On Representations of Completely Bounded Maps*

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We discuss two different representations for completely bounded maps on C*-algebras and obtain some uniqueness criteria for the representations. © 1991 Academic Press, Inc.

1. Introduction

A question of Kadison [5] asks whether or not every bounded homomorphism from a C*-algebra into the algebra of operators on a Hilbert space, L(H), is similar to a *-homomorphism. Hadwin [2] has shown that a bounded unital homomorphism from a C*-algebra into L(H) is similar to a *-homomorphism if and only if the homomorphism belongs to the span of the completely positive maps. Wittstock [10] and Paulsen [6] proved that the span of the completely positive maps from a C*-algebra into an injective C*-algebra is identical with the set of completely bounded maps. Together these two results prove that a bounded unital homomorphism from a C*-algebra into L(H) is similar to a *-homomorphism if and only if it is completely bounded. Recently, Paulsen [7] proved that a bounded linear operator on a Hilbert space is similar to a contraction if and only if it is completely polynomially bounded. This gives a partial answer to problem 6 of [3]. Therefore, the set of completely bounded maps between C*-algebras has been shown to play an important role in the study of several problems in C*-algebras and operator theory.

2. Completely Bounded Maps

Let A and B be C*-algebras. By a subspace of A we mean a (not necessarily closed) complex linear subspace. Let $M_n$ denote the C*-algebra

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of complex $n \times n$ matrices. If $\mathcal{L}$ is a subspace of $A$ and $L: \mathcal{L} \to B$ is a linear map, then we set $L_n = L \otimes I_n: \mathcal{L} \otimes M_n \to B \otimes M_n$ by

$$L_n(a \otimes b) = L(a) \otimes b.$$ 

The map $L$ is positive if $L(a) \geq 0$ whenever $a \in \mathcal{L}$ and $a \geq 0$. The map $L$ is completely positive if $L_n$ is positive for all $n = 1, 2, \ldots$. $L$ is completely bounded if $\sup_n \|L_n\|$ is finite, and we let

$$\|L\|_{cb} = \sup_n \|L_n\|.$$ 

If $\|L\|_{cb} \leq 1$, then $L$ is called a complete contraction (or completely contractive). If $\mathcal{L} \subseteq A$ is a linear subspace with $\mathcal{L} = \mathcal{L}^*$ and $L: \mathcal{L} \to B$ is linear, then we define a linear map $L^*: \mathcal{L} \to B$ by

$$L^*(a) = L(a^*)^*.$$ 

Let $\text{Re}(L): \mathcal{L} \to B$ and $\text{Im}(L): \mathcal{L} \to B$ be defined by

$$\text{Re}(L) = \frac{1}{2} (L + L^*)$$

and

$$\text{Im}(L) = \frac{1}{2i} (L - L^*)$$

so that

$$L = \text{Re}(L) + i \text{Im}(L)$$

with $\text{Re}(L)^* = \text{Re}(L)$, $\text{Im}(L)^* = \text{Im}(L)$, and $\text{Im}(L) = \text{Re}(-iL)$.

**Definition 2.1.** A $\ast$-representation $\pi$ of a C*-algebra $A$ is a $\ast$-homomorphism of $A$ into the C*-algebra $L(H)$ of all bounded operators on some Hilbert space $H$.

**Theorem 2.2.** Let $A$ be a unital C*-algebra and let $L: A \to L(H)$ be linear. Then the following are equivalent:

1. $L$ is a complete contraction,
2. there exist a Hilbert space $K$, isometries $V_i: H \to K$, for $i = 1, 2$, and a unital $\ast$-representation $\tilde{\pi}: A \to L(K)$, such that $L(a) = V_1^* \tilde{\pi}(a) V_2$, and
(3) there exist a Hilbert space $M$, an isometry $V: H \to M$, a contraction $T: M \to V(H)$, and a unital $*$-representation $\pi: A \to L(M)$ with $M = \overline{\operatorname{span}} \pi(A)^V H$, such that $L(a) = V^* T\pi(a) V$.

**Proof.** By [6, Theorem 2.7], we know that (1) implies (2) is true. So suppose that (2) is true. Let $M = \operatorname{span} \pi(A)^V H$, so that $K = M \oplus M^\perp$. As $M$ and $M^\perp$ are reducing subspace for $\pi(A)$, we may let $\pi_1$, $\pi_2$ be the restrictions of $\pi$ to $M$ and $M^\perp$, respectively. Observe that $V_2: H \to M$. Let $P_M: K \to M$ be the orthogonal projection, then

$$P_M \in \pi(A)'$$

and

$$L(A)\zeta = V_2^* V_2 V_1^* \pi(a) V_2 \zeta = V_2^* V_2 V_1^* (\pi_1 \oplus \pi_2)(a) V_2 \zeta = V_2^* V_2 V_1^* P_M \pi_1(a) V_2 \zeta = V_2^* V_2 V_1^* |_M \pi_1(a) V_2 \zeta,$$

for all $a \in A$ and $\zeta \in H$.

Let $k_1 \in M$, $k_2 \in M$, and we have that

$$\pi(I_A)(k_1 \oplus k_2) = \pi_1(I_A)k_1 \oplus \pi_2(I_A)k_2 = k_1 \oplus k_2.$$

Therefore, $\pi_1: A \to L(M)$ is a unital $*$-representation. Set $\pi_1 = \pi$, $V_2 = V$, and $T = V_2 V_1^* |_M$, then

$$L(a) = V^* T \pi(a) V$$

and

$$M = \overline{\operatorname{span}} \pi(A)^V H = \overline{\operatorname{span}} \pi(A)^V H.$$

Thus (3) is true. Suppose that (3) is true, let $(a_{ij}) \in A \otimes M_n$, then

$$\|L_n((a_{ij}))\| = \|(V^* T \pi(a_{ij}) V)\|$$

$$\leq \|V\|^2 \|T\| \|\pi\| \|(a_{ij})\| \leq \|(a_{ij})\|$$

for all positive integers $n \geq 1$. Hence (1) holds true.

**Corollary 2.3.** Let $A$ be a unital $C^*$-algebra and let $L: A \to L(H)$ be a nonzero completely bounded map. Then there exist a Hilbert space $M$, an
isometry \( V: H \to M \), a bounded linear operator \( T: M \to M \), and a unital *-representation \( \pi: A \to L(M) \) with \( M = \text{span} \{ \pi(a) \} \) \( VH \), such that
\[
L(a) = V^* T \pi(a) V \quad \text{with} \quad \| T \| = \| L \|_{cb}.
\]

**Proof.** Consider \( L/\| L \|_{cb} \).

**Definition 2.4.** Let \( A \) be a unital C*-algebra and let \( L: A \to L(H) \) be a completely bounded map. Hadwin [2] defines
\[
\beta(L) = \inf \{ \| V \| \| W \| : L(a) = V \pi(a) W, \text{for some *-representation } \pi \text{ of } A \}.
\]

**Corollary 2.5.** Let \( A \) be a unital C*-algebra and let \( L: A \to L(H) \) be a completely bounded map. Then \( \| L \|_{cb} = \beta(L) \).

**Proof.** If \( L(a) = V_1 \pi(a) V_2 \) for some *-representation \( \pi \) of \( A \), then
\[
\| L_n((a_{iy})) \| = \left\| \begin{pmatrix} V_1 & 0 & & & \\ & \ddots & & & \\ & & V_1 & & \\ 0 & & & V_2 & \end{pmatrix} \left( \begin{pmatrix} \pi(a_{ij}) \\ \vdots \\ \pi(a_{ij}) \\ 0 \\ \ast \\ V_2 \end{pmatrix} \right) \right\| 
\leq \| V_1 \| \| V_2 \| \| (a_{ij}) \|, \quad \text{for all positive integers } n \geq 1.
\]

Hence \( \| L \|_{cb} \leq \beta(L) \). On the other hand, by Corollary 2.3, there exist a Hilbert space \( M \), an isometry \( \tilde{V}: H \to M \), a bounded linear operator \( T: M \to M \), and a unital *-representation \( \tilde{\pi}: A \to L(M) \), such that
\[
L(a) = \tilde{V}^* T \tilde{\pi}(a) \tilde{V} \quad \text{with} \quad \| T \| = \| L \|_{cb}.
\]

Thus \( \| \tilde{V}^* T \| \| \tilde{V} \| \leq \| T \| = \| L \|_{cb} \), hence \( \beta(L) \leq \| L \|_{cb} \).

**Remark 2.6.** We can use Corollary 2.5 to prove that Hadwin's conjecture II [2] is true. Since by [8, Theorem 2.10], we know that \( \| L_n \| = \| L \|_{cb} \). Hence \( \beta(L) = \| L_n \| \).

**Lemma 2.7.** Let \( A \) be a C*-algebra, let \( H \) be a Hilbert space, and let \( F: A \to L(H) \) be a complete contraction. Then the map \( L: A \oplus A \oplus \cdots \oplus A \to L(H \oplus H \oplus \cdots \oplus H) \) (\( n \) copies of \( A \) and \( H \)) defined by
\[
(a_1, a_2, ..., a_n) \to \begin{pmatrix} F(a_1) \\ \sqrt{n} \\ \vdots \\ F(a_n) \\ \sqrt{n} \end{pmatrix}
\]

is a non-unital complete contraction.
Proof. Let \((a_1, a_2, \ldots, a_n) \in A \oplus A \oplus \cdots \oplus A\) with \(\|(a_1, a_2, \ldots, a_n)\|=\|a_j\|\), for \(1 \leq j \leq n\). Then

\[
\|L(a_1, a_2, \ldots, a_n)\| = \left\| \begin{pmatrix} F(a_1)^* & \cdots & F(a_n)^* \\ \sqrt{n} & \cdots & \sqrt{n} \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} F(a_1) \\ \sqrt{n} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \cdots 0 \\ \sqrt{n} \end{pmatrix} \right\|^{1/2}
\]

\[
= \left\| \sum_{i=1}^{n} \frac{F(a_i)^* F(a_i)}{n} \right\|^{1/2} \leq \left( \frac{\|F\|}{n} \sum_{i=1}^{n} \|a_i\|^2 \right)^{1/2} \leq \|a_j\|
\]

Let \(m\) be a positive integer. We know that \((A \oplus A \oplus \cdots \oplus A) \otimes M_m\) is isomorphic to \((A \otimes M_m) \oplus \cdots \oplus (A \otimes M_m)\) \((n \text{ copies of } A \text{ and } A \otimes M_m)\). Thus we can identify them. Under this identification we observe that the map

\[
L_m: (A \oplus A \oplus \cdots \oplus A) \otimes M_m \to M_n(L(H)) \otimes M_m.
\]

Let \(o_i \in A \otimes M_m\), for \(i = 1, 2, \ldots, n\). Then

\[
\|L_m(o_1 \oplus o_2 \oplus \cdots \oplus o_n)\|
\]

\[
= \left\| \begin{pmatrix} F_m(o_1) \\ \sqrt{n} \\ \vdots \\ F_m(o_n) \end{pmatrix} \begin{pmatrix} 0 \cdots 0 \\ \sqrt{n} \end{pmatrix} \begin{pmatrix} 0 \cdots 0 \end{pmatrix} \right\|^{1/2}
\]

\[
= \left\| \sum_{i=1}^{n} \frac{F_m(o_i)^* F_m(o_i)}{n} \right\|^{1/2} \leq \left( \frac{1}{n} \sum_{i=1}^{n} \|F_m(o_i)\|^2 \right)^{1/2}
\]

\[
\leq \|F_m(o_1) \oplus \cdots \oplus F_m(o_n)\| \leq \|o_1 \oplus o_2 \oplus \cdots \oplus o_n\|.
\]

Hence

\[
\|L_m\| \leq 1, \quad \text{for all } m \geq 1.
\]
Remark 2.8. Put $A = C$ and $H = C$ in the above lemma. Then the map $L: C \oplus C \oplus \cdots \oplus C \to M_n$ defined by

$$L(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_n) = \begin{pmatrix}
\frac{\lambda_1}{\sqrt{n}} & 0 & \cdots & 0 \\
\frac{\lambda_2}{\sqrt{n}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\lambda_n}{\sqrt{n}} & 0 & \cdots & 0
\end{pmatrix}$$

is a complete contraction.

Example 2.9. This is an example of Theorem 2.2. From the above lemma, we know that the map $L: C \oplus C \to M_2$ defined by

$$L(\lambda_1 \oplus \lambda_2) = \begin{pmatrix}
\frac{\lambda_1}{\sqrt{2}} & 0 \\
\frac{\lambda_2}{\sqrt{2}} & 0
\end{pmatrix}$$

is a complete contraction. If we define isometries by

$$V_1 = \begin{pmatrix}1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1\end{pmatrix}, \quad V_2 = \begin{pmatrix}1 \\
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
0 \\
1 \\
0 \\
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
1 \\
0 \\
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
1 \\
0 \\
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
1 \\
0 \\
\frac{1}{\sqrt{2}} \\
0 \end{pmatrix},$$

and a unital $\ast$-representation $\pi: C \oplus C \to M_4$ by

$$\pi(\lambda_1 \oplus \lambda_2) = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix},$$

then $L(\lambda_1 \oplus \lambda_2) = V_1^* \pi(\lambda_1 \oplus \lambda_2) V_2$, $\|V_2 V_1^*\| = \|L\|_{cb} = 1$, and $\text{span} \pi(C \oplus C)V_2C^2 = C^4$. 

3. Unitary Equivalence for Representations of Completely Bounded Maps

Let $A$ be a unital $C^*$-algebra, let $H$ be a Hilbert space, and let $\phi$ be a completely positive linear map of $A$ into $L(H)$. By Stinespring’s theorem [9], we know that $\phi$ has a representation $V^*\pi(\cdot)V$, where $\pi$ is a $*$-representation of $A$ on some Hilbert space $K$ and $V$ is a bounded linear operator from $H$ to $K$. By [11], we know that the triple $(\pi, V, K)$ is minimal under the condition $\text{span} \, \pi(A)VH=K$. That is, if $(\pi', V', K')$ is another triple for $\phi$ with $\text{span} \, \pi'(A)V'H=K'$, then there exists a unitary $U: K \rightarrow K'$, such that

$$\pi'(a)=U\pi(a)U^* \quad \text{and} \quad UV=V'.$$

Hence the two representations for $\phi$ are unitarily equivalent. In this section we attempt to develop some uniqueness criteria for the representations of completely bounded maps presented in the previous section.

**Lemma 3.1.** Let $A$ be a $C^*$-algebra and let $L: A \rightarrow L(H)$ be a completely bounded map. Let $L(a)=V^*\pi(a)V_2$ and $\text{span} \, \pi(A)V_iH=M_i$ for $i=1,2$, where $V_i$ are bounded operators of $H$ into a Hilbert space $K$ and $\pi$ is a $*$-representation of $A$ into $L(K)$. Then $M_1+M_2=K$ if and only if the map

$$\psi: A \otimes M_2 \rightarrow L(H \oplus H)$$

defined by

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} V_1^*\pi(a)V_1 & V_1^*\pi(b)V_2 \\ V_2^*\pi(c)V_1 & V_2^*\pi(d)V_2 \end{pmatrix}$$

is a minimal completely positive map.

**Proof.** If $M_1+M_2=K$, by the definition of $\psi$, we know that

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix},$$

and hence $\psi$ is a completely positive map. Set

$$\hat{\pi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \pi(a) \\ \pi(c) \end{pmatrix} \begin{pmatrix} \pi(b) \\ \pi(d) \end{pmatrix},$$

then $\hat{\pi}$ is a $*$-representation.

We claim that

$$\text{span} \, \hat{\pi}(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H) = K \oplus K.$$
Let $h_1, h_2 \in H$. We have that

$$
\hat{\pi} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (h_1 \oplus h_2) \right) = (\pi(a) V_1 h_1 + \pi(b) V_2 h_2) \oplus (\pi(c) V_1 h_1 + \pi(d) V_2 h_2).
$$

Hence

$$
\text{span } \hat{\pi}(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H) \subseteq (M_1 + M_2) \oplus (M_1 + M_2).
$$

Let $(\sum_i \pi(a_i) V_1 h_i + \sum_j \pi(b_j) V_2 \zeta_j) \oplus (\sum_k \pi(c_k) V_1 \tau_k + \sum_l \pi(d_l) V_2 t_l) \in (M_1 + M_2) \oplus (M_1 + M_2)$. Since

$$
\begin{pmatrix} \pi(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (h \oplus h) = \pi(a) V_1 h \oplus 0,
$$

$$
\begin{pmatrix} 0 & \pi(a) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (h \oplus h) = \pi(a) V_2 h \oplus 0,
$$

and

$$
\begin{pmatrix} 0 & 0 \\ \pi(a) & 0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (h \oplus h) = 0 \oplus \pi(a) V_1 h,
$$

are in $\hat{\pi}(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H)$ and

$$
\left( \sum_i \pi(a_i) V_1 h_i + \sum_j \pi(b_j) V_2 \zeta_j \right) \oplus \left( \sum_k \pi(c_k) V_1 \tau_k + \sum_l \pi(d_l) V_2 t_l \right)
\begin{align*}
= & \sum_i (\pi(a_i) V_1 h_i \oplus 0) + \sum_j (\pi(b_j) V_2 \zeta_j \oplus 0) + \sum_k (0 \oplus \pi(c_k) V_1 \tau_k) \\
& + \sum_l (0 \oplus \pi(d_l) V_2 t_l),
\end{align*}
$$

we have that

$$(M_1 + M_2) \oplus (M_1 + M_2) = \text{span } \left( \hat{\pi}(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H) \right).$$

Hence

$$
\text{span } \hat{\pi}(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H) = (M_1 + M_2) \oplus (M_1 + M_2)
$$

$$
= M_1 + M_2 \oplus M_1 + M_2 = K \oplus K.
$$
Conversely, since
\[
\text{span } \pi(A \otimes M_2) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (H \oplus H) = M_1 + M_2 + M_1 + M_2 = K \oplus K,
\]
we have that
\[
M_1 + M_2 = K.
\]

Remark 3.2. In the above lemma, if \( V_1^* \pi(\cdot) V_1 \) or \( V_2^* \pi(\cdot) V_2 \) is minimal, then \( \psi \) is minimal.

Definition 3.3. Let \( A \) be a \( C^* \)-algebra and let \( L: A \to L(H) \) be a completely bounded map. We say that \( (\pi, V_1, V_2, K) \) is a minimal representation of \( L \), provided there exist bounded operators \( V_i \) of \( H \) into a Hilbert space \( K \) for \( i = 1, 2 \), and a unital \( * \)-representation of \( A \) into \( L(K) \), such that
\[
L(a) = V_1^* \pi(a) V_1 \quad \text{and} \quad \text{span } \pi(A) V_1 H + \text{span } \pi(A) V_2 H = K.
\]

Theorem 3.4. Let \( A \) be a unital \( C^* \)-algebra and let \( L: A \to L(H) \) be a completely bounded map. If \( (\pi_1, V_1, V_2, K) \) and \( (\pi_2, V_1', V_2', K') \) of \( L \) are minimal, then \( V_1^* \pi_1(a) V_1 = V_1'^* \pi_2(a) V_1' \) and \( V_2^* \pi_1(a) V_2 = V_2'^* \pi_2(a) V_2' \) if and only if there exists a unitary \( U: K \to K' \), such that
\[
UV_1 = V_1', \quad UV_2 = V_2', \quad \text{and} \quad U \pi_1(a) U^* = \pi_2(a).
\]

Proof. If the necessary part is true, then the sufficient part is obvious. Now, if the sufficient part is true, let the map \( \psi: A \otimes M_2 \to L(H \oplus H) \) be defined by
\[
\psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} V_1^* \pi_1(a) V_1 & V_1^* \pi_1(b) V_2 \\ V_2^* \pi_1(c) V_1 & V_2^* \pi_1(d) V_2 \end{pmatrix}.
\]
Since
\[
\psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} \pi_1(a) & \pi_1(b) \\ \pi_1(c) & \pi_1(d) \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} V_1' & 0 \\ 0 & V_2' \end{pmatrix} \begin{pmatrix} \pi_2(a) & \pi_2(b) \\ \pi_2(c) & \pi_2(d) \end{pmatrix} \begin{pmatrix} V_1' & 0 \\ 0 & V_2' \end{pmatrix},
\]
\[
\text{span } \pi_1(A) V_1 H + \text{span } \pi_1(A) V_2 H = K,
\]
and
\[
\text{span } \pi_2(A) V_1' H + \text{span } \pi_2(A) V_2' H = K',
\]
by Lemma 3.1, we know that

\[
\begin{pmatrix}
V_1 & 0 \\
0 & V_2
\end{pmatrix}^* \begin{pmatrix}
\pi_1(\cdot) & \pi_1(\cdot) \\
\pi_1(\cdot) & \pi_1(\cdot)
\end{pmatrix} \begin{pmatrix}
V_1 & 0 \\
0 & V_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
V'_1 & 0 \\
0 & V'_2
\end{pmatrix}^* \begin{pmatrix}
\pi_2(\cdot) & \pi_2(\cdot) \\
\pi_2(\cdot) & \pi_2(\cdot)
\end{pmatrix} \begin{pmatrix}
V'_1 & 0 \\
0 & V'_2
\end{pmatrix}
\]

are minimal completely positive maps. By [11], there exists a unitary \( \bar{U} : K \oplus K \to K' \oplus K' \),

\[
\bar{U} \begin{pmatrix}
V_1 & 0 \\
0 & V_2
\end{pmatrix} = \begin{pmatrix}
V'_1 & 0 \\
0 & V'_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\pi_2(\cdot) & \pi_2(\cdot) \\
\pi_2(\cdot) & \pi_2(\cdot)
\end{pmatrix} = \bar{U} \begin{pmatrix}
\pi_1(\cdot) & \pi_1(\cdot) \\
\pi_1(\cdot) & \pi_1(\cdot)
\end{pmatrix} \bar{U}^*.
\]

Let \( k \in K \) and \( \bar{U}(k \oplus 0) = k'_1 \oplus k'_2 \), then

\[
k'_1 \oplus k'_2 = \bar{U} \begin{pmatrix}
I_k & 0 \\
0 & 0
\end{pmatrix} (k \oplus 0) = \bar{U} \begin{pmatrix}
\pi_1(I_k) & 0 \\
0 & 0
\end{pmatrix} (k \oplus 0)
\]

\[
= \begin{pmatrix}
\pi_2(I_A) & 0 \\
0 & 0
\end{pmatrix} \bar{U}(K \oplus 0) = \begin{pmatrix}
I_{k'} & 0 \\
0 & 0
\end{pmatrix} \bar{U}(k \oplus 0)
\]

\[
= \begin{pmatrix}
I_{k'} & 0 \\
0 & 0
\end{pmatrix} (k'_1 \oplus k'_2) = k'_1 \oplus 0.
\]

Hence \( \bar{U}(K \oplus 0) \subset K' \oplus 0 \). Similarly, \( \bar{U}(0 \oplus K) \subset 0 \oplus K' \). Thus there exist unitaries \( U_i : K \to K' \) for \( i = 1, 2 \), such that

\[
\bar{U}(k_1 \oplus k_2) = U_1(k_1) \oplus U_2(k_2) \quad \text{for} \quad k_1, k_2 \in K.
\]

We may write

\[
\bar{U} = \begin{pmatrix}
U_1 & 0 \\
0 & U_2
\end{pmatrix}.
\]

Since

\[
\begin{pmatrix}
\pi_2(\cdot) & \pi_2(\cdot) \\
\pi_2(\cdot) & \pi_2(\cdot)
\end{pmatrix} = \begin{pmatrix}
U_1 & 0 \\
0 & U_2
\end{pmatrix} \begin{pmatrix}
\pi_1(\cdot) & \pi_1(\cdot) \\
\pi_1(\cdot) & \pi_1(\cdot)
\end{pmatrix} \begin{pmatrix}
U_1^* & 0 \\
0 & U_2^*
\end{pmatrix},
\]
we have that
\[ \pi_2(a) = U_1 \pi_1(a) U_2^* \quad \text{and} \quad U_1 = U_2. \]
Set \( U_i = U \), then
\[ U \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} V_1' & 0 \\ 0 & V_2' \end{pmatrix}. \]
Thus
\[ UV_1 = V_1', \quad UV_2 = V_2', \quad \text{and} \quad \pi_2(a) = U \pi_1(a) U^*. \]

**Example 3.5.** This example shows that the condition \( V_2^* \pi_1(a) V_2 = V_2^* \pi_2(a) V_2^* \) in Theorem 3.4 is necessary. Let \( L: C \oplus C \to L(C^2) \) be the linear map defined by
\[ L(\lambda_1 \oplus \lambda_2) = \begin{pmatrix} \lambda_1 \\ \sqrt{2} \\ \lambda_2 \\ \sqrt{2} \\ 0 \end{pmatrix}. \]
We can find isometries
\[
V_1 = V_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \\ 1 \sqrt{2} \\ 0 \end{pmatrix}, \quad V_2' = \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \\ 1 \sqrt{2} \\ 0 \end{pmatrix},
\]
and the unital *-representation \( \pi: C \oplus C \to L(C^4) \) defined by
\[ \pi(\lambda_1 \oplus \lambda_2) = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_2 \end{pmatrix}, \]
such that
\[ L(\lambda_1 \oplus \lambda_2) = V_1^* \pi(\lambda_1 \oplus \lambda_2) V_2 = V_1^* \pi(\lambda_1 \oplus \lambda_2) V_2'. \]
Note that

$$V_1^*(\lambda_1 \oplus \lambda_2) V_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = V_1^* \pi(\lambda_1 \oplus \lambda_2) V_1$$

and

$$V_2^* \pi(\lambda_1 \oplus \lambda_2) V_2 = \begin{pmatrix} \frac{\lambda_1 + \lambda_2}{2} & 0 \\ 0 & \frac{\lambda_1 + \lambda_2}{2} \end{pmatrix}$$

$$\neq V_2^* \pi(\lambda_1 \oplus \lambda_2) V_2' = \begin{pmatrix} \frac{\lambda_1 + \lambda_2}{2} & 0 \\ 0 & \frac{2\lambda_1 + \lambda_2}{3} \end{pmatrix}.$$ 

It is easy to see that $(\pi, V_1, V_2, \mathcal{C}^4)$ and $(\pi, V_1', V_2', \mathcal{C}^4)$ are minimal representations of $L$. But we cannot get a unitary $U$, such that

$$UV_1 = V_1', \quad UV_2 = V_2', \quad \text{and} \quad U\pi = \pi U.$$ 

**Lemma 3.6.** Let $H$ and $M$ be Hilbert spaces, let $V: H \to M$ be an isometry, and let $T: M \to V(H)$ be a contraction. Then there exist a Hilbert space $K$ containing $M$, an isometry $V_1: H \to K$, such that

$$T = VV_1^*|_M$$ 

with $\|T\| = \|V_1^*|_M\|$.

**Proof:** Set $T_0 = V^* T$, so that $T = VT_0$. Since $T$ is a contraction, by [4], there exist a Hilbert space $K$ containing $M$, a unitary dilation $U: K \to K$ of $T$, and an orthogonal projection $P_M: K \to M$, such that

$$V T_0 P_M = P_M U P_M.$$ 

Let $V': H \to V'(H)$ be the unitary defined by $V'h = Vh$ for $h \in H$. Let $m \in M$, and we compute

$$(V')^{-1} P_{R(V)} U (m) = (V')^{-1} P_{R(V)} P_M U P_M (m) = (V')^{-1} P_{R(V)} VT_0 P_M (m)$$

$$= (V')^{-1} VT_0 (m) = T_0 (m).$$

Hence $(V')^{-1} P_{R(V)} U|_M = T_0$.

Let $h \in H$ and $S = (V')^{-1} P_{R(V)} U$, and we compute

$$\langle S^* h, S^* h \rangle = \langle (V')^{-1} h, (V')^{-1} h \rangle = \|h\|^2.$$ 

Hence $S^*$ is an isometry.
Let \( V_1 = S^* \), then \( V_1^*|_M = (V')^{-1} P_{R(V)} U|_M = T_0 \). Thus \( T = VV_1^*|_M \).

Since \( T_0 = V^* T \), we have that \( \| T_0 \| = \| V_1^* \| \leq \| T \| \). Hence \( \| T \| = \| V_1^* \| \).

**DEFINITION 3.7.** Let \( A \) be a C*-algebra and let \( L: A \to L(H) \) be a completely bounded map. We say that \((\pi, V, T, K)\) is a minimal isometric representation of \( L \), provided there exist an isometry \( V \) of \( H \) into a Hilbert space \( K \), a bounded linear operator \( T \) of \( K \) into \( VH \) with \( \| T \| = \| L \|_{cb} \), and a unital *-representation \( \pi \) of \( A \) into \( L(K) \) with \( \text{span} \pi(A)VH = K \), such that

\[
L(a) = V^* T \pi(a) V.
\]

**THEOREM 3.8.** Let \( A \) be a C*-algebra and let \( L: A \to L(H) \) be a completely bounded map. If \( (\pi, V, T, K) \) and \( (\pi', V', T', M') \) are minimal isometric representations of \( L \), then there exists a unitary \( U: M \to M' \), such that

\[
UTU^* = T',
\]

if and only if \( V^* \pi(a) V = V'^* \pi'(a) V' \).

**Proof:** If the sufficient part is true, then the necessary part is obvious. Now, if the necessary part is true then since \( V^* \pi(a) V = V'^* \pi'(a) V' \) is a minimal completely positive map, by [11], there exists a unitary \( U: M \to M' \) such that

\[
UV = V', \quad U \pi(a) U^* = \pi'(a), \quad \text{and} \quad UTU^* = T',
\]

We observe that \( T \) is a bounded operator of \( M \) into \( VH \), by Lemma 3.6, and that there exists a Hilbert space \( K \) containing \( M \), an isometry \( V_1: H \to K \), such that

\[
T = \| L \|_{cb} VV_1^*|_M.
\]

Similarly, \( T' = \| L \|_{cb} V' V_1^*|_{M'} \). Thus

\[
L(a) = V'^* T' \pi'(a) V' = (UV)^* \| L \|_{cb} (UV) V_1^*|_M (U \pi(a) U^*) (UV) - \| L \|_{cb} V_1^*|_M U \pi(a) V - \| L \|_{cb} V_1^*|_M \pi(a) V.
\]

Hence \( V_1^*|_M U = V_1^*|_M \). So we have that

\[
T' = \| L \|_{cb} V' V_1^*|_{M'} = (UV) \| L \|_{cb} V_1^*|_M = U \| L \|_{cb} VV_1^*|_M U^* = UTU^*.
\]
REFERENCES