Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 226 (2011) 5320–5337
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The geometry of p-convex intersection bodies ☆

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Received 11 June 2010; accepted 10 January 2011

Communicated by Erwin Lutwak

Dedicated to the memory of Nigel J. Kalton, 1946–2010

Abstract

Busemann's theorem states that the intersection body of an origin-symmetric convex body is also convex. In this paper we provide a version of Busemann's theorem for p-convex bodies. We show that the intersection body of a p-convex body is q-convex for certain q. Furthermore, we discuss the sharpness of the previous result by constructing an appropriate example. This example is also used to show that IK, the intersection body of K, can be much farther away from the Euclidean ball than K. Finally, we extend these theorems to some general measure spaces with log-concave and s-concave measures.
Published by Elsevier Inc.

MSC: 44A12; 52A15; 52A21

Keywords: p-Convex body; Star-shaped body; Intersection body; Busemann's theorem

☆ The first and third authors were supported in part by U.S. National Science Foundation grant DMS-0652684. The second author was supported in part by NSERC.

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0001-8708/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.aim.2011.01.011

1. Introduction and notations

A *body* is a compact set with nonempty interior. For a body K which is star-shaped with respect to the origin its *radial function* is defined by

$$\rho_K(u) = \max\{\lambda \geqslant 0:\ \lambda u \in K\} \quad \text{for every } u \in S^{n-1}.$$

A body K is called a *star body* if it is star-shaped at the origin and its radial function ρ_K is positive and continuous. The *Minkowski functional* $\|\cdot\|_K$ is defined by $\|x\|_K = \min\{\lambda \geqslant 0:\ x \in \lambda K\}$. Clearly, $\rho_K(u) = \|u\|_K^{-1}$, for $u \in S^{n-1}$. Moreover, we can assume that this identity holds for all $u \in \mathbb{R}^n \setminus \{0\}$ by extending ρ_K from the sphere to $\mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree -1.

In [35], Lutwak introduced the notion of the *intersection body IK* of a star body K. IK is defined by its radial function

$$\rho_{IK}(u) = \left|K \cap u^{\perp}\right|, \quad \text{for } u \in S^{n-1}.$$

Here and throughout the paper, $u^{\perp}$ denotes the hyperplane perpendicular to u, i.e. $u^{\perp} = \{x \in \mathbb{R}^n:\ x \cdot u = 0\}$. By $|A|_k$, or simply $|A|$ when there is no ambiguity, we denote the k-dimensional Lebesgue measure of a set A.

We refer the reader to the books [15,28,29] for more information on the definition and properties of intersection bodies, and their applications in convex geometry and geometric tomography.

In this paper we are interested in the properties of the operator I that assigns to a star body K its intersection body IK. One of the well-known results is the classical Busemann theorem ([10], see also [4,3,38]).

Theorem 1. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then its intersection body IK is convex.*

Recently, a new proof of Busemann's theorem was established by Berck [7], who also generalized the theorem to the case of L_p intersection bodies (see [16,21,41] and [28] for more information on the theory of L_p intersection bodies).

The development of the theory of intersection bodies shows that it is not natural to restrict ourselves to the class of convex bodies, and in fact, in many situations one has to deal with bodies which are not necessarily convex. How does I act on these bodies? In this paper we will answer this question for the class of p-convex bodies.

Let $p \in (0, 1]$. We say that a body K is p-convex if, for all $x, y \in \mathbb{R}^n$,

$$\|x + y\|_K^p \leqslant \|x\|_K^p + \|y\|_K^p,$$

or, equivalently $t^{1/p}x + (1-t)^{1/p}y \in K$ whenever x and y are in K and $t \in (0, 1)$. One can see that p-convex sets with $p = 1$ are just convex. Note also that a p_1-convex body is p_2-convex for all $0 < p_2 \leqslant p_1$. There is an extensive literature on p-convex bodies as well as the closely related concept of quasi-convex bodies, see for example [1,5,6,11,17,18,20,23,25,24,30–34,37] and others.

The first question that we consider is the following. *Let K be an origin-symmetric p-convex body. Is the intersection body IK necessarily q-convex for some q?*

In Section 2, we prove that if K is p-convex and symmetric then its intersection body IK is q-convex for all $q \leqslant [(\frac{1}{p}-1)(n-1)+1]^{-1}$. Furthermore, we construct an example showing that this upper bound is asymptotically sharp.

Another important question about the operator I comes from works of Lutwak [36] and Gardner [15, Prob. 8.6, 8.7] (see also [19]). It is easy to see that the intersection body of a Euclidean ball is again a Euclidean ball. A natural question is whether there are other fixed points of I. In order to measure the distance between star bodies we will be using the Banach–Mazur distance $d_{BM}(K,L)=\inf\{b/a\colon\ \exists T\in GL(n)\colon\ aK\subset TL\subset bK\}$ (see [39]). In [12] it is shown that in a sufficiently small neighborhood of the ball (with respect to the Banach–Mazur distance) there are no other fixed points of the intersection body operator. However, in general this question is still open. In view of this it is natural to ask the following question.

Does IK have to be closer to the ball than K?

In Section 2 we show that the answer is "No". There are p-convex bodies for which the intersection body is farther from the Euclidean ball.

It is worth noting that, in the convex situation, there exists an absolute constant $C>0$ such that $d_{BM}(IK,B_2^n)<C$ for all origin-symmetric convex bodies K (see [22,4,3,9,38] or Corollary 2.7 in [28]). The example in Section 2 shows that this statement is wrong if we only assume p-convexity for $p<1$, and $d_{BM}(IK,B_2^n)$, for a p-convex body K, can be as large as C_p^n, where $C_p>1$ is a certain constant that depends only on p.

In recent times a lot of attention has been attracted to the study of log-concave measures. These are measures whose densities are log-concave functions. The interest to such measures stems from the Brunn–Minkowski inequality, and many results known for convex bodies are now generalized to log-concave measures, see for example [4,3,2,27,13,40] and the references cited therein.

In Section 3 we study intersection bodies in spaces with log-concave measures. Namely, let μ be a log-concave measure on $\mathbb{R}^n$ and μ_{n-1} its restrictions to $(n-1)$-dimensional subspaces. Define the *intersection body* $I_\mu K$ of a star body K *with respect to* μ by

$$\rho_{I_\mu K}(u)=\mu_{n-1}\big(K\cap u^\perp\big),\quad u\in S^{n-1}.$$

We show that if K is an origin-symmetric p-convex body and μ is a symmetric and log-concave measure, then $I_\mu K$ is q-convex for all $q\leqslant[(\frac{1}{p}-1)(n-1)+1]^{-1}$. The proof uses a version of Ball's theorem [4,3] for p-convex bodies. Namely, we show that if f is an even non-negative log-concave function on $\mathbb{R}^n$, $k\geqslant 1$, and K is a p-convex body in $\mathbb{R}^n$, $0<p\leqslant 1$, then the body L defined by the Minkowski functional $\|x\|_L=[\int_0^{\|x\|_K^{-1}} f(rx)r^{k-1}\,dr]^{-1/k}$, $x\in\mathbb{R}^n$, is p-convex.

If the measure μ is not symmetric, the situation is different, as explained at the end of Section 3. L defined above does not have to be q-convex for any $q>0$. However, if we consider s-concave measures, $0<s<1/n$, that are not necessarily symmetric, then the above construction defines a body L that is q-convex for all $q\leqslant[(\frac{1}{p}-1)(\frac{1}{s}-n)\frac{1}{k}+\frac{1}{p}]^{-1}$. We also show that this bound is sharp. This is the content of Section 4.

2. p-Convexity and related results

Here we prove a version of Busemann's theorem for p-convex bodies.

Theorem 2. *Let K be a symmetric p-convex body in $\mathbb{R}^n$, $p\in(0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1\leqslant k\leqslant n$. Then the map*

$$u \longmapsto \frac{|u|}{|K \cap \operatorname{span}(u, E)|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q-convex body in $E^\perp$ *with* $q = [(1/p-1)k+1]^{-1}$.

Proof. We follow the general idea of the proof from [38] (see also [15, p. 311]). Let $u_1, u_2 \in E^\perp \setminus \{0\}$ be non-parallel vectors. Denote $u = u_1 + u_2$, and

$$\rho(u_1) = \frac{|K \cap \operatorname{span}\{u_1, E\}|}{|u_1|} = \int_{-\infty}^{\infty} \big|K \cap (ru_1 + E)\big|\, dr,$$

$$\rho(u_2) = \frac{|K \cap \operatorname{span}\{u_2, E\}|}{|u_2|} = \int_{-\infty}^{\infty} \big|K \cap (ru_2 + E)\big|\, dr.$$

Define the functions $r_1 = r_1(t)$ and $r_2 = r_2(t)$ by

$$\begin{aligned} t &= \frac{1}{\rho(u_1)} \int_0^{r_1} \big|K \cap (ru_1 + E)\big|\, dr \\ &= \frac{1}{\rho(u_2)} \int_0^{r_2} \big|K \cap (ru_2 + E)\big|\, dr, \quad t \in [0, 1/2]. \end{aligned}$$

Let $r = (r_1^{-p} + r_2^{-p})^{-\frac{1}{p}}$, $\lambda_1 = \frac{r_1^{-p}}{r_1^{-p}+r_2^{-p}}$ and $\lambda_2 = \frac{r_2^{-p}}{r_1^{-p}+r_2^{-p}}$. Then

$$\begin{aligned} \frac{dr}{dt} &= -\frac{1}{p}\big(r_1^{-p} + r_2^{-p}\big)^{-\frac{1}{p}-1}\left(-p\, r_1^{-p-1}\frac{dr_1}{dt} - p\, r_2^{-p-1}\frac{dr_2}{dt}\right) \\ &= r\left[\lambda_1\left(\frac{1}{r_1}\frac{dr_1}{dt}\right) + \lambda_2\left(\frac{1}{r_2}\frac{dr_2}{dt}\right)\right] \\ &\geqslant r\left(\frac{1}{r_1}\frac{dr_1}{dt}\right)^{\lambda_1}\left(\frac{1}{r_2}\frac{dr_2}{dt}\right)^{\lambda_2} = \left(\lambda_1^{1/p}\frac{dr_1}{dt}\right)^{\lambda_1}\left(\lambda_2^{1/p}\frac{dr_2}{dt}\right)^{\lambda_2} \\ &= \big(\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}\big)^{\frac{1}{p}}\left(\frac{\rho(u_1)}{|K \cap (r_1u_1 + E)|}\right)^{\lambda_1}\left(\frac{\rho(u_2)}{|K \cap (r_2u_2 + E)|}\right)^{\lambda_2}. \end{aligned}$$

On the other hand, since $r\,u = (r_1^{-p} + r_2^{-p})^{-\frac{1}{p}}(u_1 + u_2) = \lambda_1^{\frac{1}{p}} r_1u_1 + \lambda_2^{\frac{1}{p}} r_2u_2$, we have

$$\begin{aligned} K \cap (r\,u + E) &\supset \lambda_1^{\frac{1}{p}}\big(K \cap (r_1\,u_1 + E)\big) + \lambda_2^{\frac{1}{p}}\big(K \cap (r_2\,u_2 + E)\big) \\ &= \lambda_1\big(\lambda_1^{\frac{1}{p}-1} K \cap (r_1\,u_1 + E)\big) + \lambda_2\big(\lambda_2^{\frac{1}{p}-1} K \cap (r_2\,u_2 + E)\big). \end{aligned}$$

Thus, by the Brunn–Minkowski inequality (see, for example, [14]), we get

$$\begin{aligned}\left|K\cap(ru+E)\right| &\geqslant \left|\lambda_1^{\frac{1}{p}-1}K\cap(r_1u_1+E)\right|^{\lambda_1}\left|\lambda_2^{\frac{1}{p}-1}K\cap(r_2u_2+E)\right|^{\lambda_2}\\ &= \left(\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}\right)^{(\frac{1}{p}-1)(k-1)}\left|K\cap(r_1u_1+E)\right|^{\lambda_1}\left|K\cap(r_2u_2+E)\right|^{\lambda_2}.\end{aligned}$$

Finally, we have the following:

$$\begin{aligned}\rho(u_1+u_2) &= \int_{-\infty}^{\infty}\left|K\cap(ru+E)\right|dr = 2\int_0^{1/2}\left|K\cap(ru+E)\right|\frac{dr}{dt}\,dt\\ &\geqslant 2\int_0^{1/2}\left(\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}\right)^{(\frac{1}{p}-1)(k-1)+\frac{1}{p}}\rho(u_1)^{\lambda_1}\rho(u_2)^{\lambda_2}\,dt\\ &\geqslant 2\int_0^{1/2}\left[\left(\lambda_1\left[\rho(u_1)\right]^q\right)^{\lambda_1}\left(\lambda_2\left[\rho(u_2)\right]^q\right)^{\lambda_2}\right]^{1/q}dt\\ &\geqslant 2\int_0^{1/2}\left[\frac{\lambda_1}{\lambda_1[\rho(u_1)]^q}+\frac{\lambda_2}{\lambda_2[\rho(u_2)]^q}\right]^{-1/q}dt\\ &= \left[\left[\rho(u_1)\right]^{-q}+\left[\rho(u_2)\right]^{-q}\right]^{-1/q}.\end{aligned}$$

Therefore $\rho(u)$ defines a q-convex body. □

As an immediate corollary of the previous theorem we obtain the following.

Theorem 3. *Let K be a symmetric p-convex body in $\mathbb{R}^n$ for $p\in(0,1]$. Then the intersection body IK of K is q-convex for $q=[(1/p-1)(n-1)+1]^{-1}$.*

Proof. Let $L=IK$ be the intersection body of K. Let $v_1, v_2\in\operatorname{span}\{u_1,u_2\}$ be orthogonal to u_1 and u_2 correspondingly. Denote $E=\operatorname{span}\{u_1,u_2\}^{\perp}$. Then

$$\rho_L(v_1)=\left|K\cap\operatorname{span}\{u_1,E\}\right|=\rho(u_1),$$

and

$$\rho_L(v_2)=\left|K\cap\operatorname{span}\{u_2,E\}\right|=\rho(u_2).$$

Using the previous theorem with $k=n-1$, we see that L is q-convex for $q=[(1/p-1)(n-1)+1]^{-1}$. □

Remark 1. Note that the previous theorem does not hold without the symmetry assumption. To see this, use the idea from [15, Thm. 8.1.8], where it is shown that IK is not necessarily convex if K is not symmetric.

A natural question is to see whether the value of q in Theorem 3 is optimal. Unfortunately, we were unable to construct a body that gives exactly this value of q, but our next result shows that the bound is asymptotically correct.

Theorem 4. *There exists a p-convex body $K \subset \mathbb{R}^n$ such that IK is q-convex with $q \leqslant [(1/p-1)(n-1)+1+g_n(p)]^{-1}$, where $g_n(p)$ is a function that satisfies*

(1) $g_n(p) \geqslant -\log_2(n-1)$,
(2) $\lim_{p\to 1^-} g_n(p) = 0$.

Proof. Consider the following two $(n-1)$-dimensional cubes in $\mathbb{R}^n$:

$$C_1 = \{|x_1| \leqslant 1, \ \ldots, \ |x_{n-1}| \leqslant 1, \ x_n = 1\} \quad \text{and} \quad C_{-1} = \{|x_1| \leqslant 1, \ \ldots, \ |x_{n-1}| \leqslant 1, \ x_n = -1\}.$$

For a fixed $0 < p < 1$, let us define a set $K \subset \mathbb{R}^n$ as follows:

$$K = \{z \in \mathbb{R}^n \colon z = t^{1/p}x + (1-t)^{1/p}y, \ \text{for some } x \in C_1, \ y \in C_{-1}, \ 0 \leqslant t \leqslant 1\}.$$

We claim that K is p-convex. To show this let us consider two arbitrary points $z_1, z_2 \in K$,

$$z_1 = t_1^{1/p}x_1 + (1-t_1)^{1/p}y_1, \qquad z_2 = t_2^{1/p}x_2 + (1-t_2)^{1/p}y_2,$$

where $x_1, x_2 \in C_1$, $y_1, y_2 \in C_{-1}$, and $t_1, t_2 \in [0,1]$.

We need to show that for all $s \in (0,1)$ the point $w = s^{1/p}z_1 + (1-s)^{1/p}z_2$ belongs to K.

Assume first that t_1 and t_2 are neither both equal to zero nor both equal to one. Since C_1 and C_{-1} are convex sets, it follows that the points

$$\bar{x} = \frac{s^{1/p}t_1^{1/p}x_1 + (1-s)^{1/p}t_2^{1/p}x_2}{s^{1/p}t_1^{1/p} + (1-s)^{1/p}t_2^{1/p}} \quad \text{and} \quad \bar{y} = \frac{s^{1/p}(1-t_1)^{1/p}y_1 + (1-s)^{1/p}(1-t_2)^{1/p}y_2}{s^{1/p}(1-t_1)^{1/p} + (1-s)^{1/p}(1-t_2)^{1/p}}$$

belong to C_1 and C_{-1} correspondingly. Then $w = \alpha\bar{x} + \beta\bar{y}$, where

$$\alpha = s^{1/p}t_1^{1/p} + (1-s)^{1/p}t_2^{1/p} \quad \text{and} \quad \beta = s^{1/p}(1-t_1)^{1/p} + (1-s)^{1/p}(1-t_2)^{1/p}.$$

Note that $\alpha^p + \beta^p \leqslant st_1 + (1-s)t_2 + s(1-t_1) + (1-s)(1-t_2) = 1$. Therefore, there exists $\mu \geqslant 0$ such that $(\alpha+\mu)^p + (\beta+\mu)^p = 1$ and

$$w = (\alpha+\mu)\bar{x} + \bigl(\beta\bar{y} + \mu(-\bar{x})\bigr) = (\alpha+\mu)\bar{x} + (\beta+\mu)\frac{\beta\bar{y} + \mu(-\bar{x})}{\beta+\mu}.$$

Since $\bar{y} \in C_{-1}$ and $-\bar{x} \in C_{-1}$, it follows that

$$\tilde{y} = \frac{\beta\bar{y} + \mu(-\bar{x})}{\beta+\mu} \in C_{-1}.$$

Therefore w is a p-convex combination of $\bar{x} \in C_1$ and $\tilde{y} \in C_{-1}$.

If t_1 and t_2 are either both zero or one, then either $\alpha = 0$ or $\beta = 0$. Without loss of generality let us say $\alpha = 0$, then $w = \beta \bar{y}$. Now choose $\bar{x} \in C_1$ arbitrarily, and apply the considerations above to the point $w = 0\bar{x} + \beta \bar{y}$. The claim follows.

Note that K can be written as

$$\begin{aligned} K &= \{r^{1/p}x + (1-r)^{1/p}y\colon\ x \in C_1,\ y \in C_{-1},\ 0 \leqslant r \leqslant 1\} \\ &= \{r^{1/p}v + (1-r)^{1/p}w + [r^{1/p} - (1-r)^{1/p}]e_n\colon\ v, w \in B_\infty^{n-1},\ 0 \leqslant r \leqslant 1\} \\ &= \{[r^{1/p} + (1-r)^{1/p}]z + [r^{1/p} - (1-r)^{1/p}]e_n\colon\ z \in B_\infty^{n-1},\ 0 \leqslant r \leqslant 1\} \\ &= \{f(t)z + te_n\colon\ z \in B_\infty^{n-1},\ -1 \leqslant t \leqslant 1\}, \end{aligned} \tag{1}$$

where $B_\infty^{n-1} = [-1, 1]^{n-1} \subset \mathbb{R}^{n-1}$ and f is a function on $[-1, 1]$ defined as the solution $s = f(t)$ of

$$\left(\frac{s+t}{2}\right)^p + \left(\frac{s-t}{2}\right)^p = 1, \quad s \geqslant |t|,\ -1 \leqslant t \leqslant 1.$$

Let $L = IK$ be the intersection body of K. Then

$$\rho_L(e_n) = |K \cap e_n^\perp| = (2f(0))^{n-1} = \left(\frac{4}{2^{1/p}}\right)^{n-1}.$$

In order to compute the volume of the central section of K orthogonal to $(e_1 + e_n)/\sqrt{2}$, use (1) to notice that its projection onto $x_1 = 0$ coincides with $K \cap e_1^\perp$. Therefore

$$\rho_L((e_1 + e_n)/\sqrt{2}) = \sqrt{2}\rho_L(e_1) = 2\sqrt{2}\int_0^1 [2f(t)]^{n-2}\,dt.$$

Let $L = IK$ be q-convex. In order to estimate q, we will use the inequality

$$\|\sqrt{2}e_n\|_L^q \leqslant \left\|\frac{e_n + e_1}{\sqrt{2}}\right\|_L^q + \left\|\frac{e_n - e_1}{\sqrt{2}}\right\|_L^q = 2\left\|\frac{e_n + e_1}{\sqrt{2}}\right\|_L^q,$$

that is

$$\frac{\sqrt{2}}{\rho_L(e_n)} \leqslant \frac{2^{1/q}}{\rho_L((e_1 + e_n)/\sqrt{2})}.$$

Thus, we have

$$2^{1/q} \geqslant \left(\frac{2^{1/p}}{2}\right)^{n-1} 2\int_0^1 f(t)^{n-2}\,dt.$$

We now estimate the latter integral. From the definition of f it follows that $f(t) \geqslant \frac{2}{2^{1/p}}$ and $f(t) \geqslant t$ for $t \in [0,1]$. Taking the maximum of these two functions, we get

$$f(t) \geqslant \begin{cases} \frac{2}{2^{1/p}}, & \text{for } 0 \leqslant t \leqslant \frac{2}{2^{1/p}}, \\ t, & \text{for } \frac{2}{2^{1/p}} \leqslant t \leqslant 1. \end{cases}$$

Therefore,

$$\int_0^1 f(t)^{n-2}\,dt \geqslant \int_0^{\frac{2}{2^{1/p}}} \left(\frac{2}{2^{1/p}}\right)^{n-2} dt + \int_{\frac{2}{2^{1/p}}}^1 t^{n-2}\,dt$$
$$= \left(\frac{2}{2^{1/p}}\right)^{n-1} + \frac{1}{n-1}\left(1 - \left(\frac{2}{2^{1/p}}\right)^{n-1}\right). \tag{2}$$

Hence,

$$2^{1/q} \geqslant 2\left(1 + \frac{1}{n-1}\left(\left(\frac{2^{1/p}}{2}\right)^{n-1} - 1\right)\right),$$

which implies

$$q \leqslant \left[\left(\frac{1}{p} - 1\right)(n-1) + 1 + \log_2 \frac{(n-2)(\frac{2}{2^{1/p}})^{n-1} + 1}{n-1}\right]^{-1}.$$

Denoting

$$g_n(p) = \log_2 \frac{(n-2)(\frac{2}{2^{1/p}})^{n-1} + 1}{n-1},$$

we get the statement of the theorem. □

We will use the above example to show that in general the intersection body operator does not improve the Banach–Mazur distance to the Euclidean ball B_2^n.

Theorem 5. *Let $p \in (0,1)$ and let c be any constant satisfying $1 < c < 2^{1/p-1}$. Then for all large enough n, there exists a p-convex body $K \subset \mathbb{R}^n$ such that*

$$c^n d_{BM}\big(K, B_2^n\big) < d_{BM}\big(IK, B_2^n\big).$$

Proof. We will consider K from the previous theorem. One can see that $K \subset B_\infty^n \subset \sqrt{n} B_2^n$. Also note that for any $a \in B_\infty^n$, there exist $x \in C_1$, $y \in C_{-1}$ and $\lambda \in [0,1]$ such that $a = \lambda x + (1-\lambda) y$. Then we have $\|a\|_K^p \leqslant \lambda^p + (1-\lambda)^p \leqslant 2^{1-p}$. Therefore, $K \supset 2^{\frac{p-1}{p}} B_\infty^n \supset 2^{\frac{p-1}{p}} B_2^n$, and thus

$$d_{BM}\big(K, B_2^n\big) \leqslant 2^{\frac{1-p}{p}} \sqrt{n}. \tag{3}$$

Next we would like to provide a lower bound for $d_{BM}(IK, B_2^n)$. Let E be an ellipsoid such that $E \subset IK \subset dE$, for some d. Then

$$IK \subset \operatorname{conv}(IK) \subset dE \quad \text{and} \quad \frac{1}{d}\operatorname{conv}(IK) \subset E \subset IK.$$

Therefore, $1/d \leqslant 1/r$, where $r = \min\{t\colon \operatorname{conv}(IK) \subset tIK\}$. Thus,

$$d_{BM}\big(IK, B_2^n\big) \geqslant r = \max\left\{\frac{\rho_{\operatorname{conv}(IK)}(\theta)}{\rho_{IK}(\theta)}, \theta \in S^{n-1}\right\} \geqslant \frac{\rho_{\operatorname{conv}(IK)}(e_n)}{\rho_{IK}(e_n)}.$$

The convexity of $\operatorname{conv}(IK)$ gives

$$\begin{aligned}\rho_{\operatorname{conv}(IK)}(e_n) &\geqslant \left\| \frac{1}{2}\left(\rho_{IK}\left(\frac{e_n+e_1}{\sqrt{2}}\right)\frac{e_n+e_1}{\sqrt{2}} + \rho_{IK}\left(\frac{e_n-e_1}{\sqrt{2}}\right)\frac{e_n-e_1}{\sqrt{2}}\right)\right\|_2 \\ &= \frac{1}{\sqrt{2}}\rho_{IK}\left(\frac{e_n+e_1}{\sqrt{2}}\right).\end{aligned}$$

Combining the above inequalities with inequality (2) from the previous theorem, we get

$$d_{BM}\big(IK, B_2^n\big) \geqslant \frac{\rho_{IK}(\frac{e_n+e_1}{\sqrt{2}})}{\sqrt{2}\rho_{IK}(e_n)} \geqslant \left(\frac{2^{1/p}}{2}\right)^{n-1}\frac{1}{n-1}.$$

Comparing this with (3) we get the statement of the theorem. □

3. Generalization to log-concave measures

A measure μ on $\mathbb{R}^n$ is called log-concave if for any measurable $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$, we have

$$\mu\big(\lambda A + (1-\lambda)B\big) \geqslant \mu(A)^{\lambda}\mu(B)^{(1-\lambda)}$$

whenever $\lambda A + (1-\lambda)B$ is measurable.

Borell [8] has shown that a measure μ on $\mathbb{R}^n$ whose support is not contained in any affine hyperplane is a log-concave measure if and only if it is absolutely continuous with respect to the Lebesgue measure, and its density is a log-concave function.

To extend Busemann's theorem to log-concave measures on $\mathbb{R}^n$, we need the following theorem of Ball [4,3].

Theorem 6. *Let $f : \mathbb{R}^n \to [0, \infty)$ be an even log-concave function satisfying $0 < \int_{\mathbb{R}^n} f < \infty$ and let $k \geqslant 1$. Then the map*

$$x \longmapsto \left[\int_0^{\infty} f(rx)r^{k-1}\,dr\right]^{-\frac{1}{k}}$$

defines a norm on $\mathbb{R}^n$.

An immediate consequence of Ball's theorem is a generalization of the classical Busemann theorem to log-concave measures on $\mathbb{R}^n$.

Let μ be a measure on $\mathbb{R}^n$, absolutely continuous with respect to the Lebesgue measure m, and f its density function. If f is locally integrable on k-dimensional affine subspaces of $\mathbb{R}^n$, then we denote by $\mu_k = f m_k$ the restriction of μ to k-dimensional subspaces, where m_k is the k-dimensional Lebesgue measure.

Define the *intersection body* $I_\mu K$ of a star body K *with respect to* μ by

$$\rho_{I_\mu K}(u) = \mu_{n-1}\big(K \cap u^\perp\big), \quad u \in S^{n-1}.$$

Let μ be a symmetric log-concave measure on $\mathbb{R}^n$ and K a symmetric convex body in $\mathbb{R}^n$. Let f be the density of the measure μ. If we apply Theorem 6 to the log-concave function $1_K f$, we get a symmetric convex body L whose Minkowski functional is given by

$$\|x\|_L = \left[(n-1)\int_0^\infty (1_K f)(rx) r^{n-2}\,dr\right]^{-\frac{1}{n-1}}.$$

Then for every $u \in S^{n-1}$,

$$\begin{aligned}\mu_{n-1}\big(K\cap u^\perp\big) &= \int_{S^{n-1}\cap u^\perp}\int_0^\infty (1_K f)(r\theta) r^{n-2}\,dr\,d\theta \\ &= \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \|\theta\|_L^{-n+1}\,d\theta \\ &= \big|L\cap u^\perp\big|.\end{aligned}$$

Using Theorem 1 for the convex body L, one immediately obtains the following version of Busemann's theorem for log-concave measures.

Theorem 7. *Let μ be a symmetric log-concave measure on $\mathbb{R}^n$ and K a symmetric convex body in $\mathbb{R}^n$. Then the intersection body $I_\mu K$ is convex.*

In order to generalize Theorem 3 to log-concave measures, we will first prove a version of Ball's theorem (Theorem 6) for p-convex bodies.

Theorem 8. *Let $f:\mathbb{R}^n \to [0,\infty)$ be an even log-concave function, $k \geqslant 1$, and K a p-convex body in $\mathbb{R}^n$ for $0 < p \leqslant 1$. Then the body L defined by the Minkowski functional*

$$\|x\|_L = \left[\int_0^{\|x\|_K^{-1}} f(rx) r^{k-1}\,dr\right]^{-\frac{1}{k}}, \quad x \in \mathbb{R}^n,$$

is p-convex.

Proof. Fix two non-parallel vectors $x_1, x_2 \in \mathbb{R}^n$ and denote $x_3 = x_1 + x_2$. We claim that $\|x_3\|_L^p \leqslant \|x_1\|_L^p + \|x_2\|_L^p$. Consider the following 2-dimensional bodies in the plane $E = \operatorname{span}\{x_1, x_2\}$,

$$\bar{K} = \left\{ \frac{t_1 x_1}{\|x_1\|_K} + \frac{t_2 x_2}{\|x_2\|_K}\colon t_1, t_2 \geqslant 0,\ t_1^p + t_2^p \leqslant 1 \right\}$$

and

$$\bar{L} = \left\{ x \in \mathbb{R}^n\colon \|x\|_{\bar{L}} = \left[\int_0^{\|x\|_{\bar{K}}^{-1}} f(rx) r^{k-1}\, dr \right]^{-\frac{1}{k}} \leqslant 1 \right\}.$$

One can see that the boundary of $\bar{K}$ consists of a p-arc connecting the points $\frac{x_1}{\|x_1\|_K}$ and $\frac{x_2}{\|x_2\|_K}$, and two straight line segments connecting the origin with these two points. Clearly $\bar{K}$ is p-convex and $\bar{K} \subset K$. Also note that $\|x_i\|_{\bar{K}} = \|x_i\|_K$ for $i = 1, 2$, since $\frac{x_1}{\|x_1\|_K}$ and $\frac{x_2}{\|x_2\|_K}$ are on the boundary of $\bar{K}$, and $\|x_3\|_{\bar{K}} \geqslant \|x_3\|_K$ since $\bar{K} \subset K$. It follows that $\|x_i\|_{\bar{L}} = \|x_i\|_L$ $(i = 1, 2)$, and $\|x_3\|_{\bar{L}} \geqslant \|x_3\|_L$.

Consider the point $y = \frac{\|x_1\|_{\bar{L}}}{\|x_1\|_{\bar{K}}} x_1 + \frac{\|x_2\|_{\bar{L}}}{\|x_2\|_{\bar{K}}} x_2$ in the plane E. The point $\frac{y}{\|y\|_{\bar{K}}}$ lies on the p-arc connecting $\frac{x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{x_2}{\|x_2\|_{\bar{K}}}$. Consider the tangent line to this arc at the point $\frac{y}{\|y\|_{\bar{K}}}$. This line intersects the segments $[0, x_i/\|x_i\|_{\bar{K}}]$, $i = 1, 2$, at some points $\frac{t_i x_i}{\|x_i\|_{\bar{K}}}$ with $t_i \in (0, 1)$.

Since $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$, $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ and $\frac{y}{\|y\|_{\bar{K}}}$ are on the same line, it follows that the coefficients of $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ in the equality

$$\frac{y}{\|y\|_{\bar{K}}} = \frac{1}{\|y\|_{\bar{K}}} \left(\frac{\|x_1\|_{\bar{L}}}{t_1} \cdot \frac{t_1 x_1}{\|x_1\|_{\bar{K}}} + \frac{\|x_2\|_{\bar{L}}}{t_2} \cdot \frac{t_2 x_2}{\|x_2\|_{\bar{K}}} \right)$$

have to add up to 1. Therefore,

$$\|y\|_{\bar{K}} = \frac{\|x_1\|_{\bar{L}}}{t_1} + \frac{\|x_2\|_{\bar{L}}}{t_2}.$$

Note also that the line between $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$ and $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ separates $\frac{x_3}{\|x_3\|_{\bar{K}}}$ from the origin, which means that the three points $\frac{t_1 x_1}{\|x_1\|_{\bar{K}}}$, $\frac{t_2 x_2}{\|x_2\|_{\bar{K}}}$ and $\frac{x_3}{\|x_3\|_{\bar{K}}}$ are in the "convex position". Applying Ball's theorem on log-concave functions (Theorem 6) to these three points, we have

$$\left[\int_0^{\frac{1}{\|x_3\|_{\bar{K}}}} f(rx_3) r^{k-1}\, dr \right]^{-\frac{1}{k}} \leqslant \left[\int_0^{\frac{t_1}{\|x_1\|_{\bar{K}}}} f(rx_1) r^{k-1}\, dr \right]^{-\frac{1}{k}} + \left[\int_0^{\frac{t_2}{\|x_2\|_{\bar{K}}}} f(rx_2) r^{k-1}\, dr \right]^{-\frac{1}{k}}.$$

If we let $s_i = \|x_i\|_{\bar{L}} [\int_0^{\frac{t_i}{\|x_i\|_{\bar{K}}}} f(rx_i) r^{k-1}\, dr]^{\frac{1}{k}}$ for each $i = 1, 2$, the above inequality becomes

$$\|x_3\|_{\bar{L}} \leqslant \frac{\|x_1\|_{\bar{L}}}{s_1} + \frac{\|x_2\|_{\bar{L}}}{s_2}.$$

By a change of variables, we get

$$s_i = t_i \|x_i\|_{\bar{L}} \left[\int_0^{\frac{1}{\|x_i\|_{\bar{K}}}} f(t_i r x_i) r^{k-1}\, dr \right]^{\frac{1}{k}} \geqslant t_i \|x_i\|_{\bar{L}} \left[\int_0^{\frac{1}{\|x_i\|_{\bar{K}}}} f(r x_i) r^{k-1}\, dr \right]^{\frac{1}{k}} = t_i$$

for each $i = 1, 2$. The above inequality comes from the fact that an even log-concave function has to be non-increasing on $[0, \infty)$. Indeed,

$$f(t_i r x_i) = f\left(\frac{1+t_i}{2} \cdot r x_i - \frac{1-t_i}{2} \cdot r x_i\right) \geqslant f(r x_i)^{\frac{1+t_i}{2}} f(-r x_i)^{\frac{1-t_i}{2}} = f(r x_i).$$

Putting all together, we have

$$\|x_3\|_L \leqslant \|x_3\|_{\bar{L}} \leqslant \frac{\|x_1\|_{\bar{L}}}{s_1} + \frac{\|x_2\|_{\bar{L}}}{s_2} \leqslant \frac{\|x_1\|_{\bar{L}}}{t_1} + \frac{\|x_2\|_{\bar{L}}}{t_2} = \|y\|_{\bar{K}}.$$

Using the p-convexity of $\bar{K}$, we have

$$\|y\|_{\bar{K}}^p \leqslant \left\| \frac{\|x_1\|_{\bar{L}}}{\|x_1\|_{\bar{K}}} x_1 \right\|_{\bar{K}}^p + \left\| \frac{\|x_2\|_{\bar{L}}}{\|x_2\|_{\bar{K}}} x_2 \right\|_{\bar{K}}^p = \|x_1\|_{\bar{L}}^p + \|x_2\|_{\bar{L}}^p = \|x_1\|_L^p + \|x_2\|_L^p,$$

and therefore $\|x_3\|_L^p \leqslant \|x_1\|_L^p + \|x_2\|_L^p$. □

Corollary 1. *Let μ be a symmetric log-concave measure and K a symmetric p-convex body in $\mathbb{R}^n$ for $p \in (0, 1]$. Then the intersection body $I_\mu K$ of K is q-convex with $q = [(1/p - 1)(n-1) + 1]^{-1}$.*

Proof. Let f be the density function of μ. By Theorem 8, the body L with the Minkowski functional

$$\|x\|_L = \left[(n-1) \int_0^{\|x\|_K^{-1}} f(rx) r^{n-2}\, dr \right]^{\frac{-1}{n-1}}, \quad x \in \mathbb{R}^n,$$

is p-convex.

On the other hand, the intersection body $I_\mu K$ of K is given by the radial function

$$\begin{aligned}
\rho_{I_\mu K}(u) &= \mu_{n-1}\big(K \cap u^\perp\big) = \int_{\mathbb{R}^n} 1_{K \cap u^\perp}(x) f(x)\, dx \\
&= \int_{S^{n-1} \cap u^\perp} \int_0^{\|u\|_K^{-1}} f(rv) r^{n-2}\, dr\, dv = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \|v\|_L^{-n+1}\, dv \\
&= \big|L \cap u^\perp\big|_{n-1} = \rho_{IL}(u),
\end{aligned}$$

which means $I_\mu K = IL$. By Theorem 2, IL is q-convex with $q = [(1/p-1)(n-1)+1]^{-1}$, and therefore so is $I_\mu K$. $\square$

We conclude this section with an example that shows that the condition on f to be even in Theorem 8 cannot be dropped.

Example 1. Let μ be a log-concave measure on $\mathbb{R}^n$ with density

$$f(x_1,\dots,x_n) = \begin{cases} 1, & \text{if } x_1+x_2 \geqslant 2^{1-1/p}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the p-convex body $K = B_p^n$ for $p \in (0,1)$. If L is the body defined in Theorem 8, then $\|e_1+e_2\|_L = 0$ and $\|e_1\|_L = \|e_2\|_L > 0$, which means L is not q-convex for any $q > 0$.

4. Non-symmetric cases and s-concave measures

Note that Ball's theorem (Theorem 6) remains valid even if f is not even, as was shown by Klartag [26]. On the other hand, as we explained above, Theorem 8 does not hold for non-symmetric log-concave measures. However, if we restrict ourselves to the class of s-concave measures, $s > 0$, then it is possible to give a version of Theorem 8 for non-symmetric measures.

Borell [8] introduced the classes $\mathfrak{M}_s(\Omega)$ ($-\infty \leqslant s \leqslant \infty$, $\Omega \subset \mathbb{R}^n$ open convex) of s-concave measures, which are Radon measures μ on Ω satisfying the following condition: the inequality

$$\mu\big(\lambda A + (1-\lambda)B\big) \geqslant \big[\lambda\mu(A)^s + (1-\lambda)\mu(B)^s\big]^{\frac{1}{s}}$$

holds for all nonempty compact $A, B \subset \Omega$ and all $\lambda \in (0,1)$. In particular, $s = 0$ gives the class of log-concave measures.

Let us consider the case $0 < s < 1/n$. According to Borell, μ is s-concave if and only if the support of μ is n-dimensional and $d\mu = f\,dm$ for some $f \in L^1_{loc}(\Omega)$ such that $f^{\frac{s}{1-ns}}$ is a concave function on Ω.

Theorem 9. *Let μ be an s-concave measure on $\Omega \subset \mathbb{R}^n$ with density f, for $0 < s < 1/n$, and K a p-convex body in Ω, for $p \in (0,1]$. If $k \geqslant 1$, then the body L whose Minkowski functional is given by*

$$\|x\|_L = \left[\int_0^\infty 1_K(rx) f(rx) r^{k-1}\,dr\right]^{-\frac{1}{k}}, \quad x \in \mathbb{R}^n$$

is q-convex with $q = [(\frac{1}{p}-1)(\frac{1}{s}-n)\frac{1}{k} + \frac{1}{p}]^{-1}$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$ and $x_3 = x_1 + x_2$. Then, for $i = 1, 2$,

$$\|x_i\|_L^{-k} = \int_0^\infty 1_K(rx_i) f(rx_i) r^{k-1}\,dr$$

$$= \frac{1}{p} \int_0^\infty 1_K\big(s^{-\frac{1}{p}} x_i\big) f\big(s^{-\frac{1}{p}} x_i\big) s^{-\frac{k}{p}-1}\, ds$$

$$= \frac{1}{p} \int_0^\infty F_i(s)\, ds$$

$$= \frac{1}{p} \int_0^\infty \big|\big\{s \in (0,\infty):\ F_i(s) > t\big\}\big|\, dt,$$

where $F_i(s) = 1_K(s^{-\frac{1}{p}} x_i) f(s^{-\frac{1}{p}} x_i) s^{-\frac{k}{p}-1}$ for each $i = 1, 2, 3$. We claim that

$$2^{\frac{k}{q}+1} F_3(s_3) \geqslant F_1(s_1)^{\lambda_1} F_2(s_2)^{\lambda_2}$$

whenever $s_3 = s_1 + s_2$ and $\lambda_i = \frac{s_i}{s_1+s_2}$ for $i = 1, 2$. Indeed, since

$$\begin{aligned} s_3^{-\frac{1}{p}} x_3 &= \left(\frac{s_1}{s_1+s_2}\right)^{\frac{1}{p}} s_1^{-\frac{1}{p}} x_1 + \left(\frac{s_2}{s_1+s_2}\right)^{\frac{1}{p}} s_2^{-\frac{1}{p}} x_2 \\ &= \lambda_1\big(\lambda_1^{\frac{1}{p}-1} s_1^{-\frac{1}{p}} x_1\big) + \lambda_2\big(\lambda_2^{\frac{1}{p}-1} s_2^{-\frac{1}{p}} x_2\big), \end{aligned}$$

the concavity of f^γ, $\gamma = \frac{s}{1-ns}$, gives

$$\begin{aligned} f^\gamma\big(s_3^{-\frac{1}{p}} x_3\big) &\geqslant \lambda_1 f^\gamma\big(\lambda_1^{\frac{1}{p}-1} s_1^{-\frac{1}{p}} x_1\big) + \lambda_2 f^\gamma\big(\lambda_2^{\frac{1}{p}-1} s_2^{-\frac{1}{p}} x_2\big) \\ &\geqslant \big[f^\gamma\big(\lambda_1^{\frac{1}{p}-1} s_1^{-\frac{1}{p}} x_1\big)\big]^{\lambda_1} \big[f^\gamma\big(\lambda_2^{\frac{1}{p}-1} s_2^{-\frac{1}{p}} x_2\big)\big]^{\lambda_2} \\ &\geqslant \big[\lambda_1^{\frac{1}{p}-1} f^\gamma\big(s_1^{-\frac{1}{p}} x_1\big)\big]^{\lambda_1} \big[\lambda_2^{\frac{1}{p}-1} f^\gamma\big(s_2^{-\frac{1}{p}} x_2\big)\big]^{\lambda_2} \\ &= \left(\left[\left(\frac{s_1}{s_3}\right)^{\frac{1}{\gamma}(\frac{1}{p}-1)} f\big(s_1^{-\frac{1}{p}} x_1\big)\right]^{\lambda_1} \left[\left(\frac{s_2}{s_3}\right)^{\frac{1}{\gamma}(\frac{1}{p}-1)} f\big(s_2^{-\frac{1}{p}} x_2\big)\right]^{\lambda_2}\right)^{\gamma}, \end{aligned}$$

that is,

$$s_3^{\frac{1}{\gamma}(\frac{1}{p}-1)} f\big(s_3^{-\frac{1}{p}} x_3\big) \geqslant \prod_{i=1}^{2} \big[s_i^{\frac{1}{\gamma}(\frac{1}{p}-1)} f\big(s_i^{-\frac{1}{p}} x_i\big)\big]^{\lambda_i}.$$

On the other hand, note that

$$\frac{2}{s_3} = \frac{\lambda_1}{s_1} + \frac{\lambda_2}{s_2} \geqslant \left(\frac{1}{s_1}\right)^{\lambda_1} \left(\frac{1}{s_2}\right)^{\lambda_2}$$

and

$$1_K\big(s_3^{-\frac{1}{p}}x_3\big) \geqslant 1_K\big(s_1^{-\frac{1}{p}}x_1\big)1_K\big(s_2^{-\frac{1}{p}}x_2\big),$$

since $s_3^{-\frac{1}{p}}x_3 = \lambda_1^{\frac{1}{p}}(s_1^{-\frac{1}{p}}x_1) + \lambda_2^{\frac{1}{p}}(s_2^{-\frac{1}{p}}x_2)$. Thus

$$\begin{aligned}
F_1(s_1)^{\lambda_1}F_2(s_2)^{\lambda_2} &= \prod_{i=1}^{2}\big[1_K\big(s_i^{-\frac{1}{p}}x_i\big)f\big(s_i^{-\frac{1}{p}}x_i\big)s_i^{-\frac{k}{p}-1}\big]^{\lambda_i} \\
&\leqslant 1_K\big(s_3^{-\frac{1}{p}}x_3\big)\prod_{i=1}^{2}\left[s_i^{\frac{1}{\gamma}(\frac{1}{p}-1)}f\big(s_i^{-\frac{1}{p}}x_i\big)\cdot\left(\frac{1}{s_i}\right)^{\frac{1}{\gamma}(\frac{1}{p}-1)+\frac{k}{p}+1}\right]^{\lambda_i} \\
&\leqslant 1_K\big(s_3^{-\frac{1}{p}}x_3\big)f\big(s_3^{-\frac{1}{p}}x_3\big)2^{\frac{1}{\gamma}(\frac{1}{p}-1)+\frac{k}{p}+1}s_3^{-\frac{k}{p}-1} \\
&\leqslant 2^{\frac{k}{q}+1}F_3(s_3).
\end{aligned}$$

It follows that for every $t > 0$,

$$\big\{s_3\colon\ 2^{\frac{k}{q}+1}F_3(s_3) > t\big\} \supset \big\{s_1\colon\ F_1(s_1) > t\big\} + \big\{s_2\colon\ F_2(s_2) > t\big\}.$$

Applying the Brunn–Minkowski inequality, we have

$$\begin{aligned}
\|x_1 + x_2\|_L^{-k} = \|x_3\|_L^{-k} &= \frac{1}{p}\int_0^\infty F_3(s)\,ds \\
&= \frac{1}{2^{\frac{k}{q}+1}}\cdot\frac{1}{p}\int_0^\infty \big|\big\{s_3 \in (0,\infty)\colon\ 2^{\frac{k}{q}+1}F_3(s_3) > t\big\}\big|\,dt \\
&\geqslant \frac{1}{2^{\frac{k}{q}+1}}\cdot\frac{1}{p}\int_0^\infty \big(\big|\big\{s_1\colon\ F_1(s_1) > t\big\}\big| + \big|\big\{s_2\colon\ F_2(s_2) > t\big\}\big|\big)\,dt \\
&= \frac{1}{2^{\frac{k}{q}+1}}\big(\|x_1\|_L^{-k} + \|x_2\|_L^{-k}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_1 + x_2\|_L &\leqslant 2^{\frac{1}{q}}\left(\frac{\|x_1\|_L^{-k} + \|x_2\|_L^{-k}}{2}\right)^{-\frac{1}{k}} \\
&= \left[\frac{1}{2}\left(\frac{(\|x_1\|_L^{-q})^{\frac{k}{q}} + \big(\|x_1\|_L^{-q}\big)^{\frac{k}{q}}}{2}\right)^{\frac{q}{k}}\right]^{-\frac{1}{q}} \\
&\leqslant \left[\frac{1}{2}\left(\frac{\|x_1\|_L^{-q} + \|x_2\|_L^{-q}}{2}\right)\right]^{-\frac{1}{q}}
\end{aligned}$$

$$\leqslant \left[\frac{1}{2}\left(\frac{\|x_1\|_L^q + \|x_2\|_L^q}{2}\right)^{-1}\right]^{-\frac{1}{q}}$$
$$= \left(\|x_1\|_L^q + \|x_2\|_L^q\right)^{\frac{1}{q}},$$

which means that L is q-convex. □

The following example shows that the value of q in the above theorem is sharp.

Example 2. Let μ be an s-concave measure on $\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_1 \geqslant 0\}$ for $s > 0$ with density

$$f(x_1, \dots, x_n) = |x_1|^{1/s-n}$$

and let

$$K = \left\{(x_1, \dots, x_n)\colon\ x_1 \geqslant 0,\ \left|\frac{x_1 + x_2}{2}\right|^p + \left|\frac{x_1 - x_2}{2}\right|^p \leqslant 1,\ |x_i| \leqslant 1\ \forall i = 3, \dots, n\right\}.$$

Note that $\|e_1\|_K = 2^{1-1/p}$ and $\|e_1 + e_2\|_K = \|e_1 - e_2\|_K = 1$. If L is the body defined by K in the above theorem, then

$$\|e_1\|_L = \left[\int_0^{2^{1-1/p}} r^{1/s-n} r^{k-1}\, dr\right]^{-\frac{1}{k}} = \left[\frac{2^{(1-\frac{1}{p})(\frac{1}{s}-n+k)}}{\frac{1}{s} - n + k}\right]^{-\frac{1}{k}}$$

and

$$\|e_1 + e_2\|_L = \left[\int_0^1 r^{1/s-n} r^{k-1}\, dr\right]^{-\frac{1}{k}} = \left[\frac{1}{s} - n + k\right]^{\frac{1}{k}}.$$

If L is q-convex for some q, then the inequality $\|2e_1\|_L \leqslant (\|e_1 + e_2\|_L^q + \|e_1 - e_2\|_L^q)^{1/q}$ implies

$$2\left[\frac{2^{(1-\frac{1}{p})(\frac{1}{s}-n+k)}}{\frac{1}{s} - n + k}\right]^{-\frac{1}{k}} \leqslant 2^{\frac{1}{q}}\left[\frac{1}{s} - n + k\right]^{\frac{1}{k}}$$

that is,

$$q \leqslant \left[\left(\frac{1}{p} - 1\right)\left(\frac{1}{s} - n\right)\frac{1}{k} + \frac{1}{p}\right]^{-1}.$$

Note that in our construction Ω is not open, as opposed to what we said in the beginning of Section 4. This is done for the sake of simplicity of the presentation. To be more precise one would need to define $\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_1 > -\epsilon\}$ and $f(x_1, \dots, x_n) = |x_1 + \epsilon|^{1/s-n}$, for $\epsilon > 0$, and then send $\epsilon \to 0^+$.

Acknowledgments

We are indebted to Alexander Litvak and Dmitry Ryabogin for valuable discussions.

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