and $J_2$ w.r.t. $e$, and
(ii) $b$ is $\lambda$, $w$-separable on $J_1$ and $J_2$ w.r.t. $d$.

The aim of this paper is to derive H–L–P-type inequalities (1)–(2) for invex (respectively, pseudo-invex, quasi-invex) functions. We provide a framework for this purpose which is based on the notion of similar separability of vectors $V_F(y)$ and $\eta(x, y)$ with respect to some dual bases in the space $\mathbb{R}^n$.

Our results are collected in Section 2. An extension of the H–L–P theorem is given from convex functions to invex ones and from majorized vectors to similarly separable vectors. Also, some variants of (1)–(2) for pseudo-invex and quasi-invex functions are considered. The results obtained are illustrated for monotonic, monotonic in mean, and star-shaped vectors in Section 3. In Section 4, applications to relative invexity are demonstrated. A result of Pečarić and Abramovich [11] is recovered (see also [9,7]).

2. Results

For a given function $f : I \to \mathbb{R}$ with interval $I \subset \mathbb{R}$, and for $p = (p_1, \ldots, p_n)$ with $p_1 > 0, \ldots, p_n > 0$, let $F_p : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F_p(z) = \sum_{k=1}^{n} p_k f(z_k) \quad \text{for } z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$  \hspace{1cm} (11)

We are interested in conditions on $f$ and $x, y \in \mathbb{R}^n$ so that $F_p(y) \leq F_p(x)$.

In what follows, the inner product on $\mathbb{R}^n$ is given by

$$\langle a, b \rangle = \sum_{k=1}^{n} a_k b_k \quad \text{for } a = (a_1, \ldots, a_n) \text{ and } b = (b_1, \ldots, b_n).$$  \hspace{1cm} (12)

We begin with a result which extends [8, Theorem 2.2] from convex functions $f$ with $\eta(x, y) = x - y$ for $x, y \in I$ to invex functions $f$ with arbitrary $\eta$.

**Theorem 2.1.** Let $f : I \to \mathbb{R}$ be an $\eta$-invex function, where $I \subset \mathbb{R}$ is an interval and $\eta : I \times I \to \mathbb{R}$ is a continuous function. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and $p = (p_1, \ldots, p_n)$, where $x_i, y_i \in I, p_i > 0$ for $i \in J = \{1, \ldots, n\}$.

For $F_p : \mathbb{R}^n \to \mathbb{R}$ given by (11), denote

$$\nabla F_p(z) = (p_1 f'(z_1), \ldots, p_n f'(z_n)) \quad \text{for } z = (z_1, \ldots, z_n) \in \mathbb{R}^n,$$

$$\eta_0(x, y) = (\eta(x_1, y_1), \ldots, \eta(x_n, y_n)).$$  \hspace{1cm} (13)

Assume that $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ are dual bases in $\mathbb{R}^n$. Let $w, v \in \mathbb{R}^n$ with $\langle w, v \rangle > 0$.

Suppose that there exist index sets $J_1$ and $J_2$ with $J_1 \cup J_2 = J$ such that

(i) $\nabla F_p(y)$ is $v$-separable on $J_1$ and $J_2$ w.r.t. $e$,
(ii) $\eta_0(x, y)$ is $\lambda$, $w$-separable on $J_1$ and $J_2$ w.r.t. $d$, where $\lambda = (\eta_0(x, y), v)/(w, v)$.


Under the above assumptions, the following two assertions hold.

(A) If \( \langle \eta_0(x, y), v \rangle = 0 \), then
\[
\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(x_k). \tag{15}
\]

(B) If \( \langle \eta_0(x, y), v \rangle \geq 0 \) and \( \langle \nabla F_p(y), w \rangle \geq 0 \), then (15) holds.

**Proof.** If follows from (3) that
\[
f(x_k) - f(y_k) \geq f'(y_k) \cdot \eta(x_k, y_k) \quad \text{for } k = 1, \ldots, n.
\]

Hence, and from (12)–(14), we have that
\[
\sum_{k=1}^{n} p_k (f(x_k) - f(y_k)) \geq \sum_{k=1}^{n} p_k f'(y_k) \eta(x_k, y_k) = \langle \nabla F_p(y), \eta_0(x, y) \rangle.
\tag{16}
\]

Now, using the above conditions (i)–(ii) and Lemma 1.1 applied to vectors \( \nabla F_p(y) \) and \( \eta_0(x, y) \), we deduce that
\[
\langle \nabla F_p(y), \eta_0(x, y) \rangle \geq \frac{1}{\langle w, v \rangle} \langle \nabla F_p(y), w \rangle \langle \eta_0(x, y), v \rangle.
\tag{17}
\]

By combining (16) and (17) we have
\[
\sum_{k=1}^{n} p_k (f(x_k) - f(y_k)) \geq \frac{1}{\langle w, v \rangle} \langle \nabla F_p(y), w \rangle \langle \eta_0(x, y), v \rangle.
\tag{18}
\]

(A) It is now clear that the condition \( \langle \eta_0(x, y), v \rangle = 0 \) forces (15) from (18).

(B) Likewise, if \( \langle \eta_0(x, y), v \rangle \geq 0 \) and \( \langle \nabla F_p(y), w \rangle \geq 0 \), then (15) follows from (18). This completes the proof. \( \square \)

A function \( \Phi : l^n \to \mathbb{R}^n \) is said to preserve \( v \)-separability on \( J_1 \) and \( J_2 \) w.r.t. \( e \) if \( \Phi(z) \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \) whenever \( z \in l^n \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \) [8].

**Remark 2.2.** It is obvious that statement (i) in Theorem 2.1 is implied by the following two conditions:

(i') \( y \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \), and

(ii') \( \nabla F_p \) preserves \( v \)-separability on \( J_1 \) and \( J_2 \) w.r.t. \( e \).

**Theorem 2.3.** Under the hypotheses of Theorem 2.1 with the \( \eta \)-inversity of \( f \) replaced by the \( \eta_0 \)-pseudo-inversity of \( F_p \), where \( \eta_0 \) and \( F_p \) are defined by (14) and (11), respectively, the following two assertions hold.

(A) If \( \langle \eta_0(x, y), v \rangle = 0 \), then
\[
\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(x_k). \tag{19}
\]

(B) If \( \langle \eta_0(x, y), v \rangle \geq 0 \) and \( \langle \nabla F_p(y), w \rangle \geq 0 \), then (19) holds.

**Proof.** As in the proof of Theorem 2.1 (see (17)), we obtain
\[
\langle \nabla F_p(y), \eta_0(x, y) \rangle \geq \frac{1}{\langle w, v \rangle} \langle \nabla F_p(y), w \rangle \langle \eta_0(x, y), v \rangle.
\]

So, if \( \langle \eta_0(x, y), v \rangle = 0 \) as assumed in part (A), and/or if \( \langle \eta_0(x, y), v \rangle \geq 0 \) and \( \langle \nabla F_p(y), w \rangle \geq 0 \) as assumed in part (B), then
\[
\langle \nabla F_p(y), \eta_0(x, y) \rangle \geq 0.
\tag{20}
\]

Now, by combining (20) with (4), we establish
\[
\sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(y_k) = F_p(x) - F_p(y) \geq 0,
\]
as required.

Thus (19) is proved in both cases (A) and (B). \( \square \)
Theorem 2.4. Under the hypotheses of Theorem 2.1 with the \( \eta \)-invexity of \( f \) replaced by the \( \eta \)-quasi-invexity of \( F_p \), where \( \eta_0 \) and \( F_p \) are defined by (14) and (11), respectively, the following assertion holds.

If \( \langle \eta_0(x, y), v \rangle > 0 \) and \( \langle \nabla F_p(y), w \rangle > 0 \), then

\[
\sum_{k=1}^{n} p_k f'(x_k) < \sum_{k=1}^{n} p_k f'(y_k).
\]

Proof. In a similar manner as in the proof of Theorem 2.3, we obtain

\[
\langle \nabla F_p(y), \eta_0(x, y) \rangle > 0.
\] (21)

Now, by virtue of (5) and (21), we conclude that

\[
\sum_{k=1}^{n} p_k f'(x_k) - \sum_{k=1}^{n} p_k f'(y_k) = F_p(x) - F_p(y) > 0,
\]

completing the proof of Theorem 2.4. □

3. Applications

In this section, we describe the situations in which the hypotheses and assertions of Theorem 2.1 are satisfied.

The forthcoming Corollaries 3.1–3.3 extend, respectively, [8, Corollaries 2.3, 2.6, 2.10] from convex functions \( f \) with \( \eta(x, y) = x - y \) for \( x, y \in I \subset \mathbb{R} \) to \( \eta \)-invex functions \( f \) with arbitrary \( \eta \).

Corollary 3.1. Under the hypotheses of Theorem 2.1, let \( w = v = (1, \ldots, 1) \in \mathbb{R}^n \), and let \( e = d \) be the basis in \( \mathbb{R}^n \) given by

\[
e_i = d_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 1, \ldots, n.
\] (22)

Denote

\[
\lambda = \frac{1}{n} \sum_{k=1}^{n} \eta(x_k, y_k).
\] (23)

If there exist index sets \( J_1 \) and \( J_2 \) with \( J_1 \cup J_2 = J \) such that

(i) \( \nabla F_p(y) \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \), i.e.,

\[
p_k f'(y_j) \leq p_i f'(y_i) \quad \text{for} \quad i \in J_1 \quad \text{and} \quad j \in J_2,
\]

(ii) \( \eta_0(x, y) \) is \( \lambda \), \( w \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( d = e \), i.e.,

\[
\eta(x_j, y_j) \leq \lambda \leq \eta(x_i, y_i) \quad \text{for} \quad i \in J_1 \quad \text{and} \quad j \in J_2.
\] (25)

then assertions (A) and (B) of Theorem 2.1 hold.

Proof. Taking into account (7)–(8) and (22), observe that a vector \( z = (z_1, \ldots, z_n) \in \mathbb{R}^n \) is \( \mu \), \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \) iff

\[
z_j \leq \mu \leq z_i \quad \text{for} \quad i \in J_1 \quad \text{and} \quad j \in J_2.
\]

Therefore (24) and (25) hold.

Next, apply Theorem 2.1. □

For example, inequalities (24)–(25) are met for

\[
J_1 = \{1, 2, \ldots, m\} \quad \text{and} \quad J_2 = \{m + 1, \ldots, n\}
\]

for some \( m \in \{0, 1, \ldots, n\} \) if both \( \nabla F_p(y) \) and \( \eta_0(x, y) \) are monotonic nonincreasing vectors, i.e.,

\[
p_k f'(y_1) \geq \cdots \geq p_i f'(y_i) \quad \text{and} \quad \eta(x_1, y_1) \geq \cdots \geq \eta(x_n, y_n).
\]
Corollary 3.2. Under the hypotheses of Theorem 2.1, let \( w = v = (1, \ldots, 1) \in \mathbb{R}^n \), and let \( \lambda \) be as in (23). Suppose that \( e \) is the basis in \( \mathbb{R}^n \) consisting of the vectors

\[
e_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0), \quad i = 1, \ldots, n - 1, \tag{26}
e_n = (0, \ldots, 0, 1).
\]

Let \( d \) be the dual basis of \( e \), i.e.,

\[
d_i = (1, \ldots, 1, 0, \ldots, 0), \quad i = 1, \ldots, n. \tag{28}
\]

If there exist index sets \( J_1 \) and \( J_2 \) with \( J_1 \cup J_2 = J \) such that

1. \( \nabla F_p(y) \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \), i.e., there exists \( \mu \in \mathbb{R} \) satisfying

\[
p_{i,j}f'(y_j) - p_{i,j+1}f'(y_{j+1}) \leq 0 \leq p_{i,j+1}f'(y_{j+1}) - p_{i,j}f'(y_i) \quad \text{for } i \in J_1, j \in J_2 \tag{29}
\]

with the convention \( p_{n+1,j}f'(y_{n+1}) = \mu \), and

2. \( \eta_0(x, y) \) is \( \lambda \), \( w \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( d \), i.e.,

\[
\frac{1}{j} \sum_{k=1}^{j} \eta(x_k, y_k) \leq \lambda \leq \frac{1}{i} \sum_{k=1}^{i} \eta(x_k, y_k) \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{30}
\]

then assertions (A) and (B) of Theorem 2.1 hold.

**Proof.** By (8) and (26)–(27), it is not hard to check that a vector \( z = (z_1, \ldots, z_n) \in \mathbb{R}^n \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \) iff

\[
z_i - z_{i+1} \leq 0 \leq z_i - z_{i+1} \quad \text{for } i \in J_1 \text{ and } j \in J_2,
\]

with the convention \( z_{n+1} = \mu \) for some \( \mu \in \mathbb{R} \). Hence (29) follows.

Likewise, using (7) and (28), one obtains (30). Now, Corollary 3.2 is a consequence of Theorem 2.1. \( \square \)

If \( \nabla F_p(y) \) is monotonic nondecreasing, i.e.,

\[
p_1f'(y_1) \leq p_2f'(y_2) \leq \cdots \leq p_nf'(y_n),
\]

and \( \eta_0(x, y) \) is monotonic nondecreasing in mean [25, p. 318], i.e.,

\[
\frac{1}{1} \sum_{k=1}^{1} \eta(x_k, y_k) \leq \frac{1}{2} \sum_{k=1}^{2} \eta(x_k, y_k) \leq \cdots \leq \frac{1}{n} \sum_{k=1}^{n} \eta(x_k, y_k),
\]

then conditions (29)–(30) are satisfied for

\[
J_1 = \{n\} \quad \text{and} \quad J_2 = \{1, 2, \ldots, n-1\}.
\]

Corollary 3.3. Under the hypotheses of Theorem 2.1, let \( w = v = (1, 2, \ldots, n) \in \mathbb{R}^n \), and let \( e = d \) be the basis in \( \mathbb{R}^n \) given by (22). Denote

\[
\lambda = \frac{6}{n(n+1)(2n+1)} \sum_{k=1}^{n} k \eta(x_k, y_k).
\]

If there exist index sets \( J_1 \) and \( J_2 \) with \( J_1 \cup J_2 = J \) such that

1. \( \nabla F_p(y) \) is \( v \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( e \), i.e.,

\[
\frac{p_{i,j}f'(y_j)}{j} \leq \frac{p_{i,j+1}f'(y_{j+1})}{i} \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{31}
\]

2. \( \eta_0(x, y) \) is \( \lambda \), \( w \)-separable on \( J_1 \) and \( J_2 \) w.r.t. \( d = e \), i.e.,

\[
\frac{\eta(x_i, y_j)}{j} \leq \lambda \leq \frac{\eta(x_i, y_j)}{i} \quad \text{for } i \in J_1 \text{ and } j \in J_2, \tag{32}
\]

then assertions (A) and (B) of Theorem 2.1 hold.

**Proof.** Apply Theorem 2.1. \( \square \)
A vector $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is said to be star shaped if
\[
\frac{z_1}{1} \leq \frac{z_2}{2} \leq \cdots \leq \frac{z_n}{n}
\]

[25, p. 318].

It is not surprising that, if both the vectors $\nabla F_p(y)$ and $\eta_0(x, y)$ are star shaped, i.e.,
\[
\frac{p_1f'(y_1)}{1} \leq \frac{p_2f'(y_2)}{2} \leq \cdots \leq \frac{p_nf'(y_n)}{n}
\]

and
\[
\frac{\eta(x_1, y_1)}{1} \leq \frac{\eta(x_2, y_2)}{2} \leq \cdots \leq \frac{\eta(x_n, y_n)}{n}
\]

then conditions (31)–(32) are satisfied for
\[
J_2 = \{1, 2, \ldots, m\} \quad \text{and} \quad J_1 = \{m + 1, \ldots, n\}
\]

for some $m \in \{0, 1, \ldots, n\}$.

4. Relative convexity and relative $\eta$-invexity

A function $f : I \to \mathbb{R}$ is said to be relatively convex with respect to nonconstant function $g : I \to \mathbb{R}$ if
\[
\begin{bmatrix}
1 & g(x) & f(x) \\
1 & g(y) & f(y) \\
1 & g(z) & f(z)
\end{bmatrix} \geq 0,
\]

(33)

whenever $x, y, z \in I$ and $g(x) \leq g(y) \leq g(z)$ (see [7, pp. 1–2]).

Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable functions with $g' > 0$. Then a related condition to (33) is
\[
f(x) - f(y) \geq \frac{f'(y)}{g'(y)} (g(x) - g(y)) \quad \text{for} \quad x, y \in I
\]

(see [26, p. 2]). In other words, $f$ is $\eta$-invex, where
\[
\eta(x, y) = \frac{g(x) - g(y)}{g'(y)} \quad \text{for} \quad x, y \in I.
\]

By analogy, we can define relative invexity as follows.

Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable functions with $g' > 0$, and let $\eta : g(I) \times g(I) \to \mathbb{R}$ be a function of two variables. The function $f$ is said to be relatively $\eta$-invex with respect to $g$ (in short, $\eta, g$-invex) if
\[
f(x) - f(y) \geq \frac{f'(y)}{g'(y)} \eta(g(x), g(y)) \quad \text{for} \quad x, y \in I.
\]

Theorem 4.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable functions, where $I \subset \mathbb{R}$ is an interval. Let $\eta : g(I) \times g(I) \to \mathbb{R}$ be a continuous function of two variables. Assume that $g$ is strictly increasing and that $f$ is $\eta, g$-invex.

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and $p = (p_1, \ldots, p_n)$, where $x_i, y_i \in I, p_i > 0$ for $i \in J = \{1, 2, \ldots, n\}$.

Denote
\[
\Phi_p(y) = \left(\frac{f'(y_1)}{g'(y_1)}, \ldots, \frac{f'(y_n)}{g'(y_n)}\right)
\]

\[
\eta_{g, \Phi}(x, y) = \langle \eta(g(x_1), g(y_1)), \ldots, \eta(g(x_n), g(y_n)) \rangle.
\]

Assume that $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ are dual bases in $\mathbb{R}^n$. Let $w, v \in \mathbb{R}^n$ with $\langle w, v \rangle > 0$.

Suppose that there exist index sets $J_1$ and $J_2$ with $J_1 \cup J_2 = J$ such that
(i) $\Phi_p(y)$ is $w$-separable on $J_1$ and $J_2$ w.r.t. $e$, and
(ii) $\eta_{g, \Phi}(x, y)$ is $\lambda$, $w$-separable on $J_1$ and $J_2$ w.r.t. $d$, where $\lambda = \langle \eta_{g, \Phi}(x, y), v \rangle / \langle w, v \rangle$.

Under the above assumptions, the following two assertions hold.

(A) If $\langle \eta_{g, \Phi}(x, y), v \rangle = 0$, then
\[
\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(x_k).
\]

(34)

(B) If $\langle \eta_{g, \Phi}(x, y), v \rangle \geq 0$ and $\langle \Phi_p(y), w \rangle \geq 0$, then (34) holds.
**Proof.** Use an analogous argument as in the proof of Theorem 2.1. \(\square\)

A variant of Theorem 4.1 is as follows.

**Corollary 4.2.** Let \(f : I \to \mathbb{R}\) and \(g : I \to \mathbb{R}\) be differentiable functions, where \(I \subset \mathbb{R}\) is an interval. Let \(\eta : g(I) \times g(I) \to \mathbb{R}\) be a continuous function of two variables. Assume that \(g\) is strictly increasing and that \(f\) is \(\eta\)-invex.

Let \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)\) and \(p = (p_1, \ldots, p_n)\), where \(x_i, y_i \in I, p_i > 0\) for \(i \in J = \{1, 2, \ldots, n\}\).

Denote
\[
\Psi(y) = \left(\frac{f'(y_1)}{g'(y_1)}, \ldots, \frac{f'(y_n)}{g'(y_n)}\right).
\]

\(\eta_{\alpha,p,g}(x, y) = (p_1 \eta(g(x_1), g(y_1)), \ldots, p_n \eta(g(x_n), g(y_n))).\)

Assume that \(e = (e_1, \ldots, e_n)\) and \(d = (d_1, \ldots, d_n)\) are dual bases in \(\mathbb{R}^n\). Let \(w, v \in \mathbb{R}^n\) with \(\langle w, v \rangle > 0\).

Suppose that \(w, v\) in \(\mathbb{R}^n\) with \(\langle w, v \rangle > 0\), \(\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k)\) (35).

Under the above assumptions, the following two assertions hold.

(A) If \(\langle \eta_{0,p,g}(x, y), v \rangle = 0\), then
\[
\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k).
\]

(B) If \(\langle \eta_{0,p,g}(x, y), v \rangle \geq 0\) and \(\langle \Psi(y), w \rangle \geq 0\), then (35) holds.

**Proof.** Use a similar argument as in the proof of Theorem 2.1. \(\square\)

By Corollary 4.2, part (A), applied to \(\eta(a, b) = a - b\) for \(a, b \in g(I)\), \(w = v = (1, \ldots, 1) \in \mathbb{R}^n\), \(J_1 = \{1, 2, \ldots, n\}\), and \(J_2 = \{n\}\), and the dual bases \(e\) and \(d\) in \(\mathbb{R}^n\) defined in (26)–(28), we recover the following result.

**Corollary 4.3** (See [9, Theorem 2.7], [7, Theorem 3], [11, Theorem 1]). Let \(f, g : I \to \mathbb{R}\) be two differentiable functions with \(g' > 0\) such that \(f\) is convex with respect to \(g\), and consider points \(x_1, \ldots, x_n, y_1, \ldots, y_n\) in \(I\) and positive weights \(p_1, \ldots, p_n\) such that

(i) \(g(y_1) \geq \cdots \geq g(y_n)\),
(ii) \(\sum_{k=1}^{n} p_k g(x_k) = \sum_{k=1}^{n} p_k g(y_k)\) for all \(r = 1, \ldots, n\), and
(iii) \(\sum_{k=1}^{n} p_k g(x_k) \geq \sum_{k=1}^{n} p_k g(y_k)\).

Then
\[
\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k).
\]

Evidently, when \(p_1 = \cdots = p_n\), conditions (i)–(iii) imply that the vector \((g(x_1), \ldots, g(x_n))\) majorizes the vector \((g(y_1), \ldots, g(y_n))\).

**Acknowledgment**

The authors wish to thank the anonymous referees for their careful reading and for giving valuable suggestions to improve the readability of the paper.

**References**