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# Edge-antimagic graphs

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**Abstract**

For a graph $G=(V,E)$, a bijection $g$ from $V(G)\cup E(G)$ into $\{1,2,\dots,|V(G)|+|E(G)|\}$ is called $(a,d)$-edge-antimagic total labeling of $G$ if the edge-weights $w(xy)=g(x)+g(y)+g(xy)$, $xy\in E(G)$, form an arithmetic progression starting from $a$ and having common difference $d$. An $(a,d)$-edge-antimagic total labeling is called super $(a,d)$-edge-antimagic total if $g(V(G))=\{1,2,\dots,|V(G)|\}$. We study super $(a,d)$-edge-antimagic properties of certain classes of graphs, including friendship graphs, wheels, fans, complete graphs and complete bipartite graphs.


*Keywords:* Edge-antimagic labeling; Edge-weight

## 1. Introduction

We consider finite undirected graphs without loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph $G$, respectively. Let $|V(G)|=v$ and $|E(G)|=e$ be the number of vertices and the number of edges of $G$. General references for graph-theoretic notions are [9,10].

By a *labeling* we mean a one-to-one mapping that carries a set of graph elements into a set of numbers (usually integers), called labels. In this paper we deal with labelings with domain either the set of all vertices, or the set of all edges, or the set of all vertices and edges, respectively. We call these labelings a vertex labeling, or an edge labeling, or a total labeling, respectively.

The *edge-weight* of an edge $xy$ under a labeling is the sum of labels (if present) carried by that edge and the vertices $x, y$ incident with $xy$.

By an $(a,d)$-*edge-antimagic vertex labeling* we mean a one-to-one mapping from $V(G)$ into $\{1,2,\dots,v\}$ such that the set of edge-weights of all edges in $G$ is $\{a, a+d,\dots,a+(e-1)d\}$, where $a>0$ and $d\geqslant 0$ are two fixed integers.

An $(a,d)$-*edge-antimagic total labeling* is defined as a one-to-one mapping from $V(G)\cup E(G)$ into the set $\{1,2,\dots,v+e\}$ so that the set of edge-weights of all edges in $G$ is equal to $\{a,a+d,\dots,a+(e-1)d\}$, for two integers $a>0$ and $d\geqslant 0$.

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An $(a, d)$-edge-antimagic total labeling $g$ is called *super* $(a, d)$-edge-antimagic total if $g(V(G)) = \{1, 2, \dots, v\}$ and $g(E(G)) = \{v+1, v+2, \dots, v+e\}$. A graph $G$ is called $(a, d)$-edge-antimagic total or super $(a, d)$-edge-antimagic total if there exists an $(a, d)$-edge-antimagic total or a super $(a, d)$-edge-antimagic total labeling of $G$.

The $(a, 0)$-edge-antimagic total labelings are usually called *edge-magic* in the literature (see [3,4,6,7]).

Definitions of $(a, d)$-edge-antimagic total labeling and super $(a, d)$-edge-antimagic total labeling were introduced by Simanjuntak et al. [8]. These labelings are natural extensions of the notions of edge-magic labelings (see [6,7], where edge-magic labelings are called *magic valuations*) and super edge-magic labelings (introduced by Enomoto et al. [3]).

Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [2], Hartsfield and Ringel [5].

In [8], Simanjuntak et al. studied the properties of $(a, d)$-edge-antimagic vertex labeling and $(a, d)$-edge-antimagic total labeling and gave constructions of $(a, d)$-edge-antimagic total labelings for cycles and paths. Bača et al. [1] presented some relationships between $(a, d)$-edge-antimagic vertex labeling, $(a, d)$-edge-antimagic total labeling and other labelings, namely, edge-magic vertex labeling and edge-magic total labeling.

In this paper we study super $(a, d)$-edge-antimagic properties of certain classes of graphs, including friendship graphs, wheels, fans, complete graphs and complete bipartite graphs.

## 2. Friendship graphs

The *friendship graph* $F_n$ is a set of $n$ triangles having a common center vertex, and otherwise disjoint. Let $c$ denote the center vertex. For the $i$th triangle, let $x_i$ and $y_i$ denote the other two vertices.

**Theorem 1.** *If the friendship graph* $F_n$, $n \geqslant 1$, *is super* $(a, d)$*-edge-antimagic total, then* $d < 3$.

**Proof.** Assume that there exists a bijection $g$: $V(F_n) \cup E(F_n) \to \{1, 2, \dots, 5n+1\}$ which is super $(a, d)$-edge-antimagic total and $W = \{w(uv)\colon w(uv) = g(u) + g(v) + g(uv), uv \in E(F_n)\} = \{a, a+d, \dots, a+(3n-1)d\}$ is the set of edge-weights. It is easy to see that the minimum possible edge-weight in super $(a, d)$-edge-antimagic total labeling is at least $2n+5$. On the other hand, the maximum edge-weight is no more than $2n + (2n+1) + (5n+1)$.

Thus we have

$$a + (3n-1)d \leqslant 9n + 2$$

and

$$d \leqslant \frac{9n+2-a}{3n-1} \leqslant \frac{7n-3}{3n-1} < 3. \qquad \square$$

The following result is interesting because it characterizes $(a, 1)$-edge-antimagicness of friendship graphs.

**Lemma 1.** *The friendship graph* $F_n$ *has* $(a, 1)$*-edge-antimagic vertex labeling if and only if* $n \in \{1, 3, 4, 5, 7\}$.

**Proof.** First, we show that $F_n$ has $(a, 1)$-edge-antimagic vertex labeling for $n \in \{1, 3, 4, 5, 7\}$.

Trivially, $F_1$ has $(a, 1)$-edge-antimagic vertex labeling $g_1$ with $g_1(c) = 1$, $g_1(x_1) = 2$, $g_1(y_1) = 3$.

In the case $n = 3$, label $g_2(c) = 4$, $g_2(x_1) = 1$, $g_2(y_1) = 3$, $g_2(x_2) = 2$, $g_2(y_2) = 6$, $g_2(x_3) = 5$, $g_2(y_3) = 7$.

If $n = 4$ then label $g_3(c) = 6$, $g_3(x_1) = 1$, $g_3(y_1) = 5$, $g_3(x_2) = 2$, $g_3(y_2) = 3$, $g_3(x_3) = 4$, $g_3(y_3) = 8$, $g_3(x_4) = 7$, $g_3(y_4) = 9$.

If $n = 5$ then construct the vertex labeling $g_4$ in the following way: $g_4(c) = g_3(c)$, $g_4(x_i) = g_3(x_i)$ and $g_4(y_i) = g_3(y_i)$ for $i = 1, 2, 3$, and $g_4(x_4) = 7$, $g_4(y_4) = 11$, $g_4(x_5) = 9$, $g_4(y_5) = 10$.

For $n = 7$, put $g_5(c) = 8$, $g_5(x_1) = 1$, $g_5(y_1) = 5$, $g_5(x_2) = 3$, $g_5(y_2) = 4$, $g_5(x_3) = 2$, $g_5(y_3) = 6$, $g_5(x_4) = 7$, $g_5(y_4) = 9$, $g_5(x_5) = 10$, $g_5(y_5) = 14$, $g_5(x_6) = 12$, $g_5(y_6) = 13$, $g_5(x_7) = 11$ and $g_5(y_7) = 15$.

It is a matter for routine checking to see that the vertex labelings $g_i$, $1 \leqslant i \leqslant 5$, are $(a, 1)$-edge-antimagic.

For the converse, assume that there exists a one-to-one mapping $g$ from $V(F_n)$ into $\{1, 2, \dots, 2n+1\}$ such that the set of edge-weights of all edges in $F_n$ is $W = \{a, a+1, \dots, a+3n-1\}$. Let $g(c) = k$, $1 \leqslant k \leqslant 2n+1$, and $g(V(F_n)) = S_1 \cup S_2 \cup \{k\}$, where $S_1 = \{1, 2, \dots, k-2, k-1\}$ and $S_2 = \{k+1, k+2, \dots, 2n, 2n+1\}$ be the sets of consecutive integers.

Denote $W_1 = \{w(cx_i)\colon 1 \leqslant i \leqslant n\} \cup \{w(cy_i)\colon 1 \leqslant i \leqslant n\} = \{k+1, k+2, \dots, 2k-2, 2k-1, 2k+1, 2k+2, \dots, k+2n+1\}$, $W_2 = \{a, a+1, \dots, k-1, k\}$ and $W_3 = \{k+2n+2, k+2n+3, \dots, a+3n-2, a+3n-1\}$ as the sets of edge-weights where the edge-weights of $W_2$ are obtained as sums of two distinct elements in the set $S_1 - \{s_1\}$ and the edge-weights of $W_3$ are obtained as sums of two distinct elements in the set $S_2 - \{s_2\}$. There exists an edge $x_i y_i$ such that its edge-weight is $w(x_i y_i) = 2k = s_1 + s_2$, where $s_1 \in S_1$, $s_2 \in S_2$.

We can see that $k-2$ distinct elements in the set $S_1 - \{s_1\}$ lead to $(k-2)/2$ pairs (as edge-weights), which implies that $k$ must be even and $|W_2| = (k-2)/2$.

The sum of all the values in the set $S_1 - \{s_1\}$ is equal to the sum of the edge-weights in $W_2$.

Thus,

$$\frac{k(k-1)}{2} - s_1 = \frac{k-2}{2}a + \frac{k-2}{4}\left(\frac{k-2}{2} - 1\right). \tag{1}$$

Since $s_1 \in S_1$, it follows that $1 \leqslant s_1 \leqslant k-1$ and from (1) we have

$$\frac{3k}{4} \leqslant a \leqslant \frac{3k+8}{4}. \tag{2}$$

The value of the center of $F_n$ is used $2n$ times and the value of the other vertices of $F_n$ are used twice in the computation of the edge-weights. The sum of all the vertex labels used to calculate the edge-weights of $F_n$ is equal to

$$2\sum_{i=1}^{n} g(x_i) + 2\sum_{i=1}^{n} g(y_i) + 2ng(c) = 4n^2 + 6n + 2 + 2nk - 2k. \tag{3}$$

The sum of edge-weights in the set $W$ is

$$\sum_{i=1}^{n} w(cx_i) + \sum_{i=1}^{n} w(cy_i) + \sum_{i=1}^{n} w(x_i y_i) = 3na + \frac{9n^2 - 3n}{2}. \tag{4}$$

Thus, the following equation holds:

$$2\sum_{i=1}^{n} g(x_i) + 2\sum_{i=1}^{n} g(y_i) + 2ng(c) = \sum_{i=1}^{n} w(cx_i) + \sum_{i=1}^{n} w(cy_i) + \sum_{i=1}^{n} w(x_i y_i) \tag{5}$$

which is obviously equivalent to the equation

$$4nk - 4k - n^2 + 15n + 4 = 6na. \tag{6}$$

Since $k$ is even, from $2 \leqslant k \leqslant 2n$, $3k/4 \leqslant a \leqslant (3k+8)/4$ and Eq. (6) we obtain all possible integer values of parameter $n,\ k,\ a$ which are $(n, k, a) = (1, 2, 3), (3, 4, 4), (4, 2, 3), (4, 4, 4), (4, 6, 5), (4, 8, 6), (5, 6, 5), (7, 8, 6), (12, 4, 3)$.

We can see that if $n = 12$, $k = 4$ and $a = 3$, then the edge-weights 3, 4 and 5 can be expressed uniquely as sums of two distinct elements, namely, $3 = 1 + 2$, $4 = 1 + 3$ and $5 = 1 + 4$. However, it is impossible to arrange the value 1 on any vertex of $F_{12}$ to obtain edge-weights 3, 4 and 5, and this means that $F_{12}$ does not have a (3, 1)-edge-antimagic vertex labeling. □

With previous lemma in hand, we now present the following result.

**Theorem 2.** *For $n \in \{1, 3, 4, 5, 7\}$ the friendship graph $F_n$ has super $(a, 0)$-edge-antimagic total labeling and super $(a, 2)$-edge-antimagic total labeling.*

**Proof.** Label the vertices of $F_n$, $n \in \{1, 3, 4, 5, 7\}$ by the vertex labelings $g_i$, $1 \leqslant i \leqslant 5$. From the previous lemma it follows that each labeling $g_i$, $1 \leqslant i \leqslant 5$, successively assumes the values $1, 2, \dots, 2n+1$ and the edge-weights of all the edges of $F_n$ constitute an arithmetic sequence of difference 1.

If for each $F_n$, $n \in \{1, 3, 4, 5, 7\}$, we complete the edge labeling with values in the set $\{2n+2, 2n+3, \dots, 5n+1\}$, then the resulting total labeling can be

(i) super $(a, 0)$-edge-antimagic with the common edge-weight $a$, or
(ii) super $(a, 2)$-edge-antimagic, where edge-weights constitute an arithmetic sequence of difference 2. □

Now, define the vertex labeling $g_6$: $V(F_n) \to \{1, 2, \dots, 2n+1\}$ and the edge labeling $g_7$: $E(F_n) \to \{2n+2, 2n+3, \dots, 5n+1\}$ in the following way:

$$g_6(c) = n+1,$$

$$g_6(x_i) = i \quad \text{and} \quad g_6(y_i) = 2n+2-i \quad \text{for } 1 \leqslant i \leqslant n,$$

$$g_7(x_i c) = \begin{cases} 3n+3-\dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ 4n+3-\dfrac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

$$g_7(y_i c) = \begin{cases} 2n+1+\dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ 3n+2+\dfrac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

$$g_7(x_i y_i) = \begin{cases} 4n+2+i & \text{if } 1 \leqslant i \leqslant n-1, \\ \dfrac{7n+5}{2} & \text{if } i = n \text{ and } n \text{ is odd,} \\ \dfrac{5n+4}{2} & \text{if } i = n \text{ and } n \text{ is even.} \end{cases}$$

**Theorem 3.** *Every friendship graph $F_n$, $n \geqslant 1$, has super $(a, 1)$-edge-antimagic total labeling.*

**Proof.** Label the vertices and the edges of $F_n$ by $g_6$ and $g_7$, respectively. It is easy to verify that the set of edge-weights consists of the consecutive integers $\{4n+4, 4n+5, \dots, 7n+3\}$ and we arrive at the desired result. □

## 3. Fans

A *fan* $\mathbb{F}_n$, $n \geqslant 2$, is a graph obtained by joining all vertices of path $P_n$ to a further vertex called the center. Thus $\mathbb{F}_n$ contains $n+1$ vertices, say, $c, x_1, x_2, \dots, x_n$ and $2n-1$ edges, say, $cx_i$, $1 \leqslant i \leqslant n$, and $x_i x_{i+1}$, $1 \leqslant i \leqslant n-1$.

We shall find a least upper bound for a feasible value $d$ for super $(a, d)$-edge-antimagic total labeling of fans.

**Theorem 4.** *If $\mathbb{F}_n$, $n \geqslant 2$, is super $(a, d)$-edge-antimagic total, then $d < 3$.*

**Proof.** Assume that $\mathbb{F}_n$, $n \geqslant 2$, has a super $(a, d)$-edge-antimagic total labeling $f$: $V(\mathbb{F}_n) \cup E(\mathbb{F}_n) \to \{1, 2, \dots, 3n\}$ and $W = \{w(uv)\colon uv \in E(\mathbb{F}_n)\} = \{a, a+d, \dots, a+(2n-2)d\}$ is the set of edge-weights.

The sum of edge-weights in the set $W$ is

$$\sum_{uv \in E(\mathbb{F}_n)} w(uv) = (2n-1)a + d(2n-1)(n-1). \tag{7}$$

In the computation of the edge-weights of $\mathbb{F}_n$ the label of the center is used $n$ times, the labels of vertices $x_1$ and $x_n$ are used twice each and the labels of all the other vertices $x_i$, $2 \leqslant i \leqslant n-1$, are used three times each. The sum of all vertex labels and edge labels used to calculate the edge-weights is thus equal to

$$3\sum_{i=2}^{n-1} f(x_i) + nf(c) + 2(f(x_1) + f(x_n)) + \sum_{uv\in E(\mathbb{F}_n)} f(uv)$$
$$= \frac{1}{2}(11n^2 + 9n + 4) + (n-3)f(c) - f(x_1) - f(x_n). \tag{8}$$

From (7) and (8) we have the following equation:

$$\tfrac{1}{2}(11n^2 + 9n + 4) + (n-3)f(c) - f(x_1) - f(x_n) = (2n-1)a + d(2n-1)(n-1). \tag{9}$$

The minimum possible edge-weight is $a = 1 + 2 + n + 2$. The label of the center is $f(c) \leqslant n + 1$ and $f(x_1) + f(x_n) \geqslant 3$. Then we get the upper bound on the parameter $d$:

$$d = \frac{11n^2 + 9n + 4 + 2(n-3)f(c) - 2(f(x_1) + f(x_n)) - 2(2n-1)a}{2(2n-1)(n-1)},$$

$$d \leqslant \frac{9n^2 - 13n + 2}{2(2n-1)(n-1)} < 3. \qquad \square$$

**Lemma 2.** *The fan* $\mathbb{F}_n$ *has* (3, 1)-*edge-antimagic vertex labeling if and only if* $2 \leqslant n \leqslant 6$.

**Proof.** Suppose $f\colon V(\mathbb{F}_n) \to \{1, 2, \ldots, n+1\}$ is $(a, 1)$-edge-antimagic vertex labeling. It is easy to see that the minimum possible edge-weight in an $(a, 1)$-edge-antimagic vertex labeling is at least $1 + 2$. Consequently $a \geqslant 3$. On the other hand, the maximum edge-weight is no more than $n + (n + 1)$. Thus

$$a + (e-1) = a + 2n - 2 \leqslant 2n + 1$$

and $a \leqslant 3$. Therefore, $a = 3$.

Let us distinguish three cases.

*Case* 1: If $f(c) = 1$ then the edge-weights of edges $cx_i$, $1 \leqslant i \leqslant n$, are $3, 4, 5, \ldots, n+2$ and the edge-weights of edges $x_ix_{i+1}$, $1 \leqslant i \leqslant n-1$, are $n+3, n+4, \ldots, 2n+1$. In the computation of the edge-weights $n+3, n+4, \ldots, 2n+1$, the labels of vertices $x_1$ and $x_n$ are used once and the labels of the other vertices $x_i$, $2 \leqslant i \leqslant n-1$, are used twice. We have

$$2\sum_{i=1}^{n}(i+1) - f(x_1) - f(x_n) = (n+3) + (n+4) + \cdots + (2n+1),$$

consequently

$$f(x_1) + f(x_n) = \frac{4 + 5n - n^2}{2}. \tag{10}$$

Since $5 \leqslant f(x_1) + f(x_n) \leqslant 2n + 1$, then from (10) it follows that $n = 2$ or $n = 3$.

For $n = 2$, we label $f_1(c) = 1$, $f_1(x_1) = 2$, $f_1(x_2) = 3$ and for $n = 3$, we label $f_2(c) = 1$, $f_2(x_1) = 3$, $f_2(x_2) = 4$, $f_2(x_3) = 2$. We can see that the vertex labelings $f_1$ and $f_2$ are (3, 1)-edge-antimagic.

*Case* 2: If $f(c) = n+1$ then the edge-weights of edges $cx_i$, $1 \leqslant i \leqslant n$, are $n+2, n+3, \ldots, 2n+1$ and the edge-weights of edges $x_ix_{i+1}$, $1 \leqslant i \leqslant n-1$, are $3, 4, \ldots, n+1$. The edge-weights $3, 4, \ldots, n+1$ are obtained as sums of two distinct elements in the set $\{1, 2, \ldots, n\}$, where the labels of vertices $x_i$, $2 \leqslant i \leqslant n-1$, are used twice and the labels of vertices $x_1$ and $x_n$ are used once

$$2\sum_{i=1}^{n} i - f(x_1) - f(x_n) = 3 + 4 + \cdots + n + 1$$

and

$$f(x_1) + f(x_n) = \frac{n^2 - n + 4}{2}. \tag{11}$$

In this case the bounds for $f(x_1) + f(x_n)$ are

$$3 \leqslant f(x_1) + f(x_n) \leqslant 2n - 1. \tag{12}$$

From (11) and (12) we obtain that $n = 2$ or $n = 3$. It is simple to find a (3, 1)-edge-antimagic vertex labeling for $n = 2$ and 3, where $f(c) = n + 1$.

*Case* 3: If $f(c) = k$, $1 < k < n + 1$, then the labels of vertices $x_i$, $1 \leqslant i \leqslant n$, can be partitioned into two sets $S_1 = \{1, 2, \ldots, k-1\}$ and $S_2 = \{k+1, k+2, \ldots, n+1\}$. There exists an edge $x_i x_{i+1}$ such that its edge-weight is $w(x_i x_{i+1}) = 2k = s_1 + s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$.

Denote $W_1 = \{3, 4, \ldots, k\}$, $W_2 = \{n + k + 2, n + k + 3, \ldots, 2n + 1\}$ and $W_3 = \{w(cx_i)\colon 1 \leqslant i \leqslant n\} = \{k + 1, k + 2, \ldots, n + k + 1\} \backslash \{2k\}$ as the sets of edge-weights.

The sum of all values in the set $S_1$ (values $s_1$ and $f(x_1)$ are used once and the other values are used twice) is equal to the sum of edge-weights in the set $W_1$:

$$2\sum_{i=1}^{k-1} i - f(x_1) - s_1 = \sum_{j=3}^{k} j. \tag{13}$$

Since $f(x_1) + s_1 \leqslant 2k - 3$, then (13) implies that $k = 3$ or $k = 4$.

The sum of all values in the set $S_2$ (values $s_2$ and $f(x_n)$ are used once and the other values are used twice) is equal to the sum of the edge-weights in the set $W_2$:

$$2[(k+1) + (k+2) + \cdots + (n+1)] - f(x_n) - s_2 = (n+k+2) + (n+k+3) + \cdots + (2n+1).$$

Then we have

$$f(x_n) + s_2 = (n + k + 2)(n + 1 - k) - \frac{(3n + k + 3)(n - k)}{2}. \tag{14}$$

Evidently

$$2k + 3 \leqslant f(x_n) + s_2 \leqslant 2n + 1. \tag{15}$$

It is a matter for routine checking to see that

(i) if $k = 3$ then from (14) and (15) it follows that $n = 4$ or $n = 5$,
(ii) if $k = 4$ then (14) and (15) imply that $n = 5$ or $n = 6$.

Now, for $k = 3$, $n = 4$ and $n = 5$, we construct (3, 1)-edge-antimagic vertex labelings $f_3$ and $f_4$. For $k = 4$, $n = 5$ and $n = 6$ we define (3, 1)-edge-antimagic vertex labelings $f_5$ and $f_6$, $f_3(c) = f_4(c) = 3$, $f_3(x_1) = f_4(x_1) = 1$, $f_3(x_2) = f_4(x_2) = 2$, $f_3(x_3) = f_4(x_3) = 4$, $f_3(x_4) = 5$, $f_4(x_4) = 6$, $f_4(x_5) = 5$. $f_5(c) = f_6(c) = 4$, $f_5(x_1) = f_6(x_1) = 2$, $f_5(x_2) = f_6(x_2) = 1$, $f_5(x_3) = f_6(x_3) = 3$, $f_5(x_4) = f_6(x_4) = 5$, $f_5(x_5) = 6$, $f_6(x_5) = 7$, $f_6(x_6) = 6$. □

Figueroa-Centeno et al. [4] showed that fan $\mathbb{F}_n$ is super edge-magic (super $(a, 0)$-edge-antimagic in our terminology) if and only if $2 \leqslant n \leqslant 6$.

In light of Lemma 2, we get the next theorem.

**Theorem 5.** *The fan $\mathbb{F}_n$ is super $(a, d)$-edge-antimagic total if $2 \leqslant n \leqslant 6$ and $d \in \{0, 1, 2\}$.*

**Proof.** From the previous lemma it follows that the fan $\mathbb{F}_n$, $2 \leqslant n \leqslant 6$, has (3, 1)-edge-antimagic vertex labeling. Say, that $g\colon V(\mathbb{F}_n) \to \{1, 2, \ldots, n + 1\}$ is (3, 1)-edge-antimagic vertex labeling of $\mathbb{F}_n$, $2 \leqslant n \leqslant 6$, and $W_g = \{w_g(e_i) = 2 + i\colon 1 \leqslant i \leqslant 2n - 1\}$ is the set of edge-weights of edges $e_i \in E(\mathbb{F}_n)$.

Let $g_j$: $E(\mathbb{F}_n) \to \{n+2, n+3, \ldots, 3n\}$ be the edge labeling of $\mathbb{F}_n$ for $j \in \{1, 2, 3\}$ and $2 \leqslant n \leqslant 6$, where

$$g_1(e_i) = 3n + 1 - i \quad \text{if } 1 \leqslant i \leqslant 2n-1,$$

$$g_2(e_i) = \begin{cases} 2n + 2 - \dfrac{i+1}{2} & \text{if } i \text{ is odd}, \ 1 \leqslant i \leqslant 2n-1, \\ 3n + 1 - \dfrac{i}{2} & \text{if } i \text{ is even}, \ 2 \leqslant i \leqslant 2n-2, \end{cases}$$

$$g_3(e_i) = n + 1 + i \quad \text{if } 1 \leqslant i \leqslant 2n-1.$$

It can be seen that combining the vertex labeling $g$ and the edge labeling $g_j$, $j \in \{1, 2, 3\}$, gives a super $(a, j-1)$-edge-antimagic total labeling where $W_j = \{w_g(e_i) + g_j(e_i)\colon 1 \leqslant i \leqslant 2n-1\}$, $j \in \{1, 2, 3\}$, is the set of edge-weights. □

## 4. Wheels

A *wheel* $W_n$, $n \geqslant 3$, is a graph obtained by joining all vertices of cycle $C_n$ to a further vertex called the center. Thus $W_n$ contains $n+1$ vertices, say, $c, x_1, x_2, \ldots, x_n$ and $2n$ edges, say, $cx_i$, $1 \leqslant i \leqslant n$, $x_i x_{i+1}$, $1 \leqslant i \leqslant n-1$, and $x_n x_1$.

**Theorem 6.** *If wheel $W_n$, $n \geqslant 3$, is super $(a, d)$-edge-antimagic total then $d < 2$.*

**Proof.** Consider the extreme values of vertices and edges. For super $(a, d)$-edge-antimagic total labeling $h$: $V(W_n) \cup E(W_n) \to \{1, 2, \ldots, n+1, n+2, \ldots, 3n+1\}$, the maximum edge-weight is no more than $n + (n+1) + (3n+1)$. Thus,

$$a + (e-1)d = a + (2n-1)d \leqslant 5n + 2. \tag{16}$$

On the other hand, the minimum possible edge-weight is at least $1 + 2 + (n+2)$, i.e.,

$$a \geqslant n + 5. \tag{17}$$

From the inequalities (16) and (17), for wheel $W_n$ we have

$$d \leqslant \frac{5n + 2 - a}{2n-1} \leqslant \frac{4n-3}{2n-1} < 2. \qquad \square$$

Enomoto et al. [3] proved that a wheel graph $W_n$ is not super edge-magic (super $(a, 0)$-edge-antimagic total in our terminology).

Thus, we claim that:

**Theorem 7.** *The wheel $W_n$ has super $(a, d)$-edge-antimagic total labeling if and only if $d = 1$ and $n \not\equiv 1 \pmod 4$.*

**Proof.** Suppose that a bijection $h$: $V(W_n) \cup E(W_n) \to \{1, 2, \ldots, 3n+1\}$ is super $(a, 1)$-edge-antimagic total labeling. In the computation of the edge-weights of $W_n$ under the bijection $h$ the label of the center is used $n$-times, the label of each vertex $x_i$, $1 \leqslant i \leqslant n$, is used three times and the label of each edge is used once.

Thus,

$$\begin{aligned} &3\sum_{i=1}^{n} h(x_i) + nh(c) + \sum_{e \in E(W_n)} h(e) \\ &\quad = 3(1 + 2 + \cdots + n + 1) + (n-3)h(c) + (n + 2 + \cdots + 3n + 1). \end{aligned} \tag{18}$$

The sum of the edge-weights under the bijection $h$ is

$$\sum_{e \in E(W_n)} w(e) = 2na + n(2n-1). \tag{19}$$

From (18) and (19) we get

$$a = \frac{7n^2 + 17n + 6 + 2(n-3)h(c)}{4n}. \tag{20}$$

If $n \equiv 1 \pmod 4$ then from Eq. (20) it is easy to see that the value $a$ is not an integer.

Now, if $n \equiv 0 \pmod 4$ then construct the function $h_1$ of $W_n$ as follows:

$$h_1(c) = \frac{n+2}{2},$$

$$h_1(x_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ \dfrac{n+2+i}{2} & \text{if } i \text{ is even,} \end{cases}$$

$$h_1(cx_i) = \begin{cases} \dfrac{3n}{2} + 3 - \dfrac{i+3}{4} & \text{if } i \equiv 1 \pmod 4, \\ \dfrac{7n}{4} + 3 - \dfrac{i+2}{4} & \text{if } i \equiv 2 \pmod 4, \\ 2n + 3 - \dfrac{i+1}{4} & \text{if } i \equiv 3 \pmod 4, \\ \dfrac{5n}{4} + 2 - \dfrac{i}{4} & \text{if } i \equiv 0 \pmod 4, \end{cases}$$

$$h_1(x_n x_1) = \frac{5n}{4} + 2,$$

$$h_1(x_i x_{i+1}) = \begin{cases} \dfrac{5n}{2} + 3 - \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ 3n + 2 - \dfrac{i}{2} & \text{if } i \text{ is even, } i < n. \end{cases}$$

If $n \equiv 2 \pmod 4$, define the function $h_2$ of $W_n$ in the following way:

$$h_2(c) = h_1(c),$$

$$h_2(x_i) = h_1(x_i),$$

$$h_2(x_i x_{i+1}) = h_1(x_i x_{i+1}),$$

$$h_2(x_n x_1) = \frac{7n+10}{4},$$

$$h_2(cx_i) = \begin{cases} \dfrac{3n}{2} + 3 - \dfrac{i+3}{4} & \text{if } i \equiv 1 \pmod 4, \\ \dfrac{5n+2}{4} + 2 - \dfrac{i+2}{4} & \text{if } i \equiv 2 \pmod 4, \\ 2n + 3 - \dfrac{i+1}{4} & \text{if } i \equiv 3 \pmod 4, \\ \dfrac{7n+2}{4} + 2 - \dfrac{i}{4} & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

If $n \equiv 3 \pmod 4$, define the function $h_3$ of $W_n$ as follows:

$$h_3(c) = 1,$$

$$h_3(x_i) = \begin{cases} \dfrac{i+3}{2} & \text{if } i \text{ is odd,} \\ \dfrac{n+3+i}{2} & \text{if } i \text{ is even,} \end{cases}$$

$$h_3(cx_i) = \begin{cases} \dfrac{7n+11}{4} - \dfrac{i+3}{4} & \text{if } i \equiv 1 \pmod 4, \\ \dfrac{11(n+1)}{4} - \dfrac{i+2}{4} & \text{if } i \equiv 2 \pmod 4, \\ \dfrac{5(n+1)}{2} - \dfrac{i+1}{4} & \text{if } i \equiv 3 \pmod 4, \\ 3n+2-\dfrac{i}{4} & \text{if } i \equiv 0 \pmod 4, \end{cases}$$

$$h_3(x_n x_1) = \frac{3(n+1)}{2},$$

$$h_3(x_i x_{i+1}) = \begin{cases} \dfrac{9(n+1)}{4} - \dfrac{i+1}{2} & \text{if } i \text{ is odd, } i < n, \\ \dfrac{3(n+1)}{2} - \dfrac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

Label the vertices and edges of $W_n$ by $h_1$, $h_2$ and $h_3$. It is easy to verify that the labeling $h_i$, $i \in \{1, 2, 3\}$, uses each integer from the set $h_i(V(W_n)) \cup h_i(E(W_n)) = \{1, 2, \dots, n+1\} \cup \{n+2, n+3, \dots, 3n+1\}$ exactly once.

By direct computation we obtain that the sets of edge-weights $\{w_{h_1}(e): e \in E(W_n)\} = \{w_{h_2}(e): e \in E(W_n)\} = \{2n+4, 2n+5, \dots, 4n+3\}$ and $\{w_{h_3}(e): e \in E(W_n)\} = \{7(n+1)/4+3, 7(n+1)/4+4, \dots, 15(n+1)/4\}$ consist of consecutive integers.

Thus the labelings $h_1$, $h_2$ and $h_3$ are super $(a, 1)$-edge-antimagic total. □

## 5. Complete graphs

Next we shall investigate super $(a, d)$-edge-antimagic total labelings for complete graphs $K_n$.

**Theorem 8.** *If complete graph $K_n$, $n \geqslant 4$, is super $(a, d)$-edge-antimagic total then $d < 2$.*

**Proof.** Assume that a one-to-one mapping $f: V(K_n) \cup E(K_n) \to \{1, 2, \dots, |V(K_n)| + |E(K_n)|\}$ is a super $(a, d)$-edge-antimagic total labeling of complete graph $K_n$, where the set of edge-weights of all edges in $K_n$ is equal to $\{a, a+d, \dots, a+(|E(K_n)|-1)d\}$. The maximum edge-weight $a+(|E(K_n)|-1)d$ is no more than the sum of the extreme values of vertices $|V(K_n)|-1$, $|V(K_n)|$ and the extreme value of the edge $|V(K_n)|+|E(K_n)|$.

Thus,

$$a + (|E(K_n)| - 1)d \leqslant (|V(K_n)| - 1) + |V(K_n)| + (|V(K_n)| + |E(K_n)|)$$

i.e.,

$$a + \frac{n^2 - n - 2}{2} d \leqslant \frac{n^2 + 5n - 2}{2}.$$

Clearly, $a \geqslant n+4$ and so

$$d \leqslant \frac{n^2+3n-10}{n^2-n-2} < 2 \quad \text{for } n \geqslant 4. \qquad \square$$

For $n=3$, following the proof of Theorem 8 we have

**Corollary 1.** *If the complete graph $K_3$ is super $(a, d)$-edge-antimagic total then $d \leqslant 2$.*

**Theorem 9.** *The complete graph $K_n$, $n \geqslant 3$, has super $(a, d)$-edge-antimagic total labeling if and only if either*

(i) *$d=0$ and $n=3$, or*
(ii) *$d=1$ and $n \geqslant 3$, or*
(iii) *$d=2$ and $n=3$.*

**Proof.** Complete graph $K_3$ is the friendship graph $F_1$ and by Theorem 2 we know that $F_1$ has super $(a, d)$-edge-antimagic total labeling for $d=0$ and 2.

In [1] it is proved that for every complete graph $K_n$, $n \geqslant 4$, there is no super $(a, 0)$-edge-antimagic total labeling. It remains to deal with the case $d=1$.

For $n \geqslant 3$ let $K_n$ be the complete graph with $V(K_n)=\{x_i\colon 1 \leqslant i \leqslant n\}$ and $E(K_n)=\bigcup_{i=1}^{n-1}\{x_i x_{i+j}\colon 1 \leqslant j \leqslant n-i\}$.

Construct the one-to-one mapping $f\colon V(K_n) \cup E(K_n) \to \{1, 2, \ldots, n^2/2+n/2\}$ as follows:

If $1 \leqslant i \leqslant n$ then $f(x_i)=n+1-i$.

If $1 \leqslant j \leqslant n-1$ and $1 \leqslant i \leqslant n-j$ then $f(x_i x_{i+j})=nj+i+\sum_{k=1}^{j}(1-k)$ .

It is a routine procedure to verify that the set of edge-weights consists of the consecutive integers $\{2n+2, 2n+3, \ldots, (n^2+3n+2)/2\}$ which implies that $f$ is a super $(2n+2, 1)$-edge-antimagic total labeling of $K_n$. $\square$

## 6. Complete bipartite graphs

Let $K_{n,n}$ be the complete bipartite graph with $V(K_{n,n})=\{x_i\colon 1 \leqslant i \leqslant n\} \cup \{y_j\colon 1 \leqslant j \leqslant n\}$ and $E(K_{n,n})=\{x_i y_j\colon 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n\}$.

Our first result in this section provides an upper bound for the parameter $d$ for a super $(a, d)$-edge-antimagic total labeling of the complete bipartite graph $K_{n,n}$.

**Theorem 10.** *If complete bipartite graph $K_{n,n}$, $n \geqslant 4$, is super $(a, d)$-edge-antimagic total then $d<2$.*

**Proof.** Let $K_{n,n}, n \geqslant 4$, be super $(a, d)$-edge-antimagic total with a super $(a, d)$-edge-antimagic total labeling $g\colon V(K_{n,n}) \cup E(K_{n,n}) \to \{1, 2, \ldots, 2n+n^2\}$ and $W=\{w(uv)\colon uv \in E(K_{n,n})\}=\{a, a+d, \ldots, a+(n^2-1)d\}$ be the set of edge-weights. The sum of all vertex labels and edge labels used to calculate the edge-weights is equal to

$$n\sum_{i=1}^{n} g(x_i)+n\sum_{j=1}^{n} g(y_j)+\sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i y_j)=\frac{n^2}{2}(n^2+8n+3). \tag{21}$$

The sum of edge-weights in the set $W$ is

$$\sum_{uv \in E(K_{n,n})} w(uv)=\frac{n^2}{2}(2a+d(n^2-1)). \tag{22}$$

Since $a \geqslant 2n+4$, then combining (21) and (22) gives

$$d=\frac{n^2+8n+3-2a}{n^2-1} \leqslant 1+\frac{4}{n+1}<2 \tag{23}$$

for $n \geqslant 4$. $\square$

Applying the first part of inequality (23) to $K_{n,n}$, $2 \leqslant n \leqslant 3$, we obtain

**Corollary 2.** *If $K_{n,n}$, $2 \leqslant n \leqslant 3$, is super $(a, d)$-edge-antimagic total then $d < 3$.*

The next lemma shows that the complete bipartite graph $K_{n,n}$ is super $(a, 1)$-edge-antimagic total.

**Lemma 3.** *Every complete bipartite graph $K_{n,n}$, $n \geqslant 2$, has super $(a, 1)$-edge-antimagic total labeling.*

**Proof.** Define the bijective function $g$: $V(K_{n,n}) \cup E(K_{n,n}) \to \{1, 2, \ldots, |V(K_{n,n})| + |E(K_{n,n})|\}$ of $K_{n,n}$ in the following way:

$$g(x_i) = i \quad \text{for } 1 \leqslant i \leqslant n,$$

$$g(y_j) = n + j \quad \text{for } 1 \leqslant j \leqslant n,$$

$$g(x_i y_j) = (j - i + 3)n - i + 1 - \sum_{k=0}^{j-i} k \quad \text{for } 1 \leqslant i \leqslant n \text{ and } i \leqslant j \leqslant n,$$

$$g(x_i y_j) = \frac{n^2 + n}{2} + (i - j + 2)n - j + 1 - \sum_{k=0}^{i-j} k$$

for $1 \leqslant j \leqslant n - 1$ and $j + 1 \leqslant i \leqslant n$.

Let $A = (a_{ij})$ be a square matrix, where $a_{ij} = g(x_i) + g(y_j)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. The matrix $A$ is formed from the edge-weights of $K_{n,n}$ under the vertex labeling:

$$A = \begin{pmatrix} n+2 & n+3 & n+4 & n+5 & \cdots & 2n & 2n+1 \\ n+3 & n+4 & n+5 & n+6 & \cdots & 2n+1 & 2n+2 \\ n+4 & n+5 & n+6 & n+7 & \cdots & 2n+2 & 2n+3 \\ n+5 & n+6 & n+7 & n+8 & \cdots & 2n+3 & 2n+4 \\ \vdots & & & & & & \vdots \\ 2n & 2n+1 & 2n+2 & 2n+3 & \cdots & 3n-2 & 3n-1 \\ 2n+1 & 2n+2 & 2n+3 & 2n+4 & \cdots & 3n-1 & 3n \end{pmatrix}.$$

It is not difficult to see that the labels of the edges $x_i y_j$ form the square matrix $B = (b_{ij})$, where $b_{ij} = g(x_i y_j)$, for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, and $t = (n^2 + 5n)/2$, $r = n^2 + 2n$:

$$B = \begin{pmatrix} 3n & 4n-1 & 5n-3 & 6n-6 & \cdots & t-1 & t \\ \dfrac{n^2+7n}{2} - 1 & 3n-1 & 4n-2 & 5n-4 & \cdots & t-4 & t-2 \\ \dfrac{n^2+9n}{2} - 3 & \dfrac{n^2+7n}{2} - 2 & 3n-2 & 4n-3 & \cdots & t-8 & t-5 \\ \dfrac{n^2+11n}{2} - 6 & \dfrac{n^2+9n}{2} - 4 & \dfrac{n^2+7n}{2} - 3 & 3n-3 & \cdots & t-13 & t-9 \\ \vdots & & & & & & \vdots \\ r-1 & r-4 & r-8 & r-13 & \cdots & 2n+2 & 3n+1 \\ r & r-2 & r-5 & r-9 & \cdots & t+1 & 2n+1 \end{pmatrix}.$$

The vertex labeling and the edge labeling of $K_{n,n}$ combine to a total labeling where the edge-weights of edges $x_i y_j$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, are given by square matrix $C = (c_{ij})$ which is the sum of the matrices $A$ and $B$. We are setting $p=(n^2+9n)/2$ and $q=n^2+4n$,

$$C=\begin{pmatrix} 4n+2 & 5n+2 & 6n+1 & 7n-1 & \cdots & p-1 & p+1 \\ p+2 & 4n+3 & 5n+3 & 6n+2 & \cdots & p-3 & p \\ \dfrac{n^2+11n}{2}+1 & p+3 & 4n+4 & 5n+4 & \cdots & p-6 & p-2 \\ \dfrac{n^2+13n}{2}-1 & \dfrac{n^2+11n}{2}+2 & p+4 & 4n+5 & \cdots & p-10 & p-5 \\ \vdots & & & & & & \vdots \\ q-1 & q-3 & q-6 & q-10 & \cdots & 5n & 6n \\ q+1 & q & q-2 & q-5 & \cdots & \dfrac{n^2+11n}{2} & 5n+1 \end{pmatrix}.$$

We can see that the matrix $C$ is formed from consecutive integers $4n+2, 4n+3, \ldots, n^2+4n+1$. This implies that the labeling $g\colon V(K_{n,n})\cup E(K_{n,n})\to\{1,2,\ldots,n^2+2n\}$ is super $(4n+2,1)$-edge-antimagic total. □

In [3], it was proved that a complete bipartite graph $K_{m,n}$ is super edge-magic (super $(a,0)$-edge-antimagic total) if and only if $m=1$ or $n=1$. It means that for $n\geqslant 2$ there is no super $(a,0)$-edge-antimagic total labeling of $K_{n,n}$. It remains to deal with super $(a,2)$-edge-antimagic total labelings of $K_{2,2}$ and $K_{3,3}$.

**Lemma 4.** *For complete bipartite graph $K_{n,n}$, $2\leqslant n\leqslant 3$, there is no super $(a,2)$-edge-antimagic total labeling.*

**Proof.** Let us consider the two cases.

*Case* 1: $n=2$.

Assume that $K_{2,2}$ is super $(a,2)$-edge-antimagic total with a total labeling $g\colon V(K_{2,2})\cup E(K_{2,2})\to\{1,2,\ldots,8\}$. The sum of all vertex labels and edge labels used to calculate the edge-weights is equal to the sum of all edge-weights. Thus, we have

$$2\sum_{i=1}^{4} i+\sum_{i=1}^{4}(i+4)=\sum_{i=1}^{4}(a+2(i-1))$$

and we get that $34=4a$. This contradicts the fact that $a$ is an integer.

*Case* 2: $n=3$.

Suppose that $K_{3,3}$ has super $(a,2)$-edge-antimagic total labeling $g\colon V(K_{3,3})\cup E(K_{3,3})\to\{1,2,\ldots,15\}$ and $\{a, a+2, a+4, \ldots, a+16\}$ is the set of edge-weights. By direct computation we obtain that the smallest value of edge-weight, under total labeling $g$, is $a=10$. The edge-weight $a=10$ can be obtained only from the triple $(1,2,7)$ where 1 and 2 are values of adjacent vertices, say $g(x_1)=1$ and $g(y_1)=2$, and 7 is the value of edge $(x_1y_1)$. The following value of edge-weight $a+2=12$ can be obtained only from the triple $(1,3,8)$. Let $g(y_2)=3$ and $g(x_1y_2)=8$. The edge-weight $a+4=14$ can be composed by the triples $(1,4,9)$ or $(2,3,9)$. We consider only the triple $(1,4,9)$ because the vertices $y_1$ and $y_2$ labeled by values 2 and 3 are non-adjacent. Without loss of generality, we may assume that $g(y_3)=4$ and $g(x_1y_3)=9$.

On the other hand, the largest value of edge-weight $a+16=26$ can be obtained only from the triple $(5,6,15)$, but the values 5 and 6 can be given only to vertices $x_2$ and $x_3$ which are non-adjacent in $K_{3,3}$. Therefore, no edge of $K_{3,3}$ has the weight 26 and this contradicts the fact that $K_{3,3}$ is super $(a,2)$-edge-antimagic total. □

From the previous lemmas it follows that

**Theorem 11.** *The complete bipartite graph $K_{n,n}$ has super $(a,d)$-edge-antimagic total labeling if and only if $d=1$ and $n\geqslant 2$.*

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