On Finite Groups with the Cayley Isomorphism Property, II

Cai Heng Li

Department of Mathematics, University of Western Australia, Perth, Western Australia, 6907, Australia

Communicated by Sasha Ivanov

Received January 26, 1997

For a positive integer m, a group G is said to have the m-DCI property if, for any Cayley digraphs Cay(G,S) and Cay(G,T) of G of valency m (that is, |S| = |T| = m), Cay(G,S) \cong Cay(G,T) if and only if S^\sigma = T for some \sigma \in Aut(G).

This paper is one of a series of papers towards characterizing finite groups with the m-DCI property. It is shown that, for infinitely many values of m, there exist Frobenius groups with the m-DCI property but not with the k-DCI property for any k < m. Further, it is conjectured that for relative small values of m, these groups and an explicit list of groups given by C. H. Li, C. E. Praeger, and M. Y. Xu (1998, J. Combin. Theory Ser. B 73, 164–183) contain all finite groups with the m-DCI property. This conjecture is verified for the case m \leq 4.

1. INTRODUCTION

For a finite group G and a subset S of G* := G\{1\}, we define the Cayley digraph of G with respect to S to be the directed graph Cay(G,S) with vertex set G and edge set \{(a,b) | a, b \in G, ba^{-1} \in S\}. By the definition, Cay(G,S) has (out)valency |S|, and Cay(G,S) is connected if and only if \langle S \rangle = G. A Cayley digraph Cay(G,S) is called a CI-graph of G if, for any \tau \in G, Cay(G,S) \cong Cay(G,T) implies S^\tau = T for some \tau \in Aut(G).

This is the Cayley Isomorphism property. For a positive integer m, if all Cayley digraphs of G of valency m are CI-graphs, then G is said to have the m-DCI property.

The problem of characterizing finite CI-graphs is a long-standing open problem about Cayley graphs, see surveys [1, 11, 18, 19].
problem, Praeger, Xu and the author in [14] initiated a study of finite
groups with the \( m \)-DCI property, and proposed

**Problem** [14]. Characterize finite groups with the \( m \)-DCI property.

This paper contributes towards a classification of finite groups having
the \( m \)-DCI property.

In [14], a general investigation is made of the structure of Sylow sub-
groups of groups with the \( m \)-DCI property for certain values of \( m \). In [10],
it is proved that if \( G \) is an abelian group with the \( m \)-DCI property then
every Sylow subgroup of \( G \) is homocyclic. (Here a group is called
homocyclic if it is a direct product of cyclic groups of the same order.)
Further, a reasonably complete classification of the cyclic groups with the
\( m \)-DCI property is obtained in [9]. If a group \( G \) has the \( i \)-DCI property
for all \( i \leq m \), then \( G \) is called an \( m \)-DCI-group. Finite \( m \)-DCI-groups have
been investigated for a long time, see for example [2, 3, 5, 12, 15–18]. In
particular, Zhang [23] obtained a description of \( 1 \)-DCI-groups; Praeger,
Xu and the author [15] proved that if \( G \) is an \( m \)-DCI-group for \( m \geq 2 \) then
\( G = U \times V \) where \( (|U|, |V|) = 1 \), \( U \) is abelian and \( V \) lies in an explicitly
determined short list. If a group \( G \) has both the \( 1 \)-DCI property and the
\( m \)-DCI property for some \( m \geq 2 \), then \( G \) is well-characterized in [23],
which is improved in [13, Corollary 1.3]. In view point of induction on \( m \),
the problem of characterizing finite groups with the \( m \)-DCI property is
therefore reduced to the following problem:

**Problem** 1.1. For an integer \( m \geq 2 \), characterize the finite groups which
have the \( m \)-DCI property but do not have the \( k \)-DCI property for any
\( k < m \).

Regarding this problem, the first question we face is, for an integer
\( m \geq 2 \), whether there exist groups which have the \( m \)-DCI property but do
not have the \( k \)-DCI property for any \( k < m \). Theorem 1.3 of [14] gives
such examples for \( m = 2 \). The first result of this paper is, for infinitely many
values of \( m \), to construct a family of groups which have the \( m \)-DCI property but do not have the \( k \)-DCI property for any \( k < m \). Such groups are
defined as follows.

**Definition** 1.2. Let \( G = E(M, n) = M \times \langle z \rangle \) be a finite group such that

(i) \( M \) is an abelian group of odd order and all Sylow subgroups of
\( M \) are homocyclic;

(ii) \( \langle z \rangle \cong \mathbb{Z}_n \) where \( n \geq 2 \), and \( (|M|, n) = 1 \);

(iii) there exists an integer \( l \) such that for any \( x \in M \setminus \{1\} \), \( z^{-1}xz = x^l \)
and \( n \) is the smallest positive integer satisfying \( l \equiv 1 \pmod{a(x)} \).
By the definition, any non-identity element of \( \langle z \rangle \) centralizes no non-identity elements of \( M \), and hence by [20, p. 299], \( E(M, n) \) is a Frobenius group with \( M \) the Frobenius kernel and \( \langle z \rangle \) a Frobenius complement. Moreover, we have

**Theorem 1.3.** Let \( G = E(M, q) \) and \( m = q - 1 \), where \( q \) is a prime. Then \( G \) has the \( m \)-DCI property but does not have the \( k \)-DCI property for any \( k < m \).

We do not have any other examples. Moreover, as finite groups with the \( m \)-DCI property are very restricted, we are inclined to think that the groups \( E(M, q) \) are all possibilities for groups satisfying Problem 1.1 with "small" values of \( m \), that is,

**Conjecture 1.4.** Let \( m \geq 2 \) be an integer, and let \( G \) be a finite group which has a proper subgroup of order \( d \). If \( m < d - 1 \) and \( G \) has the \( m \)-DCI property but not the \( k \)-DCI property for any \( k < m \), then \( m + 1 \) is a prime and \( G = E(M, m + 1) \) for some abelian group \( M \).

If the conjecture is true, then for small values of \( m \), a group with the \( m \)-DCI property either lies in an explicit list given in [15], or is \( E(M, m + 1) \) for some abelian group \( M \). By [9, 10], the conjecture is true for abelian groups \( G \), and by [8], the conjecture is true for \( m = 2 \). The next theorem shows that the conjecture is also true for \( m = 3 \) and \( 4 \).

**Theorem 1.5.** Let \( G \) be a finite group.

1. If \( G \) has the 3-DCI property then \( G \) has the 1-DCI or the 2-DCI property.

2. \( G \) has the 4-DCI property but does not have the \( k \)-DCI property for any \( k \in \{1, 2, 3\} \) if and only if \( G = E(M, 5) \) for some abelian group \( M \).

2. **Preliminaries**

This section draws together some preliminary results which will be used. The general terminology and notation used in this paper is standard, see for example [4, 20]. For a finite group \( G \), elements \( a, b \) of \( G \) are said to be **fused** if \( a^\sigma = b \) for some \( \sigma \in \text{Aut}(G) \); similarly, subsets \( S, T \) of \( G \) are said to be **fused** if \( S^\sigma = T \) for some \( \sigma \in \text{Aut}(G) \). By the definition, the group \( G \) acting by right multiplication (that is, \( g: x \mapsto xg \)) is a subgroup of \( \text{Aut} \Gamma \) and acts regularly on \( V \Gamma \); we shall denote this regular subgroup by \( \hat{G} \). The normalizer of \( \hat{G} \) in \( \text{Aut} \Gamma \) is often used to characterize \( \Gamma \).
Lemma 2.1 (see [6, Lemma 2.1]). Let $G$ be a finite group, and let $\Gamma = \text{Cay}(G, S)$. Let $\text{Aut}(G, S) := \{ \pi \in \text{Aut}(G) \mid S^\pi = S \}$. Then $\mathcal{N}_{\text{Aut}(G)}(\Gamma) = \hat{G} = \text{Aut}(G, S)$.

Next we have a criterion for a Cayley digraph to be a CI-graph.

Theorem 2.2 (Alspach and Parsons [2], or Babai [3]). Let $\Gamma$ be a Cayley digraph of a finite group $G$. Let $\text{Sym}(G)$ be the symmetric group on $G$. Then $\Gamma$ is a CI-graph if and only if, for any $\pi \in \text{Sym}(G)$ with $\hat{G} \subseteq \text{Aut}(\pi)$, there exists $\pi \in \text{Sym}(G)$ such that $\hat{G} = \hat{G}^\pi$.

The following result of Gross, together with Theorem 2.2, can provide a lot of examples of CI-graphs.

Theorem 2.3 (Gross [7]). Let $G$ be a finite group and let $\pi$ be a set of odd primes. If $G$ has a Hall $\pi$-subgroup, then all Hall $\pi$-subgroups of $G$ are conjugate.

For a prime $p$, a finite group $G$ is said to be $p$-nilpotent if it has a normal Hall $p'$-subgroup. The following is the well-known Burnside's Theorem.

Theorem 2.4 (see [20, 10.1.7]). If, for some prime $p$, a Sylow $p$-subgroup $P$ of a finite group $G$ lies in the centre of its normalizer, then $G$ is $p$-nilpotent.

Now notice a simple fact that for a group $G$ and $S \subseteq G^*$, $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ if and only if $\text{Cay}(\langle S \rangle, S) \cong \text{Cay}(\langle T \rangle, T)$.

The next simple lemmas will often be used.

Lemma 2.5. Suppose that $G$ is a finite group with the $m$-DCI property for some positive integer $m$. Let $k$ be an integer greater than $m + 1$, and suppose that there exist $a, b \in G$ such that $o(a) = o(b) = k$. Then $a^j$ is fused to $b^j$ for every integer $j$.

Proof. Let $S = \{a, a^2, \ldots, a^m\}$ and $T = \{b, b^2, \ldots, b^m\}$. Then $\text{Cay}(\langle a \rangle, S) \cong \text{Cay}(\langle b \rangle, T)$, and so $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. Since $G$ has the $m$-DCI property, $S^\sigma = T$ for some $\sigma \in \text{Aut}(G)$. Thus $a^i = b^j$ for some integer $i$, and so $\{b^i, b^{2i}, \ldots, b^{mi}\} = \{a^i, a^{2i}, \ldots, a^{mi}\}^\sigma = S^\sigma = T = \{b, b^2, \ldots, b^m\}$. By [10, Lemma 2.1], $i = 1$ and so $a^i = b$. Thus we have that $(a^j)^\sigma = b^j$ for each integer $j$.

Lemma 2.6. For any two elements $x, y$ of a finite group $G$ and any automorphism $\alpha$ of $G$, if $x$ is conjugate to $y$ then $x^\alpha$ is conjugate to $y^\alpha$.
Proof. Suppose that \(z^{-1}xz = y\) for some \(z \in G\). Then \((z^*)^{-1}x^*z^* = (z^{-1}xz)^* = y^*\). So \(x^*\) is conjugate to \(y^*\) by \(z^*\). □

Some results about Sylow subgroups of groups with the \(m\)-DCI property are obtained in [14]. The following lemmas give some further result in the case where \(m = 3, 4\), which will be used in the proof of Theorem 1.3.

**Lemma 2.7.** Let \(m \in \{3, 4\}\), and let \(G\) be a finite group with the \(m\)-DCI property. Suppose that \(p \in \{3, 5\}\) divides \(|G|\), and let \(G_p\) be a Sylow \(p\)-subgroup of \(G\). Then either \(G_p \cong \mathbb{Z}_p\), or any two elements of \(G\) of order \(p\) are fused.

**Proof.** Suppose that there exist two elements of \(G\) of order \(p\) which are not fused. By [14, Theorem 1.4], \(G_p\) is cyclic. Thus by Lemma 2.5, \(G_p \cong \mathbb{Z}_p\). □

**Lemma 2.8.** Let \(G\) be a finite group with the \(m\)-DCI property where \(m = 3\) or \(4\). Let \(G_2\) be a Sylow 2-subgroup of \(G\). Then either \(G_2\) is elementary abelian, or \(G_2 \cong \mathbb{Z}_4\) or \(Q_8\).

**Proof.** If the order of \(G_2\) is at most 4, then the lemma holds. Thus we may assume that \(G_2\) has order greater than 4. First suppose that \(G_2\) has only one involution, then by [20, 5.3.6], \(G_2\) is either cyclic or generalized quaternion. Suppose that \(G_2\) is a generalized quaternion group, that is,

\[
G_2 = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^n-1}, y^{-1}xy = x^{-1} \rangle,
\]

where \(n \geq 2\). If \(n \geq 3\), then \(G_2\) contains elements \(a, b, c\) such that \(\langle a, b \rangle \cong Q_8\) and \(\langle c \rangle \cong \mathbb{Z}_4\). Set \(S_4 := \{c, c^3, c^7\}\) and \(T_3 := \{a, b, a^2\}\), and let \(S_3 := \langle c \rangle^* \setminus S_4\) and \(T_3 := \langle a, b \rangle^* \setminus T_3\) \(\cong \mathbb{Z}_4\). It is straightforward to check that \(\text{Cay}(\langle c \rangle, S_3) \cong \text{Cay}(\langle a, b \rangle, T_3)\), and so \(\text{Cay}(\langle c \rangle, S_3) \cong \text{Cay}(\langle a, b \rangle, T_3) \cong \text{Cay}(\langle a, b \rangle, T_4)\). Hence \(\text{Cay}(G, S_m) \cong \text{Cay}(G, T_m)\). However clearly there is no \(\sigma \in \text{Aut}(G)\) such that \(S_m \cong T_m\), which is a contradiction since \(G\) has the \(m\)-DCI property. Thus \(n = 2\) and \(G_2 = Q_8\).

Next suppose that \(G_2\) is cyclic of order greater than 4. Then there exists an element \(x \in G_2\) of order 8. Let \(S_3 = \{x, x^5, x^9\}\) and \(T_3 = \{x, x^5, x^9\}\), and let \(S_3 = \langle x \rangle^* \setminus S_3\) and \(T_3 = \langle x \rangle^* \setminus T_3\). A straightforward checking shows that \(\text{Cay}(\langle x \rangle, S_3) \cong \text{Cay}(\langle x \rangle, T_3)\) and so \(\text{Cay}(\langle x \rangle, S_3) \cong \text{Cay}(\langle x \rangle, T_4)\), but \(S_m \cong T_m\) for any \(\sigma \in \text{Aut}(\langle x \rangle)\). Now \(\text{Cay}(G, S_m) \cong \text{Cay}(G, T_m)\). Since \(G\) has the \(m\)-DCI property, \(S_m \cong T_m\) for some \(\tau \in \text{Aut}(G)\). Thus \(\langle x \rangle^* \cong \langle x \rangle\) and so \(\tau\) induces an automorphism of \(\langle x \rangle\), which is a contradiction.

Finally, suppose that \(G_2\) has more than one involution. Arguing as in the previous paragraph, \(G_2\) does not contain an element of order greater than 4. On the other hand, we suppose that \(G_2\) has exponent 4, and let \(a\)
be an element of $G_2$ of order 4. Consider $N_1 := N_{G_2}(\langle a \rangle)$. Since $|G_2| > 4$, $N_1 > \langle a \rangle$. Suppose that there exists an involution $g \in N_1 \setminus \langle a \rangle$. Then $a^g = a^{-1}$ or $a$. Let $S_2 = \{a, a^{-1}, a^2\}$ and $T_3 = \{a^2, g, a^2g\}$, and let $S_4 = \langle a, g \rangle \setminus S_3$ and $T_4 = \langle a, g \rangle \setminus T_3$. It is easy to show that Cay$(G, S_2) \cong$ Cay$(G, T_3)$ but $S_2$ is not fused to $T_3$, which is a contradiction since $G$ has the $m$-DCI property. Thus $a^2$ is the only involution of $N_1$. Since $N_1$ has exponent 4 and $N_1 > \langle a \rangle$, $N_1$ is not cyclic. It then follows from [20, 5.3.6] that $N_1 \cong Q_8$. Since $G_2$ has at least two involutions, $G_2 > N_1$. Thus $N_2 := N_{G_2}(N_1) > N_1$, and so (since $N_2$ still has exponent 4), again by [20, 5.3.6], there exists an involution $g \in N_2 \setminus N_1$. Since $g$ does not normalise $\langle a \rangle$, $b := a^g \notin \langle a \rangle$. Thus $\langle a, b \rangle = N_1$ and $b^g = a$, and hence $x := ab$ satisfies $x^g = (ab)^g = ba = (ab)^{-1} = x^{-1}$. Let $S_3 = \{x, x^{-1}, x^2\}$ and $T_3 = \{x^2, g, x^2g\}$, and let $S_4 = \langle x, g \rangle \setminus S_3$ and $T_4 = \langle x, g \rangle \setminus T_3$. Similar arguments as above show that Cay$(G, S_4) \cong$ Cay$(G, T_4)$ but $S_4$ is not fused to $T_4$, which is again a contradiction. Therefore, $G_2$ has exponent 2, and so $G_2$ is elementary abelian.

Let $\Gamma$ be a finite graph such that $G \leqslant \text{Aut} \Gamma$ is transitive on $V \Gamma$. For a normal subgroup $N$ of $G$ which is intransitive on $V \Gamma$, $\Gamma$ has a quotient graph $\Gamma_N$, for which $V \Gamma_N$ is the set of all $N$-orbits on $V \Gamma$, and two vertices $U, V \in V \Gamma_N$ are adjacent in $\Gamma_N$ if and only if there exist $u \in U$ and $v \in V$ which are adjacent in $\Gamma$. For a positive integer $s$, an $s$-arc in a digraph $\Gamma$ is a sequence $(v_0, \ldots, v_s)$ of $s + 1$ vertices of $\Gamma$ such that, for all $i$ with $1 \leq i \leq s$, $v_{i-1}$ is adjacent to $v_i$, and for all $i$ with $1 \leq i < s$, $v_{i-1} \neq v_{i+1}$. The digraph $\Gamma$ is said to be $(G, s)$-arc transitive if $G \leqslant \text{Aut} \Gamma$ is transitive on the set of $s$-arcs of $\Gamma$. In particular, a $(G, 1)$-arc transitive digraph is also called $G$-arc transitive, and if $G = \text{Aut} \Gamma$ then a $(G, s)$-arc transitive digraph is simply called $s$-arc transitive. The proof of the following lemma is easy and omitted.

**Lemma 2.9.** Let $\Gamma$ be a connected $(G, s)$-arc transitive digraph, and let $N$ be a normal subgroup of $G$ which is intransitive on $V \Gamma$. Then $\Gamma$ is connected $(G/N, s)$-arc transitive, the girth and the valency of $\Gamma_N$ divides the girth and the valency of $\Gamma$, respectively. Moreover, if $G/N$ acts faithfully on the set of $N$-orbits then the valency of $\Gamma_N$ equals the valency of $\Gamma$ if and only if $N$ acts semiregularly on $V \Gamma$.

3. THE $m$-DCI PROPERTY OF $E(M, n)$

The main aim of this section is to prove Theorem 1.3. The first lemma gives some simple properties of the groups $E(M, n)$, the proof of which is easy and omitted.
Lemma 3.1. For the group $E(M, n)$,

(i) any prime divisor of $|M|$ is greater than $n$, and $(|M|, l) = 1$, where $l$ is as in Definition 1.2;
(ii) $C_a(z) = \langle z \rangle$;
(iii) $z$ normalizes every cyclic subgroup of $M$.

The next lemma forms a part of the proof of Theorem 1.3.

Lemma 3.2. The group $E(M, n)$ does not have the $k$-DCI property for any $k < n - 1$.

Proof. Let $G = E(M, n)$. As in Definition 1.2, write $G = M \rtimes \langle z \rangle$ where $\langle z \rangle \cong \mathbb{Z}_n$. Let $k < n - 1$ be a positive integer, and let

$$S = \{z, ..., z^k\}, \quad T = \{z^{-1}, ..., z^{-k}\}.$$ 

Then $\text{Cay}(\langle z \rangle, S) \cong \text{Cay}(\langle z \rangle, T)$, so $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. Suppose that $G$ has the $k$-DCI property. Then there is an element $x$ of $\text{Aut}(G)$ such that $S^x = T$. Since $z \in S$, we have $z^i \in T$ and so $z^i = z'$ for some $i \in \{-1, ..., -k\}$. Thus $\{z^i, ..., z^{ik}\} = S^x = T = \{z^{-1}, ..., z^{-k}\}$. Let $z'^i = z^{-1}$ and $i' = -i$. Then $\{z'^{2}, (z'^i)^{2}, ..., (z'^i)^{k}\} = \{z^i, z^i, ..., z^i\}$. By [10, Lemma 2.1], $i = -i' = 1$, that is, $z^i = z^{-1}$. Since $M$ is characteristic in $G$, for any $a \in M$, we have $a^i = a'$ for some $a' \in M$ with $o(a') = o(a)$. Since $a^i = a'$, where $l$ is as in Definition 1.2,

$$z a' z^{-1} = (z^{-1}a z)^i = (a')^i,$$

and on the other hand, by Definition 1.2, $z^{-1}a z = (a')^i$. Therefore, we have $a' = z(a')^i z^{-1} = (a')^i$. Thus $P \equiv 1 \pmod{o(a')}$, which is a contradiction to Definition 1.2(iii). Consequently, $G$ does not have the $k$-DCI property.

Now we prove that a Cayley digraph $\text{Cay}(G, S)$ of $G = E(M, n)$ of valency $n - 1$ is a CI-graph if and only if $\text{Cay}(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$.

Lemma 3.3. Let $G = E(M, n) = M \rtimes \langle z \rangle$, and let $\text{Cay}(G, S)$ be of valency $n - 1$. Then $\text{Cay}(G, S)$ is a CI-graph of $G$ if and only if $\text{Cay}(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$.

Proof. Suppose that $\text{Cay}(G, S)$ is a CI-graph of $G$. Let $T \subset \langle S \rangle$ be such that $\text{Cay}(\langle S \rangle, S) \cong \text{Cay}(\langle S \rangle, T)$. Then $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. As $\text{Cay}(G, S)$ is a CI-graph of $G$, there exists $\sigma \in \text{Aut}(G)$ such that $S^\sigma = T$. Now $\langle S \rangle^\sigma = \langle S \rangle^T = \langle T \rangle = \langle S \rangle$, and so $\sigma$ induces an automorphism of $\langle S \rangle$ which sends $S$ to $T$. Thus $\text{Cay}(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$. 


Conversely, suppose that Cay$(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$, and we need to prove that Cay$(G, S)$ is a CI-graph of $G$. Let $H = \langle S \rangle$, $\Gamma = \text{Cay}(H, S)$ and $A = \text{Aut} \Gamma$. For any $T \in G$ such that Cay$(G, S) \cong \text{Cay}(G, T)$, Cay$(H, T) \cong \text{Cay}(H, S)$. Let $K = \langle T \rangle$ and $B = \text{Aut Cay}(K, T)$. Then $B \cong A$ and $|K| = |H|$. Let $q$ be a prime divisor of $|H|$, and let $H_q$ and $K_q$ be a Sylow $q$-subgroup of $H$ and $K$, respectively. We claim that $H_q \cong K_q$. If $q > n - 1$, it follows that $H_q$ is a Sylow $q$-subgroup of $A$. Since $A \cong B$, $H_q \cong K_q$. Next assume that $q \leq n - 1$. Then a Sylow $q$-subgroup $G_q$ of $G$ is cyclic, and so any two subgroups of $G_q$ of the same order are isomorphic. Since $|H| = |K|$, we have $|H_q| = |K_q|$ and so $H_q \cong K_q$. Consequently, $H_q \cong K_q$ for all $q$, and it follows that $K \cong H$. Let $\sigma$ be an isomorphism from $K$ to $H$, and let $S = T^\sigma$. Then Cay$(H, S) \cong \text{Cay}(K, T) \cong \text{Cay}(H, S)$. Since Cay$(H, S)$ is a CI-graph of $H$, $(S^\sigma)^1 = S$ for some $\sigma \in \text{Aut}(H)$. Thus $\rho := \sigma \sigma'$ is an isomorphism from $K$ to $H$ such that $T^\rho = T^\sigma = (S^\sigma)^1 = S$.

To complete the proof of the lemma we must show that $\rho$ extends to an automorphism of $G$. We do this first in the case where $|H|$ is coprime to $n$. Assume that $(|H|, n) = 1$. Then $H \subseteq M$ and $(|K|, n) = 1$, so also $K \subseteq M$. Since $M$ is abelian and each Sylow subgroup of $G$ is homocyclic, it is easy to see that there exists $\alpha \in \text{Aut}(M)$ such that $\rho$ is the restriction of $\alpha$ to $K$, that is, $\alpha|_K = \rho$. Let $\beta$ be a map from $G$ to $G$ defined by

$$(xz^i)^\beta = x^\alpha z^i \quad \text{for any} \quad x \in M \quad \text{and} \quad i \in \{0, 1, \ldots, n - 1\}.$$ 

A straightforward calculation shows that $\beta$ is an automorphism of $G$. Since $y^\beta = y^\alpha = y^\rho$ for any $y \in K$, $T^\beta = T^\rho = S$. Hence Cay$(G, S)$ is a CI-graph of $G$.

Next we consider the case where $(|H|, n) \neq 1$, and hence $(|K|, n) \neq 1$. Suppose that $H$ and $K$ are $\pi$-groups of $G$, where $\pi$ is the set of prime divisors of $|G|$ at most $n$. Since $|S| = |T| = n - 1$, we have that $H, K \cong \mathbb{Z}_n$.

By Hall's Theorem, $H$ is conjugate to $K$ (in $G$). Therefore, $S = H \setminus \{1\}$ is conjugate to $T = K \setminus \{1\}$. So Cay$(G, S)$ is a CI-graph of $G$. Thus we may assume that $H$ and $K$ are not $\pi$-groups of $G$. Then $H = J \times \langle z_1 \rangle$ and $K = J' \times \langle z_2 \rangle$ where $z_1$ and $z_2$ are of order dividing $n$, and both $J$ and $J'$ are non-trivial subgroups of $M$. By Hall's Theorem, to prove that $S$ is conjugate in $A$ to $T$, we may assume that $\langle z_1 \rangle = \langle z_2 \rangle \leq \langle z \rangle$ and that $z_2 = z^j$ for some $i \in \{1, 2, \ldots, n - 1\}$. Then $(J')^\rho = J$ and $(z')^\rho = (z^j)^i$ for some $z' \in H$ with $o(z') = o(z)$ and some integer $j \in \{1, 2, \ldots, n - 1\}$. Clearly, we may choose $z'$ such that $x^{z'} = x^z$ for all $x \in M$. For each $x \in M \setminus \{1\}$, we have, setting $x' = x^z$,

$$(x')^{u_1} = (x^z)^u = (z^{-1}xz^i)^u = (z^{-1}x'z'^i)^u = (x')^{u_1}.$$
so \( l' - l' \equiv 0 \pmod{a(x')}. \) If \( j \neq i \), say \( j > i \), then since \( l \) is coprime to \( a(x') \), we have \( l' - l' \equiv 1 \pmod{a(x')} \), which is a contradiction to Definition 1.2(iii).

Thus \( j = i \) and therefore, \( (z')^p = (z')^l \). Let \( \rho | J \) be the restriction of \( \rho \) to \( J' \), and let \( x \in \text{Aut}(M) \) be an extension of \( \rho | J \). Let \( \beta \) be a map from \( G \) to \( G \) defined as

\[
(xz^h)^\beta = x^* (z')^h \quad \text{for any} \quad x \in M \quad \text{and} \quad h \in \{0, 1, ..., n-1\}.
\]

As before, noting that \( \text{Aut}(\langle x \rangle) \) is abelian, a straightforward calculation shows that \( \beta \) is an automorphism of \( G \). It is easy to see that \( \beta | J = \alpha | J = \rho | J \) and \( \beta | J' = \rho | J' \). Consequently, \( \beta | K = \rho \) and so \( T^\beta = T^\rho = S \).

Hence \( \text{Cay}(G, S) \) is a CI-graph of \( G \). Therefore, \( G \) has the \( m \)-DCI property. \( \blacksquare \)

Now we can complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By Lemma 3.2, \( G \) does not have the \( k \)-DCI property for any \( k < m \). So we need only verify that \( G \) has the \( m \)-DCI property. Let \( \text{Cay}(G, S) \) be a Cayley digraph of \( G \) of valency \( m \). To prove that \( \text{Cay}(G, S) \) is a CI-graph of \( G \), by Lemma 3.3, we only need to prove that \( T^\prime : = \text{Cay}(\langle S \rangle, S) \) is a CI-graph of \( \langle S \rangle \). Let \( A = \text{Aut} \Gamma \) and \( A_1 \), the stabilizer of 1 in \( A \). Since \( \Gamma \) is a connected digraph of valency \( m \) (\( = q - 1 \)), it follows that all prime divisors of \( |A_1| \) are less than \( q \). As all prime divisors of \( |G| \) are at least \( q \), \( |\langle S \rangle| \) and \( |A_1| \) are coprime. Therefore, \( A_1 \) is a \( \pi \)-group and \( \langle S \rangle \) is a Hall \( \pi' \)-subgroup of \( A \), where \( \pi \) is the set of primes less than \( q \). By Theorem 2.3, all Hall \( \pi' \)-subgroups of \( A \) are conjugate to \( \langle S \rangle \). Thus by Theorem 2.2, \( \Gamma \) is a CI-graph of \( \langle S \rangle \), and so by Lemma 3.3, \( \text{Cay}(G, S) \) is a CI-graph of \( G \). Therefore, \( G \) has the \( m \)-DCI property. This completes the proof of Theorem 1.3. \( \blacksquare \)

In contrast to Theorem 1.3, we shall show that \( E(M, 4) \) has neither the 3-DCI property nor the 4-DCI property in the following proposition.

**Proposition 3.4.** For any abelian group \( M \), \( E(M, 4) \) does not have the \( m \)-DCI property for \( m = 3, 4 \).

**Proof.** Write \( E(M, 4) = M \times \langle z \rangle \). Let \( a \in M \) be a non-identity element of order a prime \( p \), and write \( G = \langle a \rangle \times \langle z \rangle \). Then \( G \cong \mathbb{Z}_p \times \mathbb{Z}_4 \) is a Frobenius group. Let

\[
S = \begin{cases} \{az, a^{-1}z, z^2\}, & \text{if } m = 3, \\ \{az, a^{-1}z, z, z^{-1}\}, & \text{if } m = 4. \end{cases}
\]

Let \( \Gamma = \text{Cay}(G, S) \), and let \( A = \text{Aut} \Gamma \). Let \( \tau \) be the inner automorphism of \( G \) induced by \( z^2 \). Then \( o(\tau) = 2 \), \( a^\tau = a^{-1} \) and \( z^\tau = z \). Thus \( \tau \) fixes \( S \), and by Lemma 2.1, \( \tau \in A \) and normalizes \( G \). Write \( \hat{G} = \langle \hat{a} \rangle \cong \langle \hat{z} \rangle \), and let \( e = \tau z^2 \).
For any $a'z' \in G$ where $i, j$ are integers (note that we are identifying $VT$ with $G$), we have that $(a'z')^{x^2} = (a'z')^{x^2} = a'z'$. Thus $c$ is of order 2. Further,

\[(a'z')^{l'e} = (a'z'/a)^{l'c} = (a^{-l}'z/a^{-1})^{l} = a^{-l}'z/a^{-1}z^2 = a^{-l}'z^{l-2}.\]

Thus $(a'z')^{l'e} = (a'z')^{l'c}$, so $a'c = c'$. Similarly, $\bar{c}c = c\bar{c}$, and so $c$ centralizes $\hat{G}$. In particular, $\hat{G} \times \langle c \rangle \leq A$.

As $\langle az, a^{-1}z \rangle = G$, $\Sigma := \text{Cay}(G, \{az, a^{-1}z\})$ is connected. It is easy to see that the edge set of $\Sigma$ is an orbit of $\text{Aut} \, \Gamma$ on the edge set of $\Gamma$. It then follows that $A$ is a subgroup of $\text{Aut} \, \Sigma$. Observe that $\Sigma$ is a digraph of valency 2 and girth 4. It follows that $\Sigma$ is not 3-arc transitive, and so $|\text{Aut} \, \Sigma : G| = 2$ or 4, in particular, $|\text{Aut} \, \Sigma : G| = 2$ or 4, where $A$ is the stabilizer of 1 in $A$. Suppose that $|\text{Aut} \, \Sigma : G| = 4$. Then $\text{Aut} \, \Sigma = A$, and $\hat{G} \leq \hat{G} \times \langle c \rangle \leq A$.

Let $\hat{G}_p$ be a Sylow $p$-subgroup of $\hat{G}$, and let $C = C_A(\hat{G}_p)$. Then $\hat{G}_p$ is a characteristic subgroup of $\hat{G} \times \langle c \rangle$, and so $\hat{G}_p \leq A$. As $\hat{G}$ is a Frobenius group and $N_d(\hat{G}_p)/C_d(\hat{G}_p) \leq \text{Aut} \, \hat{G}_p$ which is cyclic, it follows that $C = G_p \times C_2$ such that $C_2$ is of order 4. Now $C_2$ is a characteristic subgroup of $C$ and $C \leq A$, so $C_2 \leq A$. Consider the quotient graph $\Sigma_{C_2}$. It follows as $\hat{G}$ is a Frobenius group that $A/C_2 \cong \hat{G}$ and that $A/C_2$ acts faithfully on $V\Sigma_{C_2}$. Suppose that $C_2$ acts semiregularly on $V\Sigma$. Then $C_2$-orbits on $V\Sigma$ have size 4, and so $\Sigma_{C_2}$ is of order $p$. Thus $\Sigma_{C_2}$ is a Cayley digraph of $\hat{G}_p$, and by Lemma 2.9, $\Sigma_{C_2}$ is of order 2 and $(A/C_2, 2)$-arc transitive, which is not possible. Thus $C_2$ acts non-semiregularly on $V\Sigma$, and so $C_2$-orbits on $V\Sigma$ have size 2. By Lemma 2.9, $\Sigma_{C_2}$ is connected of order 2 and has girth dividing 4 and valency 1. This is not the case.

Thus $|\text{Aut} \, \Sigma : G| = 2$, so that $A = \hat{G} \times \langle c \rangle$.

Further, $\tilde{d} = \langle d \rangle' = \tilde{d}' = \tilde{d}''$ where $l$ is an integer as in Definition 1.2, so that $\tilde{G} := \langle \tilde{d}, \tilde{z}c \rangle = \langle \tilde{d} \rangle \times \langle \tilde{z}c \rangle \cong \tilde{G}$. We claim that $\tilde{G}$ acts regularly on $VT$ so that $\tilde{G} = \tilde{G}^p$ for some $p \in \text{Sym}(VT)$. Since $\tilde{G} \cong \tilde{G}$, we only need to show that no element of $\tilde{G}^*$ has fixed points in $VT$. For an element $\tilde{x} \in \tilde{G}$, either $\tilde{o}(x) = p$ so that $x \in \langle d \rangle'$ and $x$ fixes no points in $VT$, or $\tilde{o}(x) > 4$ so that $x$ is conjugate to an element of $\langle \tilde{z}c \rangle$. Thus we only need to prove that $\langle \tilde{z}c \rangle^2$ fixes no point in $VT$. Suppose that $\langle \tilde{z}c \rangle^2$ fixes $a'z'j$ for some integers $i, j$. Then

\[a'z'j = (a'z')^{(l'c)2} = (a'z')^{(l'c)^2} = a'z'j + 2.\]

Thus $\tilde{z}^2 = 1$, which is a contradiction. So $\tilde{G}$ acts regularly on $VT$. Now it is easy to see that $\tilde{G}$ is not conjugate in $A$ to $\hat{G}$. By Theorem 2.2, $\tilde{V}$ is not a CI-graph of $G$.

Hence there exists $\text{Cay}(G, T)$ such that $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ and $S^\beta \neq T$ for any $x \in \text{Aut} \, G$. Let $H = E(M, 4)$. Then we have that $\text{Cay}(H, S) \cong \text{Cay}(H, T)$. If $H$ has the $m$-DCI property, then $S^\beta = T$ for some $\beta \in \text{Aut} \, H$.
As \( G^\beta = \langle S \rangle^\beta = \langle S^\beta \rangle = \langle T \rangle = G \), \( \beta \) induces an automorphism of \( G \) which sends \( S \) to \( T \), a contradiction. Therefore, \( H \) has no the \( m \)-DCI property for \( m = 3 \) and \( 4 \). 

4. GROUPS WITH THE \( m \)-DCI PROPERTY FOR \( m = 3, 4 \)

The main aim of this section is to prove Theorem 1.5. We shall fulfill this aim in two subsections, and we shall always assume that \( m = 3 \) or \( 4 \).

4.1. Frobenius Groups

Now we consider a class of Frobenius groups with the \( m \)-DCI property, which will be shown to be the candidates of groups with the \( m \)-DCI property in the next subsection.

**Proposition 4.1.** Let \( G = M \rtimes \langle z \rangle \) where \( o(z) = 3, 4 \) or \( 5 \). Suppose that \( G \) is a Frobenius group with \( M \) the Frobenius kernel and that \( G \) has the \( m \)-DCI property where \( m = 3 \) or \( 4 \). Then each Sylow subgroup of \( M \) is homocyclic, \( o(z) = 3 \) or \( 5 \), and further \( G = E(M, o(z)) \) or \( A_4 \).

**Proof.** Note that \( G \) is soluble. Thus by [14, Theorem 5.1], each Sylow subgroup of \( G \) of odd order is homocyclic. Since \( G \) is a Frobenius group, it follows that a Sylow 2-subgroup of \( M \) is neither quaternion nor non-trivial cyclic. Thus by Lemma 2.8, if \( 2 \mid |M| \) then a Sylow 2-subgroup of \( M \) is noncyclic elementary abelian. Therefore, every Sylow subgroup of \( M \) is homocyclic.

Assume first that \( z \) normalizes every cyclic subgroup of \( M \) of prime-power order. Then it follows that \( z \) normalizes each cyclic subgroup of \( M \). Thus for any \( x \in M \setminus \{1\} \), we have \( x^2 = x^{i(x)} \) for some integer \( i(x) \in \{1, 2, ..., o(x) - 1\} \). Since \( z \) does not centralize \( x \), \( i(x) \neq 1 \). Let \( x_1, x_2 \in M \) be such that \( \langle x_1 \rangle \cap \langle x_2 \rangle = 1 \). As \( 1 \leq i(x) < o(x) \), and

\[
\lambda_1^{i(x_2)} \lambda_2^{i(x_1)} = (x_1 x_2)^{i(x_1 x_2)} = (x_1 x_2)^i = x_1^i x_2^i = x_1^{i(x)} x_2^{i(x)},
\]

we have \( i(x_1) = i(x_1 x_2) = i(x_2) = : i \) say. For any \( y \in M \), either \( \langle y \rangle \cap \langle x_1 \rangle = 1 \) or \( \langle y \rangle \cap \langle x_2 \rangle = 1 \). Thus \( i(y) = i(x_1) \) or \( i(x_2) \), and so \( i(y) = i \). It follows that \( G = E(M, o(z)) \), and so by Proposition 3.4, \( o(z) = 3 \) or \( 5 \), as required.

Assume now that \( z \) does not normalize \( \langle a \rangle \) for some \( a \in M \). We shall treat the case \( o(z) = 4 \) and the case \( o(z) = 3 \) or \( 5 \) separately.

1. Consider the case \( o(z) = 4 \). In this case, \( M \) is of odd order. If there exists \( x \in M \) such that \( y := xx^{x^{-1}} \neq 1 \), then \( z^2 \) centralizes \( y \), which is a contradiction since \( G \) is a Frobenius group. Hence \( x^{x^{-1}} = x^{-1} \) for every
\[ x \in M. \] As \( z \) does not normalize \( \langle a \rangle \), \( b := a^z \neq \langle a \rangle \). Suppose that \( \langle a \rangle \cap \langle b \rangle = 1 \). Let \( S_3 = \{ a, b, a^{-1} \} \), \( T_3 = \{ ab, a, a^{-1} \} \), \( S_4 = \{ a, b, a^{-1}, b^{-1} \} \) and \( T_4 = \{ ab, a^{-1}, (ab)^{-1} \} \). Then \( \text{Cay}(\langle a, b \rangle, S_m) \cong \text{Cay}(\langle a, b \rangle, T_m) \), and hence \( \text{Cay}(G, S_m) \cong \text{Cay}(G, T_m) \). However, all elements of \( S_m \) are conjugate, and \( a \) is not conjugate to \( ab \). By Lemma 2.6, \( S_m \) is not fused to \( T_m \), which is a contradiction since \( G \) has the \( m \)-DCI property. Thus we have that \( \langle a \rangle \cap \langle b \rangle \neq 1 \). Then \( \langle a, b \rangle = \langle a \rangle \times \langle d \rangle \) where \( 1 < o(d) < o(a) \) so that \( b = a^d \) for some integers \( i, j \) with \( j \) coprime to \( o(d) \) (as \( \langle a, b \rangle = \langle a, d \rangle \)). Since \( o(d) \geq 3 \) is odd, there exists an integer \( j_0 \in \{ 1, 2, \ldots, o(d) - 1 \} \) such that \( j_0 \neq j \) and \( (j_0, o(d)) = 1 \). Let \( b' = a^{d_0} \). Set \( S_3 = \{ a, b, a^{-1} \} \), \( T_3 = \{ a, b', a^{-1} \} \), \( S_4 = \{ a, b, a^{-1}, b^{-1} \} \) and \( T_4 = \{ a, b', a^{-1}, b'^{-1} \} \). It is easy to see that \( \langle S_m \rangle = \langle T_m \rangle = \langle a, b \rangle \) and there exists \( \sigma \in \text{Aut}(\langle a, b \rangle) \) such that \( a^\sigma = a \) and \( (d')^\sigma = d_0 \), so \( S_m \cong T_m \). It then follows that \( \text{Cay}(G, S_m) \cong \text{Cay}(G, T_m) \). However, all elements of \( S_m \) are conjugate, and \( a \) is not conjugate to \( b' \). By Lemma 2.6, \( S_m \) is not fused to \( T_m \), which is a contradiction. Therefore, \( z \) normalizes every cyclic subgroup of \( M \), which is a contradiction.

(2) Consider the case \( o(z) = 3 \) or 5. Suppose that \( M \) is of even order, and let \( M_2 \) be the Sylow 2-subgroup of \( M \). Then \( M_2 \trianglelefteq G \), and \( M_2 \) is non-cyclic elementary abelian. Let \( a_0 \) be an involution of \( M_2 \), and let \( a_i = a_0^i \) for \( 1 \leq i \leq o(z) - 1 \). Suppose that \( |M_2| > 4 \). Then there exists \( b \in M_2 \setminus a_0^{\mathbb{Z}_2} \). Let

\[
S_3 = \begin{cases} 
\{ a_0, a_1, a_0 a_1 \}, & \text{if } o(z) = 3, \\
\{ a_0, a_1, a_2 \}, & \text{if } o(z) = 5;
\end{cases}
\]

\[
T_3 = \begin{cases} 
\{ a_0, b, a_0 b \}, & \text{if } o(z) = 3, \\
\{ a_0, a_1, b \} \text{ with } b \neq a_0 a_1, & \text{if } o(z) = 5;
\end{cases}
\]

\[
S_4 = \langle a_0, a_1, a_2 \rangle \not\cong S_3;
\]

\[
T_4 = \langle a_0, a_1, b \rangle \not\cong T_3.
\]

Note that since \( z \) centralizes no elements of \( M_2 \), \( a_2 = a_0 a_1 \) if and only if \( o(z) = 3 \). It is then easy to show that \( \text{Cay}(G, S_m) \cong \text{Cay}(G, T_m) \). Since \( G \) has the \( m \)-DCI property, \( S_m \cong T_m \) for some \( \sigma \in \text{Aut}(G) \). It follows from Lemma 2.6 that \( a_0 \) is conjugate to \( b \), which is a contradiction. Hence \( |M_2| \leq 4 \), and so either \( M_2 = 1 \), or \( M_2 \cong \mathbb{Z}_2 \) and \( o(z) = 3 \). In particular, if \( M = M_2 \) then \( G \cong A_4 \).

Assume that there exists an odd prime \( p \mid |M| \), and assume that the exponent of \( M \) equals \( p^r \) for some \( r \geq 1 \), and take an element \( a \in M_2 \) of order \( p^r \). Let \( b = a^* \). We are going to prove that \( b \in \langle a \rangle \). Suppose that \( \langle a \rangle \cap \langle b \rangle = 1 \). Then there exists \( \sigma \in \text{Aut}(\langle a, b \rangle) \) such that \( a^\sigma = a^{-1} \) and \( b^\sigma = b \). Set \( S_3 = \{ a, b, ab \} \), \( T_3 = \{ a^{-1}, b, a^{-1} b \} \), \( S_4 = \{ a, b, ab, (ab)^{-1} \} \) and \( T_4 = \{ a^{-1}, b, a^{-1} b, (a^{-1} b)^{-1} \} \). Then \( S_m \cong T_m \), and so \( \text{Cay}(\langle a, b \rangle, S_m) \cong \text{Cay}(\langle a, b \rangle, T_m) \).
Since $\langle a, b, c \rangle \cong S_4$, and $(G, S_m) \cong \text{Cay}(G, T_m)$, and since $G$ has the m-DCI property, there is $x \in \text{Aut}(G)$ such that $S_m^x = T_m$. It is easy to show that $\langle a, b, c \rangle = \langle a^{-1}, b \rangle$. Since $a^2 = b$, it follows from Lemma 2.6 that $(a^{-1})^s = b$ for some integer $k \in \{1, 2, \ldots, o(z) - 1\}$. Thus $z^{-k}a^{-1}z^k = b = z^{-1}az$, so $z^{-1+k}a^{-1}z^{-k-1} = a$, which is a contradiction since both $a$ and $z$ are of odd order. Thus $\langle a \rangle \cap \langle b \rangle \neq \{1\}$, and so $\langle a, b \rangle = \langle a \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $c \in M_4$ such that $o(c) = p'$ for some $s < r$. Suppose that $b \not\in \langle a \rangle$. Then $o(c) = p^r > 1$, and $b = a c^i$ for some integers $i, j$ coprime to $p$ as $\langle a, b \rangle = \langle a, c \rangle$ with $1 \leq i \leq p - 1$. Let $S_1 = \{a, c, ac\}$, $T_1 = \{a, c^{-1}, ac^{-1}\}$, $S_a = \{a, c, ac, (ac)^{-1}\}$ and $T_a = \{a, c^{-1}, ac, (ac^{-1})^{-1}\}$. It follows that $\text{Cay}(G, S_a) \cong \text{Cay}(G, T_a)$, and as $G$ has the m-DCI property, we have $S_m^x = T_m$ for some $x \in \text{Aut}(G)$. Thus $(a, c)^s = (a, c^{-1})$. Now $z^i = a^{ke}$ for some $d \in M$ and some integer $k$ with $1 \leq k \leq o(z) - 1$. Since $a^2 = b$,

$$z^{-k}a^{ke} = (z^{-1})^s a^ke = (z^{-1}az)^s = b^s = (a^e)^s = a^{ke}.$$  

Thus $z^{-k}a^{ke} = a^{pe} = b^p = z^{-1}a^p$, that is, $z^{-(k-1)}a^p z^{k-1} = a^p$. Since $G$ is a Frobenius group, $z^{k-1} = 1$ and so $k = 1$. Thus $a^{pe} = b = z^{-1}az = z^{-k}a^{ke} = a^{ke}$, and so $b^p = 1$, which is a contradiction. Therefore, $b \not\in \langle a \rangle$ and thus $z$ normalizes every cyclic subgroup of $M$ of odd prime-power order. By our assumption, $|M|$ is even. We have shown that a Sylow 2-subgroup $M$ of $G$ is isomorphic to $\mathbb{Z}_2^2$. Let $g_1$ be an involution of $M_2$, and let $g_2 = g_1^2$. Let $a$ be an element of $M$ of order $p$. Then $a^p = a^k$ for some integer $k$. Set $u := g_1a, v := g_2a^k$, and $y := g_2a$. Let $S_3 = \{u, v, w, z\}$, $T_3 = \{x, y, x^2y\}$, $S_4 = \{u, v, w, (uw)^{-1}\}$ and $T_4 = \{x, y, xy, (xy)^{-1}\}$. It is easy to see that $\langle S_3 \rangle = \langle T_3 \rangle = \langle u, v \rangle = \langle x, y \rangle$ and there exists $\sigma \in \text{Aut}(G)$ such that $u^\sigma = y$ and $v^\sigma = x$. Thus $S_m^x = T_m$, and so Cay$(\langle S_4 \rangle, S_4) \cong \text{Cay}(G, T_m)$ and Cay$(G, S_m) \cong \text{Cay}(G, T_m)$. Since $G$ has the m-DCI property, there exists $x \in \text{Aut}(G)$ such that $S_m^x = T_m$. It follows that $\{u, v\}^x = \{x, y\}$. Since $u^v = v$, it follows from Lemma 2.6 that $(g_1a^{ke})^v = y = g_2a$ for some integer $k \in \{1, 2, \ldots, o(z) - 1\}$. Thus $z^{-1}g_1z = g_2$ and $z^{-1}a^{ke} = a$. Therefore, $z^{-1}g_1z = g_2 = z^{-1}g_1z$ and $z^{-1}a^{ke}z^{-1} = z^{-1}(z^{-1}a^{ke}z) = z^{-1}az = a^k$, so $z^{-1}$ centralizes $g_1$ and $z^{ke}$ centralizes $a^k$. Since $1 \leq i \leq o(z) - 1$ and $o(z) = 3$ or $5$, at least one of $i$ and $i + 1$ is nonzero. This is a contradiction since $G$ is a Frobenius group, completing the proof of the proposition.  

4.2. Proof of Theorem 1.5

Now we are ready to complete the proof of Theorem 1.5:

By Theorem 1.3, $E(M, S)$ has the 4-DCI property but does not have the k-DCI property for $k = 1, 2, 3$. 

Conversely, let \( m \in \{3, 4\} \), and suppose that \( G \) is a finite group which has the \( m \)-DCI property but does not have the \( k \)-DCI property for any \( k \) with \( 1 \leq k < m \). Then in particular \( G \) does not have the 1-DCI property. Thus there exists a pair of elements \( a, b \) of the same order which are not fused. By Lemma 2.5, \( o(a) \leq m + 1 \). So either \( o(a) \leq 4 \), or \( o(a) = 5 \) and \( m = 4 \). For a prime \( p \mid |G| \), let \( G_p \) be a Sylow \( p \)-subgroup of \( G \). Since now \( o(a) \) is a prime-power, by the Sylow’s Theorem, we may assume that \( a, b \) lie in the same Sylow subgroup, say \( a, b \in G_p \) for the prime \( p \) dividing \( o(a) \) (so \( p = 2, 3, \) or \( 5 \)).

**Case 1.** Assume that \( o(a) = 2 \). Then \( p = 2 \) and \( G_2 \) contains two involutions \( a, b \) which are not fused. Hence \( G_2 \) is noncyclic elementary abelian group (see Lemma 2.8), and \( N_2(G_2) \) has at least two orbits on \( G_2^* \).

Suppose that each \( N_2(G_2) \)-orbit on \( G_2^* \) has size 1. Then \( N_2(G_2) = C_2(G_2) \). By Theorem 2.4, \( G \) is 2-nilpotent, that is, \( G = G_2 \times G_2 \) where \( G_2 \) is a Hall \( 2^r \)-subgroup of \( G \). In particular, \( G \) is soluble. Let \( N \) be a minimal characteristic subgroup of \( G_2 \). Then \( N \trianglelefteq G \) and \( N \) is an elementary abelian \( q \)-group, \( q \) an odd prime. For any \( x \in N^* \), if \( x^e \in \langle x \rangle \) then \( x^e = x \) or \( x^{-e} \); if \( x^e = x^e \notin \langle x \rangle \) then \( x^e = x \) and so \( (xx')^e = xx = xx' \). Thus there always exists an element \( x \in N^* \) such that \( x^e = x \) where \( e = 1 \) or \( -1 \). Similarly, there exists \( y \in N^* \) such that \( y^e = y \) where \( e = 1 \) or \( -1 \). Let \( S_3 = \{x, x^{-1}, a\} \) and \( T_3 = \{y, y^{-1}, b\} \). If \( q > 3 \) then let \( S_q = \{x, x^{-1}, x^2, a\} \) and \( T_q = \{y, y^{-1}, y^2, b\} \); if \( q = 3 \) then let \( S_q = \langle x, a \rangle \setminus \{a\} \) and \( T_q = \langle y, b \rangle \setminus \{b\} \). It is straightforward to check that \( \text{Cay}(\langle S_m \rangle, S_m) \cong \text{Cay}(\langle T_m \rangle, T_m) \), and hence \( \text{Cay}(G, S_m) \cong \text{Cay}(G, T_m) \). Since \( G \) has the \( m \)-DCI property, \( S_m \) is fused to \( T_m \). It then follows that \( a \) is fused to \( b \), which is a contradiction.

Thus some \( N_2(G_2) \)-orbit \( O \) on \( G_2^* \) has size \( s > 1 \). Then \( s \geq 3 \), and as \( a \) and \( b \) are not fused, there exists \( y \in G_2^* \setminus O \). Let \( O = \{x_1, x_2, \ldots, x_s\} \). If there exist \( x_i, x_j, x_k \in O \) such that \( x_k = x_i x_j \), then let \( S_3 = \{x_i, x_j, x_k\} \), \( T_3 = \{x_i, y, x_k\} \), \( S_q = \{x_i, x_j, x_k, y\} \) and \( T_q = \{x_i, y, x_j, x_k\} \) if \( x_i x_j \notin O \) for any \( i \neq j \), then let \( S_q = \{x_i, x_j, x_k\} \), \( T_q = \{x_i, x_j, x_k, y\} \), \( S_q = \{x_i, x_2, x_3, x_4\} \) and \( T_q = \{x_i, x_2, x_3, x_4, y\} \). It is easy to see that \( \langle S_m \rangle \cong \langle T_m \rangle \) and that there exists an automorphism \( \alpha \) from \( \langle S_m \rangle \) to \( \langle T_m \rangle \) such that \( S_m \alpha = T_m \). Thus \( \text{Cay}(\langle S_m \rangle, S_m) \cong \text{Cay}(\langle T_m \rangle, T_m) \), and so \( \text{Cay}(G, S_m) \cong \text{Cay}(G, T_m) \). However, 3 elements of \( S_m \) lie in the same \( N_2(G_2) \)-orbit \( O \) but this is not true for \( T_m \). Since \( G_2 \) is abelian, by [22, p.143], two elements of \( G_2 \) are conjugate in \( G \) if and only if they are conjugate in \( N_2(G_2) \). It follows from Lemma 2.6 that there is no \( \sigma \in \text{Aut}(G) \) such that \( S_m \alpha = T_m \), which is a contradiction since \( G \) has the \( m \)-DCI property.

**Case 2.** Assume that \( o(a) = 3 \). Then by Lemma 2.7, \( G_3 \cong \mathbb{Z}_3 \). Thus since \( a, b \in G_3 \) and \( a \) is not fused to \( b \), we have that \( b = a^{-1} \) and \( N_2(G_3) = C_2(G_3) \). By Theorem 2.4, \( G \) is 3-nilpotent, and so we may write
$G = G_Y \cong G_3$. Moreover, it follows from Lemma 2.5 that $G_3$ centralizes no non-identity element of $G_Y$. Thus by [20, 10.5], $G$ is a Frobenius group with $G_3$ the Frobenius kernel and $G_2$ a Frobenius complement. By Proposition 4.1, $G = A_5$ or $E(G_Y, 3)$. But $A_5$ has the 1-DCI property, and by Theorem 1.2, $G$ has the 2-DCI property, which is a contradiction.

Case 3. Assume that $o(a) = 5$ so that $m = 4$. Then by Lemma 2.7, $G_5 \cong \mathbb{Z}_4$ and so $G_5 = \langle a \rangle = \langle b \rangle$. Since $|\text{Aut}(\langle a \rangle)| \cong \mathbb{Z}_4$, $|N_{G}(\langle a \rangle)\cap C_{G}(\langle a \rangle)|$ divides 4. If $|N_{G}(\langle a \rangle)\cap C_{G}(\langle a \rangle)| = 4$ then it follows that $a$ is conjugate to $b$, a contradiction. Suppose that $|N_{G}(\langle a \rangle)\cap C_{G}(\langle a \rangle)| = 2$. It follows that there exists $g \in N_{G}(\langle a \rangle)$ such that $a^g = a^{-1}$. Since $a$ is not fused to $b$, $b \neq a^{-1}$ and so $b = a^2$ or $a^3$. Let $S = \{b, a, a^{-1}, g\}$ and $T = \{a, a^2, a^{-2}, g\}$. It is easy to show that $\text{Cay}(S, S) \cong \text{Cay}(T, T)$, and thus $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. Since $G$ has the 4-DCI property, $S$ is fused to $T$ and so $a$ is fused to $b$, which is a contradiction. Thus $N_{G}(\langle a \rangle) = C_{G}(\langle a \rangle)$. By Theorem 2.4, $G$ is 5-nilpotent, and so we may write $G = G_2 \times \langle a \rangle$ where $G_2$ is a Hall $5'$-subgroup of $G$. Moreover, it follows from Lemma 2.5 that $G_3$ centralizes no non-identity elements of $G$. Thus by [20, 10.5], $G$ is a Frobenius group with $G_2$ the Frobenius kernel.

By Proposition 4.1, $G = E(G_2, 5)$.

Case 4. Assume that $o(a) = 4$. By Lemma 2.8, $G_2 \cong Q_8$ or $\mathbb{Z}_4$. Suppose that $G_2 \cong Q_8$. Since $a, b \in G_2$ and $a$ is conjugate to $a^{-1}$, $b \neq a^{-1}$. Let $S_3 = \{a, a^2, a^{-1}\}$, $T_3 = \{b, b^2, b^{-1}\}$, $S_4 = G_2\langle a \rangle$ and $T_4 = G_2\langle b \rangle$. It is easy to see that $\text{Cay}(G_2, S_m) \cong \text{Cay}(G_2, T_m)$, and thus $\text{Cay}(G, S_m) \cong \text{Cay}(G, T_m)$. Since $G$ has the $m$-DCI property, $S$ is fused to $T$ and so $a$ is fused to $b$, which is a contradiction.

Thus $G_2 \cong \mathbb{Z}_4$. Since $a, b \in G_2$ and $a$ is not fused to $b$, we have that $b = a^{-1}$. It follows that $N_{G}(G_2) = C_{G}(G_2)$. By Theorem 2.4, $G$ is 2-nilpotent, and so we may write $G = G_2 \times \langle a \rangle$. By Cases 2 and 3 of this proof, any two elements of $G$ of order 3 or 5 are fused. By Lemma 2.5, any two elements of $G$ of order $k$ are fused if $k \geq m + 2$. Since $G_2$ is of odd order, any element of $G_2$ has order 3, 5 or $k \geq m + 2$. Thus all elements of $G_2$ of the same order are fused. By [13, Corollary 1.3], $G_2 = M \rtimes L$, where $(|M|, |L|) = 1$, $M$ is abelian and $L$ is cyclic. Thus $G = G_2 \times \langle a \rangle = M \rtimes (L \times \langle a \rangle)$. Note that, by Lemma 2.5, $a$ centralizes no non-identity elements of $G_2$.

Suppose that $a^2$ centralizes some $x \in M \setminus \{1\}$ where $o(x)$ is a prime. If $y := x^{a} \neq x^{-1}$, then, since $y^{a} = x^{a^2} = x$ and $M$ is abelian, we have $(xy)^{a} = xy$, which is a contradiction. Thus $x^{a} = x^{-1}$. Let $S_3 = \{x, x^{-1}, a\}$, $T_3 = \{x, x^{-1}, a^{-1}\}$, $S_4 = \{x, x^{-1}, a, a^2\}$ and $T_4 = \{x, x^{-1}, a^{-1}, a^2\}$. Then it is easy to show that $\text{Cay}(\langle a, x \rangle, S_m) \cong \text{Cay}(\langle a, x \rangle, T_m)$, and so $\text{Cay}(G, S_m) \cong \text{Cay}(G, T_m)$. Since $G$ has the $m$-DCI property, $S_m$ is fused to $T_m$, and it then follows that $a$ is fused to $a^{-1}$, which is a contradiction.
Suppose now that $a^2$ centralizes some $x \in L$. Then $a$ normalizes $\langle x \rangle$, and similarly this leads to a contradiction. Thus $a^2$ centralizes no non-identity element of $G_2$. By [20, 10.5], $G_2$ is nilpotent, and it follows that $G$ is a Frobenius group with $G_2$ the Frobenius kernel and $\langle a \rangle$ a Frobenius complement. By Proposition 4.1, $G$ does not have the $m$-DCI property.

REFERENCES

10. C. H. Li, Finite abelian groups with the $m$-DCI property, Ars Combin. 51 (1999), 77–88.
13. C. H. Li and C. E. Praeger, On finite groups in which any two elements of the same order are fused or inverse-fused, Comm. Algebra 25 (1997), 3081–3118.
23. J. P. Zhang, On finite groups all of whose elements of the same order are conjugate in their automorphism groups, J. Algebra 153 (1992), 22–36.