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Efficient algorithms for construction of recurrence relations for the expansion and connection coefficients in series of quantum classical orthogonal polynomials

Eid H. Doha [a,*], **Hany M. Ahmed** [b]

[a] *Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*
[b] *Department of Mathematics, Faculty of Industrial Education, Helwan University, Cairo, Egypt*

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Abstract Formulae expressing explicitly the q-difference derivatives and the moments of the polynomials $P_n(x\,;q) \in T$ ($T = \{P_n(x\,;q) \in$ Askey–Wilson polynomials: Al-Salam-Carlitz I, Discrete q-Hermite I, Little (Big) q-Laguerre, Little (Big) q-Jacobi, q-Hahn, Alternative q-Charlier) of any degree and for any order in terms of $P_i(x\,;q)$ themselves are proved. We will also provide two other interesting formulae to expand the coefficients of general-order q-difference derivatives $D_q^p f(x)$, and for the moments $x^\ell D_q^p f(x)$, of an arbitrary function $f(x)$ in terms of its original expansion coefficients. We used the underlying formulae to relate the coefficients of two different polynomial systems of basic hypergeometric orthogonal polynomials, belonging to the Askey–Wilson polynomials and $P_n(x\,;q) \in T$. These formulae are useful in setting up the algebraic systems in the unknown coefficients, when applying the spectral methods for solving q-difference equations of any order.

Introduction

The expansion of a given function as a series in classical orthogonal polynomials is a matter of great interest in applied mathematics and mathematical physics. This is particularly true for the connection problem between any two families of classical orthogonal polynomials. Usually, the determination of the expansion coefficients of this series requires a deep knowledge of hypergeometric functions. It should be

* Corresponding author. Tel.: +20 2 3302329; fax: +20 2 35727556.
E-mail address: eiddoha@frcu.eun.eg (E.H. Doha).

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stressed that, even when explicit forms for these coefficients are available, it is often useful to have a linear recurrence relation satisfied by these coefficients. This recurrence relation may serve as a tool for detection of certain properties of the expansion coefficients of the given function, and for numerical evaluation of these quantities, using a judiciously chosen algorithm [1]. The construction of such recurrences attracted much interest in the last few years. Special emphasis has been given to the classical continuous orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel), the discrete cases (Hahn, Meixner, Krawtchouk and Charlier) and the basic hypergeometric orthogonal polynomials, belonging to the Askey–Wilson polynomials.

The construction of recurrence relations for the coefficients of the Fourier series expansions with respect to the classical continuous/discrete orthogonal polynomials is presented by many authors. Special emphasis by Lewanowicz [2–4] is given to solving the connection and linearization problems. Ronveaux et al. [5], Godoy et al. [6] and Area et al. [7] have developed a recurrent method for solving the connection problem for all families of classical orthogonal polynomials, as well as some special kind of linearization problems, and have used it for solving different problems related to the associated Sobolev-type polynomials, etc. [see also, Godoy et al. [8,9]]. Also, different algorithms for solving the connection problem of the four families of classical orthogonal polynomials of continuous variable (Laguerre, Hermite, Jacobi and Bessel) are presented by Doha [10–12] and Doha and Ahmed [13] respectively, and for the discrete cases (Hahn, Meixner, Krawtchouk and Charlier) by Doha and Ahmed [14,15].

Also, the construction of recurrence relations for the coefficients of the Fourier series expansions with respect to the q-classical orthogonal polynomials are presented by Lewanowicz [16,17], Lewanowicz et al. [18], and Lewanowicz and Woźny [19]. A great importance of the connection and linearization coefficients is appeared in [20–28].

Lately, there has been increasing interest in the q-orthogonal polynomials. This is due to their numerous applications in several areas of mathematics, e.g., continued fractions, Eulerian series, theta functions, elliptic functions (cf. [29–31]), quantum groups and algebras [32–34], discrete mathematics (combinatorics, graph theory) and coding theory, among others (see also [24]). There is also a connection between the representation theory of quantum algebras (Clebsch–Gordan coefficients, 3j and 6j symbols), which play an important role in physical applications, and the q-orthogonal polynomials; see [35] and the references cited there. This partially motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often requires the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its q-derivatives and moments into a set of q-classical orthogonal polynomials. This is also true for the basic hypergeometric orthogonal polynomials belonging to the Askey–Wilson polynomials. They are important in certain problems of mathematical physics; for example, the development in quantum groups has led to the so-called q-harmonic oscillators (see, for instance [36–39]). The known models of q-oscillators are closely related with q-orthogonal polynomials. The q-analogues of boson operators have been introduced explicitly in Askey and Suslov [36], where the corresponding wave functions were constructed in terms of the continuous q-Hermite polynomials of Rogers (see [40,41]), in terms of the Stieltjes-Wigert polynomials [42] and in terms of q-Charlier polynomials of Al-Salam and Carlitz [43]. Askey and Suslov [44] have shown that Al-Salam-Carlitz I polynomials are closely connected with the q-harmonic oscillator. Also, Atakishiyev and Klimyk [45] have shown that the little q-Laguerre polynomials are related to the problem of diagonalization (eigenfunctions, spectra, transition coefficients, etc.) of some classes of operators for the discrete series representations of the quantum algebra $U_q\,(su_{1,1})$.

In this paper we introduce new knowledge and explicit formulae for the expansion coefficients of general-order q-derivatives and the moments of an arbitrary function in terms of q-orthogonal polynomials. Similar formulae have been obtained by Karageorghis [46,47], Phillips [48], Doha [10–12,49,50] and Doha and Ahmed [13–15] for classical orthogonal polynomials of continuous and discrete variables, as well as Doha and Ahmed [51] for Al-Salam-Carlitz I polynomials and little (big) q-Laguerre, belonging to the Askey–Wilson polynomials, which are unknown and traceless in the literature. To obtain such formulae, we require knowledge of the so-called structure and three-term recurrence relations for the q-orthogonal polynomials.

The paper is organized as follows. In "Properties of the q-classical orthogonal polynomials in the Hahn sense" section, we give some relevant properties of the polynomials $P_n(x\,;q) \in T$. In "Relation between the coefficients $a_n^{(p)}$ and a_n and the pth q-derivative of $P_n(x\,;q)$" section, we prove a theorem that relates the $P_n(x\,;q)$ expansion coefficients of the q-derivatives of a function in terms of its original expansion coefficients. Explicit expressions for the q-derivatives of the polynomials $P_n(x\,;q)$ of any degree and for any order as a linear combination of suitable $P_n(x\,;q)$ themselves are also deduced. In "Explicit formula for the expansion coefficients of the moments of $D_q^p f(x)$" section, we prove a theorem that gives the $P_n(x\,;q)$ expansion coefficients of the moments of one single $P_n(x\,;q)$ polynomial of any degree. Another theorem that expresses the $P_n(x\,;q)$ expansion coefficients of the moments of a general-order q-derivative of an arbitrary function in terms of its $P_n(x\,;q)$ original expansion coefficients is also discussed. In "Recurrence relations for connection coefficients between different monic q-polynomials belonging to the Askey–Wilson polynomials" section, we give an application for these theorems that provides an algebraic symbolic approach (using Mathematica) in order to build and solve recursively for the connection coefficients between two different polynomial systems of basic hypergeometric orthogonal polynomials, belonging to the Askey–Wilson polynomials and $P_n(x\,;q) \in T$.

Properties of the q-classical orthogonal polynomials in the Hahn sense

The families of q-orthogonal polynomials belonging to the Askey–Wilson polynomials have the property that their derivatives form orthogonal systems, and also satisfy second-order q-difference equation of the form

$$[\sigma(x)D_q D_{1/q} + \tau(x)D_q + \lambda_{n,q}]P_n(x;q) = 0, \tag{2.1}$$

where the q-derivative operator D_q (also called Hahn operator) is defined (see Hahn [52]) by

$$D_q f(x) := \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x}, & x \neq 0, \\ f'(0), & x = 0, \quad \text{provided} \quad f'(0)\,\text{exists}, \end{cases} \tag{2.2}$$

Table 1 Polynomials $\sigma(x)$ and $\tau(x)$ in the q-difference equation (2.1).

Family		$\sigma(x)$	$\tau(x)$
Big q-Jacobi	$P_n(x\,;a,\,b,\,c\,;q)$	$(aq-x)\,(cq-x)$	$\frac{cq-x+aq(1-(b+c)q+bqx)}{q-1}$
q-Hahn	$Q_n(x\,;a,\,b,\,N\,;q)$	$(aq-x)\,(q^{-N}-x)$	$\frac{aq(1+(x-1)bq)-x+q^{-N}(1-aq)}{q-1}$
Little q-Jacobi	$p_n(x\,;a,\,b\vert q)$	$x(x-1)$	$\frac{1-x+aq(bqx-1)}{q-1}$
Big q-Laguerre	$P_n(x\,;a,\,b\,;q)$	$(x-aq)(bq-x)$	$\frac{x-(a+b)q+abq^2}{q-1}$
q-Meixner	$M_n(x\,;b,\,c\,;q)$	$c(x-bq)$	$\frac{c(bq-1)+q(x-1)}{q-1}$
q-Alternative Charlier	$K_n(x\,;a\,;q)$	$x(1-x)$	$\frac{-1+x(1+aq)}{q-1}$
Little q-Laguerre	$p_n(x\,;a\vert q)$	$x(1-x)$	$\frac{x+aq-1}{q-1}$
q-Laguerr	$L_n^{(\alpha)}(x;q)$	x	$\frac{1-q^{\alpha+1}(x+1)}{1-q}$
Stieltjes-Wigert	$S_n(x\,;q)$	x	$\frac{qx-1}{q-1}$
Al-Salam-Carlitz I	$U_n^{(\alpha)}(x;q)$	$(x-1)(x-\alpha)$	$\frac{x-\alpha-1}{1-q}$
Discrete q-Hermit I	$h_n(x\,;q)$	$1-x^2$	$\frac{x}{q-1}$
Al-Salam-Carlitz II	$V_n^{(\alpha)}(x;q)$	α	$\frac{x-\alpha-1}{q-1}$

and $\sigma(x)=ax^2+bx+c,\ \tau(x)=dx+e$ are polynomials in x of degree at most 2 and exactly 1, respectively (but depending possibly on q), and $\lambda_{n,q}=-[n]_q[n-1]_q(\sigma'/2)+[n]_q\tau'$, where the q-analogues of the real numbers, $[x]_q$, is defined by

$$[x]_q := \begin{cases} \dfrac{1-q^x}{1-q}, & 0, \\ 0, & x=0. \end{cases}$$

For a brief background, definitions for some terminology and most basic properties of these polynomials, please refer to Gasper and Rahman [[24], p. 3–6] and Koekoek and Swarttouw [[53], p.113–114].

Remark 1. For the sake of completeness, Table 1 is included at the end of this paper to give the expressions of $\sigma(x)$ and $\tau(x)$ for most monic q-polynomials belonging to the Askey–Wilson polynomials.

The following two recurrence relations (which may be found in Area et al. [23] and Medem [54]) are of fundamental importance in developing the present work. These are:

(i) Recurrence relation

$$\begin{aligned} &xP_n(x;q)=P_{n+1}(x;q)+\beta_n P_n(x;q)+\gamma_n P_{n-1}(x;q), \quad (\gamma_n\neq 0), \quad n\geq 0, \\ &P_0(x;q)=1, \quad P_{-1}(x;q)=0, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \beta_n &= -\frac{q^n(-(aq(b(1+q)+e(q-1)))-(a+d(q-1))q^{2n}(b(1+q)+e(q-1)))}{a^2q^2+(a+d(q-1))^2q^{4n}-a(a+d(q-1))q^{2n}(1+q^2)} \\ &\quad +\frac{q^{2n}(1+q)(b(a-d)+(a+d)q)+ae(q-1)q)}{a^2q^2+(a+d(q-1))^2q^{4n}-a(a+d(q-1))q^{2n}(1+q^2)}, \\ \gamma_n &= -\frac{q^{n+1}(q^n-1)(-(aq^2)+(a+d(q-1))q^n)}{(aq^2-(a+d(q-1))q^{2n})^2(aq-(a+d(q-1))q^{2n})(aq^3-(a+d(q-1))q^{2n})} \\ &\quad \times(a^2cq^4+c(a+d(q-1))^2q^{4n}-abq^{n+3}(b+e(q-1))-b(a+d(q-1))q^{3n+1}\times(b+e(q-1)) \\ &\quad +q^{2(n+1)}(-2a^2c+b^2d(q-1)+a(2b^2+2be(q-1)+(q-1)(-2cd+e^2(q-1))))). \end{aligned}$$

(ii) Structure formula

$$P_n(x;q)=\frac{D_qP_{n+1}(x;q)}{[n+1]_q}+\bar{\beta}_n\frac{D_qP_n(x;q)}{[n]_q}+\bar{\gamma}_n\frac{D_qP_{n-1}(x;q)}{[n-1]_q}, \quad n\geq 2, \tag{2.4}$$

where

$$\bar{\beta}_n = -\frac{q^n(q^n-1)}{a^2q^2+(a+d(q-1))^2q^{4n}-a(a+d(q-1))q^{2n}(1+q^2)}$$
$$\times((-(b(a+d(q-1))q^n(1+q))+aq(b+e(q-1))+(a+d(q-1))q^{2n}(b+e(q-1)))),$$
$$\bar{\gamma}_n = \frac{(a+d(q-1))q^{2n}(q^n-1)(q^n-q)}{(aq^2-(a+d(q-1))q^{2n})^2(aq-(a+d(q-1))q^{2n})(aq^3-(a+d(q-1))q^{2n})}$$
$$\times(a^2cq^4+c(a+d(q-1))^2q^{4n}-abq^{n+3}(b+e(q-1))-b(a+d(q-1))q^{3n+1}$$
$$\times(b+e(q-1))+q^{2(n+1)}(-2a^2c+b^2d(q-1)+a(2b^2+2be(q-1)+(q-1)(-2cd+e^2(q-1))))).$$

Suppose that we have a smooth function $f(x)$, which is formally expanded in an infinite series of $P_n(x;q)$. In the case of q-Hahn polynomials, we assume that f is a polynomial belonging to the Askey–Wilson polynomials,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x;q), \tag{2.5}$$

and for the pth q-derivatives of $f(x)$, i.e. $D_q^p f(x)$,

$$D_q^p f(x) = \sum_{n=0}^{\infty} a_n^{(p)} P_n(x;q), \quad a_n^{(0)} = a_n. \tag{2.6}$$

It is possible to derive a recurrence relation involving the expansion coefficients of successive q-derivatives of $f(x)$. Let us write

$$D_q\left[\sum_{n=0}^{\infty} a_n^{(p-1)} P_n(x;q)\right] = \sum_{n=0}^{\infty} a_n^{(p)} P_n(x;q),$$

then use of identity (2.4) leads to the recurrence relation

$$a_{n-1}^{(p)} + \bar{\beta}_n a_n^{(p)} + \bar{\gamma}_{n+1} a_{n+1}^{(p)} = [n]_q a_n^{(p-1)}, \quad n, \quad p \geq 1. \tag{2.7}$$

Relation between the coefficients $a_n^{(p)}$ and a_n and the pth q-derivative of $P_n(x;q)$

The aim of this section is to explicitly express the expansion coefficients $a_n^{(p)}$ in terms of a_n. It is easy to prove the following theorem:

Theorem 1.

$$D_q^p P_n(x;q) = \sum_{k=0}^{n-p} C_{p,k}(n) P_k(x;q), \quad n, \quad p \geq 0, \tag{3.1}$$

if and only if

$$a_n^{(p)} = \sum_{k=0}^{\infty} C_{p,n}(n+p+k) a_{k+n+p}, \quad n, \quad p \geq 0, \tag{3.2}$$

where the expansion coefficients $C_{p,k}(n)$ are assumed to be known.

Proof. Suppose we are given the expansion (3.1); then by applying the operator $D_q^p f(x)$ to the expansion (2.5), we obtain

$$D_q^p f(x) = \sum_{n=p}^{\infty} a_n D_q^p P_n(x;q). \quad \square \tag{3.3}$$

Substituting (3.1) into (3.3), expanding and collecting similar terms, we obtain

$$D_q^p f(x) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} C_{p,n}(n+p+k) a_{n+p+k}\right] P_n(x;q). \tag{3.4}$$

Identifying (2.6) with (3.4) gives immediately (3.2).

On the other hand, suppose we have (3.2). Substituting (3.2) into (2.6) gives (3.4). Expanding (3.4) and collecting similar terms and identifying the result with (3.3), we get (3.1), which completes the proof of the theorem.

Al-Salam-Carlitz I case $[P_n(x;q) = U_n^{(\alpha)}(x;q)]$

In this problem the recurrence relation (2.7) has the form

$$a_{n-1}^{(p)} = [n]_q a_n^{(p-1)} \quad n, p \geq 1. \tag{3.5}$$

Doha and Ahmed [51] have proved that

$$a_n^{(p)} = [p]_q! \begin{bmatrix} n+p \\ n \end{bmatrix}_q a_{n+p} \quad n, p \geq 0, \tag{3.6}$$

and

$$D_q^p U_{n+q}^{(\alpha)}(x;q) = [p]_q! \begin{bmatrix} n+p \\ n \end{bmatrix}_q U_n^{(\alpha)}(x;q), \quad n, p \geq 0. \tag{3.7}$$

Note 1. It is worth noting that the corresponding results for the case of discrete q-Hermite polynomials of the first kind, $h_n(x;q)$ can be easily deduced by taking $\alpha = -1$.

Little q-Laguerre cases $[P_n(x;q) = P_n(x;a,b;q),\ p_n(x;a|q)]$

In these two cases, the recurrence relation (2.7) takes the form

$$\frac{1}{[n]_q} a_{n-1}^{(p)} + \mu q^n a_n^{(p)} = a_n^{(p-1)}, \quad n, p \geq 1, \tag{3.8}$$

where $\mu = abq(1-q)$ and $a(1-q)$ for big and little q-Laguerre polynomials, $P_n(x;a,b;q)$ and $p_n(x;a|_q)$ respectively.

The solution of (3.8) has the form (see Doha and Ahmed)

$$a_n^{(p)} = \frac{(q^{n+1};q)_p}{(1-q)^p} \sum_{k=0}^{\infty} \left(\frac{\mu}{q-1}\right)^k q^{(n+1)k+\binom{k}{2}} \frac{(q^p, q^{n+p+1};q)_k}{(q;q)_k} a_{k+n+p}, \quad n \geq 0, \quad p \geq 0, \tag{3.9}$$

and then

$$D_q^p P_{n+q}(x;q) = \frac{1}{(1-q)^p} \sum_{k=0}^{n} \left(\frac{\mu q^k}{q-1}\right)^{n-k} q^{\binom{n-k+1}{2}} \frac{(q^{k+1};q)_p (q^p, q^{k+p+1};q)_{n-k}}{(q;q)_{n-k}} P_k(x;q), \quad n \geq 0, p \geq 0. \tag{3.10}$$

Big q-Jacobi case $[P_n(x;q) = P_n(x;a,b,c;q)]$

Theorem 2. *The pth q-derivatives of monic big q-Jacobi polynomials of any degree in terms of monic big q-Jacobi polynomials with the same parameters have the form*

$$D_q^p P_{n+q}(x;a,b,c;q) = q^{-np} \frac{(q^{n+1};q)_p}{(1-q)^p} \sum_{k=0}^{n} C_{n,k}(aq^p, bq^p, cq^p, a, b, c, p) P_k(x;a,b,c;q), \quad n \geq 0, \quad p \geq 0, \tag{3.11}$$

and the relation between $a_n^{(p)}$ *and* a_n *is given by*

$$a_n^{(p)} = \frac{1}{(1-q)^p} \sum_{i=0}^{\infty} q^{-(n+i)p} (q^{n+i+1};q)_p C_{n+i,n}(aq^p, bq^p, cq^p, a, b, c, p) a_{n+p+i}, \quad n, p \geq 0, \tag{3.12}$$

where

$$\begin{aligned} C_{n,k}(\alpha,\beta,\gamma,a,b,c,r) &= (-1)^i q^{\binom{i}{2}} \frac{q^{i(r+1)}(\alpha q, \gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n} \frac{(q^{-n}, \alpha\beta q^{n+1};q)_i}{(q, \alpha q, \gamma q;q)_i} \\ &\times \sum_{j=0}^{n-i} q^{j(r+1)} \frac{(q^{-(n-i)}, \alpha\beta q^{n+i+1}, aq^{i+1}, cq^{i+1};q)_j}{(\alpha q^{i+1}, \gamma q^{i+1}, q, abq^{2(i+1)};q)_j} {}_3\phi_2 \left[\begin{matrix} q^{-(n-i-j)}, \alpha\beta q^{n+i+j+1}, q^r \\ \alpha q^{i+j+1}, \gamma q^{i+j+1} \end{matrix} \middle| q;q \right]. \end{aligned} \tag{3.13}$$

The following lemma is needed to proceed with the proof of the theorem.

Lemma 1. *It can be shown that*

$$P_n(q^r x;\alpha,\beta,\gamma;q) = \sum_{i=0}^{n} C_{n,i}(\alpha,\beta,\gamma,a,b,c,r) P_i(x;a,b,c;q), \quad r \geq 0, \tag{3.14}$$

where the connection coefficients $C_{n,i}(\alpha,\beta,\gamma,a,b,c,r)$ *are given as in* (3.13).

Proof of Theorem 2. In view of the formula (3.5.7) of Koekoek and Swarttouw [[53], p. 74], we can show that the monic big q-Jacobi polynomials $P_n(x\,;a,\,b,\,c\,;q)$ satisfy the formula

$$D_q P_n(x;a,b,c;q) = q^{-n+1}\frac{(1-q^n)}{(1-q)}P_{n-1}(qx;aq,bq,cq;q), \quad n \geq 1, \tag{3.15}$$

and by using relation (0.8.4) of Koekoek and Swarttouw [[53], p. 20], one may obtain

$$D_q^p P_n(x;a,b,c;q) = q^{2\binom{p}{2}-(n-1)p}\frac{(q^{n-p+1};q)_p}{(1-q)^p}P_{n-p}(q^p x;aq^p,bq^p,cq^p;q), \quad n \geq p \geq 0. \quad \square \tag{3.16}$$

By making use of (3.14) and (3.16), we obtain (3.11). Relation (3.12) can be deduced immediately by the aid of Theorem 1, and this completes the proof of the theorem.

Note 2. It is worth noting that the corresponding results for the case of monic q-Hahn polynomials $Q_n(x,\,a,\,b,\,N\,;q)$ can be easily deduced by using relation (3.5) of Doha and Ahmed [[51], p. 10115].

Little q-Jacobi case [$P_n(x\,;q) = p_n(x\,;a,\,b|q)$]

Theorem 3. *The pth q-derivatives of monic little q-Jacobi polynomials of any degree in terms of monic little q-Jacobi polynomials with the same parameters have the form*

$$D_q^p p_{n+p}(x;a,b|q) = \frac{(q^{n+1};q)_p}{(1-q)^p}\sum_{i=0}^{n} C_{n,i}(aq^p,bq^p,a,b)p_i(x;a,b|q), \quad n, \quad p \geq 0, \tag{3.17}$$

and the relation between $a_n^{(p)}$ and a_n is given by

$$a_n^{(p)} = \sum_{i=0}^{\infty}\frac{(q^{n+i+1};q)_p}{(1-q)^p}C_{n+i,n}(aq^p,bq^p,a,b)a_{n+p+i}, \quad n, \quad p \geq 0, \tag{3.18}$$

where

$$C_{n,i}(\alpha,\beta,\gamma,a,b) = \frac{(-1)^n q^{i+\binom{n}{2}}(\alpha q;q)_n(q^{-n},\alpha\beta q^{n+1};q)_i}{(\alpha\beta q^{n+1};q)_n(q,\alpha q;q)_i}\,{}_3\phi_2\left[\begin{matrix} q^{-(n-i)},\alpha\beta q^{n+i+1},aq^{i+1} \\ \alpha q^{i+1},abq^{2(i+1)}\end{matrix}\middle| q;q\right]. \tag{3.19}$$

The following lemma is needed to proceed with the proof of the theorem.

Lemma 2 ((see [55])). *The connection problem between monic little q-Jacobi polynomials with different parameters is*

$$p_n(x;\alpha,\beta|q) = \sum_{i=0}^{n} C_{n,i}(\alpha,\beta,a,b)p_i(x;a,b|q), \tag{3.20}$$

where the connection coefficients $C_{n,i}(\alpha,\,\beta,\,a,\,b)$ are given as in (3.19).

Proof of Theorem 3. In view of formula (3.12.7) of Koekoek and Swarttouw [[53], p. 93], it can be shown that the monic little q-Jacobi polynomials $p_n(x\,;a,\,b|q)$ satisfy the formula

$$D_q p_n(x;a,b|q) = \frac{(1-q^n)}{(1-q)}p_n(x;aq,bq|q), \quad n \geq 1, \tag{3.21}$$

and therefore

$$D_q^p p_n(x;a,b|q) = \frac{(q^{n-p+1};q)_p}{(1-q)^p}p_{n-p}(x;aq^p,bq^p|q), \quad n \geq p \geq 0. \quad \square \tag{3.22}$$

By making use of (3.20) and (3.22), we obtain (3.17). Relation (3.18) can be deduced immediately by the aid of Theorem 1 which completes the proof of the theorem.

The monic alternative q-Charlier polynomials, $K_n(x\,;b\,;q)$, can be obtained from the monic little q-Jacobi polynomials by using the relation (55) of Doha and Ahmed [[51], p. 10118], and accordingly, one can show that

$$\lim_{a\to 0} D_q^p p_n\left(x;a,-\frac{b}{aq}\middle| q\right) = D_q^p K_n(x;b;q), \quad n \geq p \geq 0. \tag{3.23}$$

In view of relations (3.17), (3.23) and the q-analogues of the Vandermonde summation formula [see Kokoek and Swarttouw [53], p. 15, formula (0.5.8)], we obtain the following corollary.

Corollary 1. *The pth q-derivatives of monic alternative q-Charlier polynomials of any degree in terms of monic alternative q-Charlier polynomials with the same parameter have the form*

$$D_q^p K_{n+p}(x;b;q) = \frac{(q^{n+1};q)}{(1-q)^p}\sum_{i=0}^{n} C_{n,i}(b,q)K_i(x;b;q), \quad n,\ p \geq 0, \tag{3.24}$$

and the relation between $a_n^{(p)}$ and a_n is given by

$$a_n^{(p)} = \sum_{i=0}^{\infty} q^{-(n+i)p}\frac{(q^{n+i+1};q)_p}{(1-q)^p}C_{n+i,n}(b,p)a_{n+p+i}, \quad n, \quad p \geq 0, \tag{3.25}$$

where

$$C_{n,i}(b,p) = (-1)^i b^{n-i} q^{(n+i+2p)(n-i)} q^{i+\binom{n}{2}} \frac{(q^{-n}, -q^{n+2p}b;q)_i (q^{i-n-2p+1};q)_{n-i}}{(-q^{n+2p}b;q)_n(-q^{2i+1}b;q)_{n-i}(q;q)i}.$$

Remark 2. The formulae for $a_n^{(p)}$ given by (3.12), (3.18) and (3.23) are the exact solutions of the difference equation (2.7) for the cases of big (little) q-Jacobi and alternative q-Charlier polynomials respectively.

Explicit formula for the expansion coefficients of the moments of $D_q^p f(x)$

For the evaluation of the expansion coefficients of $x^\ell D_q^p f(x)$ as expanded in series of $P_n(x;q)$ polynomials, the following theorem is needed.

Theorem 4. *In the expansion*

$$x^m P_n(x;q) = \sum_{n=0}^{2m} a_{m,n}(j) P_{j+m-n}(x;q), \quad m \geq 0, \quad j \geq 0, \tag{4.1}$$

the coefficients $a_{m,n}(j)$ can be computed as follows

(i) *For Al-Salam-Carlitz I, $U_n^{(\alpha)}(x;q)$, we have*

$$a_{m,n}(j) = \sum_{i=\max(j-n,0)}^{j} (-\alpha)^{j-i} q^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q b_{m+j-n}^{m+i}\, {}_1\phi_1\left[\begin{matrix} q^{-(j-i)} \\ 0 \end{matrix}\middle| q; \frac{q}{\alpha}\right], \tag{4.2}$$

where $b_n^{(m)} = \begin{bmatrix} m \\ n \end{bmatrix}_q \sum_{r=0}^{m-n} \begin{bmatrix} m-n \\ r \end{bmatrix}_q \alpha^r.$

In particular, and for the special case $\alpha = -1$, explicit formula for the expansion coefficients of $x^m h_j(x;q)$ is obtained (see [51]).

(ii) *For big q- Jacobi, $P_n(x;a,b,c;q)$, we have*

$$\begin{aligned} a_{m,n}(j) = {} & \frac{q^{(j+m-n)(j-n)}(aq,cq;q)_j(q;q)_m}{(abq^{j+1};q)_j(q;q)_{j+m-n}} \times \sum_{i=0}^{n} \Bigg[q^{\binom{\max(j-i,0)}{2}} q^{\binom{n-i}{2}} \frac{(-q^{(j-n+1)})^{(n-i)}(-q^{-(j+m-i-1)})^{\max(j-i,0)}}{(q, abq^{2(j+m-n+1)};q)_{n-i}} \\ & \times \frac{(aq^{j+m-n+1}, cq^{j+m-n+1};q)_{n-i}(q^{m+1}, q^{-j}, abq^{j+1};q)_{\max(j-i,0)}}{(q;q)_{\max(i-j,0)}(q,aq,cq;q)_{\max(j-i,0)}} \Bigg] \\ & \times {}_4\phi_3\left[\begin{matrix} q^{\max(-i,-j)}, abq^{j+\max(j-i,0)+1}, q^{-m+\max(i-j,0)}, 0 \\ aq^{\max(j-i,0)+1}, cq^{\max(j-i,0)+1}, q^{-(j-i)+2\max(j-i,0)+1} \end{matrix}\middle| q;q\right]. \end{aligned} \tag{4.3}$$

In particular, explicit formula for the expansion coefficients of $P_n(x;a,b;q)$ is obtained by using relation (34) of Doha and Ahmed [[51], p. 10115].

(iii) *For little q-Jacobi, $p_n(x;a,b|q)$, we have*

$$\begin{aligned} a_{m,n}(j) = {} & (-1)^j \frac{(aq, abq^{2(m+1)+r+j-n}, q^{r-j+n+1};q)_j (abq^{j+1}, q^{m+j-n+1}, aq^{j+m-n+1}, q^{-j};q)_{r+n}}{(q^{m+r+1}, aq^{m+r+1}, abq^{j+1};q)_j (q,q,aq,abq^{2(j+m-n+1)}q;q)_{r+n}} \\ & \times q^{r+\binom{j}{2}} \frac{(aq^{r+1}, q^{r+1};q)}{(abq^{j+r+1}, q^{r-j};q)_n} {}_4\phi_3\left[\begin{matrix} q^{\max(-i,-j)}, abq^{j+\max(j-i,0)+1}, q^{-m+\max(i-j,0)}, 0 \\ aq^{\max(j-i,0)+1}, cq^{\max(j-i,0)+1}, q^{-(j-i)+2\max(j-i,0)+1} \end{matrix}\middle| q;q\right], \quad r = \max(j-n,0). \end{aligned} \tag{4.4}$$

In particular, explicit formulae for the expansion coefficients of $x^m p_j(x;a|q)$ and $x^m K_j(x;b;q)$ are obtained by using relations (54) and (55) of Doha and Ahmed [[51], p. 10118] respectively.

The following lemma is needed to proceed with the proof of the theorem.

Lemma 3. *It can be shown that the coefficients $a_{m,n}(j)$ of (4.2)–(4.4), satisfy the recurrence relation*

$$a_{m,n}(j) = a_{m-1,n}(j) + \beta_{j+m-n}a_{m-1,n-1}(j) + \gamma_{j+m-n+1}a_{m-1,n-2}(j), \quad n = 0, 1, \ldots, 2m, \tag{4.5}$$

with $a_{0,0}(j)=1$, $a_{m-1,-\ell}(j)=0$, $\forall\, \ell>0$, $a_{m-1,r}(j)=0$, $r=2m-1, 2m$.

Proof of Theorem 4. To prove this theorem we proceed by induction. In view of recurrence relation (2.3), we may write

$$x P_j(x;q) = a_{10}(j)P_{j+1}(x;q) + a_{11}(j)P_j(x;q) + a_{12}(j)P_{j-1}(x;q), \tag{4.6}$$

and this in turn shows that (4.1) is true for $m=1$. Proceeding by induction, assuming that (4.1) is valid for m, we want to prove that

$$x^{m+1} P_j(x;q) = \sum_{n=0}^{2m+2} a_{m+1,n}(j)P_{j+m-n+1}(x;q). \tag{4.7}$$

From (4.6) and assuming the validity for m, we have

$$x^{m+1} P_j(x;q) = \sum_{n=0}^{2m} a_{m,n}(j)\left[a_{10}(j+m-n)P_{j+m-n+1}(x;q) + a_{11}(j+m-n)P_{j+m-n}(x;q) + a_{12}(j+m-n)P_{j+m-n-1}(x;q)\right].$$

Collecting similar terms, we get

$$\begin{aligned} x^{m+1} P_j(x;q) = {} & a_{m0}(j)a_{10}(j+m)P_{j+m+1}(x;q) + [a_{m1}(j)a_{10}(j+m-1) + a_{m0}(j)a_{11}(j+m)]P_{j+m}(x;q) \\ & + \sum_{n=0}^{2m}\left[a_{mn}(j)a_{10}(j+m-n) + a_{m,n-1}(j)a_{11}(j+m-n+1) + a_{m,n-2}(j)a_{12}(j+m-n+2)\right]P_{j+m-n+1}(x;q) \\ & + [a_{m,2m}(j)a_{11}(j-m) + a_{m,2m-1}(j)a_{12}(j-m+1)]P_{j-m}(x;q) + a_{m,2m}(j)a_{12}(j-m)P_{j-m-1}(x;q). \quad \square \end{aligned} \tag{4.8}$$

Application of Lemma 3 given in (4.5) to Eq. (4.8) yields Eq. (4.7) and the proof of the theorem is complete.

According to Theorem 4, we can state the following theorem, which relates the $P_n(x\,;q)$ expansion coefficients of $x^\ell D_q^p f(x)$ in terms of $a_n^{(p)}$.

Theorem 5. *Assume that $f(x)$, $f^{(p)}(x)$ and $x^\ell P_j(x\,;q)$ have the $P_n(x\,;q)$ expansions (2.5), (2.6) and (4.1) respectively, and assume also that*

$$x^\ell D_q^p f(x) = \sum_{i=0}^{\infty} b_i^{p,\ell} P_i(x;q) = I^{p,\ell};\ say, \tag{4.9}$$

then the connection coefficients $b_i^{p,\ell}$ are given by

$$b_i^{p,\ell} = \begin{cases} \displaystyle\sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & 0 \le i \le \ell, \\ \displaystyle\sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & \ell+1 \le i \le 2\ell-1, \\ \displaystyle\sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & i \ge 2\ell. \end{cases} \tag{4.10}$$

Recurrence relations for connection coefficients between different monic q-polynomials belonging to the Askey–Wilson polynomials

Let $f(x)$ have the expansion (2.5), and assume that it satisfies the non-homogeneous linear q-difference equation of order m

$$\sum_{i=0}^{m} p_i(x)D_q^i f(x) = g(x), \tag{5.1}$$

where $p_0, p_1, \ldots, p_m \neq 0$ are polynomials in x, and the expansion coefficients of the function $g(x)$ in terms of $P_n(x;q)$ are known, then formulae (3.2), (4.1) and (4.6) enable one to construct in view of Eq. (5.1) the linear recurrence relation of order r,

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k), \quad k \geq 0, \tag{5.2}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r$ ($\alpha_0 \neq 0, \alpha_r \neq 0$) are polynomials of the variable k.

In this section, we consider the problem of determining the connection coefficients between different polynomial systems. An interesting question is how to transform the Fourier coefficients of a given polynomial corresponding to an assigned orthogonal basis into the coefficients of another basis orthogonal with respect to a different weight function. The aim is to determine the so-called connection coefficients of the expansion of any element of the first basis in terms of the elements of the second basis. Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence.

Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), p_2(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), q_2(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis.

This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$Q_n(x) = \sum_{i=0}^{n} a_i(n) P_i(x). \tag{5.3}$$

The choice of $P_n(x)$ and $Q_n(x)$ depends on the situation. For example, suppose

$$P_n(x) = x^n,\ Q_n(x) = (x;q)_n,$$

then the connection coefficients $a_i(n)$ are given by (see [24])

$$a_i(n) = \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i\, q^{i(i-1)/2}.$$

If the roles of these $P_n(x)$ and $Q_n(x)$ are interchanged, then we get (see [[23], p.774, Eq. (3.3)])

$$a_i(n) = \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i\, q^{i(i+1-2n)/2}.$$

Eq. (2.1) can be written in the form (see [[51], p. 10113–10114])

$$\tilde{\sigma}(x) D_q^2 y(x) + \tilde{\tau}(x) D_q y(x) + \lambda_{n,q} y(x) = 0, \tag{5.4}$$

where $\tilde{\sigma}(x) = q^{-1}\sigma(qx) + (q-1)\, x\, \tau(qx)$ and $\tilde{\tau}(x) = \tau(qx) + \lambda_{n,q}(q-1)x$.

Al-Salam-Carlitz I-Little q-Jacobi connection problem

The link between $U_n^{(\alpha)}(x;q)$ and $p_i(x\,; a, b|q)$ given by (5.3) can easily be replaced by a linear relation involving only $p_i(x\,; a, b|q)$, using the Al-Salam-Carlitz I q-difference equation, namely

$$[(q-1)^2 \alpha D_q^2 + (q-1) q^{1-n}[-qx + q^n(\alpha+1)] D_q + q^2(1-q^{-n})] U_n^{(\alpha)}(x;q) = 0, \tag{5.5}$$

by substituting

$$U_n^{(\alpha)}(x;q) = \sum_{i=0}^{n} a_i(n)\, p_i(x;a,b|q), \tag{5.6}$$

and by virtue of formula (4.9), Eq. (5.5) takes the form

$$(q-1)^2 \alpha b_i^{2,0} - q^{2-n}(q-1) b_i^{1,1} + q(q-1)(\alpha+1) b_i^{1,0} + q^2(1-q^{-n}) b_i^{0,0} = 0.$$

By making use of formula (4.4) and (4.10), we obtain

$$\gamma_{i\,0}^{(0)} a_i(n) + \gamma_{i,-1}^{(1)} a_{i-1}^{(1)}(n) + \gamma_{i\,0}^{(1)} a_i^{(1)}(n) + \gamma_{i\,1}^{(1)} a_{i+1}^{(1)}(n) + \gamma_{i\,0}^{(2)} a_i^{(2)}(n) = 0, \tag{5.7}$$

where

$$\gamma_{i0}^{(0)} = q^2(q^n-1), \quad \gamma_{i,-1}^{(1)} = q^2(1-q),$$
$$\gamma_{i0}^{(1)} = [(1-bcq^{2i})(1-bcq^{2i+2})]^{-1} \times q^{i+2}[aq^i(1+b)(q^2-1)-(1+a)(q-1)(1+bcq^{2i+1})] + q^{n+1}(q-1)(1+\alpha),$$
$$\gamma_{i1}^{(1)} = (1-q)q^{i+2}(1-aq^{i+1})(1-abq^{i+1})[(abq^{2i+1})_2]^{-1}, \quad \gamma_{i0}^{(2)} = q^n(q-1)^2\alpha;$$

again, making use of formula (3.18) with (5.7) enable one to obtain the following recurrence relation,

$$\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \quad i = n-1, \quad n-2, \ldots, 0, \tag{5.8}$$

where

$$\delta_{i\,0} = (q^n - q^i),$$

$$\begin{aligned}\delta_{i1} &= (q^{i+1}-1)[(1-bcq^{2(i+1)})(1-bcq^{2(i+3)})]^{-1} \times [q^i(-1-aq^{n+1}(1+q)) - abq^{3i+3}(1+q+q^2+aq^{n+2}(1+q)) + q^{n-1}(1+\alpha)\\ &\quad + a^2bq^{4i+6}(1+bq^{n+1}(1+\alpha)) + aq^{2i+1}(1+q+q^2-bq^n(1-q(1+q+q^2)+\alpha+q^4\alpha))],\end{aligned}$$

$$\begin{aligned}\delta_{i2} &= q^{n-2}(q^{i+1};q)_2(1-q)^{-1} \times \Big[-\alpha + [(abq^{2i+3};q)_5]^{-1} \times (1-abq^{2i+5})(q^{1-n}(q^n\alpha + a^4b^4q^{n+8i+20}\alpha\\ &\quad -a(1+\alpha)(q-1)q^{n+i+2}[1+b^3a^3q^{6i+15}] + aq^{2i+3}(1+a^2b^3q^{4i+10})(-1+q^2+q^n(q(q-1)(b+(a+b)q)\\ &\quad -b(1+2q+q^4)\alpha)) + a^2bq^{4i+7}((q-1)(q+1)^3 + q^n((q-1)(aq^2(q+1)^2+b(-1+q^2+2q^3)) + 2b(1+q^3+q^4)\alpha))\\ &\quad + (a^2bq^{5(i+2)} + aq^{3i+5})((a+b)(1-q^3) + ba(q-1)q^n(1+\alpha+q(1+q)(-1-2q+(q-1)q\alpha)))))\Big],\end{aligned}$$

$$\begin{aligned}\delta_{i3} &= q^{3i+6}a^2(q^{i+1};q)_3(1-bq^{i+3})(1-aq^{i+3})[(1-abq^{2(i+3)})(abq^{2i+4};q)_5]^{-1}\\ &\quad \times \Big[1+bq(-abq^{3i+8}(1+aq^{n+1}(1+q)) - q^{i+1}(1+q+q^2+aq^{n+2}(1+q)) + (1+\alpha)q^n[1+a^2b^2q^{4(i+3)}]\\ &\quad + aq^{2(i+2)}(1+q+q^2-bq^n(1-q(1+q+q^2)+\alpha(1+q^4))))\Big],\end{aligned}$$

$$\delta_{i4} = q^{5i+14}a^3b(q^{i+1};q)_4(aq^{i+3};bq^{i+3};q)_2(-1+abq^{n+i+5})[(abq^{2(i+3)};q)_3(abq^{2i+5};q)_5]^{-1},$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3$ and $a_n(n) = 1$. The solution of (5.8) is

$$a_i(n) = \frac{(-1)^n\alpha^{n-i}q^{\binom{n}{2}+i}(q^{-n};q)_i}{(q;q)_i}\sum_{j=0}^{n-i}\frac{(q/\alpha)^j(q^{-(n-i)}, aq^{i+1};q)_j}{(q, abq^{2(i+1)};q)_j}\,{}_1\phi_1\left[\begin{matrix} q^{-(n-i-j)} \\ 0\end{matrix}\,\middle|\, q; \frac{q}{\alpha}\right], \quad i = 0, 1, \ldots, n. \tag{5.9}$$

Al-Salam-Carlitz I-Big q-Jacobi connection problem

In this problem

$$U_n^{(\alpha)}(x;q) = \sum_{i=0}^{n} a_i(n)P_i(x;a,b,c;q), \tag{5.10}$$

the coefficients $a_i(n)$ satisfy the fourth-order recurrence relation

$$\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0, \quad i = n-1, n-2, \ldots, 0, \tag{5.11}$$

$$\delta_{i\,0} = (q^n - q^i),$$

$$\begin{aligned}\delta_{i1} &= q^{-1}(q^{i+1}-1)[(1-acq^{2(i+1)})(1-acq^{2(i+3)})]^{-1} \times [-q^{i+2}(a+b) - a(b+c)q^{n+i+3}(1+q) + acq^{3i+5}(-a(b+c)q^{n+2}(1+q)\\ &\quad -(a+b)(1+q+q^2)) + q^n(1+\alpha) + a^2cq^{4(i+2)}(b+c+cq^n(1+\alpha)) + aq^{2(i+1)}((b+c)q(1+q+q^2)\\ &\quad + cq^n((a+b)q^2(1+q)(1+q^2)-(1+q^4)(1+\alpha)))\Big],\end{aligned}$$

Table 2 Formulae for the connection coefficients in problem (5.13) for the case of $P_i(x\,;q) = P_i(x\,;a,\,b,\,c\,;q)$.

$\bar{P}_n(x;q)$	$a_i(n)$ $(0 \le i \le n)$
$V_n^{(\beta)}(x;q)$	$(-1)^n \beta^{n-i} q^{ni-\binom{n}{2}} \frac{(q^{-n};q)_i}{(q;q)_i}\, {}_3\phi_1 \left[\begin{matrix} q^{-(n-i)},\, aq^{i+1},\, cq^{i+1} \\ abq^{2(i+1)} \end{matrix} \,\middle\vert\, q;\, \frac{q^{n-i}}{\beta} \right]$
$P_n(x\,;\alpha,\,\beta,\,\gamma\,;q)$	$(-1)^i q^{\binom{i+1}{2}} \frac{(\alpha q,\,\gamma q;q)_n (q^{-n},\,\alpha\beta q^{n+1};q)_i}{(\alpha\beta q^{n+1};q)_n (q,\,\alpha q,\,\gamma q;q)_i}\, {}_4\phi_3 \left[\begin{matrix} q^{-(n-i)},\, \alpha\beta q^{n+i+1},\, aq^{i+1},\, cq^{i+1} \\ \alpha q^{i+1},\, \gamma q^{i+1},\, abq^{2(i+1)} \end{matrix} \,\middle\vert\, q;q \right]$
$M_n(x\,;\beta,\,\gamma\,;q)$	$(-1)^n (\gamma/q^n)^{n-i} q^{\binom{i+1}{2}} \frac{(\beta q;q)_n (q^{-n};q)_i}{(q,\,\beta q;q)_i}\, {}_3\phi_2 \left[\begin{matrix} q^{-(n-i)},\, aq^{i+1},\, cq^{i+1} \\ \beta q^{i+1},\, abq^{2(i+1)} \end{matrix} \,\middle\vert\, q;-\frac{q^{n+1}}{\gamma} \right]$
$p_n(x\,;\alpha,\,\beta \vert q)$	$(-1)^n q^{i+\binom{n}{2}} \frac{(\alpha q;q)_n (q^{-n},\,\alpha\beta q^{n+1};q)_i}{(\alpha\beta q^{n+1};q)_n (q,\,\alpha q;q)_i} \times \sum_{j=0}^{n-i} (-q^{1-i})^j q^{-\binom{j}{2}} \frac{(q^{-(n-i)},\,\alpha\beta q^{n+i+1},\,aq^{i+1},\,cq^{i+1};q)_j}{(q,\,\alpha q^{i+1},\,abq^{2(i+1)};q)_j}\, {}_2\phi_1 \left[\begin{matrix} q^{-(n-i-j)},\, \alpha\beta q^{n+i+j+1} \\ \alpha q^{i+j+1} \end{matrix} \,\middle\vert\, q;q^{1-i-j} \right]$
$L_n^{(\beta)}(x;q)$	$\frac{(-1)^n q^{\binom{i+1}{2}}}{q^{(n-i)(n+\beta)}} \frac{(q^{\beta+1};q)_n (q^{-n};q)_i}{(q^{\beta+1};q)_i (q;q)_i} \times \sum_{j=0}^{n-i} (-q^{n+\beta+1})^j \frac{(q^{-(n-i)},\,aq^{i+1},\,cq^{i+1};q)_j}{(q,\,q^{\beta+i+1},\,abq^{2(i+1)};q)_j}\, {}_1\phi_1 \left[\begin{matrix} q^{-(n-i-j)} \\ q^{\beta+i+j+1} \end{matrix} \,\middle\vert\, q;-q^{n+\beta+1} \right]$
$S_n(x\,;q)$	$\frac{(-1)^n q^{\binom{i+1}{2}}}{q^{n(n-i)}} \frac{(q^{-n};q)_i}{(q;q)_i} \sum_{j=0}^{n-i} (-q^{n+1})^j \frac{(q^{-(n-i)},\,aq^{i+1},\,cq^{i+1};q)_j}{(q,\,abq^{2(i+1)};q)_j}\, {}_1\phi_1 \left[\begin{matrix} q^{-(n-i-j)} \\ 0 \end{matrix} \,\middle\vert\, q;-q^{n+1} \right]$

Table 3 Formulae for the connection coefficients in problem (5.13) for the case of $P_i(x\,;q)=p_i(x\,;a,\,b\vert q)$.

$\bar{P}_n(x;q)$	$a_i(n)\ (0\le i\le n)$
$V_n^{(\beta)}(x;q)$	$(-1)^n\beta^{n-i}q^{ni-\binom{n}{2}}\dfrac{(q^{-n};q)_i}{(q;q)_i}\displaystyle\sum_{j=0}^{n-i}(q^n/\beta)^j\dfrac{(q^{-(n-i)},aq^{i+1};q)_j}{(q,abq^{2(i+1)};q)_j}\,{}_2\phi_0\left[\begin{matrix}q^{-(n-i-j)},0\\ \\ -\end{matrix}\,\middle\vert\,q;\dfrac{q^{n-i-j}}{\beta}\right]$
$P_n(x\,;\alpha,\,\beta,\,\gamma\,;q)$	$(-1)^iq^{\binom{i+1}{2}}\dfrac{(\alpha q,\gamma q;q)_n(q^{-n},\alpha\beta q^{n+1};q)_i}{(\alpha\beta q^{n+1};q)_n(q,\alpha q,\gamma q;q)_i}\times\displaystyle\sum_{j=0}^{n-i}(-q^i)^jq^{\binom{j+1}{2}}\dfrac{(q^{-(n-i)},\alpha\beta q^{n+i+1},aq^{i+1};q)_j}{(q,\alpha q^{i+1},\gamma q^{i+1},abq^{2(i+1)};q)_j}\,{}_3\phi_2\left[\begin{matrix}q^{-(n-i-j)},\alpha\beta q^{n+i+j+1},0\\ \\ \alpha q^{i+j+1},\gamma q^{i+j+1}\end{matrix}\,\middle\vert\,q;q\right]$
$M_n(x\,;\beta,\,\gamma\,;q)$	$\dfrac{(-1)^n\gamma^{n-i}q^{\binom{i+1}{2}}}{q^{n(n-i)}}\dfrac{(\beta q;q)_n(q^{-n};q)_i}{(q,\beta q;q)_i}\times\displaystyle\sum_{j=0}^{n-i}(q^{n+i}/\gamma)^jq^{\binom{j+1}{2}}\dfrac{(q^{-(n-i)},aq^{i+1};q)_j}{(q,\beta q^{i+1},abq^{2(i+1)};q)_j}\,{}_2\phi_1\left[\begin{matrix}q^{-(n-i-j)},0\\ \\ \beta q^{i+j+1}\end{matrix}\,\middle\vert\,q;-\dfrac{q^{n+1}}{\gamma}\right]$
$p_n(x\,;\alpha,\,\beta\vert q)$	$(-1)^nq^{i+\binom{n}{2}}\dfrac{(\alpha q;q)_n(q^{-n},\alpha\beta q^{n+1};q)_i}{(\alpha\beta q^{n+1};q)_n(q,\alpha q;q)_i}\,{}_3\phi_2\left[\begin{matrix}q^{-(n-i)},\alpha\beta q^{n+i+1},aq^{i+1}\\ \\ \alpha q^{i+1},abq^{2(i+1)}\end{matrix}\,\middle\vert\,q;q\right]$
$L_n^{(\beta)}(x;q)$	$\dfrac{(-1)^nq^{\binom{i+1}{2}}}{q^{(n-i)(n+\beta)}}\dfrac{(q^{\beta+1};q)_n(q^{-n};q)_i}{(q^{\beta+1};q)_i(q;q)_i}\,{}_2\phi_2\left[\begin{matrix}q^{-(n-i)},aq^{i+1}\\ \\ q^{\beta+i+1},abq^{2(i+1)}\end{matrix}\,\middle\vert\,q;-q^{n+\beta+i+1}\right]$
$S_n(x\,;q)$	$\dfrac{(-1)^nq^{\binom{i+1}{2}}}{q^{(n-i)n}}\dfrac{(q^{-n};q)_i}{(q;q)_i}\,{}_2\phi_2\left[\begin{matrix}q^{-(n-i)},aq^{i+1}\\ \\ abq^{2(i+1)},0\end{matrix}\,\middle\vert\,q;-q^{n+i+1}\right]$

$$\delta_{i2} = (q^{i+1};q)_2 \times \Big[q^{n-2}\alpha + aq^{i+2}(1-aq^{i+2})(1-bq^{i+2})(1-cq^{i+2})(b-acq^{i+2}) \times [(1-acq^{2i+4})(acq^{2i+3};q)_3]^{-1}[1+acq^{i+1}(q^{n+1}-2)]$$
$$- ca^2q^{2i+7}[(1-acq^{2i+6})(acq^{2i+5};q)_3]^{-1}(1-aq^{i+3})(1-bq^{i+3})(1-cq^{i+3})(b-acq^{i+3})(q^{i+2}-q^n)$$
$$+ [(1-acq^{2(i+2)})^2(1-acq^{2(i+3)})^2]^{-1} \times \Big\{aq^{2i+5}[((b+c)(1+acq^{2i+5}) - c(a+b)q^{i+2}(1+q))((1+acq^{2i+5})(a+b+a(b+c)q^{n+1})$$
$$- aq^{i+2}(1+q)(b+c+c(a+b)q^{n+1}))] - aq^{n+i+2}(1-acq^{2(i+2)})(1-acq^{2(i+3)})(1+\alpha)[(b+c)(1+acq^{2i+5})$$
$$- c(a+b)q^{i+2}(1+q)]\Big\}\Big],$$

$$\delta_{i3} = a^2q^{i+6}(q^{i+1};q)_3(1-aq^{i+3})(1-bq^{i+3})(1-cq^{i+3})(acq^{i+3}-b)[(1-acq^{2i+6})^2(acq^{2i+4};q)_5]^{-1}$$
$$\times \Big[acq^{i+4}(1+q)\,(q^n-1)(1-acq^{2i+6})[(b+c)(1+acq^{2(i+3)}) - c(a+b)q^{i+2}(1+q^2))]$$
$$+ cq^{i+4}(1-acq^{2i+4})[(a+b+a(b+c)q)(1+acq^{2i+7}) - aq^{i+3}(1+q)(b+c+c(a+b)q)]$$
$$+ (1+ac(q-2)q^{i+3})(1-acq^{2i+8})[(b+c)(1+acq^{2i+5}) - c(a+b)q^{i+2}(1+q)]$$
$$+ cq^n(1+\alpha)(1-acq^{2i+4})(1-acq^{2i+6})(1-acq^{2i+8})\Big],$$

$$\delta_{i4} = ca^3q^{3i+11}(q^{i+1};q)_4(aq^{i+3};bq^{i+3},cq^{i+3};q)_2(b-acq^{i+3})(acq^{i+4}-b)[1+acq^{i+4}(q^{n+1}-2)] \times [(abq^{2(i+3)};q)_3(abq^{2i+5};q)_5]^{-1},$$

with $a_{n+s}(n) = 0$, $s = 1,2,3$ and $a_n(n) = 1$. The solution of (5.11) is

$$a_i(n) = \frac{(-1)^n\alpha^{n-i}q^{\binom{n}{2}+i}(q^{-n};q)_i}{(q;q)_i}\sum_{j=0}^{n-i}(-q^{1-i}/\alpha)^j q^{-\binom{j}{2}}\frac{(q^{i-n},aq^{i+1},cq^{i+1};q)_j}{(q,abq^{2(i+1)};q)_j}\,{}_2\phi_1\left[\begin{matrix}q^{-(n-i-j)},q^{i+j}\\0\end{matrix}\,\middle|\,q;\frac{q^{1-i-j}}{\alpha}\right],\quad i=0,1,\ldots,n. \tag{5.12}$$

Remark 3. Similar recurrence relations that are satisfied by the connection coefficients $a_i(n)$ in the relation

$$\bar{P}_n(x;q) = \sum_{i=0}^{n} a_i(n)P_i(x;q), \tag{5.13}$$

may also be obtained for the other families belonging to the Askey–Wilson polynomials, but details are not given here. In Tables 2 and 3, we summarize the formulae of connection coefficients for most of the remaining monic families $P_i(x;q)$.

Remark 4. One of our goals is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica [56]). Symbolic language has been used.

Conclusion

In this paper, we deduced some interesting formulae associated with the $Pn(x;q)$ coefficients for the moments of $D_q^p f(x)$, $p = 0,1,2,\ldots$ and with the connection coefficients between the q-classical orthogonal polynomials belonging to the Askey–Wilson polynomials and $Pn(x;q)$ in T. These formulae are systematically used to set up the resulting algebraic systems when applying the spectral methods for solving q-difference equations with polynomials coefficients of any order.

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