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# On the existence of $m$-point boundary value problem at resonance for higher order differential equation ☆

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**Abstract**

By means of Mawhin's continuation theorem, we study $m$-point boundary value problem at resonance in the following form:

$$\begin{cases} x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) + e(t), & t \in (0,1), \\ x'(0) = 0,\ x''(0) = 0,\ \ldots,\ x^{(k-1)}(0) = 0, & x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases}$$

where $m \geqslant 3$, $k \geqslant 2$ are two integers, $a_i \in R$, $\xi_i \in (0,1)$ $(i = 1, 2, \ldots, m-2)$ are constants satisfying $\sum_{i=1}^{m-1} a_i = 1$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2}$. A new result on the existence of solutions is obtained. The interesting is that we do not need all the $a_i$'s $(1 \leqslant i \leqslant m-2)$ have the same sign, and also the degrees of some variables among $x_0, x_1, \ldots, x_{k-1}$ in the function $f(t, x_0, x_1, \ldots, x_{k-1})$ are allowable to be greater than 1. Meanwhile, we give some examples to demonstrate our result.



*Keywords:* $m$-point boundary value problem; Mawhin's continuation theorem; Higher order differential equation

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## 1. Introduction

The multi-point boundary value problem for second order ordinary differential equations has been extensively studied in papers [1–14]. For instance, Feng and Webb studied the boundary value problem in [11] as follows:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & t \in (0,1), \quad (1.1)\\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), & (1.2) \end{cases}$$

where $f \in C([0,1] \times R^2, R)$, $e \in L^1[0,1]$ and $a_i \geqslant 0$, $\eta_i \in (0,1)$ are constants with $\sum_{i=1}^{m-2} a_i = 1$. However, the linear growth condition

$$\left|f(t,x,y)\right| \leqslant a(t)|x| + b(t)|y| + r(t), \quad \forall (t,x,y) \in [0,1] \times R^2, \tag{1.3}$$

imposed on $f(t,x,y)$ is needed, where $a, b, r \in L^1[0,1]$. In [12], Feng and Webb again investigated the solvability of three-point BVP

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & t \in (0,1), \quad (1.4)\\ x'(0) = 0, \quad x(1) = x(\eta), & (1.5) \end{cases}$$

where $f$ and $e$ are defined as above, $\eta \in (0,1)$ is a constant. By assuming that $f$ has the decomposition

$$f(t,x,p) = g(t,x,p) + h(t,x,p), \tag{1.6}$$

where $g$ and $h : [0,1] \times R^2 \to R$ are two continuous functions satisfying the conditions

$$pg(t,x,p) \leqslant 0, \quad \forall (t,x,p) \in [0,1] \times [-M, M] \times R,$$

and

$$\left|h(t,x,p)\right| \leqslant a(t)|x| + b(t)|p| + u(t)|x|^r + v(t)|p|^k + c(t), \\ \forall (t,x,p) \in [0,1] \times [-M,M] \times R, \tag{1.7}$$

where $0 \leqslant r, k < 1$ are constants and $a, b, u, v, c \in L^1[0,1]$ with $|b|_1 < 1/2$, the authors obtain an existence result [3, Theorem 3.2]. But the problem corresponding to (1.1)–(1.2) subject to the case of all the $a_i$'s $(1 \leqslant i \leqslant m-2)$ not having the same sign, as far as we know, has been studied far less often. The reason for this is that if all the $a_i$'s $(1 \leqslant i \leqslant m-2)$ have the same sign, then BVP (1.1)–(1.2) can be studied via the existence of a solution for the following three-point BVP [13,14]:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & t \in (0,1),\\ x'(0) = 0, \quad x(1) = x(\eta), \end{cases}$$

which is crucial to estimate a priori bound of solution. When all the $a_i$'s $(1 \leqslant i \leqslant m-2)$ have no same sign, Liu studied BVP (1.1)–(1.2) in [15]. But the growth condition

$$\left|f(t,x,y)\right| \leqslant a(t)|x| + b(t)|y| + c(t)|y|^\theta \tag{1.8}$$

is imposed on $f$, where $\theta \in [0,1)$ is a constant.

In this paper, we consider $m$-point boundary value problem for higher order differential equation in the following form:

$$\begin{cases} x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) + e(t), \quad t \in (0,1) & (1.9) \\ x'(0) = 0,\ x''(0) = 0,\ \ldots,\ x^{(k-1)}(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), & (1.10) \end{cases}$$

where $f:[0,1] \times R^k \to R$ and $e:[0,1] \to R$ are two continuous functions, $m \geqslant 3$, $k \geqslant 2$ are two integers, $a_i \in R$, $\xi_i \in (0,1)$ $(i = 1, 2, \ldots, m-2)$ are constants satisfying $\sum_{i=1}^{m-1} a_i = 1$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2}$. As $\sum_{i=1}^{m-1} a_i = 1$, it is well known from [9, 10] that BVP (1.9)–(1.10) is at resonance. By using Mawhin's continuation theorem, we obtain a new result on the existence of solutions to BVP (1.9)–(1.10). Even if for the case of $k = 2$ and $a_i \geqslant 0$ $(i = 1, 2, \ldots, m-2)$, the conditions imposed on $f$ and the approaches to estimate a priori bound of the solutions to BVP (1.9)–(1.10) are different from the corresponding ones of the past work [11,12]. For example, we allow that the degrees of some variables among $x_0, x_1, \ldots, x_{k-1}$ in the function $f(t, x_0, x_1, \ldots, x_{k-1})$ are greater than 1. Meanwhile, we give some examples in Section 3 to demonstrate our result.

## 2. Some preliminaries

In order to use Mawhin's continuation theorem, first we recall this theorem.

Let $X$ and $Z$ be real Banach Spaces and let $L: D(L) \subset X \to Z$ be a Fredholm operator with index zero. This means that $X = \ker L \oplus X_1$ and $Z = \operatorname{Im} L \oplus Z_1$. Furthermore, let $P: X \to \ker L$ and $Q: Z \to Z_1$ be the corresponding natural projections. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by $K$ the inverse of $L_P$.

Now, let $\Omega$ be an open bounded subset of $X$ with $D(L) \cap \Omega \neq 0$. A map $N: \bar{\Omega} \to Z$ is said to be $L$-compact in $\bar{\Omega}$, if $QN(\bar{\Omega})$ bounded and the operator $K(I - Q)N: \bar{\Omega} \to X$ is compact.

**Lemma 2.1** [16,17]. *Assume that $X$, $Z$ are two Banach spaces, $L$ is a Fredholm operator with index zero and $N$ is $L$-compact on $\bar{\Omega}$. Moreover assume that*

(1) $Lx \neq \lambda Nx$, $\forall \lambda \in (0,1)$ *and* $x \in D(L) \cap \partial\Omega$.
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \ker L \cap \partial\Omega$.
(3) $\deg\{QN|_{\ker L}, \Omega \cap \ker L, 0\} \neq 0$.

*Then the equation $Lx = Nx$ admits at least one solution in $D(L) \cap \bar{\Omega}$.*

Next, let $C[0,1] = \{x\colon\ x \in C([0,1], R)\}$ with the norm $|x|_0 = \max_{t\in[0,1]} |x(t)|$; $C^i[0,1] = \{x\colon\ x \in C^i([0,1], R)\}$ with the norm $\|x\|_i = \max\{|x|_0, |x'|_0, \ldots, |x^{(i)}|_0\}$ $(i = 1, 2, \ldots, k)$. Clearly, $C[0,1]$ and $C^i[0,1]$ $(i = 1, 2, \ldots, k)$ are Banach spaces.

## 3. Main result

**Theorem 3.1.** *Suppose that there is a positive integer $j \in \{1, 2, \ldots, m-2\}$ such that $a_i > 0$, $\forall i \in \{1, 2, \ldots, j\}$, and $a_i < 0$, $\forall i \in \{j+1, j+2, \ldots, m-2\}$. Furthermore, we assume that the following conditions are satisfied*:

(H$_1$) $\sum_{i=1}^{m-2} a_i \xi_i^k \neq 1$.
(H$_2$) *There is a constant* $D > 0$ *such that*

$$f(t, x_0, x_1, \dots, x_{k-2}, 0) + e(t) > 0, \quad \forall t \in [0,1],\ x_0 > D,\ \text{and } x_1 \geqslant 0, \dots, x_{k-2} \geqslant 0,$$

*and*

$$f(t, x_0, x_1, \dots, x_{k-2}, 0) + e(t) < 0, \quad \forall t \in [0,1],\ x_0 < -D,\ \text{and } x_1 \leqslant 0, \dots, x_{k-2} \leqslant 0.$$

(H$_3$) *The function* $f$ *has the decomposition*

$$f(t, x_0, x_1, \dots, x_{k-1}) = u(t, x_0, x_1, \dots, x_{k-1}) + g(t, x_0) + \sum_{i=1}^{k-1} h_i(t, x_i)$$

*such that*

$$x_{k-1} u(t, x_0, x_1, \dots, x_{k-1}) \leqslant -\beta |x_{k-1}|^{n+1}, \quad \forall (t, x_0, x_1, \dots, x_{k-1}) \in [0,1] \times R^k, \tag{3.1}$$

$$\lim_{|x| \to +\infty} \sup_{t \in [0,1]} \frac{|g(t,x)|}{|x|^n} = r_0, \tag{3.2}$$

*and*

$$\lim_{|x| \to +\infty} \sup_{t \in [0,1]} \frac{|h_i(t,x)|}{|x|^n} = r_i, \quad \forall i \in \{1, 2, \dots, k-1\}, \tag{3.3}$$

*where* $n > 0$, $r_i \geqslant 0$, $i \in \{1, 2, \dots, k-1\}$, *are all constants,* $g(t,x)$ *and* $h_i(t,y)$, $i \in \{1, 2, \dots, k-1\}$, *are continuous on* $[0,1] \times R$.

*Then BVP* (1.9)–(1.10) *has at least one solution, if*

$$\sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} + r_{k-1} < \beta.$$

**Proof.** Let $X = C^{k-1}[0,1]$ and $Z = C[0,1]$. Considering the equation

$$Lx = \lambda N x, \quad \lambda \in (0,1), \tag{3.4}$$

where $L : D(L) \subset X \to Z$ defined by

$$Lx = x^{(k)},$$
$$D(L) := \left\{ x:\ x \in C^k[0,1],\ x'(0) = 0, \dots, x^{(k-1)}(0) = 0,\ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \right\},$$

and

$$N : X \to Z, \quad (Nx)(t) = f\big(t, x(t), x'(t), \dots, x^{(k-1)}(t)\big) + e(t), \quad t \in [0,1].$$

Clearly, $\ker L = \{c\colon\ c \in R\}$ and $\operatorname{Im} L = \{y\colon\ y \in C[0,1],\ \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^{s_1} \dots \int_0^{s_{k-1}} y(s_k) \times ds_k \dots ds_1 = 0\}$. So $L$ is a Fredholm operator with index zero. Let

$$P : X \to \ker L, \quad Px = x(0),$$
$$Q : Z \to Z_1, \quad Qy = \frac{k! \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^{s_1} \dots \int_0^{s_{k-1}} y(s_k)\, ds_k \dots ds_1}{1 - \sum_{i=1}^{m-2} a_i \xi_i^k},$$

and $L_P = L|_{\ker P \cap D(L)}$. Then $L_P$ has a unique inverse $K : \operatorname{Im} L \to \ker P \cap D(L)$ defined by

$$(Ky)(t) = \frac{\int_0^t (t-s)^{k-1} y(s)\, ds}{(k-1)!}. \tag{3.5}$$

It is easy to see from the definition of $N$ and (3.5) that $N$ is $L$-compact on $\bar{\Omega}$, where $\Omega$ is any bounded open subset of $X$.

Suppose $x \in D(L)$ is an arbitrary solution of Eq. (3.4) for some $\lambda \in (0,1)$. Then

$$x^{(k)}(t) = \lambda f\big(t, x(t), x'(t), \dots, x^{(k-1)}(t)\big) + \lambda e(t), \quad t \in (0,1). \tag{3.6}$$

First we will prove that

$$\big|x(0)\big| \leqslant D. \tag{3.7}$$

Suppose the contrary. Then $|x(0)| > D$. Without loss of generality, we may assume that $x(0) > D$. In view of $x'(0) = 0,\ x''(0) = 0, \dots, x^{(k-1)}(0) = 0$ and

$$x^{(k)}(0) = \lambda\big[f\big(t, x(0), x'(0), \dots, x^{(k-1)}(0)\big) + e(0)\big]$$
$$= \lambda\big[f\big(t, x(0), 0, \dots, 0\big) + e(0)\big],$$

we see from the first part of assumption ($H_2$) that $x^{(k)}(0) > 0$. By the continuity of $x^{(k)}(t)$ on $[0,1]$, one can find that there is $\delta \in (0,1)$ such that $x^{(k)}(t) > 0$ for $t \in (0,\delta)$, and then $\forall i \in \{1, 2, \dots, k-1\}$,

$$x^{(i)}(t) = x^{(i)}(0) + x^{(i+1)}(0)t + \dots + \frac{1}{(k-1-i)!} \int_0^t (t-s)^{(k-1-i)} x^{(k)}(s)\, ds$$
$$= \frac{1}{(k-1-i)!} \int_0^t (t-s)^{(k-1-i)} x^{(k)}(s)\, ds > 0 \quad \text{for } t \in (0,\delta]$$

and

$$x(t) = x(0) + x'(0)t + \dots + \frac{1}{(k-1)!} \int_0^t (t-s)^{(k-1)} x^{(k)}(s)\, ds$$
$$= x(0) + \frac{1}{(k-1)!} \int_0^t (t-s)^{(k-1)} x^{(k)}(s)\, ds > x(0) > D \quad \text{for } t \in (0,\delta].$$

In what follows, we will prove that

$$x^{(k-1)}(t) > 0, \quad \forall t \in (0,1). \tag{3.8}$$

Otherwise, there must be a constant $t_0 \in (0,1)$ such that

$$x^{(k-1)}(t) > 0 \quad \text{for } t \in (0, t_0), \tag{3.9}$$

and $x^{(k-1)}(t_0) = 0$, which deduce that $\forall i \in \{1, 2, \ldots, k-2\}$,

$$\begin{aligned} x^{(i)}(t) &= x^{(i)}(0) + x^{(i+1)}(0)t + \cdots + \frac{1}{(k-2-i)!}\int_0^t (t-s)^{(k-2-i)} x^{(k-1)}(s)\,ds \\ &= \frac{1}{(k-2-i)!}\int_0^t (t-s)^{(k-2-i)} x^{(k-1)}(s)\,ds > 0 \quad \text{for } t \in (0, t_0] \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \cdots + \frac{1}{(k-2)!}\int_0^t (t-s)^{(k-2)} x^{(k-1)}(s)\,ds \\ &= x(0) + \frac{1}{(k-2)!}\int_0^t (t-s)^{(k-2)} x^{(k-1)}(s)\,ds > x(0) > D \quad \text{for } t \in (0, t_0], \end{aligned} \tag{3.11}$$

and also

$$x^{(k)}(t_0) \leqslant 0. \tag{3.12}$$

But by (3.10), (3.11) and the fist part of assumption ($H_2$) we see that

$$\begin{aligned} x^{(k)}(t_0) &= \lambda\big[f\big(t_0, x(t_0), x'(t_0), \ldots, x^{(k-1)}(t_0)\big) + e(t_0)\big] \\ &= \lambda\big[f\big(t_0, x(t_0), x'(t_0), \ldots, x^{(k-2)}(t_0), 0\big) + e(t_0)\big] > 0, \end{aligned}$$

which contradicts (3.12). So (3.8) holds. However, as

$$x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) = \sum_{i \in \{1,2,\ldots,j\}} a_i x(\xi_i) + \sum_{i \in \{j+1, j+2, \ldots, m-2\}} a_i x(\xi_i),$$

it follows from (3.10) that

$$\begin{aligned} x(1) &< \sum_{i \in \{1,2,\ldots,j\}} a_i x(\xi_j) + \sum_{i \in \{j+1, j+2, \ldots, m-2\}} a_i x(\xi_j) \\ &= \sum_{i=1}^{m-2} a_i x(\xi_j) = x(\xi_j) < x(1), \end{aligned}$$

which is also a contradiction. This contradiction implies that (3.7) holds. Thus

$$\big|x(t)\big| \leqslant \big|x(0)\big| + \frac{\int_0^t (t-s)^{(k-2)}|x^{(k-1)}(s)|\,ds}{(k-2)!} \leqslant D + \frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}$$

and

$$\big|x^{(i)}(t)\big| \leqslant \frac{\int_0^t (t-s)^{(k-2-i)}|x^{(k-1)}(s)|\,ds}{(k-2-i)!} \leqslant \frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2-i)!},$$

that is

$$|x|_0 \leqslant D + \frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!} \tag{3.13}$$

and

$$|x^{(i)}|_0 \leqslant \frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2-i)!} \quad (i = 1, 2, \ldots, k-2). \tag{3.14}$$

On the other hand, multiplying the two sides of (3.6) by $x^{(k-1)}(t)$ and integrating them over [0, 1], we have

$$\begin{aligned}
\frac{1}{2}\big(x^{(k-1)}(1)\big)^2 &= \lambda \int_0^1 u\big(s, x(s), x'(s), \ldots, x^{(k-1)}(s)\big)x^{(k-1)}(s)\,ds \\
&\quad + \lambda \int_0^1 g\big(s, x(s)\big)x^{(k-1)}(s)\,ds + \lambda \int_0^1 \sum_{i=1}^{k-1} h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\,ds \\
&\quad + \lambda \int_0^1 e(s)x^{(k-1)}(s)\,ds.
\end{aligned}$$

So by (3.1), we have

$$\begin{aligned}
\lambda\beta \int_0^1 \big|x^{(k-1)}(t)\big|^{n+1}\,dt &\leqslant -\lambda \int_0^1 u\big(t, x(t), x'(t), \ldots, x^{(k-1)}(t)\big)x^{(k-1)}(t)\,dt \\
&= -\frac{1}{2}(x^{(k-1)})^2(1) + \lambda \int_0^1 g\big(s, x(s)\big)x^{(k-1)}(s)\,ds \\
&\quad + \lambda \sum_{i=1}^{k-1} \int_0^1 h_i\big(t, x^{(i)}(t)\big)x^{(k-1)}(t)\,dt + \lambda \int_0^1 e(t)x^{(k-1)}(t)\,dt,
\end{aligned}$$

that is

$$\begin{aligned}
\beta \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt
&\leqslant \int_0^1 g\big(s, x(s)\big)x^{(k-1)}(s)\, ds + \sum_{i=1}^{k-1} \int_0^1 h_i\big(t, x^{(k)}(t)\big)x^{(k-1)}(t)\, dt \\
&\quad + \int_0^1 e(t)x^{(k-1)}(t)\, dt \\
&\leqslant \int_0^1 \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds + \sum_{i=1}^{k-1} \int_0^1 \left|h_i\big(t, x^{(i)}(t)\big)x^{(k-1)}(t)\right| dt \\
&\quad + \int_0^1 \left|e(t)x^{(k-1)}(t)\right| dt. \qquad (3.15)
\end{aligned}$$

Let

$$\varepsilon = \frac{\beta - \sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} - r_{k-1}}{2k}.$$

We see from

$$\beta > \sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} + r_{k-1}$$

that $\varepsilon > 0$. For such a positive $\varepsilon$, we find from (3.2) and (3.3) that there must be a constant $\rho > D$ such that $\forall i \in \{1, 2, \ldots, k-1\}$,

$$\frac{|h_i(t, y)|}{|y|^n} < (r_i + \varepsilon) \quad \text{uniformly for } t \in [0,1],\ |y| > \rho,$$

and

$$\frac{|g(t, x)|}{|x|^n} < (r_0 + \varepsilon) \quad \text{uniformly for } t \in [0,1],\ |x| > \rho,$$

i.e., $\forall i \in \{1, 2, \ldots, k-1\}$,

$$\left|h_i(t, y)\right| < (r_i + \varepsilon)|y|^n \quad \text{uniformly for } t \in [0,1],\ |y| > \rho, \qquad (3.16)$$

and

$$\left|g(t, x)\right| < (r_0 + \varepsilon)|x|^n \quad \text{uniformly for } t \in [0,1],\ |x| > \rho. \qquad (3.17)$$

Let $\Delta_{1,i} = \{t\colon\ t \in [0,1],\ |x^{(i)}(t)| \leqslant \rho\}$, $\Delta_{2,i} = \{t\colon\ t \in [0,1],\ |x^{(i)}(t)| > \rho\}$, $\forall i \in \{1, 2, \ldots, k-1\}$, $\Delta_3 = \{t\colon\ t \in [0,1],\ |x(t)| \leqslant \rho\}$, $\Delta_4 = \{t\colon\ t \in [0,1],\ |x(t)| > \rho\}$. Then we have from (3.15) that

$$\begin{aligned}
&\beta \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad \leqslant \int_{\Delta_3} \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds + \int_{\Delta_4} \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds \\
&\qquad + \sum_{i=1}^{k-1} \int_{\Delta_{1,i}} \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \\
&\qquad + \sum_{i=1}^{k-1} \int_{\Delta_{2,i}} \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds + \int_0^1 \left|e(t)x^{(k-1)}(t)\right| dt. \qquad (3.18)
\end{aligned}$$

As we from (3.16) and (3.17) find that

$$\int_{\Delta_3} \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds \leqslant g_\rho \int_0^1 \left|x^{(k-1)}(s)\right| ds, \qquad (3.19)$$

$$\int_{\Delta_4} \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds \leqslant (r_0 + \varepsilon)|x|_0^n \int_0^1 \left|x^{(k-1)}(s)\right| ds, \qquad (3.20)$$

$$\int_{\Delta_{2,i}} \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \leqslant (r_i + \varepsilon)|x^{(i)}|_0^n \int_0^1 \left|x^{(k-1)}(s)\right| ds$$
$$(i = 1, 2, \ldots, k-2), \qquad (3.21)$$

$$\int_{\Delta_{1,i}} \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \leqslant h_{i,\rho} \int_0^1 \left|x^{(k-1)}(s)\right| ds$$
$$(i = 1, 2, \ldots, k-2), \qquad (3.22)$$

$$\int_{\Delta_{2,k-1}} \left|h_{k-1}\big(s, x^{(k-1)}(s)\big)x^{(k-1)}(s)\right| ds \leqslant (r_{k-1} + \varepsilon) \int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds, \qquad (3.23)$$

and

$$\int_{\Delta_{1,k-1}} \left|h_{k-1}\big(s, x^{(k-1)}(s)\big)x^{(k-1)}(s)\right| ds \leqslant h_{k-1,\rho} \int_0^1 \left|x^{(k-1)}(s)\right| ds, \qquad (3.24)$$

where $g_\rho = \max_{t\in[0,1],\,|x|\leqslant\rho} |g(t,x)|$, $h_{i,\rho} = \max_{t\in[0,1],\,|y|\leqslant\rho} |h_i(t,y)|$, $i \in \{1, 2, \ldots, k-1\}$. Substituting (3.13) and (3.14) into (3.20) and (3.21), respectively, we have

$$\int_{\Delta_4} \left|g\big(s,x(s)\big)x^{(k-1)}(s)\right| ds \leqslant (r_0+\varepsilon)\left[D+\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right]^n \int_0^1 \left|x^{(k-1)}(s)\right| ds \tag{3.25}$$

and

$$\int_{\Delta_{2,i}} \left|h_i\big(s,x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \leqslant \frac{r_i+\varepsilon}{[(k-2-i)!]^n}\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^{n+1}, \quad i\in\{1,2,\ldots,k-2\}. \tag{3.26}$$

Substituting (3.19) and (3.22)–(3.26) into (3.17), we obtain that

$$\begin{aligned}
&\beta\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad\leqslant (r_0+\varepsilon)\left[D+\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right]^n \int_0^1 \left|x^{(k-1)}(t)\right| dt \\
&\qquad+\sum_{i=1}^{k-2}\frac{r_i+\varepsilon}{[(k-2-i)!]^n}\left(\int_0^1 \left|x^{(k-1)}(t)\right| dt\right)^{n+1} + (r_{k-1}+\varepsilon)\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\qquad+\left[g_\rho+|e|_0+\sum_{i=1}^{k-1}h_{i,\rho}\right]\int_0^1 \left|x^{(k-1)}(t)\right| dt.
\end{aligned} \tag{3.27}$$

From the knowledge of mathematical analysis, we know that there must be a constant $\sigma\in(0,1)$ (independent of $\lambda$) such that

$$(1+x)^n < 1+(n+1)x, \quad \forall x\in(0,\sigma]. \tag{3.28}$$

Thus, we have the following cases.

*Case* 1. If $\int_0^1 |x^{(k-1)}(t)|\,dt = 0$ or $\int_0^1 |x^{(k-1)}(t)|\,dt \neq 0$ with

$$\frac{D(k-2)!}{\int_0^1 |x^{(k-1)}(t)|\,dt} \geqslant \sigma,$$

then

$$\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!} \leqslant \frac{D}{\sigma}.$$

So from (3.28),

$$\left[D+\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right]^n \int_0^1 \left|x^{(k-1)}(t)\right| dt \leqslant D^n(1+\sigma^{-1})^n \int_0^1 \left|x^{(k-1)}(s)\right| ds. \tag{3.29}$$

Substituting (3.29) and Hölder's inequalities

$$\int_0^1 \left|x^{(k-1)}(s)\right| ds \leqslant \left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{1}{n+1}},$$
$$\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^{n+1} \leqslant \int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds$$

into (3.27), we get

$$\begin{aligned}
&\beta \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad \leqslant \left[(r_0+\varepsilon)D^n(1+\sigma^{-1})^n + g_\rho + |e|_0 + \sum_{i=1}^{k-1} h_{i,\rho}\right]\left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{1}{n+1}} \\
&\qquad + \left[\sum_{i=1}^{k-2} \frac{r_i+\varepsilon}{[(k-2-i)!]^n} + (r_{k-1}+\varepsilon)\right]\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt.
\end{aligned} \tag{3.30}$$

Since $1/(n+1) < 1$ and

$$\begin{aligned}
r_{k-1}+\varepsilon+\sum_{i=1}^{k-2}\frac{r_i+\varepsilon}{[(k-2-i)!]^n} &< r_{k-1}+\sum_{i=1}^{k-2}\frac{r_i}{[(k-2-i)!]^n}+k\varepsilon \\
&= \frac{r_{k-1}+\sum_{i=1}^{k-2}\frac{r_i}{[(k-2-i)!]^n}+\beta}{2} < \beta,
\end{aligned}$$

it follows from (3.30) that there is a constant $M_1 > 0$ such that

$$\int_0^1 \left|x^{(k-1)}(t)\right|^{1+n} dt < M_1. \tag{3.31}$$

*Case* 2. If

$$\frac{D(k-2)!}{\int_0^1 |x^{(k-1)}(t)|\,dt} < \sigma,$$

then from (3.28) we get

$$\begin{aligned}
\left[D+\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right]^n &= \left(\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right)^n\left[1+\frac{D(k-2)!}{\int_0^1 |x^{(k-1)}(t)|\,dt}\right]^n \\
&\leqslant \left(\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right)^n\left[1+\frac{(n+1)D(k-2)!}{\int_0^1 |x^{(k-1)}(t)|\,dt}\right]
\end{aligned}$$

$$= \left(\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right)^n + \frac{(n+1)D}{[(k-2)!]^{n-1}}\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^{n-1}. \tag{3.32}$$

Substituting (3.32) into (3.27), we have

$$\begin{aligned}
&\beta \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\leqslant (r_0+\varepsilon)\left(\frac{\int_0^1 |x^{(k-1)}(s)|\,ds}{(k-2)!}\right)^n \int_0^1 \left|x^{(k-1)}(s)\right| ds \\
&\quad + \frac{(n+1)D}{[(k-2)!]^{n-1}}\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^n \\
&\quad + \sum_{i=1}^{k-2} \frac{r_i+\varepsilon}{[(k-2-i)!]^n}\left(\int_0^1 \left|x^{(k-1)}(t)\right| dt\right)^{n+1} + (r_{k-1}+\varepsilon)\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad + \left[g_\rho + |e|_0 + \sum_{i=1}^{k-1} h_{i,\rho}\right]\int_0^1 \left|x^{(k-1)}(t)\right| dt \\
&= \frac{r_0+\varepsilon}{[(k-2)!]^n}\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^{n+1} + \frac{(n+1)D}{[(k-2)!]^{n-1}}\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^n \\
&\quad + \sum_{i=1}^{k-2} \frac{r_i+\varepsilon}{[(k-2-i)!]^n}\left(\int_0^1 \left|x^{(k-1)}(t)\right| dt\right)^{n+1} + (r_{k-1}+\varepsilon)\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad + \left[g_\rho + |e|_0 + \sum_{i=1}^{k-1} h_{i,\rho}\right]\int_0^1 \left|x^{(k-1)}(t)\right| dt.
\end{aligned} \tag{3.33}$$

By using inequalities

$$\int_0^1 \left|x^{(k-1)}(s)\right| ds \leqslant \left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{1}{n+1}},$$

$$\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^{n+1} \leqslant \int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds$$

and

$$\left(\int_0^1 \left|x^{(k-1)}(s)\right| ds\right)^n \leqslant \left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{n}{n+1}},$$

we get from (3.33) that

$$\begin{aligned}
&\beta \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\quad \leqslant \sum_{i=0}^{k-2} \frac{r_i+\varepsilon}{[(k-2-i)!]^n} \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt + (r_{k-1}+\varepsilon) \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\qquad + \frac{(n+1)D}{[(k-2)!]^{n-1}} \left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{n}{n+1}} \\
&\qquad + \left[g_\rho + |e|_0 + \sum_{i=1}^{k-1} h_{i,\rho}\right] \left(\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}}.
\end{aligned}$$

Since

$$r_{k-1} + \sum_{i=0}^{k-2} \frac{r_i+\varepsilon}{[(k-2-i)!]^n} \leqslant \frac{\beta + r_0 + r_{k-1} + \sum_{i=1}^{k-2} \frac{r_i}{[(k-2-i)!]^n}}{2} < \beta,$$

$n/(n+1) < 1$ and $1/(n+1) < 1$, it follows from the above formula that there is a constant $M_2 > 0$ such that

$$\int_0^1 \left|x'(t)\right|^{n+1} dt \leqslant M_2.$$

Thus, in either Case 1 or 2, we obtain that

$$\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \leqslant \max\{M_2, M_1\} := M, \tag{3.34}$$

which together with (3.13) yields

$$\begin{aligned}
|x|_0 &\leqslant D + \frac{1}{(k-2)!} \left(\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \\
&\leqslant D + \frac{1}{(k-2)!} M^{\frac{1}{n+1}} := A_0
\end{aligned} \tag{3.35}$$

and

$$|x^{(i)}|_0 \leqslant \frac{1}{(k-2-i)!}\left(\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \leqslant \frac{1}{(k-2-i)!} M^{\frac{1}{n+1}} := A_i,$$
$$i = 1, 2, \ldots, k-2. \tag{3.36}$$

Multiplying the two sides of Eq. (3.6) by $x^{(k-1)}(t)$ again and integrating them over $[0, t]$, we get

$$\begin{aligned}
&\frac{1}{2}\left|x^{(k-1)}(t)\right|^2 \\
&\quad = \lambda \int_0^t u\big(s, x(s), x'(s), \ldots, x^{(k-1)}(s)\big)x^{(k-1)}(s)\, ds + \lambda \int_0^t g\big(s, x(s)\big)x^{(k-1)}(s)\, ds \\
&\qquad + \lambda \sum_{i=1}^{k-1} \int_0^t h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\, ds + \lambda \int_0^t e(s)x^{(k-1)}(s)\, ds \\
&\quad \leqslant \int_0^1 \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds + \sum_{i=1}^{k-1} \int_0^1 \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \\
&\qquad + \int_0^1 \left|e(s)x^{(k-1)}(s)\right| ds \\
&\quad = \int_0^1 \left|g\big(s, x(s)\big)x^{(k-1)}(s)\right| ds + \sum_{i=1}^{k-2} \int_0^1 \left|h_i\big(s, x^{(i)}(s)\big)x^{(k-1)}(s)\right| ds \\
&\qquad + \int_0^1 \left|e(s)x^{(k-1)}(s)\right| ds + \int_{\Delta_{1,k-1}} \left|h_{k-1}\big(s, x^{(k-1)}(s)\big)x^{(k-1)}(s)\right| ds \\
&\qquad + \int_{\Delta_{2,k-1}} \left|h_{k-1}\big(s, x^{(k-1)}(s)\big)x^{(k-1)}(s)\right| ds \\
&\quad \leqslant \left[g_{A_0} + \sum_{i=1}^{k-2} h_{i,A_i} + |e|_0\right] \int_0^1 \left|x^{(k-1)}(s)\right| ds + (r_{k-1} + \varepsilon) \int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt \\
&\qquad + h_{k-1,\rho} \int_0^1 \left|x^{(k-1)}(s)\right| ds \\
&\quad \leqslant \left[g_{A_0} + \sum_{i=1}^{k-2} h_{i,A_i} + h_{k-1,\rho} + |e|_0\right] \left(\int_0^1 \left|x^{(k-1)}(s)\right|^{n+1} ds\right)^{\frac{1}{n+1}}
\end{aligned}$$

$$+ (r_{k-1}+\varepsilon)\int_0^1 \left|x^{(k-1)}(t)\right|^{n+1} dt, \tag{3.37}$$

where $\Delta_{1,k-1}$ and $\Delta_{2,k-1}$ are defined by the explanation preceding (3.18),

$$g_{A_0} := \max_{t\in[0,1],\,|x|\leqslant A_0}\left|g(t,x)\right|,\ h_{i,A_i} := \max_{t\in[0,1],\,|x|\leqslant A_i}\left|h_i(t,x)\right| \quad (i=1,2,\ldots,k-2)$$

and

$$h_{k-1,\rho} := \max_{t\in[0,1],\,|x|\leqslant\rho}\left|h_{k-1}(t,x)\right|.$$

Substituting (3.34) into (3.37) we get

$$\begin{aligned}\left|x^{(k-1)}(t)\right|^2 &\leqslant 2\left[h_\rho + g_{A_0} + \sum_{i=1}^{k-2} h_{i,A_i} + h_{k-1,\rho} + |e|_0\right] M^{1/(n+1)} + 2(r_{k-1}+\varepsilon)M \\ &:= M_3, \quad \forall t\in[0,1],\end{aligned}$$

that is

$$|x^{(k-1)}|_0 \leqslant M_3. \tag{3.38}$$

If $c > D$, then by assumption ($H_2$) we get

$$\begin{aligned}&\sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\quad= \sum_{i=1}^{j} a_i \int_{\xi_i}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\qquad+ \sum_{i=j+1}^{m-2} a_i \int_{\xi_i}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\quad> \sum_{i=1}^{j} a_i \int_{\xi_j}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\qquad+ \sum_{i=j+1}^{m-2} a_i \int_{\xi_j}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\quad= \sum_{i=1}^{m-2} a_i \int_{\xi_j}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 \\ &\quad= \int_{\xi_j}^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[f(s_k,c,0,\ldots,0)+e(s_k)\right] ds_k \ldots ds_1 > 0.\end{aligned} \tag{3.39}$$

Similarly, if $c < -D$, then we have

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s_1} \dots \int_{0}^{s_{k-1}} \big[f(s_k, c, 0, \dots, 0) + e(s_k)\big]\, ds_k \dots ds_1 < 0. \tag{3.40}$$

Now, if $Nx \in \operatorname{Im} L$, $x \in \ker L$, then $x = c$, $c \in R$, satisfying $QNx = 0$, that is

$$\frac{k! \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_0^{s_1} \dots \int_0^{s_{k-1}} [f(s_k, c, 0, \dots, 0) + e(s_k)]\, ds_k \dots ds_1}{1 - \sum_{i=1}^{m-2} a_i \xi_i^k} = 0.$$

So by (3.39) and (3.40) we see that $|c| \leqslant D$. Let $M_4 = \max\{A_0, A_1, \dots, A_{k-2}, M_3\} + 1$ and $\Omega = \{x\colon\ x \in X,\ \|x\|_{k-1} < M_4\}$, and also set

$$H(x, \mu) = \mu \operatorname{sgn}\left(1 - \sum_{i=1}^{m-2} a_i \xi_i^k\right) x + (1 - \mu) QNx, \quad (x, \mu) \in \bar{\Omega} \times [0, 1].$$

It is to see from (3.39) and (3.40) that $H(x, \mu) \neq 0$, $(x, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1]$. Thus

$$\deg\{QN|_{\ker L}, \ker L \cap \Omega, 0\} = \deg\big\{H(x, 0), \ker L \cap \Omega, 0\big\},$$

$$\deg\big\{H(x, 1), \ker L \cap \Omega, 0\big\} = \deg\left\{\operatorname{sgn}\left(1 - \sum_{i=1}^{m-2} a_i \xi_i^k\right) I, \ker L \cap \Omega, 0\right\} \neq 0.$$

Therefore, by applying Lemma 2.1, we get BVP (1.9)–(1.10) has at least one solution. □

**Remark 3.1.** If $a_i \geqslant 0$, $\forall i \in \{1, 1, \dots, m-2\}$, then the positive integer $j$ of Theorem 3.1 can be chosen as $j = m - 2$, and then assumption $(H_1)$ of Theorem 3.1 holds. So we have the following result.

**Corollary 3.1.** *Suppose* $a_i \geqslant 0$, $\forall i \in \{1, 1, \dots, m-2\}$, *and assume that conditions* $(H_2)$ *and* $(H_3)$ *of Theorem* 3.1 *are satisfied. Then BVP* (1.9)–(1.10) *has at least one solution if*

$$\sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} + r_{k-1} < \beta.$$

**Example 3.1.** Let us consider the boundary value problem as follows:

$$\begin{cases} x''(t) = -(2 + x^2(t)) x'^3(t) + x^3(t) + x'^{8/3}(t) + t^2, & t \in (0, 1), \qquad (3.41) \\ x'(0) = 0, \qquad x(1) = \frac{1}{4} x(1/2) + \frac{3}{4} x(2/3). & \qquad (3.42) \end{cases}$$

Corresponding to BVP (1.9)–(1.10), we have $a_1 = 1/4 > 0$ and $a_2 = 3/4 > 0$ with $a_1 + a_2 = 1$, $f(t, x_0, x_1) = -(2 + x_0^2) x_1^3 + x_0^3 + x_1^{8/3} + t^2$ and $e(t) \equiv 0$. So we can chose $D = 2$ such that assumption $(H_2)$ of Theorem 3.1 is satisfied, and also we can chose $u(t, x_0, x_1) = -(2 + x_0^2) x_1^3$, $g(t, x) = x^3 + t^2$ and $h_1(t, y) = y^{8/3}$ such that $f(t, x_0, x_1) = u(t, x_0, x_1) + g(t, x_0) + h_1(t, x_1)$. Then $n = 3$, $\beta = k = 2$,

$$r_0 = \lim_{|x| \to +\infty} \sup_{t \in [0,1]} \frac{|g(t, x)|}{|x|^3} = 1 \quad \text{and} \quad r_1 = \lim_{|y| \to +\infty} \sup_{t \in [0,1]} \frac{|h_1(t, y)|}{|y|^3} = 0,$$

that is

$$\sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} + r_{k-1} = r_0 + r_1 = 1 < \beta = 2.$$

Hence, BVP (3.41)–(3.42) has at least one solution by applying Corollary 3.1.

**Remark 3.2.** From the above example, we see $a_1 = 1/4 > 0$, $a_2 = 3/4 > 0$. But the degrees of the second variable $x_0$ and the third variable $x_1$ in $f(t, x_0, x_1)$ are all equal to 3, which is different from the growth condition (1.3) assumed by [11]; and also the degree of variable $y$ in the function $h_1(t, y)$ is equal to $8/3$, which is different from the corresponding condition of (1.7) assumed by [12].

**Example 3.2.** Let us consider the following boundary value problem:

$$\begin{cases} x''(t) = -(2 + x^2(t))x'^5(t) + x^5(t) + x'^4(t) + t^2, \quad t \in (0, 1), & (3.43) \\ x'(0) = 0, \qquad x(1) = \frac{5}{4}x(1/2) - \frac{1}{4}x(2/3). & (3.44) \end{cases}$$

Corresponding to BVP (1.9)–(1.10), we have $a_1 = 5/4 > 0$ and $a_2 = -1/4 < 0$ with $a_1 + a_2 = 1$, $f(t, x_0, x_1) = -(2 + x_0^2)x_1^5 + x_0^5 + x_1^4 + t^2$ and $e(t) \equiv 0$. So assumption ($H_1$) of Theorem 3.1 holds, and we can chose $D = 2$ such that assumption ($H_2$) of Theorem 3.1 is satisfied, and also we can chose $u(t, x_0, x_1) = -(2 + x_0^2)x_1^5$, $g(t, x) = x^5 + t^2$ and $h_1(t, y) = y^4$ such that $f(t, x_0, x_1) = u(t, x_0, x_1) + g(t, x_0) + h_1(t, x_1)$. Then $n = 5$, $\beta = k = 2$,

$$r_0 = \lim_{|x|\to+\infty} \sup_{t\in[0,1]} \frac{|g(t, x)|}{|x|^5} = 1 \quad \text{and} \quad r_1 = \lim_{|y|\to+\infty} \sup_{t\in[0,1]} \frac{|h_1(t, y)|}{|y|^5} = 0,$$

that is

$$\sum_{i=0}^{k-2} \frac{r_i}{[(k-2-i)!]^n} + r_{k-1} = r_0 + r_1 = 1 < \beta = 2.$$

Hence, BVP (3.43)–(3.44) has at least one solution by applying Theorem 3.1.

**Remark 3.3.** From the above example, we see $a_1 = -1/4 < 0$. So the above result cannot be obtained by [1–14]. Also, the degrees of the second variable $x_0$ and the third variable $x_1$ in the function $f(t, x_0, x_1)$ are all equal to 5, which is different from the growth condition (1.8) assumed by [15].

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