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Global Existence and Optimal Temporal Decay Estimates for Systems of Parabolic Conservation Laws. II. The Multidimensional Case

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This paper is a continuation of our previous paper. It is concerned with the global existence and the optimal temporal decay estimates for the Cauchy problem of the following multidimensional parabolic conservation laws

$$\begin{cases} u_t + \sum_{j=1}^{N} f_j(u) z_j = D\,\Delta u, & x \in R^N,\ t > 0, \\ u(t,x)|_{t=0} = u_0(x), & x \in R^N,\ N > 1. \end{cases} \qquad (*)$$

Here $u(t,x) = (u_1(t,x), \ldots, u_n(t,x))^t$ is the unknown vector, $f_j(u) = (f_{j1}(u), \ldots, f_{jn}(u))^t$ $(j = 1, 2, \ldots, N)$ are arbitrary $n \times 1$ smooth vector-valued flux functions defined in $\overline{B}_r(\bar{u})$, a closed ball of radius r centered at some fixed vector $\bar{u} \in R^n$,

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and D is a constant, diagonalizable matrix with positive eigenvalues. Our results show that if the flux function $f_j(u)$ satisfies $f_j(u)/|u-\bar{u}|^s \in L^\infty(\overline{B}_r(\bar{u}), R^n)$, $j = 1, 2, \ldots, N$ for some $s > 2 + 1/N$, $\bar{u} \in R^n$, then for $u_0(x) - \bar{u} \in L^\infty \cap L^1(R^N, R^n)$ with $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ sufficiently small, the above Cauchy problem (*) admits a unique globally smooth solution $u(t,x)$ and $u(t,x)$ satisfies the following temporal decay estimates. For each $k = 0, 1, 2, \ldots$

$$\begin{cases} \|D^k(u(t,x) - \bar{u})\|_{L^2(R^N, R^n)} \le C(1+t)^{-(2k+N)/4}, \\ \|D^k(u(t,x) - \bar{u})\|_{L^\infty(R^N, R^n)} \le C(1+t)^{-(k+N)/2}. \end{cases}$$

Here $D^k = \sum_{|\alpha|=k}(\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N})$. The above decay estimates are optimal in the sense that they coincide with the corresponding decay estimates for the solution to the linear part of the corresponding Cauchy problem.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

This paper is a continuation of our previous paper [16]. It is concerned with the global existence and the optimal temporal decay estimates for the following multidimensional parabolic conservation laws

$$u_t + \sum_{j=1}^{N} f_j(u)_{x_j} = D\,\Delta u, \qquad x \in R^N, t > 0, \tag{1.1}$$

with initial data

$$u(t,x)|_{t=0} = u_0(x), \qquad x \in R^N, N > 1. \tag{1.2}$$

Here $u(t,x) = (u_1(t,x), \ldots, u_n(t,x))^t$ is the unknown vector, $f_j(u) = (f_{j1}(u), \ldots, f_{jn}(u))^t$ $(j = 1, 2, \ldots, N)$ are arbitrary $n \times 1$ smooth vector-valued flux functions defined in $\overline{B}_r(\bar{u})$, a closed ball of radius r centered at some fixed vector $\bar{u} \in R^n$, and D is a constant, diagonalizable matrix with positive eigenvalues (without loss of generality, we can assume $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$ with $d_{ii} > 0$, $i = 1, 2, \ldots, n$ in our following analyses).

The Cauchy problem (1.1), (1.2) has been studied by many authors and a lot of good results, especially for the case of one space dimension, have been obtained (a complete literature on these regards is beyond the scope of this paper; however, we want to mention [1–13, 16] and the references cited therein). To go directly to the main points of the present paper, in what follows we only review some former results concerning the multidimensional case (for the results on the case of $N = 1$, we refer the reader to [1, 2, 8, 10, 12, 16] and the references cited therein): First, for the global existence results, the most representative results on this regard are due to

D. Hoff and J. A. Smoller [9]. Their results showed that if the corresponding hyperbolic conservation laws, i.e., (1.1) with $D \equiv 0$, are equipped with a strictly convex (thus nontrivial) entropy $\eta(u)$ which is strongly consistent with the viscous matrix D, then the Cauchy problem (1.1), (1.2) admits a unique globally smooth solution provided that $u_0(x) - \bar{u} \in L^\infty \cap L^2(R^N, R^2)$ with $\|u_0(x) - \bar{u}\|_{L^2(R^N, R^n)}$ sufficiently small for each fixed vector $\bar{u} \in R^n$. But for $n > 2$, the corresponding entropy equation is overdetermined and the existence of a nontrivial entropy may be attributed only to a happy coincidence. Hence for general systems of type (1.1), it is necessary to give some other sufficient conditions to guarantee the existence of a unique globally smooth solution to the Cauchy problem (1.1), (1.2). Secondly, for the optimal temporal decay estimates for the global solution to the Cauchy problem (1.1), (1.2), to the knowledge of the authors, the only result concerning the case of multidimensional space variables is limited to the case of the scalar parabolic conservation laws [12, 13], i.e., (1.1) with $n = 1$. For the case of $N > 1$, $n > 1$, as we know, no results have been obtained.

Our present paper is devoted to giving some sufficient conditions on the flux functions $f_j(u)$ ($j = 1, 2, \ldots, N$) and the initial data $u_0(x)$ to guarantee the global existence and the optimal temporal decay estimates to the Cauchy problem (1.1), (1.2). Our main results can be summarized in the following

THEOREM 1 (Main Results). *Let $r > 0$ be an arbitrary constant and if there exists some fixed vector $\bar{u} \in R^n$, a constant $s > 0$ such that*

$$\frac{f_j(u)}{|u - \bar{u}|^s} \in L^\infty\big(\bar{B}_t(\bar{u}), R^n\big), \qquad j = 1, 2, \ldots, N, \tag{1.3}$$

we have

(i) *If $s \geq 1 + 1/N$, then the Cauchy problem* (1.1), (1.2) *admits a unique globally smooth solution $u(t, x)$ provided $u_0(x) - \bar{u} \in L^\infty \cap L^1(R^N, R^n)$ with $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ sufficiently small* (*without loss of generality, we may assume $\|u_0(x) - \bar{u}\|_{L^\infty(R^N, R^n)} < r$*);

(ii) *If $s > 2 + 1/N$, then, under the same conditions on the initial data $u_0(x)$ as those in* (i), *the solution $u(t, x)$ obtained in* (i) *satisfies the following temporal decay estimates. For each nonnegative integer $k = 0, 1, 2, \cdots$*

$$\begin{cases} \|D^k(u(t,x) - \bar{u})\|_{L^2(R^N, R^n)} \leq C(1+t)^{-(N+2k)/4}, \\ \|D^k(u(t,x) - \bar{u})\|_{L^\infty(R^N, R^n)} \leq C(1+t)^{-(N+k)/2}. \end{cases} \tag{1.4}$$

Here

$$D^k = \sum_{|\alpha|=k} \frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

Remarks. (1) It is easy to see that the decay rates we get in (1.4) are optimal since they coincide with the corresponding decay estimates for the solution to the linear part of the corresponding Cauchy problem.

(2) If the system (1.1) admits a strictly convex entropy $\eta(u)$ which is strongly consistent with the viscous matrix D, i.e., there exist some positive constants $\delta > 0$, $\varepsilon > 0$ such that

$$\begin{cases} \delta|u-\bar{u}|^2 \le \eta(u) \le \delta^{-1}|u-\bar{u}|^2, \\ w^t D\eta''(u)w \ge \varepsilon|w|^2, \qquad u \in \bar{B}_r(\bar{u}), w \in R^n, \end{cases} \tag{1.5}$$

then we can replace the assumption $s > 2 + 1/N$ in (ii) of Theorem 1 by $s \ge 1 + 1/N$ while the same results still hold. We will show this in Section 3.

(3) In our global existence results, we only ask the flux functions $f_j(u)$ $(j = 1, 2, \ldots, N)$ to satisfy local growth conditions, i.e., $f_j(u) = O(|u - \bar{u}|^s)$ $(j = 1, 2, \ldots, N)$ as $u \to \bar{u}$ for some fixed vector $\bar{u} \in R^n$ and some positive constant $s > 0$. It is easy to see that our global existence results can indeed solve some problems which cannot be solved by employing the results of D. Hoff and J. A. Smoller [9].

(4) From the proof of our main results, one can easily deduce that if for some $i \in \{1, 2, \ldots, N\}$, $\nabla f_i(\bar{u})$ is hyperbolic, i.e., $\nabla f_i(\bar{u})$ has n eigenvalues and n linearly independent right eigenvectors $r_i^j(\bar{u})$ $(j = 1, 2, \ldots, n)$ and $A_i(\bar{u})^{-1} D A_i(\bar{u}) = \operatorname{diag}(d_{11}, \ldots, d_{nn})$, $A_i(\bar{u}) = (r_i^1(\bar{u}), \ldots, r_i^n(\bar{u}))$, then, to get the global existence result, the assumption $f_j(u)/|u - \bar{u}|^s \in L^\infty(\bar{B}_r(\bar{u}), R^n)$ is unnecessary. When $N = 1$, the above observation means that for general $n \times n$ conservation laws, if the system under consideration is hyperbolic at some fixed point $\bar{u} \in R^n$, then the Cauchy problem to the corresponding viscous conservation laws always admits a unique globally smooth solution. This result is presented in our previous paper [16].

(5) In our main results, in addition to the assumption that D is a constant, diagonalizable matrix with positive eigenvalues, we do not ask the viscous matrix D to satisfy any other condition.

(6) In our main results, we ask the initial data $u_0(x)$ to satisfy $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ sufficiently small. This assumption makes our results unsuitable to be used to tackle the corresponding hyperbolic conservation laws. How to remove or relax this assumption, and thus make the result

suitable to be used to tackle the corresponding hyperbolic conservation laws remains an open problem.

In conclusion, we outline the key ideas used in the proof of our main results. We prove our global existence result by employing the method of extension of the local solutions. The results on the existence of local solutions to the Cauchy problem (1.1), (1.2) are well established [9] and our main contributions for the proof of the global existence result lie in how to extend the local solutions obtained above globally. The techniques used here are essentially due to D. Hoff and J. A. Smoller [9] with a slight modification. The main difference between our method and that of D. Hoff and J. A. Smoler [9] lies in the obtaining of the time independent $L^p(R^N, R^n)$ $(1 \le p < \infty)$ *a priori* estimate on the local solutions $u(T, x)$: In [9], to obtain such a time independent estimate, D. Hoff and J. A. Smoller employed the existence of a quadratic entropy consistent with the viscous matrix D, while in our paper, we exploit the integral representation (2.2) of the local solution fully to get the desired estimate. It is worth pointing out that it is in this step that we ask the nonlinear flux functions $f_j(u)$ $(j = 1, 2, \dots, N)$ to satisfy the assumption (1.3). As to the proof of the temporal decay estimates (1.4), we use Schonbek's Fourier splitting method [12, 13] and some delicate technical estimates. The estimate $\|u(t, x) - \bar{u}\|_{L^l(R^N, R^n)} \le C(1 + t)^{-1/2l}$, a by-product when deducing the time independent $L^1(R^N, R^n)$ *a priori* estimate in proving the global existence result, plays an important role in our analysis.

This paper is organized as in the following: After this introduction and the statement of the main results, which constitutes Section 1, we prove our global existence result in Section 2. The proof of our temporal decay estimates is given in Section 3.

2. THE PROOF OF THE GLOBAL EXISTENCE RESULT

In this section, we prove our global existence result, i.e., (i) of Theorem 1. We will also obtain some estimates on the global solution $u(t, x)$. These estimates are quite useful for our proof of the temporal decay estimates.

Let $K(t, x)$ be the fundamental solution associated with the operator $\partial/\partial t - D\sum_{j=1}^N (\partial^2/\partial x_j^2)$. That is, $K(t, x)$ is an n-vector whose jth component is

$$k_j(t, x) = (4\pi d_{jj} t)^{-n/2} \exp\left(-\frac{|x|^2}{4\, d_{jj} t} \right). \tag{2.1}$$

Then, the solution $u(t, x)$ of the Cauchy problem (1.1), (1.2) satisfies the integral representation

$$u(t, x) = K(t, x) * u_0(x) - \sum_{j=1}^{N} \int_0^t K_{x_j}(t-s, x) * f_j(u(s, x))\, ds, \quad (2.2)$$

where * denotes convolution in space, taken componentwise.

First, according to the well-established result on the existence of the local solutions to the Cauchy problem (1.1), (1.2) obtained by D. Hoff and J. A. Smoller in [9], we have

LEMMA 2.1 (Local Existence Result). *If the assumptions in Theorem* 1 *are satisfied, then the Cauchy problem* (1.1), (1.2) *admits a unique smooth solution* $u(t, x)$ *on the strip* $\Pi_{t_1} = \{(t, x) : 0 \le t \le t_1, x \in R^N\}$ *and* $u(t, x)$ *satisfies*

$$\|u(t, x) - \bar{u}\|_{L^\infty(R^N, R^n)} \le r. \quad (2.3)$$

Here t_1 *depends only on* $\|u_0(x) - \bar{u}\|^{L^\infty(R^N, R^n)}$.

Suppose that the solution $u(t, x)$ obtained in Lemma 2.1 can be extended up to $t = T(> t_1)$ while the regularity properties and the estimate (2.3) remain unchanged. We have

LEMMA 2.2 [9, 15]. *If the conditions of Theorem* 1 *are satisfied and* $u(t, x)$ *satisfies the assumptions stated above, then we can deduce that* $u(t, x)$ *satisfies the following estimates: For each* $1 \le p < \infty$, $k = 1, 2, \ldots, 0 < \bar{s}_0 < s_1 < \bar{s}_1 < s_2 < \bar{s}_2 < \cdots < \bar{s}_{k-1} < s_k < t \le T$

$$\|D^k(u(t, x) - \bar{u})\|_{L^\infty(R^N, R^n)} \le M_k(r, s_k - s_1; t - s_k), \quad (2.4)$$

$$\|D^k(u(t, x) - \bar{u})\|_{L^p(R^N, R^n)} \le \sup_{[0, T]} \|u(t, x) - \bar{u}\|_{L^p(R^N, R^n)} \overline{M}_k(r, s_k - \bar{s}_0; t - \bar{s}_{k-1}). \quad (2.5)$$

The next lemma deals with the obtaining of the time independent $L^1(R^N, R^n)$ *a priori* estimates on the solution $u(t, x)$ obtained in Lemma 2.1, which is one of our main contributions of this paper.

LEMMA 2.3 (Time Independent $L^1(R^N, R^n)$ a priori Estimate). *Suppose that the conditions in Lemma* 2.1 *are satisfied and the solution* $u(t, x)$ *obtained in Lemma* 2.1 *has been extended up to time* $T(> t_1 > 0)$ *while the regularity properties and* (2.3) *keep unchanged. Then if we assume further that* $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ *is sufficiently small,* $u(t, x)$ *satisfies the following time* *independent* $L^1(R^N, R^n)$ *a priori estimate*

$$\begin{aligned}\|u(t,x)-\bar{u}\|_{L^1(R^N,R^n)} &+ t^{1/2l}\|u(t,x)-\bar{u}\|_{L^l(R^n,R^n)} \\ &\leq C_1(r,D)\|u_0(x)-\bar{u}\|_{L^1(R^N,R^n)}, \qquad (2.6)\end{aligned}$$

where $l = 1 + 1/N$, $0 \leq t \leq T$.

Before proving Lemma 2.3, we first give the following result which is due to W. A. Strauss [14]

LEMMA 2.4 [14]. *Let* $M(t)$ *be a nonnegative continuous function of* t *satisfying the inequality*

$$M(t) \leq d_1 + d_2 M(t)^r \qquad (2.7)$$

in some interval containing 0, *where* d_1, d_2 *are positive constants and* $r > 1$. *If* $M(0) \leq d_1$ *and*

$$d_1 d_2^{1/(r-1)} \leq (1 - r^{-1}) r^{-(r-1)^{-1}}, \qquad (2.8)$$

then in the same interval

$$M(t) \leq \frac{d_1}{1 - r^{-1}}. \qquad (2.9)$$

Proof of Lemma 2.3. We take the fundamental space X as

$$\begin{aligned}X = \{u(t,x) | u(t,x) - \bar{u} \in C([0,T); L^1(R^N, R^n)), \\ t^{1/2l}(u(t,x) - \bar{u}) \in C([0,T); L^l(R^N, R^n))\}, \qquad (2.10)\end{aligned}$$

and define

$$\begin{aligned}\|u(t,x)-\bar{u}\|_X = \sup_{[0,T)} \{\|u(t,x)-\bar{u}\|_{L^1(R^N,R^n)} \\ + t^{1/2l}\|u(t,x)-\bar{u}\|_{L^l(R^N,R^n)}\}. \qquad (2.11)\end{aligned}$$

If we let $M(T)$ in Lemma 2.4 be the $\|u(t,x) - \bar{u}\|_X$ defined in (2.11), then according to Lemma 2.4, to prove Lemma 2.3, we only need to establish an inequality similar to (2.8). This is just what we want to do in the following.

From the integral representation (2.2), we have

$$u(t,x) - \bar{u} = K(t,x) * (u_0(x) - \bar{u}) - \sum_{j=1}^{N} \int_0^t K_{x_j}(t-s,x) * f_j(u(s,x))\, ds, \tag{2.12}$$

thus

$$\begin{aligned}\|u(t,x) - \bar{u}\|_X &\le \|K(t,x) * (u_0(x) - \bar{u})\|_X \\ &\quad + \sum_{j=1}^{N} \left\| \int_0^t K_{x_j}(t-s,x) * f_j(u(s,x))\, ds \right\|_X \\ &= I_1 + I_2. \end{aligned} \tag{2.13}$$

For I_1, we have the estimates

$$\begin{aligned} I_1 &= \sup_{[0,T)} \Big\{ \|K(t,x) * (u_0(x) - \bar{u})\|_{{}^1(R^N,R^n)} \\ &\qquad + t^{1/2l} \|K(t,x) * (u_0(x) - \bar{u})\|_{L^l(R^N,R^n)} \Big\} \\ &\le \sup_{[0,T)} \Big\{ \|u_0(x) - \bar{u}\|_{{}^1(R^N,R^n)} \\ &\qquad + C_2 t^{1/2l} \|u_0(x) - \bar{u}\|_{L^1(R^N,R^n)} t^{-(N/2)(1-1/l)} \Big\} \\ &\le C_3 \|u_0(x) - \bar{u}\|_{L^1(R^N,R^n)}. \end{aligned} \tag{2.14}$$

As to I_2, we have

$$\begin{aligned} I_2 &= \sum_{j=1}^{N} \sup_{[0,T)} \left\{ \int_0^t \left\| \frac{\partial K}{\partial x_j}(t-s,x) * f_j(u(s,x)) \right\|_{L^1(R^N,R^n)} ds \right\} \\ &\quad + \sum_{j=1}^{N} \sup_{[0,T)} \left\{ t^{1/2l} \int_0^t \left\| \frac{\partial K}{\partial x_j}(t-s,x) * f_j(u(s,x)) \right\|_{L^l(R^N,R^n)} ds \right\} \\ &= J_1 + J_2. \end{aligned} \tag{2.15}$$

Noticing

$$\begin{cases} \|u(t,x) - \bar{u}\|_{L^\infty(\Pi_T, R^n)} \le r, \\ \dfrac{f_j(u)}{|u - \bar{u}|^l} \in L^\infty(\bar{B}_r(\bar{u}), R^n), \end{cases} \tag{2.16}$$

we have from (2.15) that

$$J_1 \le C_4 \sum_{j=1}^{N} \sup_{[0,T)} \int_0^t (t-s)^{-1/2} \|f_j(u)\|_{L^1(R^N, R^n)}\, ds$$

$$\le C_5(r) \sup_{[0,T)} \int_0^t (t-s)^{-1/2} \|u(s,x) - \bar{u}\|^l_{L^l(R^N, R^n)}\, ds$$

$$\le C_5(r) B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \|u(t,x) - \bar{u}\|^l_X, \tag{2.17}$$

and

$$J_2 \le C_6 \sum_{j=1}^{N} \sup_{[0,T)} \left\{ t^{1/2l} \int_0^t (t-s)^{-(N/2)(1-1/l)-1/2} \|f_j(u)\|_{L^1(R^N, R^n)}\, ds \right\}$$

$$\le C_7(r) \sup_{[0,T)} \left\{ t^{1/2l} \int_0^t (t-s)^{-(N/2)(1-1/l)-1/2} \|u(s,x) - \bar{u}\|^l_{L^l(R^N, R^n)}\, ds \right\}$$

$$\le C_7(r) B\left(\tfrac{1}{2}, \tfrac{1}{2} - \tfrac{1}{2l}\right) \|u(t,x) - \bar{u}\|^l_X. \tag{2.18}$$

Thus

$$I_2 \le C_8(r) \|u(t,x) - \bar{u}\|^l_X. \tag{2.19}$$

Combining (2.14), (2.19) with (2.13), we get

$$\|u(t,x) - \bar{u}\|_X \le C_3 \|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)} + C_8(r) \|u(t,x) - \bar{u}\|^l_X. \tag{2.20}$$

Having obtained (2.20), if we assume further $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ sufficiently small, then from Lemma 2.4, we can get (2.6) immediately. This completes the proof of Lemma 2.3.

Using the above results, we are now in a position to prove our global existence result.

Proof of Our Global Existence Result. Under the assumptions of Theorem 1, we have from the local existence result Lemma 2.1 that there exists a sufficiently small $t_1 > 0$ such that the Cauchy problem (1.1), (1.2) admits a unique globally smooth solution $u(t,x)$ on the strip Π_{t_1} and $u(t,x)$ satisfies

$$\|u(t,x) - \bar{u}\|_{L^\infty(\Pi_{t_1}, R^n)} \le r, \tag{2.21}$$

which means that, on the strip Π_{t_1}, $u(t,x)$ satisfies all the conditions stated in Lemma 2.2, Lemma 2.3, and consequently $u(t,x)$ satisfies (2.5), (2.6) with $T = t_1$. If we choose $0 < \bar{s}'_0 < s'_1 < \bar{s}'_2 < \cdots < \bar{s}'_{N-1} < s'_N$ sufficiently small such that

$$s'_N < t_1 \qquad \text{and} \qquad t_1 - s'_n = \bar{s}'_j - s'_j = s'_j - \bar{s}_{j-1} = \beta,\ j = 1, 2, \dots, N-1, \tag{2.22}$$

where $\beta > 0$ is a sufficiently small positive constant, then from (2.5), (2.6) (let $T = t_1, \bar{s}_0 = \bar{s}'_0, \bar{s}_{j-1} = \bar{s}'_{j-1}$ and $s_j = s'_j$ $(j = 1, 2, \dots, N)$ in (2.5), (2.6)), we deduce

$$\begin{cases} \|u(t,x) - \bar{u}\|_{L^\infty(R^N, R^n)} \le r, \qquad 0 \le t \le t_1, \\ \|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)} \le C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}, \qquad 0 \le t \le t_1, \\ \|u(t_1,x) - \bar{u}\|_{W^{N,1}(R^N, R^n)} \le C_9(\beta, r, N) \sup\limits_{[0, t_1]} \|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)}. \end{cases} \tag{2.23}$$

Let C_{10} be the constant in Sobolev's inequality

$$\|u(t,x) - \bar{u}\|_{L^\infty(R^N, R^n)} \le C_{10}\|u(t,x) - \bar{u}\|_{W^{N,1}(R^n, R^n)}. \tag{2.24}$$

Then if we choose $\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}$ sufficiently small that

$$C_{10}C_9(\beta, r, N)C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)} \le \|u_0(x) - \bar{u}\|_{L^\infty(R^N, R^n)} < r, \tag{2.25}$$

we can deduce from (2.23), (2.25) that

$$\begin{aligned} \|u(t_1,x) - \bar{u}\|_{L^\infty(R^N, R^n)} &\le C_{10}\|u(t_1,x) - \bar{u}\|_{W^{N,1}(R^N, R^n)} \\ &\le C_{10}C_9(\beta, r, N)C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)} \\ &\le \|u_0(x) - \bar{u}\|_{L^\infty(R^N, R^n)} \\ &< r. \end{aligned}$$

So that, by Lemma 2.1, $u(t,x)$ can be extended up to the time $t = 2t_1$.

Now suppose that $u(t, x)$ has been defined up to the time kt_1 for some $k \in Z_+$ such that

$$\begin{cases} \|u(t,x) - \bar{u}\|_{L^\infty(R^N, R^n)} \le r, \qquad 0 \le t \le kt_1, \\ \|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)} \le C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}, \qquad 0 \le t \le kt_1, \\ \|u(kt_1,x) - \bar{u}\|_{W^{N,1}(R^N, R^n)} \le C_9(\beta, r, N) \sup\limits_{[0, kt_1]} \|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}. \end{cases} \tag{2.26}$$

Then

$$\begin{aligned} \|u(kt_1,x) - \bar{u}\|_{L^\infty(R^N, R^n)} &\le C_{10}\|u(kt_1,x) - \bar{u}\|_{W^{N,1}(R^N, R^n)} \\ &\le C_{10}C_9(\beta, r, N)C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)} \\ &\le \|u_0(x) - \bar{u}\|_{L^\infty(R^N, R^n)} \\ &< r. \end{aligned}$$

So that, by Lemma 2.1 again, $u(t, x)$ can be extended up to the time $(k+1)t_1$ with

$$\|u(t,x) - \bar{u}\|_{L^\infty(R^N, R^n)} \le r, 0 \le t \le (k+1)t_1, \tag{2.27}$$

and (2.27) means that $u(t, x)$ satisfies all the conditions stated in Lemma 2.2, and Lemma 2.3 on the strip $\Pi_{(k+1)t_1}$, and consequently, $u(t, x)$ satisfies (2.5), (2.6) with $T = (k+1)t_1$. If we let $t, \bar{s}_0, \bar{s}_j, s_{j+1}$ $(j = 1, 2, \ldots, N-1)$ in (2.5), (2.6) equal to $(k+1)t_1, kt_1 + \bar{s}'_0, kt_1 + \bar{s}'_j, kt_1 + s'_{j+1}$ $(j = 1, 2, \ldots, N-1)$, we get

$$\left\{ \begin{aligned} &\|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)} \le C_1(r, D)\|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}, \\ &\qquad 0 \le t \le (k+1)t_1, \\ &\big\|u((k+1)t_1, x) - \bar{u}\big\|_{W^{N,1}(R^N, R^n)} \le C_9(\beta, r, N) \\ &\qquad \times \sup_{[0, (k+1)t_1]} \|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)}. \end{aligned} \right\}$$

Thus (2.26) holds up to the time $(k+1)t_1$. Proceeding inductively, we thus establish the existence of the solution in all of $t \ge 0$.

Before concluding this section, we give some estimates on the global solution $u(t, x)$ obtained above; these estimates are quite useful for the proof of our temporal decay estimates:

COROLLARY 2.5. *Suppose that $u(t,x)$ is the global solution obtained above. Then for each fixed $\tau > 0$ and every positive integer $k = 1, 2, \ldots,$ we have*

$$\|u(t,x) - \bar{u}\|_{L^\infty(R_+ \times R^N, R^n)} \le r, \tag{2.28}$$

$$\sup_{[0,\infty)} \left\{ \|u(t,x) - \bar{u}\|_{L^1(R^N, R^n)} + t^{1/2l} \|u(t,x) - \bar{u}\|_{L^l(R^N, R^n)} \right\}$$
$$\le C_1(r, D) \|u_0(x) - \bar{u}\|_{L^1(R^N, R^n)}, \tag{2.29}$$

$$D^k(u(t,x) - \bar{u})\|_{L^\infty([\tau,\infty) \times R^N, R^n)} \le M_k(r, \beta), \tag{2.30}$$

$$\sup_{[\tau,\infty)} \|D^k(u(t,x) - \bar{u})\|_{L^2(R^N, R^n)} \le \overline{M}_k(r, \beta). \tag{2.31}$$

Here M_k, $\overline{M}_k$ are independent of t and β is defined by (2.22).

Relations (2.28), (2.29) follow from (2.6) and the proof of the global existence result. By employing the method of induction and Lemma 2.2, (2.30), (2.31) can be proved similarly to the proof of the global existence result and the details are omitted.

3. THE PROOF OF THE TEMPORAL DECAY ESTIMATES

In this section, we prove our temporal decay estimates (1.4). Our analyses are based on Schonbek's Fourier splitting method [12, 13] and some delicate technical estimates.

In what follows, C will denote a generic positive constant independent of t, x and without loss of generality we can assume $\bar{u} = 0$ in our following analyses.

For later use, we first give the following fundamental inequality

LEMMA 3.1 (Nirenberg's Inequality). *If $u \in L^q(R^N, R^n)$ and $D^m u \in L^r(R^N, R^n)$ with $1 \le q, r \le +\infty$, then, for any integer j such that $0 \le j \le m$, we have*

$$\|D^j u\|_{L^p(D^N, R^n)} \le C \|D^m u\|^{\alpha}_{L^r(R^N, R^n)} \|u\|^{1-\alpha}_{L^q(R^N, R^n)}, \tag{3.1}$$

where p is determined by

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + (1-\alpha)\frac{1}{q}, \qquad \frac{j}{m} \le \alpha \le 1. \tag{3.2}$$

From the time independent $L^1(R^N, R^n)$ estimate (2.29), we know that

$$\|u(t,x)\|_{L^l(R^N, R^n)} \le C(1+t)^{-1/2l}. \tag{3.3}$$

Our next lemma is devoted to obtaining a similar estimate for $u(t,x)$ in the $L^2(R^N, R^n)$ norm.

LEMMA 3.2. *Suppose that $u(t,x)$ is the global solution obtained in Section* 2. *Then we have*

$$\|u(t,x)\|_{L^2(R^N, R^n)} \le C(1+t)^{-1/2l+\varepsilon}, \qquad t \ge \tau > 0. \tag{3.4}$$

Here $\varepsilon > 0$ is a sufficiently small constant.

Proof. Firstly, it is easy to see that $l = 1 + 1/N < 2$. Thus we have

$$\begin{aligned}\int_{R^N} |u(t,x)|^2\,dx &\le \|u(t,x)\|_{L^\infty(R^N, R^n)}^{1-1/N} \int_{R^N} |u(t,x)|^l\,dx \\ &\le C(1+t)^{-1/2} \|u(t,x)\|_{L^\infty(R^N, R^n)}^{1-1/N}. \end{aligned} \tag{3.5}$$

Secondly, from Nirenberg's inequality, we get

$$\begin{aligned}\|u(t,x)\|_{L^\infty(R^N, R^n)} &\le C\|D^m u(t,x)\|_{L^2(R^N, R^n)}^{\alpha} \|u(t,x)\|_{L^l(R^N, R^n)}^{1-\alpha} \\ &\le C(1+t)^{-(1-\alpha)/2l} \|D^m u(t,x)\|_{L^2(R^N, R^n)}^{\alpha}, \end{aligned} \tag{3.6}$$

where

$$\alpha = \frac{N/(N+1)}{N/(N+1) + m/N - 1/2}, \qquad m > \frac{N}{2}. \tag{3.7}$$

Substituting (3.6) into (3.5), we deduce from (2.31) that

$$\int_{R^N} |u(t,x)|^2\,dx \le C(1+t)^{-1/l + ((N-1)/2(N+1))/\alpha}, \qquad t \ge \tau > 0.$$

Thus

$$\|u(t,x)\|_{L^2(R^N, R^n)} \le C(1+t)^{-1/2l+\varepsilon}, \qquad t \ge \tau > 0,$$

where $\varepsilon = ((N-1)/4(N+1))\alpha$ and it's easy to see that if we choose m sufficiently large, ε can be arbitrarily small. This completes the proof of Lemma 3.2.

LEMMA 3.3. *Under the conditions of Lemma* 3.2, *we have*

$$\sup_{(t,\xi)\in B_C(t)} |\hat{u}(t,\xi)| \le C. \tag{3.8}$$

Here

$$B_C(t) = \left\{\xi \in R^N : |\xi|^2(1+t) \le C\right\}. \tag{3.9}$$

Proof. From (3.1), one can deduce

$$\begin{pmatrix} \hat{u}_1(t,\xi) \\ \vdots \\ \hat{u}_n(t,\xi) \end{pmatrix} = \begin{pmatrix} e^{-d_{11}|\xi|^2 t}\hat{u}_{01}(\xi) \\ \vdots \\ e^{-d_{nn}|\xi|^2 t}\hat{u}_{0n}(\xi) \end{pmatrix} + i\sum_{j=1}^{N}\int_0^t \begin{pmatrix} e^{d_{11}|\xi|^2(s-t)}\xi_j \hat{f}_{j1}(u(s,\xi)) \\ \vdots \\ e^{d_{nn}|\xi|^2(s-t)}\xi_j \hat{f}_{jn}(u(s,\xi)) \end{pmatrix} ds,$$

and so from Lemma 3.2, we have

$$\begin{aligned} |\hat{u}(t,\xi)| &\le |\hat{u}_0(\xi)| + \sum_{j=1}^{N}\int_0^t |\xi|\,|\hat{f}_j(u)|\,ds \\ &\le \int_{R^N}|u_0(x)|\,dx + C\int_0^t |\xi|\,\|u(s,x)\|^2_{L^2(R^N,R^n)}\,ds \\ &\le \int_{R^N}|u_0(x)|\,dx + C|\xi|\int_0^t (1+s)^{-1/2+2\varepsilon}\,ds \\ &\le \int_{R^N}|u_0(x)|\,dx + C|\xi|(1+t)^{1-1/l+2\varepsilon}. \end{aligned}$$

Thus, if $|\xi|^2(1+t) \le C$, we can easily deduce from the above inequality that

$$|\hat{u}(t,\xi)| \le C.$$

This is (3.8) and completes the proof of Lemma 3.3.

LEMMA 3.4. *Under the conditions of Lemma* 3.2, *we have*

$$\frac{d}{dt}\int_{R^N}|u(t,x)|^2\,dx + \frac{d}{2}\int_{R^N}|Du(t,x)|^2\,dx \le C\|u(t,x)\|^{2s-\varepsilon}_{L^2(R^N,R^n)},$$
$$t \ge \tau > 0. \tag{3.10}$$

Here $d = \min\{d_{11},\dots,d_{nn}\}$, $\varepsilon > 0$, *is a sufficiently small positive constant.*

The proof of this lemma is similar to that of Lemma 3.2 and hence, we omit the detail.

The purpose of the following lemma is to improve the $L^2(R^N, R^n)$-norm temporal decay estimates obtained in Lemma 3.2.

LEMMA 3.5. *Suppose that $u(t,x)$ is the global solution obtained in Section 2 and $f_j(u)$ $(j = 1,2,\dots,N)$ satisfies* (1.3) *with $s > 2 + 1/N$. We have*

$$\int_{R^N} |u(t,x)|^2\, dx \le C(1+t)^{-P_n}, \qquad t \ge \tau > 0. \tag{3.11}$$

Here $P_n = \min\{N/2, a_n - \varepsilon\}$, $a_n = sa_{n-1} - 1$, $a_1 = 2/l - 1$, and $\varepsilon > 0$ is a sufficiently small constant.

Proof. We prove (3.11) by the method of induction.

First, we prove that (3.11) is true for $n = 1$.

From Lemma 3.2, Lemma 3.4, we have

$$\frac{d}{dt}\int_{R^N} |u(t,x)|^2\, dx + \frac{d}{2}\int_{R^N} |Du(t,x)|^2\, dx \le C(1+t)^{-1/l+\varepsilon'},$$

$$t \ge \tau > 0. \tag{3.12}$$

here $\varepsilon' = \varepsilon/2l + 2s\varepsilon - \varepsilon^2$ is also a sufficiently small constant.

Setting

$$B(t) = \left\{ \xi \in R^N : |\xi|^2 (1+t) \le \frac{4N}{d} \right\}$$

and noticing

$$\sup_{B(t)} |\hat{u}(t,\xi)| \le C,$$

we have

$$\begin{aligned}
&\frac{d}{dt}\left\{ (1+t)^{2N} \int_{R^N} |u|^2\, dx \right\} \\
&\quad \le 2N(1+t)^{2N-1}\int_{R^N} |u|^2\, dx + C(1+t)^{2N-s/l} + \varepsilon' \\
&\qquad - \frac{1}{2} d(1+t)^{2N} \int_{R^N} |Du|^2\, dx \\
&\quad \le 2N(1+t)^{2N-1}\int_{R^N} |u|^2\, dx + C(1+t)^{2N-s/l+\varepsilon'} \\
&\qquad - \frac{1}{2} d(1+t)^{2N} \int_{B(t)^C} |\xi|^2 |\hat{u}|^2\, d\xi \\
&\quad \le C(1+t)^{2N-s/l+\varepsilon'} + 2N(1+t)^{2N-1}\int_{B(t)} |\hat{u}|^2\, d\xi \\
&\quad \le C(1+t)^{2N-s/l+\varepsilon'} + C(1+t)^{(3/2)N-1}.
\end{aligned} \tag{3.13}$$

Integrating (3.13) with respect to t over $[\tau, t]$, we can easily deduce that (3.11) is true for $n = 1$.

Now suppose that (3.11) is true for $n = m$, i.e.,

$$\int_{R^N} |u|^2 \, dx \le C(1+t)^{-P_m}. \tag{3.14}$$

We are now in a position to prove (3.11) is true for $n = m + 1$.

Substituting (3.14) into (3.10), we get

$$\frac{d}{dt}\int_{R^N} |u|^2 \, dx + \frac{d}{2}\int_{R^N} |Du|^2 \, dx \le C(1+t)^{-sP_m+\varepsilon'}, \tag{3.15}$$

where ε' is a sufficiently small constant.

With (3.15) in hand, similar to the proof of (3.13), we have

$$\frac{d}{dt}\left\{(1+t)^{2N}\int_{R^N} |u|^2 \, dx\right\} \le C(1+t)^{(3/2)N-1} + C(1+t)^{2N-sP_m+\varepsilon'}.$$

Thus

$$\int_{R^N} |u|^2 \, dx \le C(1+t)^{-\min\{N/2,\, sP_m-1-\varepsilon'\}}. \tag{3.16}$$

If $a_m > N/2$, then $P_m = N/2$. Since $N \ge 2$, $s > 2 + 1/N > 2$, we have $sP_m = (N/2)s > 1 + N/2$. Thus $\min\{N/2, sP_m - 1 - \varepsilon'\} = N/2$.

If $a_m \le N/s$, then $P_m = a_m - \varepsilon$ and thus $sP_m - 1 - \varepsilon' = sa_m - 1 - \varepsilon' = a_{m+1} - \varepsilon'$, where ε' is a sufficiently small constant.

In summary, we have $\min\{N/2, sP_m - 1 - \varepsilon'\} = P_{m+1}$. Combining this observation with (3.16) proves that (3.11) is true for $n = m + 1$.

Thus by employing the principle of induction, we get that (3.11) is true for each positive integer n. This completes the proof of Lemma 3.5.

From Lemma 3.5, we can easily get the optimal $L^2(R^N, R^n)$-norm temporal decay estimate for $u(t, x)$, i.e.,

COROLLARY 3.6. *Under the conditions of Lemma* 3.5, *we have*

$$\int_{R^N} |u|^2 \, dx \le C(1+t)^{-N/2}. \tag{3.17}$$

Proof. From Lemma 3.5, we only need to show

$$\lim_{n\to\infty} a_n = +\infty. \tag{3.18}$$

This can be seen from

$$a_n = \frac{s^n(s - l - 1) + l}{l(s - 1)}$$

and

$$s > l + 1 > 2.$$

We now turn to deduce the optimal $H^s(R^N, R^n)$-norm temporal decay estimates for $u(t, x)$. First for $N \geq 3$, we have

THEOREM 3.7. *Suppose that* $N \geq 3$ *and the conditions in Lemma* 3.5 *are satisfied. Then we have*

$$\|D^k u(t, x)\|_{L^2(R^N, R^n)} \leq C(1 + t)^{-(N+2k)/4}, \qquad k = 1, 2, \ldots. \tag{3.19}$$

Proof. We prove (3.19) by the method of induction and it is divided into two major steps.

The First Step: Relation (3.19) is true for $k = 1$.

Multiplying (1.1) by $2(\Delta u)^t$ and integrating the result over R^N, after some integrations by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{R^N} |Du|^2\, dx + 2d\int_{R^N} |\Delta u|^2\, dx \\
&\quad = 2\int_{R^N} \Delta u \sum_{j=1}^{n} f_j(u)_{x_j} dx \\
&\quad \leq d\int_{R^N} |\Delta u|^2\, dx + \frac{1}{d}\sum_{j=1}^{N}\int_{R^N} |f_j(u)_{x_j}|^2\, dx \\
&\quad \leq d\int_{R^N} |\Delta u|^2\, dx + C\|u\|^2_{L^\infty(R^N, R^n)}\int_{R^N} |Du|^2\, dx.
\end{aligned} \tag{3.20}$$

Since

$$\|Du\|_{L^2(R^N, R^n)} \leq C\|D^2 u\|^{1/2}_{L^2(R^N, R^n)}\|u\|^{1/2}_{L^2(R^N, R^n)}, \tag{3.21}$$

we have from (3.20), (3.21) that

$$\begin{aligned}
&\frac{d}{dt}\int_{R^N} |Du|^2\, dx + d\int_{R^N} |\Delta u|^2\, dx \\
&\quad \leq C\|u\|^2_{L^\infty(R^N, R^n)}\|\Delta u\|_{L^2(R^N, R^n)}\|u\|_{L^2(R^N, R^n)} \\
&\quad \leq \frac{d}{2}|\Delta u|^2_{L^2(R^N, R^n)} + C\|u\|^4_{L^\infty(R^N, R^n)}\|u\|^2_{L^2(R^N, R^n)}.
\end{aligned}$$

Thus

$$\frac{d}{dt}\int_{R^N}|Du|^2\,dx + \frac{d}{2}\int_{R^N}|\Delta u|^2\,dx \le C(1+t)^{-N/2}\|u\|^4_{L^\infty(R^N,R^n)}. \quad (3.22)$$

Due to

$$\|u\|_{L^\infty(R^N,R^n)} \le C\|D^m u\|^{e(m)/4}_{L^2(R^N,R^n)}\|u\|^{1-\varepsilon(m)/4}_{L^2(R^N,R^n)}, \quad (3.23)$$

where $\varepsilon(m)/4 = N/2m \in (0,1)$ can be chosen sufficiently small, we get from (3.22), (3.23) that

$$\begin{aligned}
&\frac{d}{dt}\int_{R^N}|Du|^2\,dx + \frac{d}{2}\int_{R^N}|\Delta u|^2\,dx \\
&\quad \le C(1+t)^{-N/2}\|D^m u\|^{\varepsilon(m)}_{L^2(R^N,R^n)}\|u\|^{4-\varepsilon(m)}_{L^2(R^N,R^n)} \\
&\quad \le C(1+t)^{-(3/2)N+(N/4)\varepsilon(m)}, \qquad t \ge \tau > 0. \quad (3.24)
\end{aligned}$$

From (3.24), one can deduce

$$\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{2N}\int_{R^N}|Du|^2\,dx\right\} \\
&\quad \le 2N(1+t)^{2N-1}\int_{R^N}|Du|^2\,dx - \frac{d}{2}(1+t)^{2N}\int_{R^N}|\Delta u|^2\,dx \\
&\qquad + C(1+t)^{N/2+(N/4)\varepsilon(m)}, \qquad t \ge \tau > 0. \quad (3.25)
\end{aligned}$$

Let

$$B(t) = \left\{\xi \in R^N : |\xi|^2(1+t) \le \frac{4N}{d}\right\}$$

and due to

$$\begin{aligned}
\frac{d}{2}(1+t)^{2N}\int_{R^N}|D^2u|^2\,dx &= \frac{d}{2}(1+t)^{2N}\int_{R^N}|\xi|^4|\hat{u}|^2\,d\xi \\
&\ge \frac{d}{2}(1+t)^{2N}\int_{B(t)^c}|\xi|^4|\hat{u}|^2\,d\xi \\
&\ge 2N(1+t)^{2N-1}\int_{R^N}|Du|^2\,d\xi \\
&\quad - 2N(1+t)^{2N-1}\int_{B(t)}|\xi|^2|\hat{u}|^2\,d\xi,
\end{aligned}$$

we have from (3.25) and $N \geq 3$ that

$$
\begin{aligned}
\frac{d}{dt}&\left\{(1+t)^{2N}\int_{R^N}|Du|^2\,dx\right\} \\
&\leq 2N(1+t)^{2N-1}\int_{B(t)}|\xi|^2|\hat{u}|^2\,d\xi \\
&\quad + C(1+t)^{N/2+(N/4)\varepsilon(m)} \\
&\leq C(1+t)^{(3/2)N-2} + C(q+t)^{N/2+(N/4)\varepsilon(m)} \\
&\leq C(1+t)^{(3/2)N-2}, \qquad t \geq \tau > 0. \qquad (3.26)
\end{aligned}
$$

Integrating (3.26) with respect to t over $[\tau, t]$, we get

$$
\int_{R^N}|Du|^2\,dx \leq \frac{C\left(1+(1+t)^{(3/2)N-1}\right)}{(1+t)^{2N}} \leq C(1+t)^{-(N+2)/2}, \qquad t \geq \tau > 0,
$$

which means that (3.19) holds for $k = 1$ and completes the proof of the first step.

The Second Step: Suppose that (3.19) is true for $k \leq m-1$ ($m \in Z_+$, $m \geq 2$). We then prove (3.19) is true for $k = m$.

First, we can get the differential inequality

$$
\begin{aligned}
\frac{d}{dt}&\int_{R^N}|D^m u|^2\,dx + d\int_{R^N}|D^{m+1}u|^2\,dx \\
&\leq C\|u\|^2_{L^\infty(R^N,R^n)}\int_{R^N}|D^m u|^2\,dx \\
&\quad + C\sum_{\substack{\sum_{i=1}^s i\alpha_i = m\\ 1\leq s<m\\ \alpha_s\neq 0}}\prod_{i=1}^{s-1}\|D^i u\|^{2\alpha_i}_{L^\infty(R^N,R^n)}\|D^s u\|^{2(\alpha_s-1)}_{L^\infty(R^N,R^n)}\int_{R^N}|D^s u|^2\,dx.
\end{aligned}
\qquad (3.27)
$$

Since

$$
\begin{aligned}
\|D^j u\|_{L^\infty(R^N,R^n)} &\leq C\|D^{\overline{m}+j}u\|^{\varepsilon}_{L^2(R^N,R^n)}\|D^j u\|^{1-\varepsilon}_{L^2(R^N,R^n)}. \\
&\leq C(\overline{m})(1+t)^{-(N+2j)/4+((N+2j)/4)\varepsilon}, \qquad (3.28)
\end{aligned}
$$

where

$$\varepsilon = \frac{N}{2\overline{m}} \in (0,1), \qquad j = 0,1,\dots,s,$$

we have from (3.27), (3.28), and the assumption that (3.19) is true for $k \le m-1$ that

$$\begin{aligned}
&\frac{d}{dt}\int_{R^N}|D^m u|^2\,dx + d\int_{R^N}|D^{m+1}u|^2\,dx \\
&\qquad \le C(1+t)^{-N/2+(N/4)\varepsilon}\int_{R^N}|D^m u|^2\,dx \\
&\qquad\quad + C\sum_{\substack{\sum_{i=1}^s i\alpha_i = m\\ 1\le s<m\\ \alpha_s\ne 0}}(1+t)^{-\sigma} \\
&\qquad \le C(1+t)^{-N/2+(N/4)\varepsilon}\int_{R^N}|D^m u|^2\,dx \\
&\qquad\quad + c(1+t)^{-(N/2)\sum_{j=1}^s\alpha_j - m + ((N/2)\sum_{j=1}^s\alpha_j + m - (N+2s)/2)\varepsilon}. \qquad (3.29)
\end{aligned}$$

where $\sigma = -\sum_{j=1}^{s-1}((N+2j/2)\alpha_j - ((N+2j)/2)\alpha_j\varepsilon) - ((N+2s)/2)\times(\alpha_s - 1) + ((N+2s)/2)(\alpha_s-1)\varepsilon - (N+2s)/2$.

From $\sum_{i=1}^s i\alpha_i = m, 1 \le s < m$, we can easily deduce that

$$\frac{N}{2}\sum_{j=1}^s \alpha_j > \frac{N}{2} + 1;$$

thus, if we choose $\varepsilon = N/2\overline{m}$ sufficiently small such that

$$\begin{cases} \dfrac{N}{2} - \dfrac{N}{4}\varepsilon \ge 1, \\ \dfrac{N}{2}\displaystyle\sum_{j=1}^s\alpha_j + m - \left(\dfrac{N}{2}\sum_{j=1}^s\alpha_j + m - \dfrac{N+2s}{2}\right)\varepsilon \ge \dfrac{N}{2} + 1 + m, \end{cases} \qquad (3.30)$$

we have from (3.29), (3.30) that

$$\begin{aligned}
&\frac{d}{dt}\int_{R^N}|D^m u|^2\,dx + d\int_{R^N}|D^{m+1}u|^2\,dx \\
&\qquad \le C_1(1+t)^{-1}\int_{R^N}|D^m u|^2\,dx + C_2(1+t)^{-N/2-m-1}. \qquad (3.31)
\end{aligned}$$

Setting

$$B(t) = \left\{\xi \in R^N : |\xi|^2(1+t) \le \frac{\alpha + C_1}{d}\right\} \qquad \left(\alpha > m + \frac{N}{2}\right),$$

we have from (3.31) that

$$\frac{d}{dt}\left\{(1+t)^{\alpha}\int_{R^N}|D^m u|^2\,dx\right\}$$
$$\leq (\alpha + C_1)(1+t)^{\alpha-1}\int_{R^N}|D^m u|^2\,dx$$
$$+ C_2(1+t)^{\alpha-N/2-m-1} - d(1+t)^{\alpha}\int_{R^N}|D^{m+1}u|^2\,dx. \quad (3.32)$$

Since

$$d(q+t)^{\alpha}\int_{R^N}|D^{m+1}u|^2\,dx = d(1+t)^{\alpha}\int_{R^N}|\xi|^{2(m+1)}|\hat{u}|^2\,d\xi$$
$$\geq d(1+t)^{\alpha}\int_{B(t)^C}|\xi|^{2(m+1)}|\hat{u}|^2\,d\xi$$
$$\geq (\alpha + C_1)(1+t)^{\alpha-1}\int_{R^N}|D^m u|^2\,d\xi$$
$$-(\alpha + C_1)(1+t)^{\alpha-1}\int_{B(t)}|\xi|^{2m}|\hat{u}|^2\,d\xi, \quad (3.33)$$

we have from (3.32), (3.33) that

$$\frac{d}{dt}\left\{(1+t)^{\alpha}\int_{R^N}|D^m u|^2\,dx\right\}$$
$$\leq (\alpha + C_1)(1+t)^{\alpha-1}\int_{B(t)}|\xi|^{2m}|\hat{u}|^2\,d\xi + C_2(1+t)^{\alpha-N/2-m-1}$$
$$\leq C(1+t)^{\alpha-m-N/2-1}. \quad (3.34)$$

Integrating the above differential inequality over $[\tau, t]$ with respect to t, we can immediately deduce that (3.19) holds for $k = m$ and this completes the proof of the second step.

Based on the above two steps and from the principle of induction, we can easily deduce that (3.19) is true for each positive integer k. Thus the proof of Theorem 3.7 is complete.

For the case of $N = 2$, we have the following result

THEOREM 3.8. *Relation* (3.19) *also holds for* $N = 2$.

Proof. We first prove the following assertion:

Assertion A. If (3.19) with $N = 2$ is true for $k \leq 4$, then (3.19) with $N = 2$ is true for k ≥ 5.

In fact, if (3.19) with $N = 2$ is true for $k \leq 4$, we have

$$\|u(t,x)\|_{L^\infty(R^2, R^n)} \leq C(1+t)^{-1}, \tag{3.35}$$

and its easy to see that to prove the above assertion, we only need to prove that if (3.19) with $N = 2$ is true for $k \leq m - 1 (m \geq 5)$, then (3.19) with $N = 2$ is true for $k = m$. In what follows, we will prove this observation.

First, noticing that $m \geq 5$, we have that the nonnegative integers $\alpha_1, \dots, \alpha_s$ in (3.27) must satisfy

$$s \leq m - 3, \tag{3.36}$$

or s can be equal to $m - 2$ and $m - 1$ but $\alpha_{m-2}, \alpha_{m-1}$ must satisfy

$$\alpha_{m-1} + \alpha_{m-2} = 1. \tag{3.37}$$

Consequently

$$s - 1 \leq m - 3 \qquad \text{and} \qquad \alpha_s = 1 \text{ with } s = m - 1 \text{ or } s = m - 2. \tag{3.38}$$

On the other hand, since $N = 2$, we have the inequalities

$$\begin{cases} \|D^j u\|_{L^\infty(R^2, R^n)} \leq C\|D^{j+2}u\|^{1/2}_{L^2(R^2, R^n)} \|D^j u\|^{1/2}_{L^2(R^2, R^n)}, \\ \qquad j = 1, 2, \dots, \\ \|D^j u\|_{L^2(R^2, R^n)} \leq C\|D^{j+1}u\|^{1/2}_{L^2(R^2, R^n)} \|D^{j-1} u\|^{1/2}_{L^2(R^2, R^n)}, \\ \qquad j = 1, 2, \dots . \end{cases} \tag{3.39}$$

Combining (3.38), (3.39) with the assumption that (3.19) is true for $k \leq m - 1$ $(m \geq 5)$, we have

$$\begin{cases} \|D^j u\|_{L^\infty(R^2, R^n)} \leq C\|D^{j+2}u\|^{1/2}_{L^2(R^2, R^n)} \|D^j u\|^{1/2}_{L^2(R^2, R^n)} \\ \qquad\qquad \leq C(1+t)^{-(2+j)/2}, \qquad j = 1, 2, \dots, s-1, \\ \|D^s u\|_{L^2(R^2, R^n)} \leq C(1+t)^{(1+s)/2}. \end{cases} \tag{3.40}$$

Thus from the above discussions, (3.27) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_{R^2}|D^m u|^2\,dx &+ d\int_{R^2}|D^{m+1}u|^2\,dx\\ &\le C\|u\|^2_{L^\infty(R^2,R^n)}\int_{R^2}|D^m u|^2\,dx\\ &\quad + C\sum_{\substack{\sum_{i=1}^s i\alpha_i=m\\ 1\le s<m\\ \alpha_s=1}}\prod_{i=1}^{s-1}\|D^i u\|^{2\alpha_i}_{L^\infty(R^2,R^n)}\int_{R^2}|D^2u|^2\,dx\\ &\le C(1+t)^{-2}\int_{R^2}|D^m u|^2\,dx + C(1+t)^{-2\sum_{i=1}^s\alpha_i-m+1}. \quad (3.41)\end{aligned}$$

Due to

$$2\sum_{i=1}^{s}\alpha_i + m - 1 \ge m + 2,$$

then, similar to the proof of Theorem 3.7, we can prove that (3.19) with $N = 2$ is true for $k = m$. This proves Assertion A.

From Assertion A, to prove Theorem 3.8, we only need to show that (3.19) with $N = 2$ is true for $k = 1, 2, 3, 4$. In the following, we only give the proof of the cases of $k = 1, 2$. For $k = 3, 4$, the proof is similar and the details are omitted.

Similar to the proof of (3.27), we have

$$\begin{aligned}\frac{d}{dt}\int_{R^2}|Du|^2\,dx + d\int_{R^2}|D^2u|^2\,dx &\le C\|u\|^2_{L^\infty(R^2,R^n)}\int_{R^2}|Du|^2\,dx\\ &\le C\|u\|^2_{L^2(R^2,R^n)}\int_{R^2}|D^2u|^2\,dx\\ &\le C(1+t)^{-1}\int_{R^2}|D^2u|^2\,dx. \quad (3.42)\end{aligned}$$

If we choose t_0 sufficiently large such that

$$C(1+t)^{-1} \le \frac{d}{2} \qquad \text{for } t \ge t_0, \tag{3.43}$$

then we have from (3.42), (3.43) that

$$\frac{d}{dt}\int_{R^2}|Du|^2\,dx + \frac{d}{2}\int_{R^2}|D^2u|^2\,dx \le 0, \qquad t \ge t_0. \tag{3.44}$$

Let

$$B(t) = \left\{ \xi \in R^2 : |\xi|^2 (q + t) \le \frac{2\alpha}{d} \right\} \qquad (\alpha > 2).$$

Then

$$\begin{aligned} &\frac{d}{dt}\left\{ (1+t)^{\alpha} \int_{R^2} |Du|^2 \, dx \right\} \\ &\quad \le \alpha (1+t)^{\alpha-1} \int_{R^2} |Du|^2 \, dx - \frac{d}{2} (1+t)^{\alpha} \int_{R^2} |D^2 u|^2 \, dx \\ &\quad \le \alpha (1+t)^{\alpha-1} \int_{R^2} |Du|^2 \, dx - \frac{d}{2} (1+t)^{\alpha} \int_{B(t)^C} |\xi|^4 |\hat{u}| d\xi \\ &\quad \le \alpha (1+t)^{\alpha-1} \int_{B(t)} |\xi|^2 |\hat{u}|^2 \, d\xi \\ &\quad \le C (1+t)^{\alpha-3}, \qquad t \ge t_0. \end{aligned}$$

Integrating the above differential inequality with respect to t over $[t_0, t]$, we get

$$BI_{R^2} |Du|^2 \, dx \le C(1+t)^{-2},$$

which means (3.19) with $N = 2$ is true for $k = 1$.

We now turn to prove that (3.19) with $N = 2$ is true for $k = 2$.

Similar to (3.42), we have

$$\begin{aligned} &\frac{d}{dt} \int_{R^2} |D^2 u|^2 \, dx + d \int_{R^2} |D^3 u|^2 \, dx \\ &\quad \le C \|u\|^2_{L^\infty(R^2, R^n)} \int_{R^2} |D^2 u|^2 \, dx \\ &\qquad + C \|Du\|^2_{L^\infty(R^2, R^n)} \int_{R^2} |Du|^2 \, dx \\ &\quad \le \|u\|_{L^2(R^2, R^n)} \|D^2 u\|^3_{L^2(R^2, R^n)} \\ &\qquad + \|D^3 u\|_{L^2(R^2, R^n)} \|Du\|^3_{L^2(R^2, R^n)} \\ &\quad \le \frac{d}{2} \int_{R^2} |D^3 u|^2 \, dx + C\Big(\|Du\|^6_{L^2(R^2, R^n)} \|u\|^4_{L^2(R^2, R^n)} + \|Du\|^6_{L^2(R^2, R^n)} \Big) \\ &\quad \le \frac{d}{2} \int_{R^2} |D^3 u|^2 \, dx + C(1+t)^{-6}, \end{aligned}$$

i.e.,

$$\frac{d}{dt}\int_{R^2}|D^2u|^2\,dx + \frac{d}{2}\int_{R^2}|D^3u|^2\,dx \le C(1+t)^{-6}. \tag{3.45}$$

Setting

$$B(t) = \left\{\xi \in R^2 : |\xi|^2(1+t) \le \frac{2\alpha}{d}\right\} \qquad (\alpha > 3),$$

we have

$$\begin{aligned}\frac{d}{dt}\left\{(1+t)^{\alpha}\int_{R^2}|D^2u|^2\,dx\right\} &\le \alpha(1+t)^{\alpha-1}\int_{B(t)}|\xi|^4|\hat{u}|^2\,d\xi + C(1+t)^{\alpha-6}\\ &\le C(1+t)^{\alpha-4}. \end{aligned}\tag{3.46}$$

From (3.46), we can easily deduce that (3.19) with $N = 2$ is true for $k = 2$ and as a direct corollary of the above results, we can easily deduce that (3.35) holds.

Having obtained (3.35), for $k = 3, 4$, we can get the differential inequalities

$$\frac{d}{dt}\int_{R^2}|D^3u|^2\,dx + d\int_{R^2}|D^4u|^2\,dx \le C(1+t)^{-1}\int_{R^2}|D^3u|^2\,dx + C(1+t)^{-7}, \tag{3.47}$$

and

$$\frac{d}{dt}\int_{R^2}|D^4u|^2\,dx + d\int_{R^2}|D^5u|^2\,dx \le C(1+t)^{-1}\int_{R^2}|D^4u|^2\,dx + C(1+t)^{-7}, \tag{3.48}$$

and from the above two differential inequalities, we can also deduce that (3.19) with $N = 2$ is true for $k = 3, 4$. This completes the proof of Theorem 3.8.

As a direct corollary of Theorem 3.7, Theorem 3.8, we have

COROLLARY 3.9. *Under the conditions of Theorem* 3.7, *Theorem* 3.8, *we have*

$$\|D^k u(t,x)\|_{L^\infty(R^N, R^n)} \le C(1+t)^{-(N+k)/2}, \qquad k = 0, 1, 2, \ldots . \tag{3.49}$$

Remark. From the proof of Theorem 3.7, Theorem 3.8, we can deduce that the assumption $s > 2 + 1/N$ needed in (ii) of Theorem 1 is used only to get the optimal $L^2(R^N, R^n)$-norm temporal decay estimate. In this remark, we will show that if the system (1.1) is equipped with a quadratic strictly convex entropy $\eta(u)$ which is strongly consistent with the viscous matrix D, i.e., (1.5) holds, then we can replace the assumption $s > 2 + 1/N$ in (ii) of Theorem 1 by $s \geq 1 + 1/N$ while the same results also hold.

To show that the above assertion is true, we only need to get the optimal $L^2(R^N, R^n)$-norm temporal decay estimate for the global solution $u(t, x)$. This is just what we want to do in the following.

Multiplying (1.1) by $\nabla\eta(u)^t$ and integrating the results over R^N with respect to x, after some integrations by parts, we get from (1.5) that

$$\frac{d}{dt}\int_{R^N}\eta(u)\,dx + \varepsilon\int_{R^N}|Du|^2\,dx \leq 0. \tag{3.50}$$

If we let $B(t) = \{\xi \in R^N : |\xi|^2(1+t) \leq 2N/\delta\varepsilon\}$, we have from (3.50) and (1.5) that

$$\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{2N}\int_{R^N}\eta(u)\,dx\right\}\\
&\quad = 2N(1+t)^{2n-1}\int_{R^N}\eta(u)\,dx + (1+t)^{2N}\frac{d}{dt}\int_{R^N}\psi(u)\,dx\\
&\quad \leq \frac{2N}{\delta}(1+t)^{2N-1}\int_{R^N}|u|^2\,dx - \varepsilon(1+t)^{2N}\int_{R^N}|\xi|^2|\hat{u}|^2\,d\xi\\
&\quad \leq \frac{2N}{\delta}(1+t)^{2N-1}\int_{B(t)}|\hat{u}|^2\,d\xi\\
&\quad \leq C(1+t)^{2N-1}\int_{|\xi|^2\leq 2N/\delta\varepsilon(1+\varepsilon)}d\xi\\
&\quad \leq C(1+t)^{(3/2)N-1}.
\end{aligned}$$

From the above inequality, we can easily get

$$\int_{R^N}|u|^2\,dx \leq C(1+t)^{-N/2},$$

and this completes the proof of this remark.

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