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About Systems of Equations, X-Separability, and Left-Invertibility in the λ-Calculus*

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A system of equations in the λ-calculus is a pair (Γ, X), where Γ is a set of formulas of Λ (the equations) and X is a finite set of variables of Λ (the unknowns.) A system $\mathscr{S} = (\Gamma, X)$ is said to be solvable in the theory **T** (**T**-solvable) iff there exists a suitable simultaneous substitution for the unknowns that makes the equations of $\mathscr{S}$ theorems in the theory **T**. For any finite system and within any semisensible (sms) theory **T** (e.g., β, $\beta\eta$, $\mathscr{H}^*$) a necessary condition for **T**-solvability is proved. A class of systems for which this condition also becomes sufficient is shown and the sufficiency is proved constructively. This class properly contains the systems $\mathscr{S} = (\Gamma, \{x_1, ..., x_w\})$ that satisfy 0, 1 or 0, 2 of the following hypotheses:

Hp.0. (0) If Q is a proper subterm of a LHS term of an equation and the head of Q is an unknown then the degree of Q is not *too large*.

(1) The initial part of a LHS term never *collapses* with another LHS term.

Hp.1. The equations of **S** have the shape $xM_1 \cdots M_n = yx_1 \cdots x_w M_1 \cdots M_n$, where $x \in \{x_1, ..., x_w\}$ and y does not occur in the LHS terms of the equations of $\mathscr{S}$.

Hp.2. The equations of $\mathscr{S}$ have the shape $xM_1 \cdots M_n = N$, where $x \in \{x_1, ..., x_w\}$ and N is a $\beta\eta$-normal form whose free variables do not occur in the LHS terms of the equations of $\mathscr{S}$.

With some caution we can also mix equations having the shape in Hp.1 with equations having the shape in Hp.2. A typical result is the constructive characterization of the **T**-solvability (**T** sms) of systems having the shape $\mathscr{S} = (\{xx = N_0, xM_1 = N_1, ..., xM_n = N_n\}, \{x\})$, where $M_1, ..., M_n$ are closed λ-terms and $N_0, ..., N_n$ are $\beta\eta$-normal forms which do not contain the unknown x. When the equations of a system $\mathscr{S} = (\Gamma, X)$ have the shape $M = y$, with the RHS variables fresh and pairwise distinct, we have te X-separability problem for the LHS terms. For a class of λ-free sets (see Hp.0) the X-separability is constructively characterized within any sms theory. A single equation can be solved via a system of equations. Using this idea we characterize the $\beta\eta$-left-invertibility for a class of λ-terms.

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0. Motivations

A lot of problems that arise in an equational theory, such as the λ-calculus, can be recast as the solution of systems of equations.

0.0. Example. The search for a singleton basis (Δ) of Λ^0 (the set of closed λ-terms) can be transformed into the search for solutions of a suitable system of equations. It is known [Bar 84, p. 184] that $\Delta \in \Lambda$ s.t. $\Delta\Delta =_\beta \mathbf{K}$ and $\Delta\mathbf{K} =_\beta \mathbf{S}$ is a singleton basis for Λ^0 (see 2.0 for the definition of $\mathbf{K}$, $\mathbf{S}$, $\mathbf{U}_i^n$, $\mathbf{\Omega}$). A solution, e.g., $\Delta =_\beta \lambda t \cdot t\ (t\mathbf{U}_3^3(\mathbf{U}_1^7\mathbf{K})\,\mathbf{\Omega}\,\mathbf{\Omega}\,\mathbf{S})\,\mathbf{\Omega}$, is known, but no systematic method is given in the literature to solve these kinds of equations.

The set of λ-terms modulo $\beta\eta$-convertibility forms a monoid with $\mathbf{I}$ as an identity and composition defined by $M \cdot N \equiv \mathbf{B}MN$ (where $\mathbf{B}$ is defined in 2.0) [Chu 37]. The $\beta\eta$-invertibility problem was first raised in [CFC 58, pp. 167, 168] and solved in [Dez 76, BK 80]. The one side $\beta\eta$-invertibility problems (left and right) are still open. The $\beta\eta$-left-invertibility problem can be presented as a system of equations. Self-application is widely involved in this problem.

0.1. Example. Let $M \in \Lambda$; we wish to find $L \in \Lambda$ s.t. $L(My) =_\eta y$ (where y does not occur in LM). Let $M \equiv \lambda yx \cdot y(xy(\lambda t \cdot x(xt)))(x(xxxx)(\lambda t \cdot x(xx))(x(x\mathbf{\Omega})))$. We transform the equation $L(My) =_\eta y$ into a system of equations. Let $C[\ ] \equiv (\lambda x \cdot [\ ])\ \Delta \in \Lambda[\ ]$ (see Section 2 for $\Lambda[\ ]$) s.t.

$$C[xy(\lambda t \cdot x(xt))] =_\eta y_1, \qquad C[x(xxxx)(\lambda t \cdot x(xx))(x(x\mathbf{\Omega}))] =_\eta y_2$$

(self-application occurs on x). If we set $L \equiv \lambda t y_1 y_2 \cdot t\Delta$ then it is easy to verify that $L(My) =_\eta y$. A possible choice for Δ is

$$\Delta \equiv D[y_1 := \mathbf{U}_1^3\, y_1][y_2 := \mathbf{U}_1^4\, y_2],$$
$$D \equiv \lambda t_1 t_2 \cdot (\mathbf{U}_3^3\, y_1)(\lambda abcd \cdot c(\lambda abcd \cdot c\mathbf{\Omega}(\mathbf{U}_1^5\, y_2)))\ t_1 t_2.$$

0.2. Example. Let $\mathscr{F}$ be a subset of Λ. We say that $\Delta \in \Lambda^0$ is a right identity for $\mathscr{F}$ iff $\forall M \in \mathscr{F}\ M\Delta =_\beta M$. A right identity for a set $\mathscr{F}$ does not always exist. It is easy to verify that the set $(\lambda x \cdot x,\ \lambda x \cdot x\mathbf{\Omega})$ does not have a right identity. Let $\mathscr{F} = \{M_1, M_2\}$, where $M_1 \equiv \lambda x \cdot xa_1(\lambda t \cdot x(xt))$, $M_2 \equiv \lambda x \cdot x\mathbf{\Omega}(\lambda t \cdot x(xx))(x(xa_2))$. We must find $\Delta \in \Lambda^0$ s.t.

$$M_i\Delta =_\beta M_i, \qquad i = 1, 2.$$

If we are able to find $Q \in \Lambda$ s.t.

$$\begin{aligned}
M_1 Q &=_\beta Q a_1(\lambda t \cdot Q(Qt)) =_\beta y_1 a_1(\lambda t \cdot Q(Qt)),\\
M_2 Q &=_\beta Q\boldsymbol{\Omega}(\lambda t \cdot Q(QQ))(Q(Qa_2))\\
&=_\beta y_2 a_2 \boldsymbol{\Omega}(\lambda t \cdot Q(QQ))(Q(Qa_2)),
\end{aligned}$$

then it is sufficient to replace the variables y_1, y_2 by suitable combinators that reconstruct the terms M_1, M_2. We have

$$\Delta \equiv Q[y_1 := \lambda abx \cdot xa(\lambda t \cdot x(xt))][y_2 := \lambda abcdx \cdot x\boldsymbol{\Omega}(\lambda t \cdot x(xx))(x(xa))].$$

It is easy to verify that Δ is a right identity for $\mathscr{F}$. We can set (D as in 0.1)

$$\begin{aligned}
Q &\equiv D[y_2 := \lambda abc \cdot y_2(Ec)\, abc],\\
E &\equiv \langle \mathbf{U}_3^4, \mathbf{U}_3^4 \rangle.
\end{aligned}$$

A system of equations can be viewed as the specification of a functional in an equational programming language. The solution then is just a program that satisfies the equations. In this respect a theory of systems of equations can be regarded as a theory of compilers for equational programming languages. For this topic we refer the reader to [O'D 85].

0.3. Example. Find a program f s.t.

(0) $f\underline{0} = y_0$,

(1) $f(\underline{s}a) = y_1 a f$,

(2) $ff = \mathbf{K}$,

(3) $f\,\mathbf{K} = \mathbf{S}$,

where $\underline{0} \equiv \lambda xy \cdot y$ and $\underline{s} \equiv \lambda ax \cdot xax$ (the numeral system $(\underline{0}, \underline{s})$ was introduced in [Ber 83]). Equations (0)–(3) amount to saying that f is a certain recursive function on the numeral system $(\underline{0}, \underline{s})$ (Eq. (0), (1) and f is a singleton basis for Λ^0 (Eq. (2), (3)). A possible solution for the system of Eq. (0)–(3) is

$$\begin{aligned}
G &\equiv \lambda t_1 t_2 \cdot t_2 \mathbf{P}_3(\mathbf{U}_1^7 y_0)(\lambda u_1 \cdots u_8 \cdot y_1 u_1(u_7 u_7))(\mathbf{U}_1^{15}\mathbf{K})\, \boldsymbol{\Omega}(\mathbf{U}_1^6 \mathbf{S})\, t_1 t_2,\\
f &\equiv GG =_\beta \lambda t \cdot t\mathbf{P}_3(\mathbf{U}_1^7 y_0)(\lambda u_1 \cdots u_8 \cdot y_1 u_1(u_7 u_7))(\mathbf{U}_1^{15}\mathbf{K})\, \boldsymbol{\Omega}(\mathbf{U}_1^6 \mathbf{S})\, Gt.
\end{aligned}$$

The major difficulty that we have had to surmount has been the treatment of self-application (see Examples 0.1, 0.2, and 0.3). This has been transformed into the search for a common solution of suitable separability problems (in the sense of [CDR 78]).

1. Summary

In this section we describe the structure of the paper.

Section 2 gives some notions about the λ-calculus. For an exhaustive treatment the reader is referred to [Bar 84]. Section 3 introduces an equivalence relation that models the indistinctness between pairs of λ-terms in a finite set $\mathscr{F}$. This relation will be a fundamental tool for stating a necessary condition of solvability for finite systems of equations (Section 5). Section 4 shows that substitutive contexts that preserve the relation introduced in Section 3 are the core of the solution strategy for a class of systems of equations (Section 6). Here a family of such contexts is constructed. Section 5 gives the notion of a system of equations and some of its easy properties. A necessary condition for the **T**-solvability (**T** sms) of a finite system of equations is proved. Sections 5.0–5.6 may also be read independent of Sections 3 and 4. Section 6 constructively proves for a class of systems a necessary and sufficient condition for **T**-solvability (**T** sms). Section 7 presents an application of the results of Section 6 to the X-separability problem. Finally, Section 8 applies the result of Section 7 to the $\beta\eta$-left-invertibility problem.

2. The λ-Calculus

Syntax. The λ-calculus is a formal theory whose language we denote by Λ. The elements of the set $\mathbf{V} = \{v_0, v_1, ...\}$ are said to be variables (of Λ). The symbols $x, y, z, ...$ denote arbitrary variables. The set Λ is the least set U s.t. $\mathbf{V} \subseteq U$; $M \in U \Rightarrow (\lambda x M) \in U$; $M, N \in U \Rightarrow (MN) \in U$.We call λ-terms the elements of Λ. The symbols $M, N, L, ...$ denote λ-terms.

We adopt the following conventions: the symbol $\equiv$ denotes the syntactic equality; $\vec{x} \equiv x_1, x_2, ..., x_n$; $\{\vec{x}\} \equiv \{x_1, x_2, ..., x_n\}$; $|\vec{x}| = n$; $\lambda\vec{x} \cdot M \equiv \lambda x_1 x_2 \cdots x_n \cdot M \equiv (\lambda x_1(\lambda x_2 \cdots (\lambda x_n M))$; $M_1 M_2 M_3 \cdots M_n \equiv (\cdots((M_1 M_2) M_3) \cdots M_n)$. We give some examples of λ-terms.

2.0. Example. The following are λ-terms: $\mathbf{U}_i^n \equiv \lambda x_1 \cdots x_n \cdot x_i$ with $n, i \in \mathbf{N}^+$ and $i \leqslant n$; $\mathbf{I} \equiv \mathbf{U}_1^1$; $\mathbf{K} \equiv \mathbf{U}_1^2$; $\mathbf{S} \equiv \lambda xyz \cdot xz(yz)$; $\mathbf{B} \equiv \lambda xyz \cdot x(yz)$; $\omega \equiv \lambda x \cdot xx$; $\mathbf{\Omega} \equiv \omega\omega$; $\mathbf{W} \equiv \lambda xy \cdot xyy$; $\mathbf{P}_q \equiv \lambda x_1 \cdots x_q x_{q+1} \cdot x_{q+1} x_1 \cdots x_q$ with $q \in \mathbf{N}$; $\langle M_1, ..., M_n \rangle \equiv \mathbf{P}_n M_1 \cdots M_n$ with $n \in \mathbf{N}$ and $M_1, ..., M_n \in \Lambda$.

A subterm of M is a string which occurs in M and belongs to Λ. The variable x is said to be bound in M if it occurs in the scope of a λx; x is said to be free otherwise. $\mathrm{FV}(M)$ is the set of the free variables of M. If $\mathrm{FV}(M) = \varnothing$ we say that M is closed (or M is a combinator). Let $X \subseteq \mathbf{V}$; we define $\Lambda^0 = \{M \in \Lambda \mid \mathrm{FV}(M) = \varnothing\}$, $\Lambda^0(X) = \{M \in \Lambda \mid \mathrm{FV}(M) \subseteq X\}$.

We identify λ-terms that are different only with regard to bound variables (i.e., α-convertible λ-terms). If $M_1, ..., M_n$ are λ-terms occurring in a mathematical context we suppose that all the bound variables in them are (or have been made) mutually distinct and differ from all the other variables occurring in that context.

We denote $M[x := N]$ as the λ-term obtained from M by substituting N for all the free occurrences of x in M.

A context is a λ-term with holes. More precisely the set $\Lambda[\ ]$ is the least set U s.t. $\mathbf{V} \subseteq U$; $[\ ] \in U$; $C[\ ] \in U \Rightarrow (\lambda x \cdot C[\ ]) \in U$; $C[\ ]$, $D[\ ] \in U \Rightarrow (C[\ ]\, D[\ ]) \in U$. The elements of $\Lambda[\ ]$ are called contexts. The symbols $C[\ ]$, $D[\ ]$, ... denote contexts. Let $C[\ ] \in \Lambda[\ ]$ and $M \in \Lambda \cdot C[M]$ is obtained from $C[\ ]$ by substituting M for any occurence of $[\ ]$ in $C[\ ]$. Let $\mathscr{F} \subseteq \Lambda$; we define $C[\mathscr{F}] = \{C[M] \mid M \in \mathscr{F}\}$.

Theories. Let $\Gamma \subseteq \Lambda$; we set $\text{Form}(\Gamma) = \{M = N \mid M, N \in \Gamma\}$. We call formulas (of Λ) the elements of $\text{Form}(\Lambda)$. A theory $\mathbf{T}$ is a subset of $\text{Form}(\Lambda)$. If $\mathbf{T}$ is a theory and $M = N \in \mathbf{T}$ we also write $M =_{\mathbf{T}} N$.

The theory $\boldsymbol{\lambda}$ (we also write β) is axiomatized by the following axioms and rules: $M = M$; $M = N \Rightarrow N = M$; $M = N$, $N = L \Rightarrow M = L$; $M = N \Rightarrow MZ = NZ$; $M = N \Rightarrow ZM = ZN$; $(\lambda x \cdot M)\, N = M[x := N]$; $M = N \Rightarrow \lambda x \cdot M = \lambda x \cdot N$. If $M = N \in \boldsymbol{\lambda}$ we also write $M =_{\beta} N$, but never $M = N$.

The theory $\boldsymbol{\lambda\eta}$ (we also write $\beta\eta$) is obtained by adding to the axioms and rules of $\boldsymbol{\lambda}$ the axiom schema $\lambda x \cdot Mx = M$, where $x \notin FV(M)$. We also write $M =_{\eta} N$ for $M =_{\boldsymbol{\lambda\eta}} N$. We say that $\lambda x \cdot Mx$ with $x \notin FV(M)$ is an η-expansion of M. We denote with $\boldsymbol{\lambda}\mathbf{T}$ ($\boldsymbol{\lambda\eta}\mathbf{T}$) the theory obtained by adding to the axioms and rules of $\boldsymbol{\lambda}$ ($\boldsymbol{\lambda\eta}$) the formulas of $\mathbf{T}$. A theory $\mathbf{T}$ is called a λ-theory if $\mathbf{T} \neq \text{Form}(\Lambda^0)$ and $\mathbf{T} = \boldsymbol{\lambda}\mathbf{T}$. In the following, unless otherwise stated, we consider λ-theories.

Semisensible Theories. We say that $M \in \Lambda$ is: a β-normal form (β-nf) if it does not have subterms of the shape $(\lambda x \cdot P)\, Q$; a $\beta\eta$-normal form ($\beta\eta$-nf) if it does not have subterms of the shape $(\lambda x \cdot P)\, Q$ or $(\lambda x \cdot Rx)$ with $x \notin FV(R)$. We say that $M \in \Lambda$ has β-nf if $\exists N \in \Lambda$ ($M =_{\beta} N$ and N is a β-nf), and $\beta\eta$-nf if $\exists N \in \Lambda$ ($M =_{\eta} N$ and N is a $\beta\eta$-nf). We define $\text{HNF} \equiv \{\lambda \vec{x} \cdot y M_1 M_2 \cdots M_m \mid M_1, M_2, ..., M_m \in \Lambda\}$. The elements of HNF are called head normal forms. We define $\text{SOL} \equiv \{M \in \Lambda \mid \exists N \in \text{HNF}\ M =_{\beta} N\}$. The elements of the set SOL are called solvable. If $M \notin \text{SOL}$ then M is called unsolvable. An example of an unsolvable term that we frequently use is $\boldsymbol{\Omega}$. A theory $\mathbf{T}$ is called semisensible (sms) iff $\mathbf{T} \subseteq \boldsymbol{\lambda}\mathscr{H}^*$, where $\mathscr{H}^* = \{M = N \in \text{Form}(\Lambda^0) \mid \forall C[\ ] \in \Lambda[\ ]\ (C[M] \in \text{SOL} \text{ iff } C[N] \in \text{SOL})\}$. We have

(0) $\boldsymbol{\lambda}$, $\boldsymbol{\lambda\eta}$, $\boldsymbol{\lambda}\mathscr{H}^*$ are λ-theories;

(1) $\boldsymbol{\lambda\eta}\mathscr{H}^* = \boldsymbol{\lambda}\mathscr{H}^*$.

We write $M =_{\mathscr{H}^*} N$ for $M =_{\boldsymbol{\lambda}\mathscr{H}^*} N$. We observe that β and $\beta\eta$ are semisensible theories.

Böhm trees. We define the Böhm tree (BT) of a λ-term M. We set

$$\mathrm{BT}(M) \equiv \bot \qquad \text{if} \quad M \notin \mathrm{SOL};$$

$$\begin{array}{ccc} & \mathrm{BT}(M) \equiv \lambda x_1 x_2 \cdots x_n \cdot y & \\ & \diagup \quad \diagdown & \\ \mathrm{BT}(M_1) & \cdots & \mathrm{BT}(M_m) \end{array} \qquad \text{if} \quad M =_\beta \lambda x_1 x_2 \cdots x_n \cdot y M_1 \cdots M_m.$$

We say that a λ-term is $\bot$-free if its BT does not have any node with label $\bot$. To any node in $\mathrm{BT}(M)$ it is possible to bind, in the usual way, a sequence of positive integers α ($\alpha \in \mathrm{Seq}$) ($\langle\ \rangle$ corresponds to the root of the tree, $\langle 1 \rangle$ corresponds to the most left son, and so on). If $\alpha \in \mathrm{Seq}$ corresponds to some node in $\mathrm{BT}(M)$ we write $\alpha \in \mathrm{BT}(M)$. We denote with $*$ the concatenation symbol for sequences.

2.1. Definition [Bar 84]. (0) Let $M, N \in \Lambda$, $\mathscr{F} \subseteq \Lambda$, and $\alpha \in \mathrm{BT}(M)$. The λ-term M_α is recursively defined by $M_\alpha \equiv M$ if $M \notin \mathrm{SOL}$; $M_{\langle\ \rangle} \equiv M$; $M_{\langle i \rangle * \rho} \equiv (M_i)_\rho$ if $M =_\beta \lambda x_1 x_2 \cdots x_n \cdot y M_1 \cdots M_m$ and $i \leqslant m$.

(1) We write $\beta < \alpha$ for $\exists \xi \neq \langle\ \rangle\ \beta * \xi = \alpha$.

(2) We write $\alpha \in_\eta \mathrm{BT}(M)$ for $\forall \beta < \alpha$ ($\beta \in \mathrm{BT}(M) \Rightarrow M_\beta \in \mathrm{SOL}$).

(3) We write $\alpha \in_\eta \mathrm{BT}(\mathscr{F})$ for $\forall M \in \mathscr{F}$ ($\alpha \in_\eta \mathrm{BT}(M)$).

(4) If $\alpha \in_\eta \mathrm{BT}(M)$ then M^α is the least η-expansion of M s.t. $\alpha \in \mathrm{BT}(M^\alpha)$.

(5) Let $\alpha \in_\eta \mathrm{BT}(M)$. We define $M_\alpha \equiv (M^\alpha)_\alpha$.

(6) If $\alpha \in_\eta \mathrm{BT}(\mathscr{F})$ then we set $\mathscr{F}_\alpha = (M_\alpha \mid M \in \mathscr{F}\}$ (analogously for $\mathscr{F}^\alpha$).

(7) We write $M|\alpha\!\downarrow$ for ($\alpha \in_\eta \mathrm{BT}(M)$ and $M_\alpha \in \mathrm{SOL}$, $M|\alpha\!\uparrow$ otherwise.

(8) We write $\mathscr{F}|\alpha\!\downarrow$ for $\forall M \in \mathscr{F}$ $M|\alpha\!\downarrow$, $\mathscr{F}|\alpha\!\uparrow$ otherwise.

(9) We define the functions deg, ord, head.
If $M =_\beta \lambda x_1 x_2 \cdots x_n \cdot y M_1 M_2 \cdots M_m$ we set $\deg(M) = m$; $\mathrm{ord}(M) = n$; $\mathrm{head}(M) \equiv y$. If $M \notin \mathrm{SOL}$ we set $\deg(M) = 0$; $\mathrm{ord}(M) = 0$; $\mathrm{head}(M)\!\uparrow$.

(10) $M \sim N$ iff ($M, N \notin \mathrm{SOL}$ or ($M, N \in \mathrm{SOL}$ and $\mathrm{head}(M) \equiv \mathrm{head}(N)$ and $\deg(M) - \mathrm{ord}(M) = \deg(N) - \mathrm{ord}(N)$)) [Böh 68; Bar 84, Sect. 10.2.19].

(11) $M \sim_\alpha N$ iff ($\alpha \notin_\eta \mathrm{BT}(M)$, $\mathrm{BT}(N)$ or ($\alpha \in_\eta \mathrm{BT}(M)$, $\mathrm{BT}(N)$ and $M_\alpha \sim N_\alpha$)) [CDR 78; Bar 84, Sect. 10.2.21].

(12) (0) $M \approx N$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$.

(1) $M \approx_\eta N$ iff $\forall \alpha \in \mathrm{Seq}\ M \sim_\alpha N$.

(2) $M \sqsubseteq N$ iff $\mathrm{BT}(M) \subseteq \mathrm{BT}(N)$.

2.2. THEOREM [Bar 84]. (0) *$M \approx_\eta N$ iff $M =_{\mathscr{H}^*} N$.*

(1) *$\forall C[\ ] \in \Lambda[\ ]\ \forall M, N \in \Lambda\ (M \sqsubseteq N \Rightarrow C[M] \sqsubseteq C[N])$.*

(2) *If $C[M] =_\beta N$ and N has β-nf then $\forall Q \in \Lambda\ (M \sqsubseteq Q \Rightarrow C[Q] =_\beta N)$.*

Note that for all $N \in \Lambda$ it holds that $\mathbf{\Omega} \sqsubseteq N$.

Separability and Distinction. The relations USF and AGT and the notions of distinction and separability for a set of λ-terms have been introduced in [CDR 78] (also see [Bar 84, pp. 256 et seq.]).

2.3. DEFINITION. Let $\alpha \in \mathrm{Seq}$, $\mathscr{F} = \{M_1, ..., M_n\} \subset \Lambda$ and $\mathbf{T}$ be a theory.

(0) We say that α is useful for $\mathscr{F}$ $((\mathscr{F}, \alpha) \in \mathrm{USF})$ iff

$$\mathscr{F}|\alpha\downarrow \qquad \text{and} \qquad \exists M, N \in \mathscr{F}\ (M \not\sim_\alpha N).$$

(1) We say that $\mathscr{F}$ agrees up to α ($(\mathscr{F}, \alpha) \in \mathrm{AGT}$ or α is agt for $\mathscr{F}$) iff

$$\forall M, N \in \mathscr{F}\ \forall \beta < \alpha \qquad (M \sim_\beta N).$$

(2) We say that $\mathscr{F}$ is distinct iff

$$\mathrm{Card}(\mathscr{F}) = 1 \qquad \text{or}$$

$$\exists \alpha \in \mathrm{Seq}\ (\alpha \text{ is useful for } \mathscr{F} \text{ and } \forall \mathscr{P} \in \mathscr{F}/\sim_\alpha \mathscr{P} \text{ is distinct}).$$

(3) We say that $\mathscr{F}$ is $\mathbf{T}$-separable iff $\exists C[\ ] \in \Lambda[\ ]$ s.t.

$$\forall i \in \{1, ..., n\} \qquad C[M_i] =_\mathbf{T} y_i \in (\mathbf{V} - FV(\mathscr{F}));$$
$$\forall i, j \in \{1, ..., n\} \qquad (y_i \equiv y_j \Rightarrow i = j).$$

2.4. THEOREM [CDR 78]. *Let $\mathscr{F} = \{M_1, ..., M_n\} \subset \Lambda$ and $\mathbf{T}$ be a sms theory.*

(0) *$\mathscr{F}$ is $\mathbf{T}$-separable iff $\mathscr{F}$ is distinct.*

(1) *Let $\alpha \in \mathrm{Seq}$ with $\mathscr{F}|\alpha\downarrow$ and $(\mathscr{F}, \alpha) \in \mathrm{AGT}$. Then $\exists C[\ ] \in \Lambda[\ ]$ s.t.*

$$\forall i \in \{1, ..., n\} \qquad C[M_i] =_\beta y_i \in (\mathbf{V} - FV(\mathscr{F}));$$
$$\forall M, N \in \mathscr{F} \qquad (C[M] =_\beta C[N] \text{ iff } M \sim_\alpha N).$$

3. $\mathscr{F}$-INDISTINCTNESS

Generalizing the concept of non-distinction for finite sets [CDR 78] we introduce an equivalence relation that models the indistinctness of pairs of

λ-terms in a finite set. The principal feature of this relation is that it cannot be refined by any context (Theorem 3.4.0). This relation is a fundamental tool for stating a necessary condition for solvability of finite systems of equations.

We write $A \subset_f B$ for $A \subset B$ and A is finite. In the following, unless otherwise stated, $\mathscr{F}$ and $\mathscr{P}$ denote finite subsets of Λ.

3.0. Definition. The relation $\simeq_{\mathscr{F}} \subseteq \mathscr{F} \times \mathscr{F}$ is defined as follows:

$$P \simeq_{\mathscr{F}} Q \qquad \text{iff} \quad \forall \alpha \in \text{Seq}\ ((\mathscr{F}, \alpha) \in \text{USF} \cap \text{AGT}$$
$$\Rightarrow (P \sim_\alpha Q \text{ and } P \simeq_{\{M \in \mathscr{F} \mid M \sim_\alpha P\}} Q)).$$

If $P \simeq_{\mathscr{F}} Q$ we say that P and Q are $\mathscr{F}$-indistinct.

It is easy to verify that the relation $\simeq_{\mathscr{F}}$ is well defined.
Note that $\text{Card}(\mathscr{F}/\simeq_{\mathscr{F}}) = 1$ iff there does not exist any α useful for $\mathscr{F}$.

3.1. Example. Let $\mathscr{F} \equiv (\lambda t \cdot t\mathbf{\Omega}(\lambda a \cdot xa(xt)),\ \lambda t \cdot tx(\lambda a \cdot x\mathbf{\Omega}(x\mathbf{B})),\ xx,\ \lambda t \cdot tx(\lambda a \cdot xx(x\mathbf{\Omega})),\ \lambda t \cdot xx\mathbf{\Omega}t\}$. We have $\mathscr{F}/\simeq_{\mathscr{F}} = \{\{\lambda t \cdot xx\mathbf{\Omega}t\},\ \{xx\},\ \{\lambda t \cdot t\mathbf{\Omega}(\lambda a \cdot xa(xt)),\ \lambda t \cdot tx(\lambda a \cdot x\mathbf{\Omega}(x\mathbf{B})),\ \lambda t \cdot tx(\lambda a \cdot xx(x\mathbf{\Omega}))\}\}$.

The following proposition states some properties of $\simeq_{\mathscr{F}}$.

3.2. Proposition. (0) *$\simeq_{\mathscr{F}}$ is an equivalence relation on $\mathscr{F}$.*

(1) *Let $\mathscr{F} \subseteq \mathscr{P} \subset_f \Lambda$. Then $\forall P, Q \in \mathscr{F}$ $(P \simeq_{\mathscr{F}} Q \Rightarrow P \simeq_{\mathscr{P}} Q)$.*

(2) *Let $\mathscr{F}$ be a finite set of $\perp$-free λ-terms. Then $\forall M, N \in \mathscr{F}$ $(M \simeq_{\mathscr{F}} N$ iff $M =_{\mathscr{H}^*} N)$.*

(3) *$\mathscr{F}$ is distinct iff* $\text{Card}(\mathscr{F}) = \text{Card}(\mathscr{F}/\simeq_{\mathscr{F}})$.

(4) *Let $g: \mathscr{F} \to \Lambda$ s.t. $\forall M \in \mathscr{F}$ $g(M) =_{\mathscr{H}^*} M$. Then $\forall M, Q \in \mathscr{F}$ $(M \simeq_{\mathscr{F}} Q$ iff $g(M) \simeq_{g(\mathscr{F})} g(Q))$.*

(5) *Let $\mathscr{F} = \{M_1, ..., M_n\} \subset \Lambda$ and $C[\ ] \in \Lambda[\ ]$.*

(0) *If $(\forall i \in \{1, ..., n\}\ C[M_i] =_\beta y_i \in (\mathbf{V} - FV(\mathscr{F}))$ and $\exists i, j \in \{1, ..., n\}\ y_i \not\equiv y_j)$ then $\exists \alpha$ useful for $\mathscr{F}$.*

(1) *If $\exists \alpha$ useful for $C[\mathscr{F}]$ then $\exists \xi$ useful for $\mathscr{F}$.*

Proof. (0)–(3) By induction on $\text{Card}(\mathscr{F})$.

(4) From 2.2.0 we have $\forall M \in \mathscr{F}$ $g(M) \simeq_\eta M$. We proceed then by induction on $\text{Card}(\mathscr{F})$ considering that $\forall \alpha \in \text{Seq}$ we have: $\mathscr{F}|\alpha\downarrow$ iff $g(\mathscr{F})|\alpha\downarrow$, $\forall M, N \in \mathscr{F}$ $(M \sim_\alpha N$ iff $g(M) \sim_\alpha g(N))$, $(\mathscr{F}, \alpha) \in \text{USF}$ iff $(g(\mathscr{F}), \alpha) \in \text{USF}$.

(5.0) Refer to the proof of 14.4.13 in [Bar 84]. (5.1) From 2.4.1 and 5.0. ∎

3.3. COUNTER EXAMPLE. The converse of 3.2.1 does not hold. In fact let $\mathscr{F} \equiv \{xy, x\mathbf{K}\}$ and $\mathscr{P} \equiv \{xy, x\mathbf{K}, x\mathbf{\Omega}\}$. We have $xy \simeq_{\mathscr{P}} x\mathbf{K}$, but $xy \not\simeq_{\mathscr{F}} x\mathbf{K}$. This is because $x\mathbf{\Omega}$ make *unuseful* (in $\mathscr{P}$) the node $\langle 1 \rangle$ that is useful in $\mathscr{F}$.

The interest in the relation $\simeq_{\mathscr{F}}$ lies principally in Theorem 3.4.0. It states that no contexts can refine the relation $\simeq_{\mathscr{F}}$, or equivalently, any context $C[\ ]$ is a morphism from $(\mathscr{F}, \simeq_{\mathscr{F}})$ to $(C[\mathscr{F}], \simeq_{C[\mathscr{F}]})$.

3.4. THEOREM. *Let $\mathscr{F} \subset_f \Lambda$. Then*

(0) $\forall C[\ ] \in \Lambda[\ ]\ \forall P, Q \in \mathscr{F}\ (P \simeq_{\mathscr{F}} Q \Rightarrow C[P] \simeq_{C[\mathscr{F}]} C[Q])$.

(1) $\forall C[\ ] \in \Lambda[\ ]\ \mathrm{Card}(\mathscr{F}/\simeq_{\mathscr{F}}) \geqslant \mathrm{Card}(C[\mathscr{F}]/\simeq_{C[\mathscr{F}]})$.

Proof. (0) By induction on $\mathrm{Card}(\mathscr{F})$. If $\mathrm{Card}(\mathscr{F}) = 1$ the result is trivial. Let $\mathrm{Card}(\mathscr{F}) > 1$. If α useful for $C[\mathscr{F}]$ does not exist the result is trivial. Now we suppose that α useful and agt for $C[\mathscr{F}]$ exist (see 2.3). Then, by 3.2.5.1, ξ useful and agt for $\mathscr{F}$ exist and we have

$$(P \sim_{\xi} Q \text{ and } P \simeq_{\{M \in \mathscr{F} \mid M \sim_{\xi} P\}} Q).$$

Because $\mathrm{Card}(\{M \in \mathscr{F} \mid M \sim_{\xi} P\}) < \mathrm{Card}(\mathscr{F})$ by inductive hypothesis it follows that $C[P] \simeq_{C[\{M \in \mathscr{F} \mid M \sim_{\xi} P\}]} C[Q]$. Hence from $C[\{M \in \mathscr{F} \mid M \sim_{\xi} P\}] \subseteq C[\mathscr{F}]$ and 3.2.1, we have $C[P] \simeq_{C[\mathscr{F}]} C[Q]$.

(1) It follows immediately from (0). ∎

Intuitively $\mathrm{Card}(\mathscr{F}/\simeq_{\mathscr{F}})$ represents the dimension of the *space* spanned by $\mathscr{F}$. Theorem 3.4.1 states that this dimension cannot be augmented.

The following example shows a typical application of 3.2.4 and 3.4.0.

3.5. EXAMPLE. Let $\mathscr{F} = \{x(x\,\mathbf{\Omega})\,x\,\mathbf{K}, x(x\,\mathbf{K})\,\mathbf{\Omega}\,\mathbf{K}, x\,\mathbf{\Omega}\,\mathbf{\Omega}(xxxxx)\}$. There does not exist $C[\ ]$ s.t. $C[x(x\mathbf{\Omega})\,x\mathbf{K}] =_{\eta} \lambda t \cdot ty\mathbf{\Omega}(tt)\,t$; $C[x(x\mathbf{K})\,\mathbf{\Omega}\mathbf{K}] =_{\eta} \mathbf{W}$; $C[x\,\mathbf{\Omega}\,\mathbf{\Omega}(xxxxx)] =_{\eta} \mathbf{B}$. In fact we have

$$\begin{aligned}
&\text{from 3.2.4:} \quad \lambda t \cdot ty\mathbf{\Omega}(tt)\,t \not\simeq_{\{\lambda t \cdot ty\mathbf{\Omega}(tt)t, \mathbf{W}, \mathbf{B}\}} \mathbf{W} \\
&\Rightarrow C[x(x\mathbf{\Omega})\,x\mathbf{K}] \not\simeq_{C[\mathscr{F}]} C[x(x\mathbf{K})\,\mathbf{\Omega}\mathbf{K}]; \\
&\text{from 3.4.0:} \quad C[x(x\mathbf{\Omega})\,x\mathbf{K}] \not\simeq_{C[\mathscr{F}]} C[x(x\mathbf{K})\,\mathbf{\Omega}\mathbf{K}] \\
&\Rightarrow x(x\mathbf{\Omega})\,x\mathbf{K} \not\simeq_{\mathscr{F}} x(x\mathbf{K})\,\mathbf{\Omega}\mathbf{K}.
\end{aligned}$$

This is absurd because $x(x\,\mathbf{\Omega})\,x\,\mathbf{K} \not\simeq_{\mathscr{F}} x(x\,\mathbf{K})\,\mathbf{\Omega}\,\mathbf{K}$.

Using 3.4.0 we can restate Theorem 2.4.0.

3.6. THEOREM. *Let $\mathscr{F} = \{M_1, ..., M_n\} \subset \Lambda$ and* $\mathbf{T}$ *be a sms theory. Then*

$$\exists C[\ ] \in \Lambda[\ ]\ \forall i \in \{1, ..., n\}\ C[M_i] =_{\mathbf{T}} y_i \in (\mathrm{V} - \mathrm{FV}(\mathscr{F}))$$
$$\textit{iff}\ \forall i, j \in \{1, ..., n\}\ (M_i \simeq_{\mathscr{F}} M_j \Rightarrow y_i \equiv y_j).$$

Theorem 3.6 suggests the following notation.

3.7. *Notation.* Let $\mathscr{F} \subset_f \Lambda$. We denote by $\mathscr{F}^{\#}: \mathscr{F} \to (\mathrm{V} - \mathrm{FV}(\mathscr{F}))$ a (arbitrary) function s.t.: $\forall M, N \in \mathscr{F}$ ($M \simeq_{\mathscr{F}} N$ iff $\mathscr{F}^{\#}(M) \equiv \mathscr{F}^{\#}(N)$). In a given mathematical context we suppose that the variables of $\mathscr{F}^{\#}(\mathscr{F})$ are different from all the others variables occurring in that context. ▮

4. $\mathscr{F}$-QUALIFIED CONTEXTS

Substitutive contexts s.t. $\mathrm{Card}(\mathscr{F}/\simeq_{\mathscr{F}}) = \mathrm{Card}(C[\mathscr{F}]/\simeq_{C[\mathscr{F}]})$ play a fundamental role in finding a solution for a system of equations. In Sections 4.1–4.4 we construct an infinite family of these contexts. In 4.0 we illustrate by an example the utility of such contexts.

4.0. A problem containing self-application can be transformed into a suitable separability problem. The following example may clarify the matter.

4.0.0. EXAMPLE. Determine $C[\ ] \equiv (\lambda x \cdot [\ ])\Delta \in \Lambda[\ ]$ s.t.:

$$C[xx] =_\beta \Delta\Delta =_\beta y_1, \qquad C[x\omega] =_\beta \Delta\omega =_\beta y_2 \qquad (\omega \equiv \lambda t \cdot tt).$$

Note that Δ cannot be $\langle H_1, ..., H_n\rangle$ for any n. In fact we have

$$\langle H_1, ..., H_n\rangle\langle H_1, ..., H_n\rangle =_\beta H_1 H_1 \cdots H_n H_2 \cdots H_n =_\beta y_1,$$
$$\langle H_1, ..., H_n\rangle\ \omega =_\beta H_1 H_1 \cdots H_n =_\beta y_2,$$

which is absurd. We search for a solution having the shape $\Delta =_\beta \lambda t \cdot t(\Delta_1 t)\,\mathbf{\Omega}$ with $t \notin FV(\Delta_1)$. Note that Δ and ω have different shapes. We have

$$\Delta\Delta =_\beta \Delta_1\Delta(\Delta_1(\Delta_1\Delta))\,\mathbf{\Omega}\,\mathbf{\Omega},$$
$$\Delta\omega =_\beta \Delta_1\omega(\Delta_1\omega)\,\mathbf{\Omega}.$$

Note that now the self-application $(\Delta\Delta)$ is in some sense weakened ($\Delta_1\Delta$ instead of $\Delta\Delta$). If Δ_1 can separate Δ from ω we have solved our problem. Unfortunately Δ depends on Δ_1. We try to eliminate this dependence by ignoring it; i.e., we replace $\Delta_1 t$ by $\mathbf{\Omega}$ in Δ. This amounts to searching for a G s.t.

$$G(\lambda t \cdot t\mathbf{\Omega\Omega}) =_\beta y_1 \qquad \text{and} \qquad G\omega =_\beta y_2.$$

The solution of this problem is well known [Böh 68, BDPR 79, CDR 78]. A possible choice for G is

$$G \equiv \lambda t \cdot t\mathbf{U}_3^3(\mathbf{U}_1^2 y_1)\, y_2.$$

In order to solve our original problem it is sufficient to replace Δ_1 in Δ by a variant G^* of G able to erase the superfluous information. We have

$$G^* \equiv G[y_1 := \mathbf{U}_1^4 y_1][y_2 := \mathbf{U}_1^3 y_2] =_\beta \lambda t \cdot t\mathbf{U}_3^3(\mathbf{U}_1^5 y_1)(\mathbf{U}_1^3 y_2),$$
$$\Delta \equiv \lambda t \cdot t(G^* t)\, \mathbf{\Omega} =_\beta \lambda t \cdot t(t\mathbf{U}_3^3(\mathbf{U}_1^5 y_1)(\mathbf{U}_1^3 y_2))\, \mathbf{\Omega}.$$

The family of contexts $\{(\lambda x \cdot [\ ])\, H \mid (\lambda t \cdot t\, \mathbf{\Omega}\, \mathbf{\Omega}) \precsim H\}$ contains the solution and preserves the distinction of the set $\{x, \omega\}$. The discovery of such a family has been the fundamental step in solving the problem in 4.0.0. A generalization of this strategy leads to the notion of $\mathscr{F}$-qualification.

4.1. We introduce the notion of qualification and some of its fundamental properties. A context $C[\ ]$ is said to be qualified for a set $\mathscr{F}$ if $C[\ ]$ does not lose any *information* contained in $\mathscr{F}$.

4.1.0. Definition. Let $\mathscr{F} \subset_f \Lambda$. $C[\ ]$ is said to be $\mathscr{F}$-qualified ($C[\ ] \in Q(\mathscr{F})$) iff $\forall M, N \in \mathscr{F}$ $(C[M] \simeq_{C[\mathscr{F}]} C[N] \Rightarrow M \simeq_{\mathscr{F}} N)$.

Note that from 3.4.0 we have $C[\ ] \in Q(\mathscr{F})$ iff $\mathrm{Card}(\mathscr{F}/\simeq_{\mathscr{F}}) = \mathrm{Card}(C[\mathscr{F}]/\simeq_{C[\mathscr{F}]})$.

4.1.1. Example. Let $\mathscr{F} = \{x, xx\}$. Then $D[\ ] \equiv (\lambda x \cdot [\ ])\, \mathbf{U}_1^2 \in Q(\mathscr{F})$ and $D'[\ ] \equiv (\lambda x \cdot [\ ])\, \mathbf{U}_1^1 \notin Q(\mathscr{F})$.

From a single element of $Q(\mathscr{F})$ we can obtain an infinite collection of them.

4.1.2. Notation. (0) Let $C[\ ], D[\ ] \in \Lambda[\ ]$ and $z \in \mathbf{V}$ s.t. z does not occur (free or bound) in $C[\ ]$ or $D[\ ]$. We write $C[\ ] \precsim D[\ ]$ for $C[z] \precsim D[z]$.

(1) $\mathscr{U}(D[\ ]) = \{C[\ ] \in \Lambda[\ ] \mid D[\ ] \precsim C[\ ]\}$.

4.1.3. *Remark.* Let $D[\ ] \in \Lambda[\ ]$. Then $D[\ ] \in Q(\mathscr{F}) \Rightarrow \mathscr{U}(D[\ ]) \subseteq Q(\mathscr{F})$.

In order to neutralize the effects of self-application of unknown terms which occurs unavoidably in a system of equations, it is important to find a separator for a set of finite sequences of λ-terms (4.1.6.2). Definition 4.1.4 is a tool for stating the assumptions we need to reach the desired goal. The intuitive meaning of 4.1.4.2 is $\mathscr{F} \in \mathrm{PFR}$ ($\mathscr{F}$ satisfies the prefix rule) iff an initial part of an element of $\mathscr{F}$ never *collapses* with another element of $\mathscr{F}$. This property is related to [BP 88a, 2.1].

4.1.4. DEFINITION. Let $X \subset_f \mathbf{V}$ and $\mathscr{F} \subset_f \Lambda$.

(0) $\mathrm{Sub}(X) = \{(\lambda x_1 \cdots x_w \cdot [\ ])\, \Delta_1 \cdots \Delta_w \mid \{x_1, ..., x_w\} = X$ and $\{\Delta_1, ..., \Delta_w\} \subset \Lambda\}$.

(1) $\mathscr{F}^+ = \{\lambda \vec{z} \cdot t\vec{M} \mid \lambda \vec{z} \cdot t\vec{M}\vec{Q} \in \mathscr{F}$ with $\vec{M}$ non-empty$\}$.

(2) $\mathrm{PFR} = \{\mathscr{F} \mid \mathscr{F} \subset_f \Lambda$ and $\forall\ \lambda \vec{z} \cdot tM_1 \cdots M_m \cdots M_n \in \mathscr{F}\ \ \forall m < n$ $\forall N \in \mathscr{F}\ \ N \not\simeq_{\mathscr{F}^+} \lambda \vec{z} \cdot tM_1 \cdots M_m\}$.

4.1.5. EXAMPLE. (0) Let $\mathscr{F}_1 = \{\langle x, x\rangle, xx, x\,\mathbf{K}\,x\}$. Then $\mathscr{F}_1^+ = \{\langle x\rangle, \langle x, x\rangle, xx, x\,\mathbf{K}, x\,\mathbf{K}x\}$ and $\mathscr{F}_1 \in \mathrm{PFR}$.

(1) Let $\mathscr{F}_2 = \{xxx, x\mathbf{\Omega}xx\}$. Then $\mathscr{F}_2^+ = \{xx, xxx, x\,\mathbf{\Omega}, x\,\mathbf{\Omega}\,x, x\,\mathbf{\Omega}\,xx\}$ and $\mathscr{F}_2 \notin \mathrm{PFR}$.

Note that if $\forall M, N \in \mathscr{F}(\mathrm{head}(M) = \mathrm{head}(N) \Rightarrow (\deg(M) = \deg(N)$ and $\mathrm{ord}(M) = \mathrm{ord}(N)))$ then $\mathscr{F} \in \mathrm{PRF}$. Refer to Example 4.0.0 for an application of 4.1.6.1.

4.1.6. LEMMA. *Let $\mathscr{F} \subset_f \Lambda$ and $\mathbf{T}$ be a sms theory (see 3.7).*

(0) *$D[\ ] \in Q(\mathscr{F})$ iff $\exists C[\ ] \in \Lambda[\ ]\ \ \forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \ \forall M \in \mathscr{F}$ $C[\Delta[M]] =_{\mathbf{T}} \mathscr{F}^{\#}(M)$.*

(1) *Let $D[\ ] \in \Lambda[\ ]$ s.t. $D[\mathscr{F}] \subset \Lambda^0$. Then $D[\ ] \in Q(\mathscr{F})$ iff $\exists F \in \Lambda \forall \Delta[\ ] \in \mathscr{U}(D[\ ])\, \forall M \in \mathscr{F}\ F\Delta[M] =_{\mathbf{T}} \mathscr{F}^{\#}(M)$.*

(2) *Let $X \subset_f \mathbf{V}$, $\mathscr{F} \subset_f \{\langle M_1, ..., M_n\rangle \mid n \in \mathbf{N}$ and $\{M_1, ..., M_n\} \subset \Lambda\}$ with $\mathscr{F} \in \mathrm{PFR}$ and $D[\ ] \in \mathrm{Sub}(X)$ s.t. $D[\mathscr{F}] \subset \Lambda^0$. If $D[\ ] \in Q(\mathscr{F}^+)$ then $\exists G \in \Lambda\ \forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall \langle M_1, ..., M_n\rangle \in \mathscr{F}\ \ \Delta[\langle M_1, ..., M_n\rangle]\, G =_{\mathbf{T}} \mathscr{F}^{\#}(\langle M_1, ..., M_n\rangle)\ \Delta[M_1] \cdots \Delta[M_n]$.*

Proof. (0) ($\Leftarrow$) Let $M, N \in \mathscr{F}$. Taking into account 3.4.0 we have

$$D[M] \simeq_{D[\mathscr{F}]} D[N] \Rightarrow C[D[M]] \simeq_{C[D[\mathscr{F}]]} C[D[N]]$$
$$\Rightarrow \mathscr{F}^{\#}(M) \equiv \mathscr{F}^{\#}(N) \Rightarrow M \simeq_{\mathscr{F}} N.$$

($\Rightarrow$) Because $\forall M, N \in \mathscr{F}(D[M] \simeq_{D[\mathscr{F}]} D[N] \Rightarrow M \simeq_{\mathscr{F}} N)$ from 3.6 we have

$$\exists C[\ ] \in \Lambda[\ ]\, \forall M \in \mathscr{F} \qquad C[D[M]] =_{\beta} \mathscr{F}^{\#}(M).$$

The thesis follows from 2.2.2 and $\forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall M \in \mathscr{F}\ (D[M] \precsim \Delta[M])$.

(1) $(\Leftarrow)$ As in (0) $(\Rightarrow)$. As in (0) observing that now

$$\exists C[\ ] \in \Lambda[\ ]\ \forall M \in \mathscr{F}$$
$$C[D[M]] =_\beta (\lambda t \cdot C[t])\, D[M] =_\beta FD[M] =_\beta \mathscr{F}^{\#}(M).$$

(2) By induction on Card(deg($\mathscr{F}$)).

Case 0. Card(deg($\mathscr{F}$)) = 1.

Case 0.0. Card($\mathscr{F}/\simeq_{\mathscr{F}}$) = 1. We choose $G =_\beta \lambda z_1 \cdots z_n \cdot \mathscr{F}^{\#}(\langle M_1, ..., M_n\rangle)\, z_1 \cdots z_n$, where $\langle M_1, ..., M_n\rangle \in \mathscr{F}$.

Case 0.1. Card($\mathscr{F}/\simeq_{\mathscr{F}}$) > 1. By (1) we have

$$\exists F \in \Lambda\ \forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall M \in \mathscr{F} \qquad F\Delta[M] =_\beta \mathscr{F}^{\#}(M).$$

Choose $G =_\beta \lambda z_1 \cdots z_n \cdot F\langle z_1, ..., z_n\rangle\, z_1 \cdots z_n$.

Case 1. Card(deg($\mathscr{F}$)) > 1. Let $m = \min \deg(\mathscr{F})$ and $\mathscr{P} = \{\langle M_1, ..., M_m\rangle \mid \langle M_1, ..., M_m, ..., M_n\rangle \in \mathscr{F}\}$.

We have Card(deg($\mathscr{P}$)) = 1, Card(deg($\mathscr{F} - \mathscr{P}$)) < Card(deg($\mathscr{F}$)), $D[\ ] \in Q(\mathscr{P}^+)$, and $D[\ ] \in Q((\mathscr{F} - \mathscr{P})^+)$. By inductive hypothesis $G, H \in \Lambda$ exist s.t.

$$\forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall \langle M_1, ..., M_m\rangle \in \mathscr{P}$$
$$\Delta[\langle M_1, ..., M_m\rangle]\, G =_\beta \mathscr{P}^{\#}(\langle M_1, ..., M_m\rangle)\, \Delta[M_1] \cdots \Delta[M_m];$$
$$\forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall \langle M_1, ..., M_n\rangle \in (\mathscr{F} - \mathscr{P})$$
$$\Delta[\langle M_1, ..., M_n\rangle]\, H =_\beta (\mathscr{F} - \mathscr{P})^{\#}\ (\langle M_1, ..., M_n\rangle)\, \Delta[M_1] \cdots \Delta[M_n].$$

We set $G^* \equiv G[\mathscr{P}^{\#}(M) := H \mid M \in (\mathscr{P} - \mathscr{F})]$. Taking into account that $\mathscr{F} \in$ PFR it is easy to verify that

$$\forall \Delta[\ ] \in \mathscr{U}(D[\ ])\ \forall \langle M_1, ..., M_n\rangle \in \mathscr{F}$$
$$\Delta[\langle M_1, ..., M_n\rangle]\, G^* =_\beta \mathscr{F}^{\#}(\langle M_1, ..., M_n\rangle)\, \Delta[M_1] \cdots \Delta[M_n]. \quad \blacksquare$$

4.1.7. Counter example. (0) The hypothesis $\mathscr{F} \in$ PFR in 4.1.6.2 is essential. Let $\mathscr{F} = \mathscr{F}^+ = \{\langle x\rangle, \langle x, x\rangle\} \notin$ PFR and $D[\ ] \equiv (\lambda x \cdot [\ ])\, \mathbf{U}_1^2 \in Q(\mathscr{F}^+)$. Then we have

$$D[\langle x\rangle]\, G =_\beta G\mathbf{U}_1^2 \qquad \text{and} \qquad D[\langle x, x\rangle]\, G =_\beta G\mathbf{U}_1^2\mathbf{U}_1^2.$$

Clearly $G \in \Lambda$ does not exist s.t. $G\mathbf{U}_1^2 =_\beta y_1$ and $G\mathbf{U}_1^2\mathbf{U}_1^2 =_\beta y_2$.

(1) The converse of 4.1.6.2 does not hold. Let $\mathscr{F} = \{\langle x\rangle,$

$\langle xx, \mathbf{K}\rangle$, $\langle x\mathbf{K}, x\rangle$ and $D[\ ] \equiv (\lambda x \cdot [\ ])\, \mathbf{U}_2^2$. Then $\mathscr{F}^+ = \{\langle x\rangle, \langle x, x\rangle, \langle x\mathbf{K}\rangle, \langle xx, \mathbf{K}\rangle, \langle x\mathbf{K}, x\rangle\}$ and $D[\mathscr{F}^+] = \{\langle \mathbf{U}_2^2\rangle, \langle \mathbf{U}_2^2\mathbf{U}_2^2\rangle, \langle \mathbf{U}_2^2\mathbf{K}\rangle, \langle \mathbf{U}_2^2\mathbf{U}_2^2, \mathbf{K}\rangle, \langle \mathbf{U}_2^2\mathbf{K}, \mathbf{U}_2^2\rangle\}$. Since $\mathbf{U}_2^2\mathbf{U}_2^2 =_\beta \mathbf{U}_2^2\mathbf{K} =_\beta \mathbf{I}$ then $D[\ ] \notin Q(\mathscr{F}^+)$. Now let $G \equiv \lambda t \cdot t(\mathbf{U}_1^2(\lambda ab \cdot by_2 y_3 ab))\, y_1 t$. We have $D[\langle x\rangle]\, G =_\beta y_1\, D[x]$, $D[\langle xx, \mathbf{K}\rangle]\, G =_\beta y_2 D[xx]\, D[\mathbf{K}]$, $D[\langle x\mathbf{K}, x\rangle]\, G =_\beta y_3 D[x\mathbf{K}]\, D[x]$.

4.2. According to 4.0 in Sections 4.2–4.4 for any set $\mathscr{F}$ we construct an infinite collection of substitutive $\mathscr{F}$-qualified contexts.

For constructing a subset of $Q(\mathscr{F})$ we need some results about useful nodes of $\mathscr{F}$. A node α is *adherent* for a set $\mathscr{F}$ if the number of η-expansions in $\mathscr{F}$ needed to reach α is not *too large.*

4.2.0. Definition. Let $\mathscr{F} \subseteq \Lambda$. The node α is said to be $\mathscr{F}$-adherent $((\mathscr{F}, \alpha) \in \mathrm{ADH})$ iff $(\alpha \in_\eta \mathrm{BT}(\mathscr{F})$ and $\exists M \in \mathscr{F}\, \alpha \in \mathrm{BT}(M))$.

4.2.1. Example. Let $\mathscr{F} \equiv \{\lambda a \cdot aa, \lambda ab \cdot xa(axbx)\}$. Then $(\mathscr{F}, \langle 3\rangle) \notin \mathrm{ADH}$, but $(\mathscr{F}, \langle 2\rangle) \in \mathrm{ADH}$ and we have $\mathscr{F}^{\langle 2\rangle} \equiv \{\lambda ab \cdot aab, \lambda ab \cdot xa(axbx)\}$.

4.2.2. Proposition. $\mathrm{USF} \cap \mathrm{AGT} \subseteq \mathrm{ADH} \cap \mathrm{AGT}$ (see 2.3).

Proposition 4.2.2. states that the shortest useful nodes of $\mathscr{F}$ are $\mathscr{F}$-adherent.

It will be useful to know the maximum degree of subterms whose head is a free variable (tdg) and the maximum degree of subterms whose head is a bound variable (bdg).

4.2.3. *Notation.* Let $M \in \Lambda$, $\mathscr{F} \subset_f \Lambda$, $\alpha \in \mathrm{Seq}$, and $x \in \mathbf{V}$. We define (see 2.1, 2.3)

(0) $\mathrm{tdg}(x, M, \alpha) = \max\{\deg(M_\beta) \mid \beta \leqslant \alpha$ and $M|\beta\!\downarrow$ and $\mathrm{head}(M_\beta) \equiv x\}$.

(1) $\mathrm{tdg}(x, \mathscr{F}, \alpha) = \max\{\mathrm{tdg}(x, M, \alpha) \mid M \in \mathscr{F}\}$.

(2) $\mathrm{bdg}(M, \alpha) = \max\{\deg(M_\beta) \mid \beta \leqslant \alpha$ and $M|\beta\!\downarrow$ and $\mathrm{head}(M_\beta) \notin FV(M_\beta)\}$.

(3) $\mathrm{bdg}(\mathscr{F}, \alpha) = \max\{\mathrm{bdg}(M, \alpha) \mid M \in \mathscr{F}\}$.

The $\mathscr{F}$-adherent nodes preserve the value of the functions just defined.

4.2.4. Proposition. *Let* $(\mathscr{F}, \alpha) \in \mathrm{ADH} \cap \mathrm{AGT}$ and $x \in \mathbf{V}$.

(0) $\mathrm{tdg}(x, \mathscr{F}^\alpha, \alpha) = \mathrm{tdg}(x, \mathscr{F}, \alpha)$.

(1) $\mathrm{bdg}(\mathscr{F}^\alpha, \alpha) = \mathrm{bdg}(\mathscr{F}, \alpha)$.

4.3. We introduce a suitable class of sequence transformations (deformations) for constructing a subset of $Q(\mathscr{F})$.

4.3.0. DEFINITION. (0) Let $d: \Lambda \times \mathrm{Seq} \to \mathrm{Seq}$. We call d a deformation iff:

(0) $\forall M, N \in \Lambda\ ((\{M, N\}, \alpha) \in \mathrm{AGT} \Rightarrow d(M, \alpha) = d(N, \alpha))$;

(1) $\forall M \in \Lambda\ d(M, \langle\ \rangle) = \langle\ \rangle$.

(1) Let $(\mathscr{F}, \alpha) \in \mathrm{AGT}$ and d be a deformation. We set $d(\mathscr{F}, \alpha) = d(M, \alpha)$ with $M \in \mathscr{F}$. This notation is legitimate because if $(\mathscr{F}, \alpha) \in \mathrm{AGT}$ then $\forall M, N \in \mathscr{F}\ d(M, \alpha) = d(N, \alpha)$.

(2) Let $\mathscr{F} \subseteq \Lambda$, $\alpha \in \mathrm{Seq}$, and d be a deformation.

(0) $\mathrm{FTH}(\mathscr{F}, \alpha, d) \equiv \{C[\ ] \in \Lambda[\ ] \mid \forall M \in \mathscr{F}(M|\alpha\downarrow$ iff $C[M] | d(M, \alpha)\downarrow)$ and $\forall M, N \in \mathscr{F}(M \sim_\alpha N$ iff $C[M] \sim_{d(M,\alpha)} C[N])\}$.

(1) $\mathrm{FTH}(\mathscr{F}, d) \equiv \bigcap \{\mathrm{FTH}(\mathscr{F}, \alpha, d) \mid \alpha \in \mathrm{Seq}$ and $\exists M, N \in \mathscr{F}$ $(\{M, N\}, \alpha) \in \mathrm{USF} \cap \mathrm{AGT}\}$. $C[\ ] \in \mathrm{FTH}(\mathscr{F}, d)$ is said to be *faithful* with respect to the pair $(\mathscr{F}, d)$.

The importance of faithful contexts lies in the following proposition.

4.3.1. PROPOSITION. *Let $\mathscr{F} \subset_f \Lambda$ and d be a deformation. Then* $\mathrm{FTH}(\mathscr{F}, d) \subseteq Q(\mathscr{F})$ *(see* 4.1.0).

Proof. By induction on $\mathrm{Card}(\mathscr{F})$.

4.4. Finally we define a particular class of contexts contained in $Q(\mathscr{F})$. We prove that this class is $\mathscr{F}$-qualified by showing that it is faithful. In 4.4.2.0 we prove that if ε satisfies certain constrains then $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq \mathrm{FTH}(\mathscr{F}, d)$ for a suitable deformation d. Then, using 4.3.1, we obtain $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq Q(\mathscr{F})$ (4.4.2.1).

4.4.0. *Notation.* Let $w \in \mathbf{N}^+$; $\vec{x} \equiv x_1, ..., x_w$; $\varepsilon: \{0, 1\} \times \{\vec{x}\} \to \mathbf{N}^+$, $x \in \{\vec{x}\}$, $\mathscr{F} \subset_f \Lambda$, $X \subset_f \mathbf{V}$, and $\alpha \in \mathrm{Seq}$ (see 4.2.3).

(0) $\Delta_{x,\varepsilon} \equiv \lambda t_1 \cdots t_{\varepsilon(0,x)} \cdot t_{\varepsilon(0,x)} \mathbf{\Omega}_1 \cdots \mathbf{\Omega}_{\varepsilon(1x)-1} \langle t_1, ..., t_{\varepsilon(0,x)} \rangle$, where $\mathbf{\Omega}_1 \equiv \cdots \equiv \mathbf{\Omega}_{\varepsilon(1,x)-1} \equiv \mathbf{\Omega}$.

(1) $\Delta_{\vec{x},\varepsilon}[\ ] \equiv (\lambda \vec{x} \cdot [\ ])\, \Delta_{x_1,\varepsilon} \cdots \Delta_{x_w,\varepsilon}$.

(2) $\mathrm{QL}(X, \mathscr{F}, \alpha) = \{\varepsilon \mid \varepsilon: \{0, 1\} \times X \to \mathbf{N}^+$ and $\forall x \in X\ \ \varepsilon(0, x) > \mathrm{tdg}(x, \mathscr{F}, \alpha)$ and $\forall x \in X\ \ \varepsilon(1, x) > \mathrm{bdg}(\mathscr{F}, \alpha)$ and $\forall x, x' \in X\ \ (\varepsilon(1, x) = \varepsilon(1, x') \Rightarrow x \equiv x')\}$.

(3) $\mathrm{QL}(X, \mathscr{F}) = \bigcap \{\mathrm{QL}(X, \mathscr{F}, \alpha) \mid \alpha \in \mathrm{Seq}$ and $\exists M, N \in \mathscr{F}(\{M, N\}, \alpha) \in \mathrm{USF} \cap \mathrm{AGT}\}$.

(4) $\mathrm{hd}(X, \varepsilon): \Lambda \to \mathrm{Seq}$ is defined as follows:

$$\mathrm{hd}(X, \varepsilon)(M) = \textbf{if } \mathrm{head}(M) \equiv x \in X \textbf{ then } \langle \varepsilon(1, x) \rangle \textbf{ else } \langle\ \rangle.$$

(5) $\delta(X, \varepsilon)\colon \Lambda \times \mathrm{Seq} \to \mathrm{Seq}$ is defined as follows:

$$\delta(X,\varepsilon)(M, \langle\ \rangle) = \langle\ \rangle$$

$$\delta(X, \varepsilon)(M, \langle j\rangle * \beta) = \mathrm{hd}(X, \varepsilon)(M) * \langle j\rangle * \delta(X, \varepsilon)(M_{\langle j\rangle}, \beta).$$

If $(\mathscr{F}, \alpha) \in \mathrm{AGT} \cap \mathrm{ADH}$ then $\mathrm{QL}(X, \mathscr{F}^{\alpha}, \alpha) = \mathrm{QL}(X, \mathscr{F}, \alpha)$. The deformation $\delta(X, \varepsilon)$ will model the effect of $\Delta_{\vec{x},\varepsilon}[\ ]$ on $M \in \Lambda$ (4.4.1.1.0).

4.4.1. LEMMA. (0) *Let* $\mathscr{F} \subset_f \Lambda$ *and* $\varepsilon \in \mathrm{QL}(\{\vec{x}\}, \mathscr{F}, \langle\ \rangle)$.

(0) $\forall \Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])\ \forall M \in \mathscr{F}\ (M \in \mathrm{SOL}$ *iff* $\Delta[M] \in \mathrm{SOL})$.

(1) $\forall \Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])\ \forall M, N \in \mathscr{F}\ (M \sim N$ *iff* $\Delta[M] \sim \Delta[N])$.

(1) *Let* $(\mathscr{F}, \alpha) \in \mathrm{AGT} \cap \mathrm{ADH}$ *and* $\varepsilon \in \mathrm{QL}(\{\vec{x}\}, \mathscr{F}, \alpha)$.

(0) $\forall \Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])\ \forall M \in \mathscr{F}\ \Delta[M_\alpha] =_\beta \Delta[M]_{\delta(\{\vec{x}\},\varepsilon)(M,\alpha)}$.

(1) $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq \mathrm{FTH}(\mathscr{F}, \alpha, \delta(\{\vec{x}\}, \varepsilon))$.

Proof. (0.0, 0.1) By easy computations (as in [BT 87, 1.2.2, 1.2.3]).

(1.0) By induction on the length of α (as in [BT 87, 1.2.5]).

(1.1) From (0.0) and (0.1) using (1.0) (as in [BT 87, 1.2.6]). ∎

Proposition 4.4.1.0 is still valid substituting $\mathbf{\Omega}$ for the last component of $\Delta_{x,\varepsilon}$. This component becomes important only if in the $\mathrm{BT}(\mathscr{F})$ (see 2.1) nodes of positive length must be considered (4.4.1.1).

Finally, we can produce a class of $\mathscr{F}$-qualified contexts. This result is essential to the solution (6.11) of a class of systems.

4.4.2. THEOREM. *Let* $\mathscr{F} \subset_f \Lambda$ *and* $\varepsilon \in \mathrm{QL}(\{\vec{x}\}, \mathscr{F})$ (*see* 4.4.0.3).

(0) $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq \mathrm{FTH}(\mathscr{F}, \delta(\{\vec{x}\}, \varepsilon))$.

(1) $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq Q(\mathscr{F})$.

Proof. (0) From 4.4.1.1.1 and by an easy induction on $\mathrm{Card}(\mathscr{F})$.

(1) We have $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq \mathrm{FTH}(\mathscr{F}, \delta(\{\vec{x}\}, \varepsilon))$ (from (0)) and $\mathrm{FTH}(\mathscr{F}, \delta(\{\vec{x}\}, \varepsilon)) \subseteq Q(\mathscr{F})$ (from 4.3.1). Hence $\mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ]) \subseteq Q(\mathscr{F})$. ∎

4.5. An interesting consequence of 4.4.1.1.0 is a kind of *filtered* Böhm-out [Bar 84, 10.3]. We can Böhm-out a subterm of $M \in \mathscr{F}$ through $\Delta[\ ] \in Q(\mathscr{F})$.

4.5.0. *Notation.* Let $M \in \Lambda$ and $\alpha \in_\eta \mathrm{BT}(M)$. We set

$$\mathrm{Tr}(M, \alpha) = \{u \mid M_\rho =_\beta \lambda z_1 \cdots z_p \cdot u Q_1 \cdots Q_r \text{ and } \rho < \alpha\}.$$

4.5.1. PROPOSITION. *Let* $(\mathscr{F}, \alpha) \in \mathrm{AGT} \cap \mathrm{ADH}$ *and* $\varepsilon \in \mathrm{QL}(\{\vec{x}\}, \mathscr{F}, \alpha)$.

(0) $\exists C[\ ] \in \Lambda[\ ]\ \forall \Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])\ \forall M \in \mathscr{F}$:

(0) $C[\Delta[M]] =_\beta \Delta[M_\alpha]^*$.

(1) *If* $\mathrm{FV}(\Delta[\ ]) \cap \mathrm{FV}(\mathscr{F}) = \emptyset$ *then* $C[\Delta[M]] =_\beta \Delta[M^*_\alpha]$.

(1) *If* $\forall M \in \mathscr{F}(\mathrm{Tr}(M, \alpha) \cap \mathrm{FV}(M) \subseteq \{\vec{x}\})$ *then* $\exists F \in \Lambda^0\ \forall \Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])\ \forall M \in \mathscr{F}$:

(0) $F\Delta[M] =_\beta \Delta[M_\alpha]^*$.

(1) *If* $\mathrm{FV}(\Delta[\ ]) \cap \mathrm{FV}(\mathscr{F}) = \emptyset$ *then* $F\Delta[M] =_\beta \Delta[M^*_\alpha]$ (*where* * *represents a suitable substitution for the variables* $(\mathrm{Tr}(M, \alpha) - \{\vec{x}\})$).

Proof. (0) Thesis (0.0) follows from the structure of $C[\ ]$ (see [Bar 84, 10.3.7]). Thesis (0.1) follows immediately from thesis (0.0).

(1) From (0) and the construction of $C[\ ]$. ▮

4.5.2. *Remark.* If M satisfies the assumptions of 4.5.1.1 then from $M_\alpha \equiv z \in \mathrm{FV}(M)$ it follows $\Delta[M_\alpha]^* =_\beta z$. Hence 4.5.1.1 gives us a constructive method for extracting, through $\Delta[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])$, a free variable of M.

4.5.3. EXAMPLE. Let $\mathscr{F} = \{\lambda t \cdot x(\lambda z \cdot z(\lambda ab \cdot xxa(yxtzab\mathbf{K})\,x)),\ \lambda t \cdot x(\lambda z \cdot z(\lambda ab \cdot x\mathbf{K}xux)),\ \lambda t \cdot x(\lambda z \cdot z(xxx))]$ and ε: $\{0, 1\} \times \{x\} \to \mathbf{N}^+$ s.t. $\varepsilon(0, x) = 5$, $\varepsilon(1, x) = 2$. Then $(\mathscr{F}, \langle 1, 1, 3\rangle) \in \mathrm{AGT} \cap \mathrm{ADH}$ and $\varepsilon \in \mathrm{QL}(\{x\}, \mathscr{F}, \langle 1, 1, 3\rangle)$. Let $\Delta_{x,\varepsilon}[\ ] \equiv (\lambda x \cdot [\ ])\,\Delta_{x,\varepsilon}$, $\Delta_{x,\varepsilon} \equiv \lambda t_1 t_2 t_3 t_4 t_5 \cdot t_5\ \mathbf{\Omega}\langle t_1, t_2, t_3, t_4, t_5\rangle$ and $F \equiv \langle a_1, a_2, a_3, a_4, \mathbf{U}^2_2, \mathbf{U}^5_1, \mathbf{I}, a_8, a_9, \mathbf{U}^2_2, \mathbf{U}^5_3\rangle$. We have

$$F\Delta_{\vec{x},\varepsilon}[\lambda t \cdot x(\lambda z \cdot z(\lambda ab \cdot xxa(yxtzab\mathbf{K})\,x))] =_\beta y\Delta_{x,\varepsilon}\, a_1 \mathbf{I} a_8 a_9 \mathbf{K}$$

$$F\Delta_{\vec{x},\varepsilon}[\lambda t \cdot x(\lambda z \cdot z(\lambda ab \cdot x\mathbf{K}xux))] =_\beta u$$

$$F\Delta_{\vec{x},\varepsilon}[\lambda t \cdot x(\lambda z \cdot z(xxx))] =_\beta a_8.$$

5. SYSTEMS OF EQUATIONS IN THE λ-CALCULUS

In this section we introduce the notion of *systems of equations* and derive some of its easy properties.

A system is a set of formulas in which some free variables are considered unknowns.

5.0. DEFINTION. Let $\Gamma \subseteq \mathrm{Form}(\Lambda)$ and $X \subset_f \mathbf{V}$.

(0) The pair (Γ, X) is said to be a system of equations on Λ in the unknowns X.

(1) Let $\mathscr{S} = (\Gamma, X)$ be a system. The formula $M = N \in \Gamma$ is said to be

an equation of $\mathscr{S}$. By abuse of language we often write $M = N \in \mathscr{S}$. Unless otherwise stated we assume that $\mathscr{S}$ is finite.

5.1. EXAMPLE. Let $\Gamma = \{x\mathbf{KK} = y, xzx = z\}$. The following are systems:

$$\mathscr{S}_1 = (\Gamma, \{x\}), \qquad \mathscr{S}_2 = (\Gamma, \{y, z\}).$$

5.2. *Notation.* Let $\mathscr{F} \subseteq \Lambda$, $\Gamma \subseteq \mathrm{Form}(\Lambda)$, $\mathscr{S} = (\Gamma, X)$ be a system, and $Z \subseteq \mathbf{V}$.

(0) $L(\Gamma) = \{P \mid P = Q \in \Gamma\}$.

(1) $R(\Gamma) = \{Q \mid P = Q \in \Gamma\}$.

(1) $\Gamma[z := M] = \{P[z := M] = Q[z := M] \mid P = Q \in \Gamma\}$.

(2) $\mathscr{F}_Z \equiv \{M[y := \mathbf{\Omega} \mid y \in (\mathrm{FV}(\mathscr{F}) - Z)] \mid M \in \mathscr{F}\}$.

(3) $\Gamma_Z \equiv \{M[y := \mathbf{\Omega} \mid y \in (\mathrm{FV}(L(\Gamma)) - Z)]$
$= N[y := \mathbf{\Omega} \mid y \in (\mathrm{FV}(L(\Gamma)) - Z)] \mid M = N \in \Gamma\}$.

(4) $\mathscr{S}_Z = (\Gamma_{X \cup Z}, X)$.

If $\mathscr{S} = (\Gamma, X)$ we also write $L(\mathscr{S})$ $(R(\mathscr{S}))$ for $L(\Gamma)$ $(R(\Gamma))$.

5.3. EXAMPLE. Let $\mathscr{S} = (\{xa\mathbf{K} = y,\ xzx = z\}, \{x\})$. Then we have

$$\mathscr{S}_\phi = (\{x\mathbf{\Omega K} = y,\ x\mathbf{\Omega}x = \mathbf{\Omega}\}, \{x\}),$$

$$\mathscr{S}_{\{z\}} = (\{x\mathbf{\Omega K} = y,\ xzx = z\}, \{x\}),$$

$$\mathscr{S}_{\{z,a\}} = (\{xa\mathbf{K} = y,\ xzx = z\}, \{x\}). \quad \blacksquare$$

A solution for $\mathscr{S} = (\Gamma, X)$ in a theory $\mathbf{T}$ is a suitable simultaneous substitution for the unknowns that makes the equations of $\mathscr{S}$ theorems in the theory $\mathbf{T}$.

5.4. DEFINITION. Let $\mathscr{S} = (\Gamma, X)$ be a system and $\mathbf{T}$ be a theory (see 4.1.4.0).

(0) $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) = \{\Delta[\ ] \in \mathrm{Sub}(X) \mid \mathrm{FV}(\Delta[\ ]) \cap \mathrm{FV}(L(\Gamma)) = \varnothing$ and $\forall M = N \in \Gamma\ \Delta[M] =_{\mathbf{T}} \Delta[N]\}$.

(1) $\mathscr{S}$ is said to be $\mathbf{T}$-solvable iff $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) \neq \varnothing$.

If $\mathscr{S}$ is a system and $\mathbf{T}_1 \subseteq \mathbf{T}_2$ are theories then $\mathrm{Sol}(\mathscr{S}, \mathbf{T}_1) \subseteq \mathrm{Sol}(\mathscr{S}, \mathbf{T}_2)$.

5.5. EXAMPLE. Let $\mathscr{S}_1, \mathscr{S}_2$ be as in 5.1. It is easy to verify that $(\lambda x \cdot [\ ])(\lambda ab \cdot by(\mathbf{U}_1^4 ab)) \in \mathrm{Sol}(\mathscr{S}_1, \beta)$; $(\lambda x \cdot [\ ])(\lambda ab \cdot by(\mathbf{U}_1^4 zb)) \notin \mathrm{Sol}(\mathscr{S}_1, \beta)$; $\mathrm{Sol}(\mathscr{S}_2, \beta) = \varnothing$.

Any system of equations can be transformed in a single equation with just one unknown (5.6.2).

5.6. PROPOSITION. *Let $\mathscr{S}=(\Gamma, \{x_1, ..., x_w\})$ be a system and* $\mathbf{T}$ *be a theory. Then*:

(0) *There exists $\mathscr{S}'=(\Gamma', \{u\})$ s.t.* $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) \neq \varnothing$ *iff* $\mathrm{Sol}(\mathscr{S}', \mathbf{T}) \neq \varnothing$.

(1) *There exists $\mathscr{S}'=(\{M=N\}, \{x_1, ..., x_w\})$ s.t.* $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) = \mathrm{Sol}(\mathscr{S}', \mathbf{T})$.

(2) *There exists $\mathscr{S}'=(\{M=N\}, \{u\})$ s.t.* $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) \neq \varnothing$ *iff* $\mathrm{Sol}(\mathscr{S}', \mathbf{T}) \neq \varnothing$.

(3) *Let $Z \subseteq \mathbf{V}$. Then* $\mathrm{Sol}(\mathscr{S}, \mathbf{T}) \subseteq \mathrm{Sol}(\mathscr{S}_Z, \mathbf{T})$.

Proof. (0) Let u be a fresh variable and $\Gamma' \equiv \Gamma[x_1 := u\mathbf{U}_1^w] \cdots [x_w := u\mathbf{U}_w^w]$. We set $\mathscr{S}'=(\Gamma', \{u\})$.

($\Rightarrow$) Let $\Delta[\ ] \equiv (\lambda x_1 \cdots x_w \cdot [\ ])\, \Delta_1 \cdots \Delta_w \in \mathrm{Sol}(\mathscr{S}, \mathbf{T})$; then

$$\Delta'[\ ] \equiv (\lambda u \cdot [\ ])\langle \Delta_1, ..., \Delta_w \rangle \in \mathrm{Sol}(\mathscr{S}', \mathbf{T}).$$

($\Leftarrow$) Let $\Delta'[\ ] \equiv (\lambda u \cdot [\ ])\, \Delta' \in \mathrm{Sol}(\mathscr{S}', \mathbf{T})$; then

$$\Delta[\ ] \equiv (\lambda x_1 \cdots x_w \cdot [\ ])(\Delta'\mathbf{U}_1^w) \cdots (\Delta'\mathbf{U}_w^w) \in \mathrm{Sol}(\mathscr{S}, \mathbf{T}).$$

(1) Let $\Gamma = \{M_i = N_i \mid i = 1, ..., n\}$. We set $M \equiv \langle M_1, ..., M_n \rangle$ and $N \equiv \langle N_1, ..., N_n \rangle$.

(2) From (0) and (1).

(3) Easy considering that if $\Delta[\ ] \in \mathrm{Sol}(\mathscr{S}, \mathbf{T})$ then $\mathrm{FV}(\Delta[\ ]) \cap \mathrm{FV}(L(\Gamma)) \neq \varnothing$. ∎

In spite of Proposition 5.6.2 it seems more interesting to transform an equation into a system of equations rather than the converse. Unfortunately the transformation is not trivial. An example of this technique is shown in Section 8.

The next theorem states a necessary condition for solvability.

5.7. THEOREM. *Let $\mathscr{S}=(\Gamma, X)$ be a system,* $\mathbf{T}$ *be a sms theory, and* $Z \subseteq \mathbf{V}$.

(0) *If* $\Delta[\ ] \in \mathrm{Sol}(\mathscr{S}, \mathbf{T})$ *then*

$$\forall M = N, M' = N' \in \mathscr{S}_Z \qquad (M \simeq_{L(\mathscr{S}_Z)} M' \Rightarrow \Delta[N] \simeq_{\Delta[R(\mathscr{S}_Z)]} \Delta[N']).$$

(1) *Let* $\mathscr{S}=(\Gamma, X)$ *be a system with* $\mathrm{FV}(R(\mathscr{S}))\cap X=\varnothing$. *If* $\mathrm{Sol}(\mathscr{S}, \mathbf{T})\neq\varnothing$ *then*

$$\forall M=N,\ M'=N'\in\mathscr{S}_Z \qquad (M\simeq_{L(\mathscr{S}_Z)} M'\Rightarrow N\simeq_{R(\mathscr{S}_Z)} N').$$

Proof. (0) From 5.6.3 $\Delta[\ ]\in\mathrm{Sol}(\mathscr{S}_Z, \mathbf{T})$. Let $M=N,\ M'=N'\in\mathscr{S}_Z$. From 3.4.0 we have $(M\simeq_{L(\mathscr{S}_Z)} M'\Rightarrow \Delta[M]\simeq_{\Delta[L(\mathscr{S}_Z)]}\Delta[M'])$. Besides, from $\Delta[M]=_{\mathbf{T}}\Delta[N]$ and $\Delta[M']=_{\mathbf{T}}\Delta[N']$ it follows that $\Delta[M]=_{\mathscr{H}^*}\Delta[N]$ and $\Delta[M']=_{\mathscr{H}^*}\Delta[N']$. Hence, from 3.2.4, we have $\Delta[N]\simeq_{\Delta[R(\mathscr{S}_Z)]}\Delta[N']$.

(1) In this case $\forall\Delta[\ ]\in\mathrm{Sol}(\mathscr{S}, \mathbf{T})\ \Delta[R(\mathscr{S}_Z)]\equiv R(\mathscr{S}_Z)$. ∎

5.8. Example. (0) The system $\mathscr{S}=(\{x\,\mathbf{\Omega}\,\mathbf{K}\,x\,\mathbf{B}=\mathbf{K},\ x\,x\,\mathbf{\Omega}\,\mathbf{K}\,\mathbf{B}=\mathbf{B},\ x\,\mathbf{K}\,x\,\mathbf{\Omega}\,\mathbf{B}=\mathbf{B}\}, \{x\})$ is not $\mathscr{H}^*$-solvable.

(1) The system $\mathscr{S}=(\{xzx=z,\ xyx=a\}, \{x\})$ is not $\mathscr{H}^*$-solvable (use 5.7.1 with $Z=\{z\}$).

5.9. Counter example. The converse of 5.7.0,1 does not hold in general. Let $\mathscr{S}=(\{x=y, xx=z\}, \{x\})$. We have $x\not\simeq_{\{x,xx\}} xx$, but $\mathrm{Sol}(\mathscr{S}, \beta)=\varnothing$.

6. Regular Systems

There are systems for which Theorem 5.7.1 is a necessary and sufficient condition for solvability. We call these systems regular systems. In this section we present a class of regular systems (reg. systems) and construct a solution (6.11) for them (if it exists).

6.0. Definition. The system $\mathscr{S}$ is said to be $\mathbf{T}$-regular ($\mathscr{S}\in\mathrm{Reg}(\mathbf{T})$) iff

$$\mathrm{Sol}(\mathscr{S}, \mathbf{T})\neq\varnothing \qquad \text{iff} \quad \forall M=N,\ M'=N'\in\mathscr{S}_\phi$$
$$(M\simeq_{L(\mathscr{S}_\phi)} M'\Rightarrow N\simeq_{R(\mathscr{S}_\phi)} N').$$

A large class of reg. systems can be constructed starting from reg. systems whose equations have the shape $xM_1\cdots M_m=yM_1\cdots M_m$ (6.1–6.3).

6.1. *Notation.* (0) Let $M\in\Lambda$ and $\{\vec{x}\}\subset_f \mathbf{V}$. We set $M^{\vec{x}}\equiv M[x:=x\vec{x}\mid x\in\{\vec{x}\}]$.

(1) Let $\mathscr{S}=(\Gamma, \{\vec{x}\})$ be a system. We set (see 3.7)

$$\mathscr{S}^*=([x\vec{x}M_1^{\vec{x}}\cdots M_m^{\vec{x}}=y\vec{x}M_1^{\vec{x}}\cdots M_m^{\vec{x}}\,|\,xM_1\cdots M_m\in L(\mathscr{S})\text{ and }x\in\{\vec{x}\}$$
$$\text{and } y\equiv L(\mathscr{S}_\phi)^{\#}((xM_1\cdots M_m)[z:=\mathbf{\Omega}\,|\,z\in(\mathrm{FV}(L(\mathscr{S}))-\{\vec{x}\})])\}, \{\vec{x}\}).$$

Note that, thanks to the choice suggested by 3.7, $\mathscr{S}^*$ satisfies the RHS of Definition 6.0. Hence $\mathscr{S}^*\in\mathrm{Reg}(\mathbf{T})$ iff $\mathscr{S}^*$ is $\mathbf{T}$-solvable.

6.2. Example. Let $\mathscr{S}=(\{xa=\mathbf{K},\ x\mathbf{K}=\mathbf{B}, yx=z\},\ \{x,y\})$. Then we have $\mathscr{S}^*=(\{xxya=z_1xya,\quad xxy\mathbf{K}=z_1xy\mathbf{K},\quad yxy(xxy)=z_2xy(xxy)\}, \{x,y\})$.

6.3. Theorem. *Let* $\mathbf{T}$ *be a sms theory and* $\mathscr{S}=(\Gamma_0\cup\Gamma_1,\ \{\vec{x}\})$ *s.t.:*

Hp.0. *The equations of* Γ_0 *have the shape* $xM_1\cdots M_m=y\vec{x}M_1\cdots M_m$ *with* $x\in\{\vec{x}\}$ *and* $y\notin(\mathrm{FV}(L(\mathscr{S}))\cup\{\vec{x}\})$.

Hp.1. *The equations of* Γ_1 *have the shape* $xM_1\cdots M_m=N$, *where* $x\in\{\vec{x}\}$ *and* N *is a* $\beta\eta$*-nf with* $\mathrm{FV}(N)\cap(\mathrm{FV}(L(\mathscr{S}))\cup\{\vec{x}\})=\varnothing$.

Hp.2. $\forall M=N\in(\Gamma_0)_{\{\vec{x}\}}\ \ \forall M'=N'\in(\Gamma_1)_{\{\vec{x}\}}\ \ ((M\simeq_{L(\mathscr{S}_\phi)}M'$ *and* $\mathrm{head}(N)\equiv\mathrm{head}(N'))\Rightarrow\deg(N)\neq\deg(N')-\mathrm{ord}(N'))$.

Hp.3. $\mathscr{S}^*\in\mathrm{Reg}(\mathbf{T})$.

Then $\mathscr{S}\in\mathrm{Reg}(\mathbf{T})$.

Proof. From Hp.0, 1, 2 and 5.7.0 it follows immediately that

$$\mathrm{Sol}(\mathscr{S},\mathbf{T})\neq\varnothing\Rightarrow\forall M=N,\ M'=N'\in\mathscr{S}_\phi\qquad(M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow N\simeq_{R(\mathscr{S}_\phi)}N').$$

Now we prove that

$$(\forall M=N,\ M'=N'\in\mathscr{S}_\phi\,(M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow N\simeq_{R(\mathscr{S}_\phi)}N'))\Rightarrow\mathrm{Sol}(\mathscr{S},\mathbf{T})\neq\varnothing.$$

Since $\mathscr{S}^*\in\mathrm{Reg}(\mathbf{T})$ then $\mathrm{Sol}(\mathscr{S}^*,\mathbf{T})\neq\varnothing$. Let $\vec{x}\equiv x_1,\dots,x_w$ and $D[\ ]\equiv(\lambda x_1\cdots x_w\cdot[\ ])\,D_1\cdots D_w\in\mathrm{Sol}(\mathscr{S}^*,\mathbf{T})$. We define

$$\begin{aligned}D^*[\ ]&\equiv(\lambda x_1\cdots x_w\cdot[\ ])\,D_1^*\cdots D_w^*\\&\equiv D[\ ][y:=\lambda t_1\cdots t_w\cdot y(t_1t_1\cdots t_w)\cdots(t_wt_1\cdots t_w)\,|\,y\in\mathrm{FV}(D[\ ])\end{aligned}$$

and

$$E[\ ]\equiv(\lambda x_1\cdots x_w\cdot[\ ])(D_1^*D_1^*\cdots D_w^*)\cdots(D_w^*D_1^*\cdots D_w^*).$$

Finally, we define

$$E_0[\ ] \equiv E[\ ][L(\mathscr{S}_\phi)^{\#}\ (xM_1 \cdots M_m) := y \mid xM_1 \cdots M_m$$
$$= y\vec{x}M_1 \cdots M_m \in (\Gamma_0)_{\{\vec{x}\}}]$$

and

$$E_1[\ ] \equiv E_0[\ ][L(\mathscr{S}_\phi)^{\#}\ (xM_1 \cdots M_m) := \mathbf{U}_1^{m+w+1} N \mid xM_1 \cdots M_m$$
$$= N \in (\Gamma_1)_{\{\vec{x}\}}\}.$$

It is easy to verify that $E_1[\ ] \in \mathrm{Sol}(\mathscr{S}, \mathbf{T})$. ∎

6.4. *Remark.* If $\mathscr{S} = (\{M_1 = y_1, ..., M_n = y_n\}, X)$ satisfies the hypotheses of 6.3 and $(y_i \equiv y_j \Rightarrow i = j)$ then we have

$\mathscr{S}$ is $\mathbf{T}$-solvable iff $(\{M_1 = N_1, ..., M_n = N_n\}, X)$ is $\mathbf{T}$-solvable,

where $N_1, ..., N_n$ are pairwise distinct closed $\beta\eta$-normal forms.

6.5. Example. (0) Let $\mathscr{S} = (\{xx = \mathbf{K}, x\mathbf{K} = ax\mathbf{K}, x\omega\mathbf{\Omega} = bx\omega\mathbf{\Omega}\}, \{x\})$. We have

$$\mathscr{S}^* = (\{xx(xx) = y_1 x(xx), xx\mathbf{K} = y_2 x\mathbf{K}, xx\omega\mathbf{\Omega} = y_3 x\omega\mathbf{\Omega}\}, \{x\}).$$

If $\mathscr{S}^* \in \mathrm{Reg}(\mathbf{T})$ then $\mathscr{S} \in \mathrm{Reg}(\mathbf{T})$. We see (6.11) that $\mathscr{S}^* \in \mathrm{Reg}(\beta)$.

(1) Let $\mathscr{S} = (\{x(xx)\,x = y, xxx = z\}, \{x\})$. If $\mathscr{S}^* \in \mathrm{Reg}(\mathbf{T})$ then

$\mathscr{S}$ is $\mathbf{T}$-solvable iff $(\{x(xx)\,x = \mathbf{S}, xxx = \mathbf{K}\}, \{x\})$ is $\mathbf{T}$-solvable.

We see (6.11) that $\mathscr{S}^* \in \mathrm{Reg}(\beta)$.

6.6. Counter example. The property stated in Remark 6.4 does not hold in general. The system $(\{x = \mathbf{U}_1^2, xx = \mathbf{U}_2^3\}, \{x\})$ is β-solvable (its solution is $D[\ ] \equiv (\lambda x \cdot [\ ])\,\mathbf{U}_1^2$), but the system $(\{x = y, xx = z\}, \{x\})$ is not β-solvable.

So far we have assumed the existence of a class of regular systems. The next step is to introduce, by a constructive definition, a candidate class for regularity.

6.7. Defintion. Let $X \subset_f \mathbf{V}$. The set $\mathscr{F}$ is said to be X-regular ($\mathscr{F} \in \mathrm{reg}(X)$) iff:

(0) $\mathscr{F} \subset_f \mathrm{SOL}$ and $\mathscr{F}$ is λ-free and $\mathrm{head}(\mathscr{F}) \subseteq X$;

(1) $\exists e: X \to \mathbf{N}^+$ s.t.:

(0) $(\forall x \in X\ e(x) \leqslant \min \deg(M \in \mathscr{F} \mid \mathrm{head}(M) \equiv x\}))$;

(1) $(\forall x M_1 \cdots M_{e(x)} \cdots M_n \in \mathscr{F}\ (M_{e(x)} \in \mathrm{SOL}$ and $\mathrm{head}(M_{e(x)}) \notin (\mathrm{FV}(M_{e(x)}) - X)))$;

(2) $(\forall M \in \mathscr{F}\ \forall \alpha \in \mathrm{Seq}\ \ (\alpha \neq \langle\ \rangle$ and $\mathrm{head}(M_\alpha) \in X \Rightarrow \deg(M_\alpha) < e(\mathrm{head}(M_\alpha))))$.

Intuitively we can say that $\mathscr{F}$ is X-regular iff the internal occurrences in $M \in \mathscr{F}$ of the variables in X do not have too many arguments.

6.8. EXAMPLE. $\mathscr{F}_1 = \{xx, x\mathbf{K}, x\mathbf{B}, x\mathbf{S}x, x(\lambda t \cdot txxxx)\} \in \mathrm{reg}(\{x\})$, $\mathscr{F}_2 = \{x\mathbf{\Omega}, xxx\} \notin \mathrm{reg}(\{x\})$; $\mathscr{F}_3 = \{x(x\mathbf{\Omega}x)\,x, xxxx\} \notin \mathrm{reg}(\{x\})$; $\mathscr{F}_4 = \{x(x\mathbf{\Omega})\,x, xxxx\} \in \mathrm{reg}(\{x\})$.

In order to solve a system we may eliminate subterms that do not yield any essential information.

6.9. *Notation.* Let $\mathscr{F} \subset_f \Lambda$. We define (see 2.1.12.2, 4.1.4.2)

$$\begin{aligned}\mathscr{F}^- = \{f(\mathscr{F}) \mid f \colon \mathscr{F} \to \Lambda \text{ s.t.: } [(\forall M \in \mathscr{F}\ f(M) \precsim M)\\ \text{and } (\mathscr{F} \in \mathrm{PFR} \Rightarrow f(\mathscr{F}) \in \mathrm{PFR})\\ \text{and } \forall M, N \in \mathscr{F}\ (f(M) \simeq_{f(\mathscr{F})} f(N) \Rightarrow M \simeq_{\mathscr{F}} N)\}.\end{aligned}$$

6.1.0. EXAMPLE. Let $\mathscr{F} = \{x(xxxx)\,\mathbf{\Omega}x, x\mathbf{\Omega}xx, x\mathbf{\Omega}\mathbf{\Omega}(xx)\,x, x\mathbf{\Omega}\mathbf{\Omega}(xx)(xxxx)\}$. Then

$$\{x\mathbf{\Omega}\mathbf{\Omega}x, x\mathbf{\Omega}xx, x\mathbf{\Omega}\mathbf{\Omega}(xx)\,x, x\mathbf{\Omega}\mathbf{\Omega}(xx)(xxxx)\} \in \mathscr{F}^-;$$

$$\{x(xxxx)\,\mathbf{\Omega}x, x\mathbf{\Omega}xx, x\mathbf{\Omega}\mathbf{\Omega}(xx)\,x, x\mathbf{\Omega}\mathbf{\Omega}(xx)\,\mathbf{\Omega}\} \notin \mathscr{F}^-;$$

$$\{x(xxxx)\,\mathbf{\Omega}x, x\mathbf{\Omega}xx, x\mathbf{\Omega}\mathbf{\Omega}\mathbf{\Omega}x, x\mathbf{\Omega}\mathbf{\Omega}(xx)(xxxx)\} \notin \mathscr{F}^-.$$

The next theorem proves that via our candidate class we obtain a class of regular systems. Moreover, since the proof is constructive, at the same time we have, for such a class, a method of finding a solution, if any.

6.11. THEOREM. *Let $\mathscr{S} = (\Gamma, X)$ s.t.:*

Hp.0. *The equations of $\mathscr{S}$ have the shape $xM_1 \cdots M_n = yM_1 \cdots M_n$ with $x \in X$ and $y \notin (\mathrm{FV}(L(\mathscr{S})) \cup X)$.*

Hp.1. $L(\mathscr{S}_\phi)^- \cap \mathrm{reg}(X) \cap \mathrm{PFR} \neq \varnothing$ *(see* 4.1.4.2, 5.2.4, 6.7, 6.9).

Then for any semisensible theory $\mathbf{T}$, *we have* $\mathscr{S} \in \mathrm{Reg}(\mathbf{T})$.

Proof. From 5.7.0 it follows immediately that

$$\mathrm{Sol}(\mathscr{S}, \mathbf{T}) \neq \varnothing \Rightarrow \forall M = N, M' = N' \in \mathscr{S}_\phi \qquad (M \simeq_{L(\mathscr{S}_\phi)} M' \Rightarrow N \simeq_{R(\mathscr{S}_\phi)} N').$$

Hence it is sufficient to prove that

$$(\forall M = N, M' = N' \in \mathscr{S}_\phi\ (M \simeq_{L(\mathscr{S}_\phi)} M' \Rightarrow N \simeq_{R(\mathscr{S}_\phi)} N')) \Rightarrow \mathrm{Sol}(\mathscr{S}, \mathbf{T}) \neq \varnothing.$$

Let, w.l.o.g., $\mathscr{S} = \mathscr{S}_\phi$, $L(\mathscr{S}) \in (\mathrm{reg}(X) \cap \mathrm{PFR})$ and all the equations of $\mathscr{S}$ have the shape $xM_1 \cdots M_n = L(\mathscr{S})^\# \ (xM_1 \cdots M_n)\, M_1 \cdots M_n$. If we prove that $\mathrm{Sol}(\mathscr{S}, \beta) \neq \varnothing$, then the thesis follows.

Step 0. We set $X = \{x_1, ..., x_w\} = \{\vec{x}\}$, $\mathscr{F} = L(\mathscr{S})$ and e as in 6.7.1 (e exists since $L(\mathscr{S}) \in (\mathrm{reg}(X) \cap \mathrm{PFR})$).

Step 1. For any $x \in X$ let

$$\mathscr{F}^0_{x,h} \equiv \{xM_1 \cdots M_{e(x)} \cdots M_n \in \mathscr{F} \mid \mathrm{head}(M_{e(x)}) \in X \quad \text{and}$$
$$\mathrm{ord}(M_{e(x)} + e(\mathrm{head}(M_{e(x)})) - \deg(M_{e(x)}) = h\};$$
$$\mathscr{F}^1_{x,h} \equiv \{xM_1 \cdots M_{e(x)} \cdots M_n \in \mathscr{F} \mid M_{e(x)}$$
$$=_\beta \lambda z_1 \cdots z_p \cdot z_h Q_1 \cdots Q_r \text{ with } h \leqslant p\}.$$

It holds that

$$\mathscr{F}^0_{x,h} \cap \mathscr{F}^1_{x,h} = \varnothing.$$

Let $\mathscr{F}(x, h) = \mathscr{F}^0_{x,h} \cup \mathscr{F}^1_{x,h}$ and $H(x) \equiv \{h \in \mathbb{N} \mid \mathscr{F}(x, h) \neq \varnothing\}$.

Step 2. Let $b_x = \max\{\max\{\mathrm{ord}(M_{e(x)}) \mid xM_1 \cdots M_{e(x)} \cdots M_n \in \mathscr{F}^1_{x,h}\} \mid h \in H(x)\}$ and $r_x = \max\{\max H(x), b_x\}$. Of course it holds that

$$\forall x \in X \qquad \max H(x) \leqslant r_x.$$

Step 3. Let $\varepsilon \in \mathrm{QL}(X, \{\langle M_1, ..., M_n\rangle \mid xM_1 \cdots M_n \in \mathscr{F}\})$ s.t.:

$$\forall x \in X \qquad (\varepsilon(0, x) = e(x) \text{ and } \varepsilon(1, x) > r_x) \text{ (see 4.4.0.3)}.$$

By the assumptions in Step 0 ε exists.

Step 4. $\forall x \in X$ let

$$\Delta_{x,\varepsilon(1,x)} \equiv \mathbf{P}_{e(x)};$$
$$\Delta_x \equiv \lambda t_1 \cdots t_{e(x)} \cdot t_{e(x)}(\Delta_{x,1} t_1 \cdots t_{e(x)}) \cdots (\Delta_{x,\varepsilon(1,x)} t_1 \cdots t_{e(x)}).$$

We define

$$\Delta^*[\ ] \equiv (\lambda x_1 \cdots x_w \cdot [\ ])\, \Delta_{x_1} \cdots \Delta_{x_w}.$$

Clearly $\Delta^*[\ ] \in \mathscr{U}(\Delta_{\vec{x},\varepsilon}[\ ])$.

Step 5. Let $xM_1, ..., M_{e(x)}M_{e(x)+1}, ..., M_n \in \mathscr{F}$. We set

$$\vec{M} \equiv M_1, ..., M_{e(x)};$$
$$\vec{Q} \equiv M_{e(x)+1}, ..., M_n; \operatorname{ord}(M_{e(x)}) = p; \deg(M_{e(x)}) = b;$$
$$\Delta^*[\vec{M}] \equiv \Delta^*[M_1], ..., \Delta^*[M_{e(x)}]; \Delta^*[\vec{Q}] \equiv \Delta^*[M_{e(x)+1}], ..., \Delta^*[M_n];$$
$$\text{Vect } 0 \equiv (\Delta_{x,1}\Delta^*[\vec{M}]), ..., (\Delta_{x,p}\Delta^*[\vec{M}]).$$

Note that $\mathscr{F} \equiv U_{x\in X} U_{h\in H(x)} \mathscr{F}(x, h)$. We have two cases.

Case 0. $x\vec{M}\vec{Q} \in \mathscr{F}^0_{x,h}$. Let $M_{e(x)} =_\beta \lambda z_1 \cdots z_p \cdot x'(N_1 z_1 \cdots z_p) \cdots (N_b z_1 \cdots z_p)$; $\text{Vect } 1 \equiv (\Delta^*[N_1]\text{Vect } 0), ..., (\Delta^*[N_b]\text{Vect } 0), (\Delta_{x,p+1}\Delta^*[\vec{M}], ..., (\Delta_{x,p+e(x')-b}\Delta^*[\vec{M}])$. We have

$$\begin{aligned}
\Delta^*[x\vec{M}\vec{Q}] &=_\beta \Delta_x \Delta^*[\vec{M}]\, \Delta^*[\vec{Q}] \\
&=_\beta \Delta^*[M_{e(x)}](\Delta_{x,1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}] \\
&=_\beta (\lambda z_1 \cdots z_p \cdot \Delta_{x'}(\Delta^*[N_1]\, z_1 \cdots z_p) \cdots (\Delta^*[N_b]\, z_1 \cdots z_p)) \\
&\qquad (\Delta_{x,1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}] \\
&=_\beta \Delta_{x'}(\Delta^*[N_1] \text{ Vect } 0) \cdots (\Delta^*[N_b] \text{ Vect } 0)(\Delta_{x,p+1}\Delta^*[\vec{M}]) \\
&\qquad \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}] \\
&=_\beta \Delta_{x,p+e(x')-b}\Delta^*[\vec{M}](\Delta_{x',1} \text{ Vect } 1) \cdots (\Delta_{x',\varepsilon(1,x')} \text{Vect } 1) \\
&\qquad (\Delta_{x,p+e(x')-b+1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}].
\end{aligned}$$

Using 4.1.6.2 choose $\Delta_{x,p+e(x')-b}$ s.t.

$$\begin{aligned}
&\Delta_{x,p+e(x')-b}\Delta_{\vec{x},\varepsilon}[\vec{M}]\, \mathbf{\Omega}_1 \cdots \mathbf{\Omega}_{\varepsilon(1,x)+b-p+\varepsilon(1,x')-e(x')}\Delta_{\vec{x},\varepsilon}[\vec{Q}] \\
&\quad =_\beta \mathscr{F}^{\#}(x\vec{M}\vec{Q})\, \Delta_{\vec{x},\varepsilon}[\vec{M}]\, \Delta_{\vec{x},\varepsilon}[\vec{Q}].
\end{aligned}$$

Case 1. $x\vec{M}\vec{Q} \in \mathscr{F}^1_{x,h}$. We have

$$\begin{aligned}
\Delta^*[x\vec{M}\vec{Q}] &=_\beta \Delta_x \Delta^*[\vec{M}]\, \Delta^*[\vec{Q}] \\
&=_\beta \Delta^*[M_{e(x)}](\Delta_{x,1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}] \\
&=_\beta (\lambda z_1 \cdots z_p \cdot z_h(\Delta^*[N_1]\, z_1 \cdots z_p) \cdots (\Delta^*[N_b]\, z_1 \cdots z_p)) \\
&\qquad (\Delta_{x,1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}] \\
&=_\beta \Delta_{x,h}\Delta^*[\vec{M}](\Delta^*[N_1] \text{ Vect } 0) \cdots (\Delta^*[N_b] \text{ Vect } 0) \\
&\qquad \cdots (\Delta_{x,p+1}\Delta^*[\vec{M}]) \cdots (\Delta_{x,\varepsilon(1,x)}\Delta^*[\vec{M}])\, \Delta^*[\vec{Q}].
\end{aligned}$$

Using 4.1.6.2 choose $\Delta_{x,h}$ s.t.

$$\Delta_{x,h}\Delta_{\vec{x},\varepsilon}[\vec{M}]\,\mathbf{\Omega}_1\cdots\mathbf{\Omega}_{\varepsilon(1,x)+h-p}\Delta_{\vec{x},\varepsilon}[\vec{Q}]=_\beta \mathscr{F}^{\#}(x\vec{M}\vec{Q})\,\Delta_{\vec{x},\varepsilon}[\vec{M}]\,\Delta_{\vec{x},\varepsilon}[\vec{Q}].$$

Again using 4.1.6.2 we have

$$\forall x\vec{M}=y\vec{M}\in\mathscr{S}\qquad \Delta^*[x\vec{M}]=_\beta \Delta^*[y\vec{M}].\ \blacksquare$$

6.12. Example. Let $\mathscr{S}=(\{xa(\lambda t\cdot x(xt))=y_1a(\lambda t\cdot x(xt)),\ x(xxxx)(\lambda t\cdot x(xx))(x(x\mathbf{\Omega}))=y_2(xxxx)(\lambda t\cdot x(xx))(x(x\mathbf{\Omega}))\},\ \{x\})$. Since $L(\mathscr{S}_\phi)^-\cap \mathrm{reg}(\{x\})\cap \mathrm{PFR}\neq\varnothing$ then $\mathscr{S}\in \mathrm{Reg}(\beta)$. It is sufficient to solve the system

$$\mathbf{Q}=(\{x\mathbf{\Omega}(\lambda t\cdot x(xt))=y_1\mathbf{\Omega}(\lambda t\cdot x(xt)),$$
$$x\mathbf{Q}(\lambda t\cdot x(xx))(x(x\mathbf{\Omega}))=y_2\mathbf{\Omega}(\lambda t\cdot x(xx))(x(x\mathbf{\Omega}))\},\ \{x\}).$$

A possible solution for $\mathbf{Q}$ (and $\mathscr{S}$) is

$$D[\ ]\equiv(\lambda x\cdot[\ ])\,D\qquad\text{with}\quad D\text{ as in Example 0.1.}$$

The systems studied in 6.11 can be useful for solving other system schemas. From 4.5.1.1, 6.3, and 6.11 we may state the following corollary.

6.13. Corollary. *Let* $\mathscr{S}=(\Gamma_0\cup\Gamma_1,\ \{\vec{x}\})$ *s.t.:*

Hp.0. *The equations of* Γ_0 *have the shape* $xM_1\cdots M_m=y(xM_1\cdots M_m)^*_{\alpha_1}\cdots(xM_1\cdots M_m)^*_{\alpha_p}\vec{x}M_1\cdots M_m$ *with* $x\in\{\vec{x}\}$, $y\notin(\mathrm{FV}(L(\mathscr{S}))\cup\{\vec{x}\})$, $*$ *as in* 4.5.1 *(see* 4.5.0), $\forall i\in\{1,...,p\}$ $(\mathrm{Tr}(xM_1\cdots M_m,\alpha_i)\cap \mathrm{FV}(xM_1\cdots M_m)\subseteq\{\vec{x}\})$ *and* $\mathrm{Card}(\Gamma_0)=\mathrm{Card}(\mathrm{head}(R(\Gamma_0)))$.

Hp.1. *The equations of* Γ_1 *have the shape* $xM_1\cdots M_m=N$, *where* $x\in\{\vec{x}\}$ *and* N *is a* $\beta\eta$*-nf with* $\mathrm{FV}(N)\cap(\mathrm{FV}(L(\mathscr{S}))\cup\{\vec{x}\})=\varnothing$.

Hp.2. $\forall M=N\in(\Gamma_0)_{\{\vec{x}\}}\ \forall M'=N'\in(\Gamma_1)_{\{\vec{x}\}}$

$$(M\simeq_{L(\mathscr{S}_\phi)}M'\ \textit{and}\ \mathrm{head}(N)\equiv\mathrm{head}(N')\Rightarrow\deg(N)\neq\deg(N')-\mathrm{ord}(N')).$$

Hp.3. $L(\mathscr{S}_\phi)^-\cap\mathrm{reg}(\{\vec{x}\})\cap\mathrm{PFR}\neq\varnothing$.

Then for any semisensible theory $\mathbf{T}$, *we have* $\mathscr{S}\in\mathrm{Reg}(\mathbf{T})$.

Proof. From 5.7.0 it follows immediately that

$$\mathrm{Sol}(\mathscr{S},\mathbf{T})\neq\varnothing\Rightarrow\forall M=N,\ M'=N'\in\mathscr{S}_\phi\qquad(M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow N\simeq_{R(\mathscr{S}_\phi)}N').$$

We prove that

$$(\forall M=N,\ M'=N'\in\mathscr{S}_\phi\ (M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow N\simeq_{R(\mathscr{S}_\phi)}N'))\Rightarrow\mathrm{Sol}(\mathscr{S},\mathbf{T})\neq\varnothing.$$

An example is sufficient.

We solve the system

$$\mathscr{S} = (\{x\mathbf{\Omega}(xa) = yax\mathbf{\Omega}(xa),\ x\mathbf{\Omega}\mathbf{U}_2^2 = zx\mathbf{\Omega}\mathbf{U}_2^2\},\ \{x\}).$$

Step 0. Using 6.11 we solve

$$\mathbf{Q}_0 = (\{xx\mathbf{\Omega}(xxa) = yx\mathbf{\Omega}(xxa),\ xx\mathbf{\Omega}\mathbf{U}_2^2 = zx\mathbf{\Omega}\mathbf{U}_2^2\},\ \{x\}).$$

Let $Q_0 \equiv \lambda tuv \cdot v(\mathbf{U}_1^7 y)\, ztuv$; then $D_0[\ ] \equiv (\lambda x \cdot [\ ])\, Q_0$ is a solution for $\mathbf{Q}_0$.

Step 1. Using 4.5.1.1 we solve

$$\mathbf{Q}_1 = (\{xx\mathbf{\Omega}(xxa) = ya(xx)\,\mathbf{\Omega}(xxa),\ xx\mathbf{\Omega}\mathbf{U}_2^2 = z(xx)\,\mathbf{\Omega}\mathbf{U}_2^2\},\ \{x\}).$$

Let $Q_1 \equiv Q_0[y := \lambda abc \cdot y(\langle \mathbf{U}_4^5 \rangle\, c)(aa)\, bc][z := \lambda a \cdot z(aa)]$; then $D_1[\ ] \equiv (\lambda x \cdot [\ ])\, Q_1$ is a solution for $\mathbf{Q}_1$ ($\langle \mathbf{U}_4^5 \rangle$ extracts a from (xxa)).

Step 2. Reasoning as in 6.3 we see that $D[\ ] \equiv (\lambda x \cdot [\ ])(Q_1 Q_1)$ is a solution for $\mathscr{S}$. ∎

Examples 0.2 and 0.3 show an application of 6.13 (in 0.2 the term E extracts a_2). Of course 6.4 holds with $\mathscr{S}^* \in \mathrm{Reg}(\mathbf{T})$ replaced by $L(\mathscr{S}_\phi^*)^- \cap \mathrm{reg}(\{\vec{x}\}) \cap \mathrm{PFR} \neq \varnothing$.

Corollary 6.14.0 extends [CDR 78] and Corollary 6.14.1 extends [Böh 68] and its generalization [BDPR 79].

6.14. Corollary. *Let* $\mathbf{T}$ *be a sms theory,* $M_1, ..., M_n \in \Lambda^0$, $N_0, ..., N_n$ *be* $\beta\eta$*-nf, and* $\mathscr{S} = (\{xx = N_0,\ xM_1 = N_1, ..., xM_n = N_n\},\ \{x\})$ *be a system. We have:*

(0) $\mathscr{S} \in \mathrm{Reg}(\mathbf{T})$.

(1) *If* $N_0, ..., N_n$ *are pairwise distinct then*

$\mathscr{S}$ *is* $\mathbf{T}$*-solvable* *iff* $\{xx, xM_1, ..., xM_n\}$ *is distinct.*

Proof. Immediately from 6.13, 3.2.2, and 3.2.3. ∎

6.15. Example. The system $\mathscr{S} = (\{xx = \mathbf{K},\ x\mathbf{K} = \mathbf{S},\ x\mathbf{S} = \omega,\ x\omega = y\},\ \{x\})$ is β-solvable because $L(\mathscr{S}_\phi)$ is distinct. A possible solution is

$$D[\ ] \equiv (\lambda x \cdot [\ ])\, D \qquad \text{where}$$

$$D \equiv \lambda t \cdot t(t\mathbf{U}_4^4(\mathbf{U}_1^{10}\mathbf{K})\,\mathbf{\Omega}(\mathbf{U}_1^6 y)(\mathbf{U}_1^4\omega)(\mathbf{U}_1^4\mathbf{S}))\,\mathbf{\Omega}\mathbf{\Omega}.$$

7. X-SEPARABILITY

The study of the separability by substitutive contexts $\Delta[\ ]\in\mathrm{Sub}(X)$ leads to the study of systems $\mathscr{S}=(\Gamma, X)$ with equations of the shape $M=y$, where the RHS variables are pairwise distinct. This is the X-separability problem (7.0) (cf. with the separability introduced in 2.3.3). This problem is also equivalent to studying the global surjectivity of $L(\mathscr{S})$ with respect to the variables of X [BP 88a, b]. Of course for these systems Corollary 6.13 applies. However, in this particular case we can drop the assumption $L(\mathscr{S}_\phi)^-\cap\mathrm{PFR}\neq\varnothing$.

Proposition 7.1.2 characterizes the X-separability for a class of λ-free sets. Proposition 7.1.3 shows that if a separator for a finite set $\mathscr{F}$ exists then there exists a separator for $\mathscr{F}$ that recognizes itself from the objects that it is separating. (Compare 7.1.3.0 with 2.4.0 or [CDR 78] and 7.1.3.1 with [Böh 68, and BDPR 79].) Example 4.0.0 gives an easy application of 7.1.3.

7.0. DEFINITION. Let $X\subset_f\mathbf{V}$, $\mathscr{F}=\{M_1, ..., M_n\}\subset\Lambda$, and $\mathbf{T}$ be a theory. The set $\mathscr{F}$ is said to be $\mathbf{T}$-X-separable iff

$$(\exists\Delta[\ ]\in\mathrm{Sub}(X)\text{ s.t.: }(\forall i\in\{1, ..., n\}\ \Delta[M_i]=_{\mathbf{T}}y_i\in(\mathbf{V}-\mathrm{FV}(\mathscr{F})))\text{ and }$$
$$(\forall i,j\in\{1, ..., n\}(y_i\equiv y_j\Rightarrow i=j))).$$

7.1. PROPOSITION. *Let* $\mathbf{T}$ *be a sms theory and* $\mathscr{S}=(\Gamma, X)$ *be a system with equations having the shape* $M=y$, *where* $y\notin(FV(L(\mathscr{S}))\cup X)$.

(0) *If* $\mathscr{S}$ *is* $\mathscr{H}^*$*-solvable then*:

(0) $L(\mathscr{S}_\phi)\subseteq\mathrm{SOL}$.

(1) $\mathrm{head}(L(\mathscr{S}_\phi))\subseteq X$.

(2) $L(\mathscr{S}_\phi)\in\mathrm{PFR}$ (*see* 4.1.4.2).

(3) $\forall M=y,\ M'=y'\in\mathscr{S}_\phi\ (M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow y\equiv y')$.

(1) *If* $L(\mathscr{S}_\phi)^-\cap\mathrm{reg}(X)\neq\varnothing$ (*see* 4.1.4.2, 6.7, 6.9) *then* $\mathscr{S}$ *is* $\mathbf{T}$*-solvable iff* $L(\mathscr{S}_\phi)\in\mathrm{PFR}$ *and* $\forall M=y,\ M'=y'\in\mathscr{S}_\phi\ (M\simeq_{L(\mathscr{S}_\phi)}M'\Rightarrow y\equiv y')$.

(2) *Let* $X\subset_f\mathbf{V}$ *and* $\mathscr{F}\subset_f\Lambda$ *with* $(\mathscr{F}_X)^-\cap\mathrm{reg}(X)\neq\varnothing$. *Then* $\mathscr{F}$ *is* $\mathbf{T}$-X*-separable iff* ($\mathscr{F}_X\in\mathrm{PFR}$ *and* $\mathscr{F}_X$ *is distinct*).

(3) *Let* $\mathscr{F}\equiv\{xx, xM_1, ..., xM_n\}$.

(0) *If* $\mathrm{FV}(M_1\cdots M_n)\cap\{x\}=\varnothing$ *then* $\mathscr{F}$ *is* $\mathbf{T}$-$\{x\}$*-separable iff* $\mathscr{F}_{\{x\}}$ *is distinct.*

(1) *If* $M_1\cdots M_n$ *are closed* $\beta\eta$*-normal forms then* $\mathscr{F}$ *is* $\mathbf{T}$-$\{x\}$*-separable* iff $M_1\cdots M_n$ *are pairwise distinct.*

Proof. (0) Per absurdum. Let $D[\ ] \in \mathrm{Sol}(\mathscr{S}, \mathscr{H}^*)$ and $M \in L(\mathscr{S}_\phi)$.

(0.0) If $M \notin \mathrm{SOL}$ then $D[M] \notin \mathrm{SOL}$, which is absurd.

(0.1) If $\mathrm{head}(M) \notin X$ then $\mathrm{head}(M) \equiv \mathrm{head}(D[M]) \notin (\mathrm{FV}(L(\mathscr{S}_\phi)) \cup X)$, which is absurd.

(0.2) Let $N, M \equiv \lambda\vec{t} \cdot xM_1 \cdots M_m \cdots M_n \in L(\mathscr{S}_\phi)$ s.t. $m < n$ and $\lambda\vec{t} \cdot xM_1 \cdots M_m \simeq_{L(\mathscr{S}_\phi)^+} N$. Then from 3.4.0 $D[\lambda\vec{t} \cdot xM_1 \cdots M_m] \simeq_{D[L(\mathscr{S}_\phi)^+]} D[N]$. Hence $\deg(D[\lambda\vec{t} \cdot xM_1 \cdots M_m]) - \mathrm{ord}(D[\lambda\vec{t} \cdot xM_1 \cdots M_m]) = 0$ and $\deg(D[M]) - \mathrm{ord}(D[M]) > 0$, which is absurd.

(0.3) Immediately from 5.7.1.

(1) From (0) and 6.13.

(2) From (1) and 3.2.3.

(3) The same as 6.14. ∎

7.2. EXAMPLE. (0) The systems $\mathscr{S} = (\{xxx = y, x\mathbf{\Omega}x = y, x\mathbf{\Omega}(x\mathbf{\Omega})\, x = z, x\mathbf{\Omega}(x\mathbf{\Omega})\, \mathbf{\Omega} = z\}, \{x\})$ is β-solvable. A possible solution is

$$D[\ ] \equiv (\lambda x \cdot [\ ])\, D, D \equiv \lambda t_1 t_2 \cdot t_2(\mathbf{U}_1^5 z)(\mathbf{U}_1^3 y).$$

(1) The set $\mathscr{F} = \{xa(\lambda t \cdot x(xt)), x(xxxx)(\lambda t \cdot x(xx))(x(x\mathbf{\Omega}))\}$ is β-$\{x\}$-separable. A β-$\{x\}$-separator is

$$D[\ ] \equiv (\lambda x \cdot [\ ])\, \Delta \qquad \text{with } \Delta \text{ as in Example 0.1.}$$

(2) It is possible to find two λ-terms each recognizing itself and each other. Let $\mathscr{F} \equiv \{x_1x_1, x_1x_2, x_2x_1, x_2x_2\}$ and $X \equiv \{x_1, x_2\}$. Because $\mathscr{F}_X \in \mathrm{PFR}$ and $\mathscr{F}_X$ is distinct then $\mathscr{F}$ is β-X-separable. A possible solution is

$$D[\ ] \equiv (\lambda x_1 x_2 \cdot [\ ])\, D_1 D_2$$

where

$$D_1 \equiv \lambda t \cdot t(G_1 t); \qquad D_2 \equiv \lambda t \cdot t(G_2 t)\, \mathbf{\Omega};$$

$$G_1 \equiv G[y_1 := \mathbf{U}_1^2 y_1][y_2 := \mathbf{U}_1^3 y_2];$$

$$G_2 \equiv G[y_1 := \mathbf{U}_1^3 y_3][y_2 := \mathbf{U}_1^4 y_4];$$

$$G \equiv \langle \mathbf{U}_3^3, \mathbf{U}_1^2 y_2, y_1 \rangle.$$

8. LEFT-INVERTIBILITY

It is always possible to transform a system of equations into a single equation with only one unknown (5.6.2). However, if we are searching for a solution, it can be more useful to transform an equation into a system of equations with as many unknowns as possible. Following this idea we transform a left-invertibility problem (solve the equation $(\{x(My)=y\}, \{x\}))$ into an X-separability problem.

8.0. DEFINITION. Let $\mathbf{T}$ be a theory. We set:

(0) $\mathbf{L}(\mathbf{T})=\{M\in\Lambda \mid \exists L\in\Lambda\ \forall y\in\mathbf{V}\ L(My)=_{\mathbf{T}} y\}$. If $M\in\mathbf{L}(\mathbf{T})$ we say that M is $\mathbf{T}-1-$invertible.

(1) $\mathbf{R}(\mathbf{T})=\{M\in\Lambda \mid \exists R\in\Lambda\ \forall y\in\mathbf{V}\ M(Ry)=_{\mathbf{T}} y\}$. If $M\in\mathbf{R}(\mathbf{T})$ we say that M is $\mathbf{T}$-r-invertible.

The sets $\mathbf{L}(\beta)$ and $\mathbf{R}(\beta)$ have been characterized in [BD 74, and MZ 83]. The set $\mathbf{L}(\beta\eta)\cap\mathbf{R}(\beta\eta)$ has been characterized in [Dez 76, BK 80]. Here we characterize (Corollary 8.2) a subset of $\mathbf{L}(\mathbf{T})$, where $\mathbf{T}\supseteq\boldsymbol{\lambda\eta}$ is a sms theory.

8.1. THEOREM. *Let* $\mathbf{T}\supseteq\boldsymbol{\lambda\eta}$ *be a sms theory and* $M\equiv\lambda x_0\vec{x}\cdot x_0 M_1\cdots M_n\in\Lambda$. *Then*

$$M\in\mathbf{L}(\mathbf{T}) \quad \textit{iff} \quad \{M_1,\ldots,M_n\} \textit{ is } \mathbf{T}\text{-}\{\vec{x}\}\textit{-separable}.$$

Proof. Let $\vec{x}\equiv x_1,\ldots,x_w$.

($\Rightarrow$) Let $L\in\Lambda$ and $y\notin \mathrm{FV}(LM)$ s.t. $L(My)=_{\mathbf{T}} y$. Then

$$L(\lambda\vec{x}\cdot yM_1[x_0:=y]\cdots M_n[x_0:=y])=_{\mathbf{T}} y.$$

Hence it holds that

$$L=_{\mathbf{T}}\lambda t_0\vec{t}\cdot t_0(L_1t_0\vec{t})\cdots(L_rt_0\vec{t}),$$

where $\vec{t}\equiv t_1\cdots t_p$ and $\{t_0,\vec{t}\}\cap\mathrm{FV}(L_1\cdots L_r)=\varnothing$. Suppose $w>r$. We have $n=p+w-r$ and

$$L(My)=_{\mathbf{T}}\lambda\vec{t}t_{p+1}\cdots t_{p+w-r}\cdot y\Delta[M_1[x_0:=y]]\cdots\Delta[M_n[x_0:=y]]=_{\mathbf{T}} y,$$

where $\Delta[\ ]\equiv(\lambda x_1\cdots x_w\cdot[\ ])(L_1(My)\vec{t})\cdots(L_r(My)\vec{t})\,t_{p+1}\cdots t_{p+w-r}$. Hence $\forall i\in\{1,\ldots,n\}$ $\Delta[M_i[x_0:=y]]=_{\mathbf{T}} t_i$ and also $\forall i\in\{1,\ldots,n\}$ $\Delta[M_i]=_{\mathbf{T}} t_i$. Then $\{M_1,\ldots,M_n\}$ is $\mathbf{T}$-$\{\vec{x}\}$-separable. The case $w\leqslant r$ is analogous.

($\Leftarrow$) Let $\Delta[\ ]\equiv(\lambda x_1\cdots x_w\cdot[\ ])\,\Delta_1\cdots\Delta_w$ s.t. $\forall i\in\{1,\ldots,n\}$ $\Delta[M_i]=_{\mathbf{T}} t_i$. Then $L\equiv\lambda t_0t_1\cdots t_n\cdot t_0\Delta_1\cdots\Delta_w$ is a left-inverse of M. ∎

8.2. COROLLARY. *Let* $\mathrm{T} \supseteq \lambda\eta$ *be a sms theory and* $M \equiv \lambda x_0 \vec{x} \cdot x_0 M_1 \cdots M_n \in \Lambda$ *with* $(\{M_1, ..., M_n\}_{\{\vec{x}\}})^- \cap \mathrm{reg}(\{\vec{x}\}) \neq \varnothing$ *(see* 5.2.2, 6.7, 6.9*). Then* $M \in \mathrm{L}(\mathrm{T})$ *iff* $(\{M_1, ..., M_n\}_{\{\vec{x}\}} \in \mathrm{PFR}$ *and* $\{M_1, ..., M_n\}_{\{\vec{x}\}}$ *is distinct).* ∎

Proof. From 8.1. and 7.1.2.

Refer to 0.1 for an application of 8.2.

9. CONCLUDING REMARKS AND FURTHER DEVELOPMENT

In summary, the results of this paper show how a disciplinated use of self-application can be employed in the solution of functional equations without degenerating into infinite computations. The next step seems to be to try to discover a larger class of regular systems.

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REFERENCES

[Bar 84] BARENDREGT, H. P. (1984), "The Lambda Calculus," North-Holland, Amsterdam.

[BD 74] BÖHM, C., AND DEZANI-CIANCAGLINI, M. (1974), Combinatorial problems, combinator equations and normal forms, *in* "Second International Colloquium on Automata, Languages, and Programming," Lecture Notes in Computer Science, Vol. 14, pp. 185–199, Springer-Verlag, New York/Berlin.

[BDPR 79] BÖHM, C., DEZANI-CIANCAGLINI, M., PERETTI, P., AND RONCHI DELLA ROCCA, S. (1979), A discrimination algorithm inside λ-β-calculus, *Theoret. Comput. Sci.* **8**, 271–291.

[Ber 83] BERARDUCCI, A. (1983), Masters thesis in mathematics (developed under the supervision of C. Böhm), Università "La Sapienza," 1983, Roma, Italy.

[Böh 68] BÖHM, C. (1968), "Alcune proprietà delle forme β-η-normali nel λ-K-calcolo," IAC Publ. No. 696, Roma.

[BK 80] BERGESTRA, J. A., AND KLOP, J. W. (1980), Invertible terms in the lambda calculus, *Theoret. Comput. Sci.* **11**, 19–37.

[BP 88a] BÖHM, C., AND PIPERNO, A. (1987), Surjectivity for finite sets of combinators by weak reduction, *in* "CSL 87, 1st Workshop on Computer Science, Oct. 87," (Börger, Kleine Büning, and Richeter, Eds.), Lecture Notes in Computer Science, Vol. 329, pp. 27–43, Springer-Verlag, New York/Berlin.

[BP 88b] BÖHM, C., AND PIPERNO, A. (1987), Characterizing X-separability and one-side invertibility in λ-β-Ω-calculus, *in* "Proceedings, 3rd IEEE Symposium on 'Logic in Computer Science,' Edinburgh, July 4–8, 1988," to appear in IEEE 1988.

[BT 87] Böhm, C., and Tronci, E. (1987), X-separability and left-invertibility in λ-calculus, *in* "Proceedings, 2nd IEEE Symposium on 'Logic in Computer Science,' June 22–25, 1987, Ithaca, New York, IEEE 1987," pp. 320–328.

[CDR 78] Coppo, M., Dezani-Ciancaglini, M., and Ronchi della Rocca, S. (1978), (Semi-)separability of finite sets of terms in Scott's D_∞-models of the λ-calculus, *in* "Fifth International Colloquium on Automata, Languages, and Programming," Lecture Notes in Computer Science, Vol. 62, pp. 142–164, Springer-Verlag, New York/Berlin.

[CFC 58] Curry, H. B., Feys, R., and Craig, W. (1958), "Combinatory Logic," Vol. 1, North-Holland, Amsterdam.

[Chu 37] Church, A. (1937), Combinatory logic as a semigroup, *Bull. Amer. Math. Soc.* **43**, 333. [abstract]

[Dez 76] Dezani-Ciancaglini, M. (1976), Characterization of normal forms possessing inverse in the λ-β-η-calculus, *Theoret. Comput. Sci.* **2**, 323–337.

[MZ 83] Margaria, I., and Zacchi, M. (1983), Right and left invertibility in λ-β-calculus, *RAIRO Inform. Theor. Appl.* **17**, 71–88.

[O'D 85] O'Donnell, M. J. (1985), Equational logic as a programming language, *in* "MIT Press Series in the Foundation of Computation, 1985."

[Tro 87] Tronci, E. (1987), "Risolubilità di sistemi di equazioni nel λ-calcolo," Masters thesis in electronics engineering (developed under the supervision of C. Böhm), Università "La Sapienza," May 25, 1987, Roma, Italy.