Global well-posedness of the Navier–Stokes-omega equations

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First, we prove that the local solution to the Navier–Stokes-omega equations is unique when the spatial dimension \( n \) satisfies \( 3 \leq n \leq 6 \). Then, a regularity criterion is established for any \( n \geq 3 \). As a corollary, it is proved that the smooth solution exists globally when \( 3 \leq n \leq 6 \).

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\textbf{1. Introduction}

In this work, we study the following Navier–Stokes-omega equations [1]:

\begin{align}
\partial_t v - \Delta v + (v \cdot \nabla)u + \sum_j u_j \nabla v_j + \nabla \pi &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \quad (1.1) \\
v &= (1 - \alpha^2 \Delta) u, \quad \alpha > 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \quad (1.2) \\
\text{div}v &= \text{div}u = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n, \quad (1.3) \\
v|_{t=0} &= v_0(x), \quad x \in \mathbb{R}^n (n \geq 3). \quad (1.4)
\end{align}

Here \( v \) is the fluid velocity field, \( u \) is the “filtered” fluid velocity, and \( \pi \) is the pressure. \( \alpha > 0 \) is the length scale parameter that represents the width of the filter. For simplicity we will take \( \alpha = 1 \).

When \( n = 3 \), the global well-posedness of the problem has been proved in [1]. In this work, we will prove:

**Theorem 1.1.** Suppose that \( 3 \leq n \leq 6 \). Suppose that \( v_0 \in L^2(\mathbb{R}^n) \) with \( \text{div}v_0 = 0 \) in \( \mathbb{R}^n \). Then for any \( T > 0 \), the problem (1.1)–(1.4) has a unique weak solution \( v \in C([0, T]; L^2) \) satisfying

\begin{equation}
\frac{1}{2} \int v^2 \, dx + \int_0^T \int |\nabla v|^2 \, dx \leq \frac{1}{2} \int v_0^2 \, dx.
\end{equation}

**Theorem 1.2.** Suppose that \( 3 \leq n \leq 6 \). Suppose that \( v_0 \in H^s(\mathbb{R}^n) \) with \( s > 0 \) and \( \text{div}v_0 = 0 \) in \( \mathbb{R}^n \). Then for any \( T > 0 \), the problem (1.1)–(1.4) has a unique smooth solution \( v \) satisfying

\begin{equation}
v \in L^\infty(0, T; H^4) \cap L^2(0, T; H^{s+1}).
\end{equation}
**Theorem 1.3.** Suppose that \( n \geq 3 \). Suppose that \( \nu_0 \in H^1(\mathbb{R}^n) \) with \( \text{div}\nu_0 = 0 \) in \( \mathbb{R}^n \). Let \( \nu \) be a unique local strong solution to the problem (1.1)–(1.4) in \( L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \). If \( \text{curl}\ \nu \) satisfies
\[
\text{curl}\ \nu \in L^p(0, T; L^q(\mathbb{R}^n)) , \quad \frac{2}{p} + \frac{n}{q} = 2, \quad \frac{n}{2} < q \leq n, \tag{1.7}
\]
then \( \nu \) can be extended beyond \( T > 0 \).

**Remark 1.1.** By (1.5) and (1.7), one can find that when \( 3 \leq n \leq 6 \), (1.7) holds true, which implies that the smooth solution exists globally. Also, it is easy to see that \( n = 6 \) is the largest dimension.

In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3.

**2. Proof of Theorem 1.1**

This section is devoted to the proof of Theorem 1.1.

Testing (1.1) with \( \nu \), using (1.3) and
\[
\int (\nu \cdot \nabla u + \sum_j u_j \nabla v_j) \nu dx = 0,
\]
we easily get (1.5).

It is easy to show that
\[
v \cdot \nabla u \in L^2(0, T; H^{-1}), \quad \sum_j u_j \nabla v_j \in L^2(0, T; H^{-1}),
\]
\[
\nabla \pi \in L^2(0, T; H^{-1}), \quad \partial_t v \in L^2(0, T; H^{-1}),
\]
and thus
\[
v \in C([0, T]; L^2).
\]

By (1.2), this gives
\[
u \in C([0, T]; H^2) \subset C([0, T]; L^n). \tag{2.1}
\]

Then the existence part can be proved by the same method as was used in [1]; we only need to establish the uniqueness. By (2.1) we can decompose \( u \) as follows: for any \( 0 < \epsilon < 1 \),
\[
u = u_{\epsilon} + u_h, \quad u_{\epsilon} \in C([0, T]; L^n), \quad u_h \in L^\infty((0, T) \times \mathbb{R}^n), \tag{2.2}
\]
with
\[
\|u_{\epsilon}\|_{L^\infty(0, T; L^n)} \leq \epsilon.
\]

Let \( \bar{\nu} \) be any two weak solutions of (1.1)–(1.4) on the interval \([0, T]\) with the same initial value \( \nu_0 \). Let us define
\[
\bar{\nu} = (1 - \Delta)\bar{u}, \quad \delta u = u - \bar{u}, \quad \delta v = v - \bar{v}.
\]

Then from (1.1), we see that
\[
\delta_t \delta u - \Delta \delta u + v \cdot \nabla \delta u + \delta u \cdot \nabla v + \sum_j u_j \cdot \nabla \delta v_j + \sum_j \delta u_j \cdot \nabla \bar{v} + \nabla (\pi - \bar{\pi}) = 0. \tag{2.3}
\]

Testing (2.3) with \( \delta v \), using (2.2) and the divergence free condition, we find that
\[
\frac{1}{2} \frac{d}{dt} \int |\delta v|^2 dx + \int |
abla \delta v|^2 dx \leq \|v\|_{L^{n/2}} \|\nabla \delta u\|_{L^{n/2}} \|\nabla \delta v\|_{L^{n/2}} + \|\bar{u}\|_{L^n} \|\nabla \delta u\|_{L^{n/2}} \|\nabla \delta v\|_{L^{n/2}}
\]
\[
+ \|\bar{u}\|_{L^n} \|\nabla \delta v\|_{L^{n/2}} \|\nabla \delta v\|_{L^{n/2}} + \|u_{\epsilon}\|_{L^n} \|\nabla \delta u\|_{L^{n/2}} \|\nabla \delta v\|_{L^{n/2}}
\]
\[
+ \|u_{\epsilon}\|_{L^n} \|\nabla \delta v\|_{L^{n/2}} \|\nabla \delta v\|_{L^{n/2}} + \|\nabla \bar{v}\|_{L^2} \|\nabla \delta u\|_{L^2} \|\nabla \delta v\|_{L^2}
\]
\[
\leq C \|v\|_{H^1} \|\nabla \delta v\|_{L^2} + C \|v\|_{H^1} \|\nabla \delta v\|_{L^2} + C \|\nabla \delta v\|_{L^2} \|\nabla \delta v\|_{L^2}
\]
\[
+C \|\nabla \bar{v}\|_{L^2} \|\delta u\|_{L^2} \|\nabla \delta v\|_{L^2} + 2C \|\nabla \delta v\|_{L^2} + C (\|v\|_{H^1}^2 + \|\bar{v}\|_{H^1}^2 + 1) \|\delta v\|_{L^2}^2.
\]

Taking \( \epsilon \) small enough and using Gronwall’s inequality, we have
\[
\|\delta v\|_{L^2} = 0
\]
and thus
\[
\delta u = 0.
\]

This completes the proof. \( \Box \)
3. Proof of Theorem 1.2

Since it is easy to prove that the problem (1.1)–(1.4) has a unique local smooth solution, we only need to prove the a priori estimate (1.6).

In the following calculations, we will use the bilinear commutator and product estimates due to Kato and Ponce [2]:
\[
\begin{align*}
\|A^s(fg) - f A^s g\|_p & \leq C\|\nabla f\|_{p_1} \|A^{s-1} g\|_{L^{p_1}} + \|A^s f\|_{p_2} \|g\|_{p_2}, \\
\|A^s(fg)\|_p & \leq C\|\nabla f\|_{p_1} \|A^s g\|_{L^{p_1}} + \|A^s f\|_{p_2} \|g\|_{p_2},
\end{align*}
\]
with \(s > 0, A := (-\Delta)^{1/2}\) and \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} \). Applying \(A^s\) to (1.1), testing with \(A^s v\), and using (1.3), (3.1), (3.2) and (2.2), we deduce that
\[
\frac{d}{dt} \int |\nabla v|^2 dx + \int |\Delta v|^2 dx = - \int A^s (v \cdot \nabla v) \cdot A^s v dx - \sum_i \int [A^s (u_i \partial_j v_j) - u_j \nabla A^s v_j] A^s v dx
\]
\[
\leq C(\|\nabla u\|_{L^2(0,T;H^1)} + \|\nabla u\|_{L^{p_1}} \|A^s \nabla u\|_{L^{p_2}}) \|A^s v\|_{L^\infty(0,T;L^{p_2})}
\]
\[
\leq C\|u\|_{L^\infty(0,T;L^2)} \|A^s v\|_{L^2} \|\Delta v\|_{L^2} + C\|A^s v\|_{L^2} \|\Delta v\|_{L^2},
\]
Taking \(\epsilon\) small enough and using Gronwall’s inequality, we obtain (1.6).

This completes the proof. \(\square\)

4. Proof of Theorem 1.3

Since it is easy to show that the problem (1.1)–(1.4) has a unique local smooth solution, we only need to establish the a priori estimate.

Since the proof of the case \(q = n\) is easier, we will assume that \(\frac{n}{2} < q < n\).

Testing (1.1) with \(-\Delta v\), using (1.3), we see that
\[
\frac{d}{dt} \int |\nabla v|^2 dx + \int |\Delta v|^2 dx = \sum_{ij} \int (v_i \partial_j u_j \Delta v_j + u_j \partial_i v_j \Delta v_i) dx
\]
\[
= \sum_{ij} \int \partial_j u_j (v_i \Delta v_j - v_j \Delta v_i) dx
\]
\[
= - \sum_{ij} \int \partial_j u_i (v_i \Delta v_j - v_j \Delta v_i) dx
\]
\[
= \frac{1}{2} \sum_{ij} \int (\partial_i u_j - \partial_j u_i) (v_i \Delta v_j - v_j \Delta v_i) dx \leq C\|\nabla u\|_{L^2(0,T;L^2)} \|\Delta v\|_{L^2} \|\nabla v\|_{L^2} \|\Delta v\|_{L^2}
\]
which yields
\[
\|v\|_{L^\infty(0,T;H^1)} + \|v\|_{L^2(0,T;H^2)} \leq C.
\]
Here we have used the Gagliardo–Nirenberg inequality
\[
\|v\|_{L^\infty(0,T;H^1)} \leq C\|\nabla v\|_{L^2}^{1-\theta} \|\Delta v\|_{L^2}^{1+\theta}
\]
with
\[
\theta = \frac{n}{q} - 1 = 1 - \frac{2}{q}.
\]
This completes the proof. \(\square\)
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