Idempotency of linear combinations of two idempotent matrices

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Abstract

A complete solution is established to the problem of characterizing all situations, where a linear combination of two different idempotent matrices \( P_1 \) and \( P_2 \) is also an idempotent matrix. Including naturally three such situations known in the literature, viz., if the combination is either the sum \( P_1 + P_2 \) or one of the differences \( P_1 - P_2, P_2 - P_1 \) (and appropriate additional conditions are fulfilled), the solution asserts that in the particular case where \( P_1 \) and \( P_2 \) are complex matrices such that \( P_1 - P_2 \) is Hermitian, these three situations exhaust the list of all possibilities and that this list extends when the above assumption on \( P_1 \) and \( P_2 \) is violated. A statistical interpretation of the idempotency problem considered in this note is also pointed out. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

It is assumed throughout that \( c_1, c_2 \) are any nonzero elements of a field \( \mathcal{F} \) and \( P_1, P_2 \) are two different nonzero idempotent matrices over \( \mathcal{F} \), i.e., \( P_1 = P_1^2, P_2 = P_2^2 \), and \( P_1 \neq P_2 \). The symbols \( \gamma_1, \gamma_2 \) and \( Q_1, Q_2 \) are used instead of \( c_1, c_2 \) and \( P_1, P_2 \) when considerations are concerned with complex scalars and matrices.

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The purpose of this paper is to establish a complete solution to the problem of characterizing all situations, where the operation of combining linearly $P_1$ and $P_2$ preserves the idempotency property. Three such situations are known in the literature, viz., if the combination is either the sum $P_1 + P_2$ or one of the differences between $P_1$ and $P_2$, and appropriate additional conditions are fulfilled; cf. Theorems 5.1.2 and 5.1.3 in [4]. The solution obtained asserts that these three situations exhaust the list of all possibilities when attention is restricted to complex idempotent matrices $Q_1$ and $Q_2$ such that $Q_1 - Q_2$ is Hermitian and that this list extends when the problem is considered in the general case.

The idempotency problem considered in this note admits a statistical interpretation due to the fact that if $A$ is an $n \times n$ real symmetric matrix and $x$ is an $n \times 1$ real random vector having the multivariate normal distribution $N_n(0, I)$, where $I$ stands for the identity matrix, then a necessary and sufficient condition for the quadratic form $x'Ax$ to be distributed as a chi-square variable is that $A = A^2$; cf. Theorem 5.1.1 in [3] or Lemma 9.1.2 in [4].

2. Results

As already pointed out, the main result deals with the idempotency of linear combinations of any idempotent matrices.

**Theorem.** Given two different nonzero idempotent matrices $P_1$ and $P_2$, let $P$ be their linear combination of the form

$$P = c_1P_1 + c_2P_2$$

with nonzero scalars $c_1$ and $c_2$. Then there are exactly four situations, where $P$ is an idempotent matrix:

(a) $P_1P_2 = P_2P_1$ holds along with either one of the following sets of conditions:

(i) $c_1 = 1$, $c_2 = 1$, $P_1P_2 = 0$;
(ii) $c_1 = 1$, $c_2 = -1$, $P_1P_2 = P_2$;
(iii) $c_1 = -1$, $c_2 = 1$, $P_1P_2 = P_1$;

(b) $P_1P_2 \neq P_2P_1$ holds along with the conditions $c_1 \in \mathbb{R} \setminus \{0, 1\}$, $c_2 = 1 - c_1$, $(P_1 - P_2)^2 = 0$.

**Proof.** Direct calculations show that $P$ of form (1) is idempotent if and only if

$$c_1(1 - c_1)P_1 - c_1c_2P_1P_2 - c_1c_2P_2P_1 + c_2(1 - c_2)P_2 = 0.$$  

Hence the sufficiency part of the theorem follows easily. The proof of necessity is split into two complementary cases (a) and (b) specified by $P_1P_2 = P_2P_1$ and $P_1P_2 \neq P_2P_1$, respectively.

In the former case, premultiplying equality (2) first by $P_1$ and then by $P_2$ leads, respectively, to
\[c_1(1 - c_1)P_1 + c_2(1 - 2c_1 - c_2)P_1P_2 = 0, \quad (3)\]
\[c_2(1 - c_2)P_2 + c_1(1 - c_1 - 2c_2)P_1P_2 = 0, \quad (4)\]

and postmultiplying \((3)\) by \(P_2\) or premultiplying \((4)\) by \(P_1\) yields
\[\left[c_1 + c_2 - (c_1 + c_2)^2\right]P_1P_2 = 0. \quad (5)\]

From \((3)\) and \((4)\) it is seen that if \(P_1P_2 = 0\), then in view of \(P_1 \neq 0, P_2 \neq 0\) and \(c_1 \neq 0, c_2 \neq 0\) both \(c_1\) and \(c_2\) must be equal to 1, which is the situation \((i)\). On the other hand, if \(P_1P_2 \neq 0\), then \((5)\) implies that either
\[c_1 + c_2 = 1 \quad (6)\]
or
\[c_1 + c_2 = 0. \quad (7)\]

Then equalities \((3)\) and \((4)\) simplify to
\[c_1c_2(P_1 - P_1P_2) = 0 \quad \text{and} \quad c_1c_2(P_2 - P_1P_2) = 0 \quad (8)\]

when substituting \((6)\) and to
\[c_1(1 - c_1)(P_1 - P_1P_2) = 0 \quad \text{and} \quad c_1(1 + c_1)(P_2 - P_1P_2) = 0 \quad (9)\]

when substituting \((7)\). In view of the assumption that \(P_1 \neq P_2\), the equalities \(P_1P_2 = P_1\) and \(P_1P_2 = P_2\) cannot hold simultaneously. Consequently, since \(c_1 \neq 0\) and \(c_2 \neq 0\), it follows that the pair of conditions \((8)\) can never be fulfilled, whereas the pair of conditions \((9)\) is fulfilled if and only if either \(c_1 = 1\) (implying \(c_2 = -1\)) and \(P_1P_2 = P_2\), which is the situation \((ii)\), or \(c_1 = -1\) (implying \(c_2 = 1\)) and \(P_1P_2 = P_1\), which is the situation \((iii)\).

We now return to the necessary and sufficient condition \((2)\). Premultiplying and postmultiplying it by \(P_1\) leads, respectively, to
\[c_1(1 - c_1)P_1 + c_2(1 - c_1 - c_2)P_1P_2 - c_1c_2P_1P_2P_1 = 0, \quad (10)\]
\[c_1(1 - c_1)P_1 + c_2(1 - c_1 - c_2)P_2P_1 - c_1c_2P_1P_2P_1 = 0. \quad (11)\]

Hence
\[c_2(1 - c_1 - c_2)(P_1P_2 - P_2P_1) = 0, \quad (10)\]

and since in the case \((b)\) the condition \(c_2 \neq 0\) is accompanied by \(P_1P_2 \neq P_2P_1\), it is clear that \((10)\) is equivalent to \((6)\). Substituting \((6)\) to \((2)\) simplifies the latter to the equality
\[c_1c_2(P_1 - P_1P_2 - P_2P_1 + P_2) = 0, \quad (12)\]

which in view of \(c_1c_2 \neq 0\) yields the last condition in \((b)\), thus concluding the proof. \(\square\)

Other functions of idempotent matrices \(P_1\) and \(P_2\) studied (quite intensively) in the literature are the products \(P_1P_2\) and \(P_2P_1\); cf. Ref. [1] containing investigations.
of their idempotency in the case where $P_1$ and $P_2$ are Hermitian and Refs. [2,5] containing recent investigations of this type in the general case. In this context it seems interesting to notice the following relationship.

**Corollary 1.** Under the assumptions of the theorem, a necessary condition for $P = c_1P_1 + c_2P_2$ to be an idempotent matrix is that each of the products $P_1P_2$ and $P_2P_1$ is an idempotent matrix.

**Proof.** In case (a) of the theorem the assertion of this corollary is obvious. In case (b) we have

$$P_1 - P_1P_2 - P_2P_1 + P_2 = 0,$$

and premultiplying (11) by $P_1$ leads to the equality

$$P_1 = P_1P_2P_1,$$

which implies the idempotency of both $P_1P_2$ and $P_2P_1$. □

The second corollary refers to a special case where $P_1$ and $P_2$ are complex matrices such that their difference is a Hermitian matrix. This requirement obviously covers the situation where both $P_1$ and $P_2$ are Hermitian.

**Corollary 2.** Given two different nonzero complex idempotent matrices $Q_1$ and $Q_2$ such that the difference $Q_1 - Q_2$ is Hermitian, let $Q$ be their linear combination of the form $Q = \gamma_1Q_1 + \gamma_2Q_2$ with nonzero complex numbers $\gamma_1$ and $\gamma_2$. Then there are exactly three situations, where $Q$ is also idempotent:

(i) $Q = Q_1 + Q_2$ and $Q_1Q_2 = 0 = Q_2Q_1$,  
(ii) $Q = Q_1 - Q_2$ and $Q_1Q_2 = Q_2Q_1$,  
(iii) $Q = -Q_1 + Q_2$ and $Q_1Q_2 = Q_1 = Q_2Q_1$.

**Proof.** In view of the theorem, it suffices to show that the situation described in its part (b) is void when $Q_1 - Q_2$ is a Hermitian matrix. But this is indeed the fact, for the equality $(Q_1 - Q_2)^2 = 0$ can then be reexpressed as

$$(Q_1 - Q_2)(Q_1 - Q_2)^* = 0,$$

which is impossible except merely for the trivial case where $Q_1 = Q_2$. □

According to the statistical interpretation of the idempotency problem pointed out at the end of Section 1, Corollary 2 asserts that when $q$ is a linear combination of two quadratic forms $q_1 = x'A_1x$ and $q_2 = x'A_2x$, each following a $\chi^2$ distribution, then $q$ is also distributed as a $\chi^2$ variable if and only if either it is the sum of $q_1$ and $q_2$, and the distributions of $q_1$ and $q_2$ are independent, or it is one of the differences $q_i - q_j$, $i, j = 1, 2; i \neq j$, and the distributions of $q$ and $q_j$ are independent.
In conclusion, we first show that the fourth possibility indicated in the theorem is a real extension of the list of three possibilities common for the theorem and Corollary 2. A simple example is provided by the matrices

\[
P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Since they are both idempotent and, in view of the equalities \(P_1 P_2 = P_2\) and \(P_2 P_1 = P_1\), satisfy \((P_1 - P_2)^2 = P_1 - P_1 P_2 - P_2 P_1 + P_2 = 0\) along with \(P_1 P_2 \neq P_2 P_1\), part (b) of the theorem asserts that, in addition to \(P_1\) and \(P_2\), also every matrix of the form

\[
P = c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (1 - c) \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 - c & 1 \end{pmatrix},
\]

with any \(c\) different from 0 and 1, is idempotent. The second example, in which

\[
Q_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}
\]

shows that the assumption in Corollary 2 that the difference \(Q_1 - Q_2\) is a Hermitian matrix is essentially weaker than the requirement that both \(Q_1\) and \(Q_2\) are Hermitian matrices. Consequently, the conditions given therein must refer to both products \(Q_1 Q_2\) and \(Q_2 Q_1\).

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