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Resultants and Moving Surfaces

CARLOS D'ANDREA†

*Departamento de Matemática, F.C.E. y N., Universidad de Buenos Aires, Pabellón I,
Ciudad Universitaria (1428) Buenos Aires, Argentina*

We prove a conjectured relationship between resultants and the determinants arising in the formulation of the method of moving surfaces for computing the implicit equation of rational surfaces formulated by Sederberg. In addition, we extend the validity of this method to the case of not properly parametrized surfaces without base points.

1. Introduction

Given four polynomials in two variables $x_1(s,t)$, $x_2(s,t)$, $x_3(s,t)$ and $x_4(s,t)$, the equations

$$X_1 = \frac{x_1(s,t)}{x_4(s,t)}, \qquad X_2 = \frac{x_2(s,t)}{x_4(s,t)}, \qquad X_3 = \frac{x_3(s,t)}{x_4(s,t)}, \tag{1}$$

define a parametrization of a rational surface.

The implicitization problem consists in finding another polynomial $F(X_1, X_2, X_3)$ such that $F(X) = 0$ is the equation of the smallest algebraic surface containing (1).

A classical method for finding this implicit equation is to eliminate the variables s and t by computing the bivariate resultant of the polynomials

$$x_1(s,t) - X_1\, x_4(s,t), \qquad x_2(s,t) - X_2\, x_4(s,t), \qquad x_3(s,t) - X_3\, x_4(s,t). \tag{2}$$

There are several types of bivariate resultant (Dixon, 1908; Sturmfels, 1993; Gelfand *et al.*, 1994). They are related with different compactifications of the affine space where the input polynomials are defined. For example, if we view the polynomials x_i as general polynomials of degree less than or equal to n, the *multivariate resultant* may be taken, and its vanishing means that the system (2) has a common root in the projective space $\mathbb{P}^2$. This is the so-called *dense* or *triangular case*.

Another situation is when we regard the polynomials x_i as having degree less than or equal to m in s, and less than or equal to n in t. Here, one can use the *bihomogeneous resultant*. This is the *tensor product case*, and the vanishing of the resultant means that the system has a solution in $\mathbb{P}^1 \times \mathbb{P}^1$.

In both cases, the resultant of (2) gives the implicit equation (actually, a power of it) in the absence of *base points*, i.e. when there are no (s_0, t_0) in the corresponding space ($\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$) such that

$$x_1(s_0,t_0) = x_2(s_0,t_0) = x_3(s_0,t_0) = x_4(s_0,t_0) = 0.$$

†E-mail: cdandrea@dm.uba.ar

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Sederberg and Chen (1995) introduced a new technique called *the moving quadric method* for finding the implicit equation of (1). It uses smaller determinants than the classical methods and often works in the presence of base points.

A detailed analysis of this technique is given in Cox *et al.* (2000), where sufficient conditions are established for the validity of implicitization by the method of moving quadrics for rectangular tensor product surfaces and triangular surfaces in the absence of base points, and when the surface is properly parametrized, i.e. in when the parametrization is not necessarily one-to-one. In that paper the authors conjectured a relationship between the moving plane and the moving quadric coefficient matrices for both the tensor product and the triangular case (Conjectures 6.1 and 6.2 in Cox *et al.*, 2000). The intuition behind this conjecture was a similar relationship valid in the plane case (see, for instance, Zhang *et al.*, 1999).

This paper presents a general relationship between the moving plane and the moving surfaces of degree d coefficient matrices. As a special case, when $d = 2$, it provides a proof for both conjectures. In addition, the validity of the method when the surface has no base points, but is not necessarily properly parametrized, is proven.

We will approach the problem by factorizing the moving surface coefficient matrices as products of simpler matrices associated to linear maps. This, combined with the well-known formulation of the resultant as the determinant of a Koszul complex (Gelfand *et al.*, 1994), gives the desired results which recover, as a particular case, Theorems 4.1 and 5.1 in Cox *et al.* (2000). Adapting these techniques to the planar case, one can also produce alternative proofs of similar relationships given in Sederberg *et al.* (1997) and Zhang *et al.* (1999).

The paper is organized as follows: in Section 2, some geometric definitions are established in order to provide a better understanding of the relationships which will follow. In Section 3, the relations between maximal minors of moving surfaces matrices and the resultant are proven for surfaces parametrized by bihomogeneous polynomials. In the following section, the validity of the method of moving quadrics is extended to the case of surfaces without base points but not properly parametrized. Finally, in the last section, the same situation is considered for surfaces parametrized by homogeneous polynomials.

2. Moving Surfaces

This section reviews some basic notions used in the "method of moving conics and quadrics", as stated in Cox *et al.* (2000), Sederberg and Chen (1995), Zhang *et al.* (1999), in order to provide a geometric meaning of the algebraic tools to be developed in the following paragraphs.

Let $\mathbb{K}$ be a field. A d-surface is an implicit homogeneous equation in the variables X_1, X_2, X_3 and X_4 of degree d:

$$\sum_{|\gamma|=d} c_\gamma X^\gamma = 0, \qquad c_\gamma \in \mathbb{K}.$$

A moving d-surface of bi-degree (σ_1, σ_2) is a family of d-surfaces parametrized by s, u, t and v as follows:

$$\sum_{i=0}^{\sigma_1} \sum_{j=0}^{\sigma_2} \left(\sum_{|\gamma|=d} A_\gamma^{ij} X^\gamma \right) s^i\, u^{\sigma_1 - i}\, t^j\, v^{\sigma_2 - j} = 0, \qquad A_\gamma^{ij} \in \mathbb{K}. \tag{3}$$

For each fixed value of the parameters, equation (3) is an implicit equation of degree d in $\mathbb{K}^3$.

Similarly, a moving d-surface of degree σ, is defined by

$$\sum_{i+j\leq\sigma}\left(\sum_{|\gamma|=d} A_\gamma^{ij} X^\gamma\right) s^i\, t^j\, u^{\sigma-i-j} = 0, \qquad A_\gamma^{ij} \in \mathbb{K}. \tag{4}$$

In both cases, a moving 1-surface will be called a "moving plane". If $d = 2$, it is a "moving quadric" (cf. Cox *et al.*, 2000).

Given a family of four bihomogeneous (resp. homogeneous) polynomials $x_i(s,u;v,t)$ (resp. $x_i(s,t,u)$) of bi-degree (m,n) (resp. degree n) with coefficients in $\mathbb{K}$, the moving d-surface (3) (resp. (4)) is said to follow the rational surface $\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right)$ if

$$\sum_{i=0}^{\sigma_1}\sum_{j=0}^{\sigma_2}\left(\sum_{|\gamma|=d} A_\gamma^{ij} x^\gamma\right) s^i\, u^{\sigma_1-i}\, t^j\, v^{\sigma_2-j} = 0$$

resp.

$$\sum_{i+j\leq\sigma}\left(\sum_{|\gamma|=d} A_\gamma^{ij} x^\gamma\right) s^i\, t^j\, u^{\sigma-i-j} = 0.$$

In order to find the $\mathbb{K}$-vector space of all d-surfaces that follow the rational surface of a fixed bi-degree (resp. degree), set the coefficients of all the monomials $s^\alpha\, u^{\sigma_1-\alpha}\, t^\beta\, v^{\sigma_2-\beta}$ (resp. $s^\alpha\, t^\beta\, u^{\sigma-\alpha-\beta}$) in the implicit equation equal to zero, and solve the linear system of equations in the indeterminates $\{A_\gamma^{ij}\}$.

Example 2.1. Consider the following family of homogeneous polynomials:

$$\begin{aligned} x_1 &= s^3 \\ x_2 &= t^3, \\ x_3 &= u^3, \\ x_4 &= s^3 + t^3 + u^3. \end{aligned} \tag{5}$$

They define a parametric surface contained in the hyperplane

$$X_1 + X_2 + X_3 - X_4 = 0$$

which is a moving plane of degree 0. Note that there exists a moving plane of degree zero if and only if the surface is contained in a plane. In this case, this is the only plane which contains (5), so it is a basis of the moving planes of degree 0. Also, it is straightforward to compute a family of generators for the moving quadrics of the same degree:

$$\begin{aligned} X_1(X_1 + X_2 + X_3 - X_4) &= 0, \\ X_2(X_1 + X_2 + X_3 - X_4) &= 0, \\ X_3(X_1 + X_2 + X_3 - X_4) &= 0, \\ X_4(X_1 + X_2 + X_3 - X_4) &= 0. \end{aligned}$$

Example 2.2. This example appears in Cox *et al.* (2000). Set

$$x_1 = s\,t + u\,v,$$

$$x_2 = s\,v,$$
$$x_3 = u\,t,$$
$$x_4 = s\,v + u\,t + u\,v,$$

and $\sigma_1 = \sigma_2 = 1$. A basis of the space of moving planes of bidegree $(1,1)$ which follow the parametric surface is given by

$$(X_4 - X_1 - X_2 - X_3)\,u\,v + s\,v\,X_3 = 0,$$
$$(X_4 - X_1 - 2\,X_2 - X_3)\,u\,v + s\,v(X_4 - X_2) = 0.$$

With the aid of Maple, a basis of 24 moving quadrics of the same bidegree was found, 8 of which come from the moving planes by multiplication by $X_1, \ldots, X_4$.

3. The Tensor Product Surface Case

3.1. NOTATION

Let x_1, x_2, x_3 and x_4 be four generic bihomogeneous polynomials in two variables of bi-degree (m, n), i.e.

$$x_i(s, u; t, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} c_{jk}^{i}\; s^j\, u^{m-j}\, t^k\, v^{n-k} \qquad i = 1, \ldots, 4.$$

Set $\mathbb{K} := \mathbb{Q}(c_{jk}^{i})$, and let $S_{k,l}$ denote the space of polynomials of bi-degree (k, l) with coefficients in $\mathbb{K}$.

CONVENTION 3.1. *All spaces to be considered have a monomial basis. Suppose all these bases have a fixed order. Then, matrices "in the monomial bases" may be defined with no ambiguity.*

Let ϕ be the $\mathbb{K}$-linear map

$$\begin{array}{cccc} \phi: & {S_{m-1,n-1}}^4 & \to & S_{2m-1,2n-1} \\ & (p_1, p_2, p_3, p_4) & \mapsto & \sum_{i=1}^{4} p_i\, x_i, \end{array} \tag{6}$$

and, following (Cox *et al.*, 2000), denote by MP the matrix of ϕ in the monomial bases. It is square, of size $4mn$.

REMARK. With the definitions stated in the previous section, it is not hard to check that MP is the coefficient matrix of the linear system generated by the moving planes of bi-degree $(m-1, n-1)$ that follow the rational surface given by $\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right)$.

Let d be a positive integer and set

$$\Gamma := \{\gamma \in {\mathbb{Z}_{\geq 0}}^4 : |\gamma| = d\}.$$

Consider the map

$$\Psi^d : {S_{m-1,n-1}}^{\Gamma} \to S_{(d+1)m-1,(d+1)n-1}$$

which sends the sequence $(p_\gamma)_{\gamma \in \Gamma}$ to the polynomial

$$\sum_{\gamma \in \Gamma} p_\gamma x^\gamma. \tag{7}$$

Let MQ^d be the matrix of Ψ^d in the monomial bases. Also, set

$$\Gamma_0 := \{\gamma \in {\mathbb{Z}_{\geq 0}}^4 : \ |\gamma| = d, \ \gamma_4 \leq 1\}.$$

One can check that its cardinality is $(d+1)^2$. Consider the map ψ^d, the restriction of Ψ^d to ${S_{m-1,n-1}}^{\Gamma_0}$.

Denote by MS^d the matrix of ψ^d in the monomial bases. It is a square matrix of size $(d+1)^2 mn$.

Remark. If $d = 1$, then $\psi^d = \phi$ and $MS^1 = MP$. For $d = 2$, the matrices MQ^2 and MS^2 are denoted by MQ and MQ_w respectively in Cox *et al.* (2000).

Remark. It is straightforward to check that MS^d is a maximal minor in MQ^d. Furthermore, $\ker(MQ^d)$ is the $\mathbb{K}$-vector space of moving d-surfaces of bi-degree $(m-1, n-1)$ that follow the rational surface.

Consider the subset Γ_1 of Γ_0 defined by those γ such that $\gamma_1 = 0$. Set

$$\rho^d : {S_{m-1,n-1}}^{\Gamma_1} \oplus S_{dm-1,dn-1} \to S_{(d+1)m-1,(d+1)n-1} \tag{8}$$

the linear mapping which sends $((p_\gamma)_{\gamma \in \Gamma_1}, q)$ to

$$\sum_{\gamma \in \Gamma_1} p_\gamma x^\gamma + q\, x_1. \tag{9}$$

Denote by MT^d the matrix of ρ^d in the monomial bases. One can check that it is a square matrix of the same size as MS^d.

For a a square matrix A, its determinant will be denoted by $|A|$.

3.2. computing resultants using Koszul complexes

We begin by reviewing the computation of the determinant of a short exact sequence of vector spaces as given by Gelfand *et al.* (1994, Appendix A). Consider the following exact complex:

$$0 \longrightarrow A \xrightarrow{d_0} B \xrightarrow{d_1} C \longrightarrow 0\,. \tag{10}$$

Let $\{a_1, \dots, a_p\}$, $\{b_1, \dots, b_q\}$, $\{c_1, \dots, c_r\}$ be bases in A, B, C respectively $(p + r = q)$. In this case, the determinant of the complex with respect to these bases is equal to the coefficient of proportionality

$$\frac{b_1 \wedge \cdots \wedge b_q}{d_0(a_1) \wedge \cdots \wedge d_0(a_p) \wedge \widehat{c_1} \wedge \cdots \wedge \widehat{c_r}} \tag{11}$$

where $\widehat{c_1}, \dots, \widehat{c_r} \in B$ satisfy $d_1(\widehat{c_i}) = c_i$.

One can make an explicit computation of (11) as follows:

(1) Compute the matrices D_0 and D_1 corresponding to d_0 and d_1 respectively in the chosen bases.
(2) Let $\overline{D_1}$ be the submatrix of D_1 given by all the r rows and the first r columns. Denote by $\overline{D_0}$ the submatrix of D_0 given by the last p rows and all the p columns.

(3) It turns out that $|\overline{D_0}| \neq 0 \iff |\overline{D_1}| \neq 0$ (cf. Gelfand *et al.*, 1994). If this is the case, then

$$\det(\text{complex}) = \frac{|\overline{D_1}|}{|\overline{D_0}|} \tag{12}$$

(4) If $|\overline{D_0}| = |\overline{D_1}| = 0$, then another maximal minor may be chosen in D_0 as follows: let $I = \{i_1, \ldots, i_r\}$ be an ordered subset of r integers chosen from $\{1, \ldots, q\}$. Set $\overline{D_{0,I}}$, (resp. $\overline{D_{1,I}}$) the submatrix of D_0 (resp. D_1) obtained by choosing all the columns (resp. rows) and the rows (resp. columns) indexed by $\{1, \ldots, q\} \setminus I$ (resp. I).
Change the order in the basis of B in such a way that the last r elements are now indexed by I. Using Proposition 9 in Gelfand *et al.* (1994, Appendix 10), it is straightforward to check that

$$|\overline{D_{0,I}}| \,.\, \det(\text{complex}) = (-1)^{\sigma} \,.\, |\overline{D_{1,I}}|, \tag{13}$$

σ being the parity of the permutation

$$\{i_1, \ldots, i_r, 1, 2, \ldots, i_1 - 1, i_1 + 1, \ldots, i_r - 1, i_r + 1, \ldots, q\}.$$

Recall the definition of $\mathrm{Res}_{m,n}(f_1, f_2, f_3)$, the bihomogeneous resultant associated with a sequence of three generic polynomials of bi-degree (m, n) (see, for instance, Dixon, 1908; Gelfand *et al.*, 1994): it is an irreducible polynomial in the coefficients of f_i which vanishes after a specialization of the coefficients in a field k if and only if the specialized system $f_i = 0$ has a solution in $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$.

One may compute powers of $\mathrm{Res}_{m,n}(x_1, x_2, x_3)$ as a determinant of a three-term exact Koszul complex as follows: consider the complex of $\mathbb{K}$-vector spaces:

$$\begin{aligned} 0 \longrightarrow {S_{m-1,n-1}}^2 \oplus S_{(d-1)m-1,(d-1)n-1} &\xrightarrow{\psi_0} {S_{dm-1,dn-1}}^2 \oplus S_{2m-1,2n-1} \xrightarrow{\psi_1} \\ &S_{(d+1)m-1,(d+1)n-1} \to 0, \end{aligned} \tag{14}$$

where ψ_1 and ψ_0 are the Koszul morphisms

$$\psi_1(p, q, r) := p\, x_1 + q\, x_2 + r\, x_3^{d-1}; \tag{15}$$

$$\psi_0(p, q, r) := (q\, x_3^{d-1} + r\, x_2, p\, x_3^{d-1} - r\, x_1, \, - p\, x_2 - q\, x_1). \tag{16}$$

Proposition 3.1. *The complex* (14) *is exact, and after a specialization of the coefficients in a field k it will remain exact* (*as a complex of k-vector spaces*) *if and only if the bihomogeneous resultant of the specialized polynomials does not vanish. The determinant of the complex with respect to the monomial bases equals* $\pm\mathrm{Res}_{m,n}(x_1, x_2, x_3)^{d-1}$.

Proof. Let $y_3(s, u; t, v)$ be a generic bihomogeneous polynomial of bi-degree $((d-1)m, (d-1)n)$. Consider the modified complex which is made by replacing x_3^{d-1} with y_3 in (15), (16).

Because the polynomials x_1, x_2 and y_3 are bihomogeneous but do not have the same bi-degree, the bihomogeneous resultant cannot be taken. However, there is another elimination operator available: the *mixed resultant* associated with the sequence (x_1, x_2, y_3) (Gelfand *et al.*, 1994, Chapter 3). It is an irreducible polynomial in the coefficients of x_1, x_2, y_3 which vanishes if and only if these polynomials have a common root in $\mathbb{P}^1 \times \mathbb{P}^1$.

In order to compute it, we may apply the Cayley method for the study of resultants (see, Gelfand *et al.*, 1994, Chapter 3). Let $\mathcal{O}(d_1, d_2)$ denote the line bundle on $X := \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$

whose sections are homogeneous polynomials of degree d_1 in coordinates $(s:u)$ and degree d_2 in coordinates $(t:v)$ on each $\mathbb{P}^1$.

Let $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{O}(m,n)$ and $\mathcal{L}_3 = \mathcal{O}((d-1)m,(d-1)n)$. Each $\mathcal{L}_i$ is very ample, and we may regard polynomials of bi-degree (m,n) with coefficients in $\overline{k}$ as elements of $H^0(X,\mathcal{L}_i)$, $i=1,2$, and polynomials of bidegree $((d-1)m,(d-1)n)$ as belonging to $H^0(X,\mathcal{L}_3)$.

Then, every specialization of (x_1,x_2,y_3) defines a section s of the vector bundle $E := \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$. If we set $\mathcal{M} := \mathcal{O}((d+1)m-1,(d+1)n-1)$, and construct the complex $C^{\bullet}_{-}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3|\mathcal{M})$ (for a definition of this complex, see Gelfand *et al.*, 1994, Chapter 3), then we recover the modified complex (14). Moreover, it is not hard to check that this complex is stably twisted (i.e. has no higher cohomology), so Proposition 4.1 and Theorem 4.2 of Gelfand *et al.* (1994, Chapter 3) hold, and we have that the complex will be exact if and only if the resultant of the specialized (x_1,x_2,y_3) is not zero. Furthermore, the determinant of the complex with respect to the monomial bases is equal to $\pm\mathrm{Res}(x_1,x_2,y_3)$.

The original complex is recovered by specializing y_3 to x_3^{d-1}. Keeping in mind that

$$\mathrm{Res}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3^{d-1}) = 0 \iff \mathrm{Res}_{m,n}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) = 0,$$

we have that the determinant of (14) is equal to a power of $\mathrm{Res}_{m,n}(x_1,x_2,x_3)$.

Comparing the degrees of both the bihomogeneous and the mixed resultant in the coefficients of x_3 and y_3 respectively (cf. Gelfand *et al.*, 1994; Cox *et al.*, 1998) we get that the determinant of the complex (14) equals

$$\pm\mathrm{Res}(x_1,x_2,y_3)\big|_{y_3=x_3^{d-1}} = \pm\mathrm{Res}_{m,n}(x_1,x_2,x_3)^{d-1}. \ \square$$

Using the recipe given above, $\pm\mathrm{Res}_{m,n}(x_1,x_2,x_3)^{d-1}$ may be computed using the following algorithm:

(1) Construct the matrices corresponding to the linear maps ψ_0 and ψ_1 with respect to the monomial bases;
(2) Choose a non vanishing maximal minor $m_{1,I}$ in the matrix corresponding to ψ_1, I being a set of $(d+1)^2mn$ columns corresponding to vectors in the monomial basis of $\oplus_{i=1}^2 S_{dm-1,dn-1} \oplus S_{2m-1,2n-1}$.
(3) Compute $m_{0,I}$, the maximal minor in the matrix representing ψ_0, which consists of all rows *not* indexed by I.
(4) It turns out that $m_{0,I} \neq 0$. Compute $\frac{m_{1,I}}{m_{0,I}}$. This quotient is equal to the determinant of the complex.

Remark. The following equality holds for every subset of indices I:

$$m_{1,I} = (-1)^{\sigma}\mathrm{Res}_{m,n}(x_1,x_2,x_3)^{d-1}\, m_{0,I}. \tag{17}$$

3.3. The relationship between $|MS^d|$ and $|MP|$

In order to prove the main result of this section, a preliminary lemma is needed. Recall that MT^d is the matrix associated with the linear map (8), and MP is the matrix associated with (6).

Lemma 3.1. *The following equality holds:*

$$\pm|MT^d| = |MP|^d \operatorname{Res}_{m,n}(x_1, x_2, x_3)^{d(d-1)/2}.$$

Proof. The proof will be by induction on d. For $d = 1$, it is clear than ρ^1 and ϕ are the same functions, so the proposition follows straightforwardly.

Suppose then $d \geq 2$. The morphism ρ^d may be factored as follows:

$$S_{m-1,n-1}{}^{\Gamma_1} \oplus S_{dm-1,dn-1} \overset{\psi_2}{\to} S_{dm-1,dn-1}{}^2 \oplus S_{2m-1,2n-1} \overset{\psi_1}{\to} S_{(d+1)m-1,(d+1)n-1} \tag{18}$$

where ψ_1 is the morphism defined in (14) and

$$\psi_2(p_\gamma, q) = \left(q, \sum_{\gamma_2 \geq 1} p_\gamma x_2^{\gamma_2 - 1} x_3^{\gamma_3} x_4^{\gamma_4}, p_{(0,0,d,0)} x_3 + p_{(0,0,d-1,1)} x_4 \right). \tag{19}$$

Denote by M_i the matrix corresponding to ψ_i in the monomial bases for $i = 1, 2$. These are not square matrices (they have sizes $(d+1)^2 mn \times (2d^2+4)mn$ and $(2d^2+4)mn \times (d+1)^2 mn$ respectively), but applying the Cauchy–Binet formula (see, for instance, Horn and Johnson, 1985), there is a relationship between their maximal minors and $|MT^d|$:

$$|MT^d| = \sum_I |M_{1,I}|\,|M_{2,I}|, \tag{20}$$

the summation made over all sequences of integers

$$I = (i_1, \ldots, i_{(d+1)^2 mn})$$

with $1 \leq i_1 \leq \cdots \leq i_{(d+1)^2 mn} \leq (2d^2+4)mn$, and $M_{1,I}$ (resp. $M_{2,I}$) denotes the square submatrix of M_1 (resp. M_2) which is made by choosing the $(d+1)^2 mn$ columns (resp. rows) indexed by I. $\square$

Remark. Note that M_1 is the matrix corresponding to ψ_1 in the monomial bases and, for each I, the maximal minor $m_{1,I}$ in step 2 of the algorithm outlined in the previous paragraph is denoted $|M_{1,I}|$ in (20).

Using formulas (17) and (20), one has

$$|MT^d| = (\operatorname{Res}_{m,n}(x_1, x_2, x_3))^{d-1} \sum_I (-1)^\sigma\, |M_{2,I}|\, m_{0,I}. \tag{21}$$

An explicit computation of the $(2d^2+4)mn \times (d+1)^2 mn$ matrix M_2 reveals the following structure:

$$M_2 = \begin{bmatrix} \mathbb{I} & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{bmatrix}, \tag{22}$$

where B_1 and B_2 have sizes $d^2 mn \times (2d-1)mn$ and $4mn \times 2mn$ respectively, and $\mathbb{I}$ denotes the identity matrix of size $d^2 mn$.

Gluing M_2 and the $(2d^2+4)mn \times ((d-1)^2+2)mn$ matrix M_0 corresponding to ψ_0, one gets a square matrix $M := [M_2, M_0]$ of size $2d^2 + 4mn$ which has the following structure:

$$M = \left[\begin{array}{ccc|ccc} \mathbb{I} & 0 & 0 & 0 & q\,x_3^{d-1} & r\,x_2 \\ 0 & B_1 & 0 & p\,x_3^{d-1} & 0 & -r\,x_1 \\ 0 & 0 & B_2 & -p\,x_2 & -q\,x_1 & 0 \end{array}\right]; \tag{23}$$

where the block $[p\,x_3^{d-1}]$ denotes the matrix corresponding to the linear map $S_{m-1,n-1} \to S_{dm-1,dn-1}$ which maps p to $p\,x_3^{d-1}$, and the other blocks have the same meaning.

It is easy to check that $[B_2, -p\,x_2, -q\,x_1]$ is a square matrix. Moreover,

$$|[B_2, -p\,x_2, -q\,x_1]| = \pm|MP|.$$

In the same way, it holds that

$$|[B_1, -r\,x_1]| = \pm|MT^{d-1}|.$$

Then, the determinant of M equals $\pm|MT^{d-1}||MP|$, and then the inductive hypothesis yields the following equality:

$$|M| = \pm(\mathrm{Res}_{m,n}(x_1, x_2, x_3))^{(d-1)(d-2)/2}\,|MP|^d. \tag{24}$$

This determinant may also be computed as a sum of maximal minors of M_2 times their complementary minor in M. This is exactly the sum which appears in (21), i.e.

$$|M| = \sum_I (-1)^\sigma\,|M_{2,I}|\,m_{0,I}.$$

Replacing (24) in (21), the lemma follows. □

Theorem 3.2.

$$\pm|MS^d| = |MP|^{(d+1)d/2}\,\mathrm{Res}_{m,n}(x_1, x_2, x_3)^{(d+1)d(d-1)/6}.$$

Proof. As in the lemma, the proof will be by induction on d. For $d = 1$, it happens that $\psi^1 = \phi$, so the statement holds straightforwardly.

Take $d \geq 2$, and factor ψ^d as follows:

$$S_{m-1,n-1}{}^{\Gamma_0} \xrightarrow{\widetilde{\psi}_2} S_{dm-1,dn-1}{}^2 \oplus S_{2m-1,2n-1} \xrightarrow{\psi_1} S_{(d+1)m-1,(d+1)n-1} \tag{25}$$

where

$$\widetilde{\psi}_2(p_\gamma) = \left(\sum_{\gamma_1 \geq 1} p_\gamma x_1^{\gamma_1 - 1} x_2^{\gamma_2} x_3^{\gamma_3} x_4^{\gamma_4}, \sum_{\gamma \in \Gamma_1, \gamma_2 \geq 1} p_\gamma x_2^{\gamma_2 - 1} x_3^{\gamma_3} x_4^{\gamma_4}, p_{(0,0,d,0)} x_3 + p_{(0,0,d-1,1)} x_4 \right). \tag{26}$$

Denote by $\widetilde{M}_2$ the matrix corresponding to $\widetilde{\psi}_2$ in the monomial bases. The Cauchy–Binet formula gives the following relationship:

$$|MS^d| = \sum_I |M_{1,I}|\,|\widetilde{M}_{2,I}|, \tag{27}$$

where, as before, I runs through all sequences of integers $(i_1, \ldots, i_{(d+1)^2mn})$ satisfying $1 \leq i_1 \leq \cdots \leq i_{(d+1)^2mn} \leq (2d^2 + 4)mn$, and $M_{1,I}$ (resp. $\widetilde{M}_{2,I}$) have the same meaning as in (20).

Proceeding as in the proof of the previous lemma, one gets

$$|MS^d| = (\mathrm{Res}_{m,n}(x_1, x_2, x_3))^{d-1} \sum_I (-1)^\sigma\,|\widetilde{M}_{2,I}|\,m_{0,I}. \tag{28}$$

Gluing $\widetilde{M}_2$ and matrix M_0, one gets a square matrix with the following structure:

$$\widetilde{M} := \left[\begin{array}{ccc|ccc} MS^{d-1} & 0 & 0 & 0 & q\,x_3^{d-1} & r\,x_2 \\ 0 & B_1 & 0 & p\,x_3^{d-1} & 0 & -r\,x_1 \\ 0 & 0 & B_2 & -p\,x_2 & -q\,x_1 & 0 \end{array}\right]. \tag{29}$$

Here, B_1 and B_2 are the same blocks which appear in (22), and it is easy to check that the block $[B_1, r\,x_1]$ is the matrix MT^{d-1}.

Then, using inductive hypothesis and the previous lemma, the following equalities hold:

$$\begin{aligned}|\widetilde{M}| &= \pm|MS^{d-1}|\,|MT^{d-1}|\,|MP|\\ &= \pm(\mathrm{Res}_{m,n}(x_1,x_2,x_3))^{(d-1)(d-2)(d+3)/6}\,|MP|^{d(d+1)/2}.\end{aligned}$$

Computing this determinant as sum of maximal minors of $\widetilde{M}_2$ times their complementary minor in $\widetilde{M}$, it appears the summation in (28). Replacing it with this last expression, the theorem follows straightforwardly. □

Corollary 3.1. (Conjecture 6.1 in Cox *et al.*, 2000)

$$|MS^2| = |MP|^3\,\mathrm{Res}_{m,n}(x_1,x_2,x_3).$$

Corollary 3.2. (General version of Theorem 4.1 in Cox *et al.*, 2000) *Given four bihomogeneous polynomials $\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3$ and $\tilde{x}_4$ of bi-degree (m,n) and coefficients in $\mathbb{C}$. If $\mathrm{Res}_{m,n}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\neq 0$, then $|\tilde{MS^d}|=0$ implies $|\tilde{MP}|=0$.*

Here, $\tilde{M}$ means the matrix M where the generic coefficients have been specialized with the coefficients of the $\tilde{x}_i$. In the language of moving surfaces, Theorem 3.2 reads as follows.

If $\mathrm{Res}_{m,n}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\neq 0$, and there are no moving planes of bi-degree $(m-1,n-1)$ which follow the rational surface, then the dimension of the $\mathbb{C}$-vector space of d-surfaces of bidegree $(m-1,n-1)$ that follow the rational surface is equal to $\frac{(d+1)d(d-1)}{6}\,mn$.

4. The Validity of the Method of Moving Quadrics When the Surface is not Properly Parametrized

In this section, we are going to discuss the validity of implicitization by moving quadrics with no base points, without requirements on the parametrization of the surface. The main result will be an improvement of Theorem 4.2 in Cox *et al.* (2000), extending the validity of the method to the case when the parametrization is not generically one-to-one.

We will set $d=2$ and $\mathbb{K}=\mathbb{Q}(c^i_{jk})$. If $|MS^2|\neq 0$, then $\ker(MQ^2)$ has dimension equal to mn. Suppose without loss of generality that $MQ^2=[MS^2,R]$, where R is a submatrix of MQ^2 of size $9mn\times mn$.

In Cox *et al.* (2000) (Proof of Theorem 4.2), a particular basis of the kernel of MQ^2 (i.e. a matrix T of size $10mn\times mn$ such that $MQ^2\cdot T=0$) is considered:

$$T := \begin{bmatrix}\bar{T}\\ \mathbb{I}\end{bmatrix};$$

here, $\mathbb{I}$ denotes the identity matrix of order mn.

Solving the linear equation $MQ\cdot T=0$, one gets

$$T = \begin{bmatrix}-MS^{2^{-1}}\cdot R\\ \mathbb{I}\end{bmatrix}.$$

T has its rows indexed by the monomials

$$s^i\,t^j x^\gamma,\qquad 0\le i\le m-1,\qquad 0\le j\le n-1,\qquad |\gamma|=2,$$

and the last mn rows corresponds to the monomials

$$s^i\, t^j x_4^2, \qquad 0 \le i \le m-1, \qquad 0 \le j \le n-1.$$

Consider $\mathbb{T}$, the matrix which results reordering the rows of T as follows:

$$\begin{gathered} x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1\,x_2,\ x_1\,x_3,\ x_1\,x_4,\ x_2\,x_3,\ x_2\,x_4,\ x_3\,x_4, \\ s(x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1\,x_2,\ x_1\,x_3,\ x_1\,x_4,\ x_2\,x_3,\ x_2\,x_4,\ x_3\,x_4), \\ s\,t(x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1\,x_2,\ x_1\,x_3,\ x_1\,x_4,\ x_2\,x_3,\ x_2\,x_4,\ x_3\,x_4), \\ s\,t^2(x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1\,x_2,\ x_1\,x_3,\ x_1\,x_4,\ x_2\,x_3,\ x_2\,x_4,\ x_3\,x_4), \\ s\,t^3(x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1\,x_2,\ x_1\,x_3,\ x_1\,x_4,\ x_2\,x_3,\ x_2\,x_4,\ x_3\,x_4), \\ \dots \end{gathered} \tag{30}$$

The third row of $\mathbb{T}$, for example, corresponds to the monomial x_3^2; its eleventh row is indexed by $s\,x_1^2$.

Let X_1, X_2, X_3 and X_4 be indeterminates over $\mathbb{K}$. Consider the following vector in $\mathbb{Z}[X_1, X_2, X_3, X_4]^{10}$:

$$C := (X_1^2,\, X_2^2,\, X_3^2,\, X_4^2,\, X_1\,X_2,\, X_1\,X_3,\, X_1\,X_4,\, X_2\,X_3,\, X_2\,X_4,\, X_3\,X_4). \tag{31}$$

Let $M \in \mathbb{Z}[X_1, X_2, X_3, X_4]^{mn\times 10mn}$ be the matrix defined as follows:

$$M := \begin{bmatrix} C & C & C & \dots & C \\ 0 & C & C & \dots & C \\ 0 & 0 & C & \dots & C \\ & & \dots & & \\ 0 & 0 & 0 & \dots & C \end{bmatrix}.$$

Recall the construction given in the proof of Theorem 4.2 in Cox *et al.* (2000) for computing the implicit equation of the parametric surface: each column $T_{\alpha,\beta}$ of $\mathbb{T}$ encodes the coordinates (in the monomial basis) of a moving quadric of bidegree (m, n) which follows the surface. Write $T_{\alpha,\beta}$ as follows:

$$T_{\alpha,\beta} = \sum_{i\le m-1,\ j\le n-1} T_{i,j}^{\alpha,\beta}\, s^i\, t^j,$$

where

$$T_{i,j}^{\alpha,\beta} = \sum_{|\gamma|=2} a_{i,j}^{\gamma} x^{\gamma}.$$

Set $\widetilde{T}_{i,j}^{\alpha,\beta} := \sum_{|\gamma|=2} a_{i,j}^{\gamma} X^{\gamma}$, and consider the square matrix

$$\widetilde{T} := [T_{i,j}^{\alpha,\beta}] \in \mathbb{K}[X_1, X_2, X_3, X_4]^{mn\times mn},$$

where its rows are indexed by (i, j) and columns by (α, β).

Remark. $\widetilde{T}$ is the matrix called M in the proof of Theorem 4.2 in Cox *et al.* (2000).

Proposition 4.1. $|\,\widetilde{T}\,| = \pm|M \cdot \mathbb{T}|$.

Proof. Set $\mathcal{R} := M \cdot \mathbb{T}$. Computing explicitly the last row of $M \cdot \mathbb{T}$, because of the order given to the rows of $\mathbb{T}$ in (30), one obtains

$$[\widetilde{T}_{m-1,n-1}^{\alpha,\beta}]_{\alpha,\beta},$$

which coincides with one of the rows in $\widetilde{T}$.

In the same way, one can check that the row immediately before the last in $\mathcal{R}$, is the following:

$$[\widetilde{T}^{\alpha,\beta}_{m-1,n-2} + \widetilde{T}^{\alpha,\beta}_{m-1,n-1}]_{\alpha,\beta},$$

so, substracting from it the last row, one gets another row of $\widetilde{T}$.

A similar situation happens in all rows. This implies that the matrix $\mathcal{R}$ may be transformed in the matrix $\widetilde{T}$ applying operations on its rows which do not change the determinant. □

The following proposition will be useful in what follows.

Proposition 4.2. *Set*

$$P(X_1, X_2, X_3) := \mathrm{Res}_{m,n}(x_1 - X_1\, x_4, x_2 - X_2\, x_4, x_3 - X_3\, x_4).$$

Then, $P(X)$ is an irreducible polynomial in $\mathbb{K}[X_1, X_2, X_3]$ and its degree in the variables X_i is equal to mn.

Proof. The fact that the degree of P in X_i is mn can be easily checked in Dixon's matrix (cf. Dixon, 1908).

Set $\mathbf{Z} := \mathbb{Z}[c^i_{jk}, X_1, X_2, X_3]$. Suppose

$$P = A(c, X) \,.\, B(c, X)$$

where $A, B \in \mathbf{Z}$. The polynomial P is homogeneous in the variables c^i_{ij}, which implies that A and B are also homogenous in the c^i_{jk}.

Specializing $X_i \mapsto 0$, one has that

$$\mathrm{Res}_{m,n}(x_1, x_2, x_3) = A(c, 0) \,.\, B(c, 0).$$

But the left-hand side is irreducible. This implies that one of the factors must have degree 0 in the variables c^i_{jk}, lets say B. The factorization now reads as follows:

$$P = A(c, X) \,.\, B(X),$$

where $B \in \mathbb{Z}[X_1, X_2, X_3]$. But if $\deg_X B \geq 1$, then the variety $(B = 0) \neq \emptyset$ in $\mathbb{C}^3$. On the other hand, it is well known that, for a given family of bihomogeneous polynomials $\tilde{x}_i(s, u; t, v)$ with no base points, the equation

$$\mathrm{Res}_{m,n}(\tilde{x}_1 - X_1\, \tilde{x}_4, \tilde{x}_2 - X_2\, \tilde{x}_4, \tilde{x}_3 - X_3\, \tilde{x}_4) = 0,$$

is a power of the implicit equation of the rational surface defined by $\left(\frac{\tilde{x}_1}{\tilde{x}_4}, \frac{\tilde{x}_2}{\tilde{x}_4}, \frac{\tilde{x}_3}{\tilde{x}_4}\right)$ (actually, if we use Dixon matrices for computing this resultant, we will find moving planes encoded in their rows; see the comment at the end of Example 3 in Cox *et al.*, 2000). If B had a nonempty zero locus, this would imply that *every* rational parametric surface of bi-degree (m, n) without base points will contain the zero locus of B, which is impossible. □

Corollary 4.1. *Let $P^h(X_1, X_2, X_3, X_4)$ be the homogenization of P up to degree mn. Then, P^h is irreducible in $\mathbb{Z}[c^i_{jk}, X_1, X_2, X_3, X_4]$.*

The following may be regarded as the main result of this section.

THEOREM 4.1.

$$\mathrm{Res}_{m,n}(x_1, x_2, x_3) \,.\, |M \cdot \mathbb{T}| = P^h,$$

PROOF. To begin, it will be proven that the ratio between $|M \cdot \mathbb{T}|$ and P^h is in $\mathbb{K}$. Note that there is a positive integer z such that

$$\mathcal{R}^{'} := |MS|^z \,.\, \det(M \cdot \mathbb{T}) \in \mathbb{Z}[c^i_{jk}, X_1, X_2, X_3, X_4]$$

(this is due to the fact that, in the construction of $\mathbb{T}$, one needs the inverse of MS.)

Set $\mathcal{R}^{''} := |MS| \,.\, \mathcal{R}^{'}$. Both $R^{''}$ and P^h are homogeneous in X and of the same degree, which is equal to mn, so it will be enough to show that one of them is a polynomial multiple of the other.

Suppose that the variables have been specialized:

$$(c^i_{jk}, X) \mapsto (\tilde{c}^i_{jk}, \widetilde{X})$$

with the following conditions:

(1) $|\widetilde{MS}| \neq 0$, where $\widetilde{MS}$ denotes the matrix MS after specializing $c \mapsto \tilde{c}$.
(2) $P^h(\tilde{c}^i_{jk}, \widetilde{X})$ vanishes.

This means that the projective point $(\widetilde{X}_1 : \widetilde{X}_2 : \widetilde{X}_3 : \widetilde{X}_4)$ belongs to the rational surface defined by $\left(\frac{\tilde{x}_1}{\tilde{x}_4}, \frac{\tilde{x}_2}{\tilde{x}_4}, \frac{\tilde{x}_3}{\tilde{x}_4}\right)$.

Using $\widetilde{MS} \neq 0$ and the same argument as in the proof of Theorem 4.2 in Cox *et al.* (2000), one can verify that, if $(\widetilde{X}_1 : \widetilde{X}_2 : \widetilde{X}_3 : \widetilde{X}_4)$ belongs to that rational surface, the determinant of $\widetilde{T}$ must vanish. Proposition 4.1 implies that $\mathcal{R}^{''}$ must vanish. Hence,

$$(P^h = 0) \subset (\mathcal{R}^{''} = 0) \qquad \text{if} \qquad |MS| \neq 0.$$

If $|MS| = 0$, the inclusion holds trivially and, using the Hilbert's Nullstellensatz, one can conclude that a power of $\mathcal{R}^{''}$ must be a multiple of P^h. As P^h is irreducible, $\mathcal{R}^{''}$ must be a multiple of P^h, so

$$|M \cdot \mathbb{T}| = c \,.\, P^h,$$

where $c \in \mathbb{K}$ as expected.

In order to compute c, do the following replacement:

$$X_1, X_2, X_3 \mapsto 0, X_4 \mapsto 1.$$

Then, it will hold that

$$\begin{array}{ccc} P^h & \mapsto & \mathrm{Res}_{m,n}(x_1, x_2, x_3) \\ \widetilde{T} & \mapsto & \mathbb{I}_{mn} \end{array}$$

if the columns of $\widetilde{T}$ are properly ordered (cf. Cox *et al.*, 2000, proof of Theorem 4.2). So, c must be $\pm \frac{1}{\mathrm{Res}_{m,n}(x_1,x_2,x_3)}$ □

COROLLARY 4.2. *Given a family of bihomogeneous polynomials* $\tilde{x}_i(s, u; t, v) \in \mathbb{C}[s, u, t, v]$, $1 \leq i \leq 4$. *Suppose that the parametrization*

$$\left(\frac{\tilde{x}_1}{\tilde{x}_4}, \frac{\tilde{x}_2}{\tilde{x}_4}, \frac{\tilde{x}_3}{\tilde{x}_4}\right)$$

defines a surface which has no base points, and that there are no moving planes of bi-degree $(m-1, n-1)$ *following the surface. Then, the method of moving quadrics always computes a power of the implicit equation of this surface.*

PROOF. In order to make the method work correctly, one can suppose without loss of generality that $\mathrm{Res}_{m,n}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \neq 0$.

The method computes $|\widetilde{T}| = \pm|M \cdot \mathbb{T}|$ which is equal to a constant times $\tilde{P^h}$, because of Theorem 3.2. But $\tilde{P^h}$ vanishes if and only if the projective point (X_1, X_2, X_3, X_4) belongs to the implicit surface.

This, combined with the fact that the implicit equation of the parametric surface is always irreducible, completes the proof. □

5. The Triangular Surface Case

In the triangular case, similar results hold. We shall denote polynomials, linear maps and its matrices as in Section 3, though the reader should keep in mind that everything will now be homogeneous rather than bi-homogeneous.

Let

$$x_i(s,t,u) = \sum_{j+k \leq n} c^i_{jk}\, s^j\, t^k\, u^{n-j-k} \qquad i = 1, \ldots, 4$$

be four generic polynomials in three variables of degree n.

Set, as before, $\mathbb{K} := \mathbb{Q}(c^i_{jk})$, and let S_l be the space of homogeneous polynomials in three variables of degree l, with coefficients in $\mathbb{K}$.

Consider now:

$$\begin{array}{cccc} \phi: & {S_{n-1}}^4 & \rightarrow & S_{2n-1} \\ & (p_1, p_2, p_3, p_4) & \mapsto & \sum_{i=1}^4 p_i\, x_i, \end{array} \tag{32}$$

and, let MP be the matrix of ϕ in the monomial bases (cf. Cox *et al.*, 2000).

MP has size $(2n^2 + n) \times (2n^2 + 2n)$. In order to have a square submatrix, let $\mathcal{I} \subset \{(j,k) : j + k \leq n\}$ with $|\mathcal{I}| = n$. Define $MP_{\mathcal{I}}$ by removing the n columns in MP corresponding to

$$s^i\, t^j\, u^{n-i-j}\, x_4, \qquad (i,j) \in \mathcal{I}.$$

As in Section 3, $MP_{\mathcal{I}}$ corresponds to a maximal square submatrix of the coefficient matrix of the system generated by the moving planes of degree $n-1$ that follow the rational surface.

Set Γ and Γ_0 as before, and consider the maps

$$\Psi^d : {S_{n-1}}^{\Gamma} \rightarrow S_{(d+1)n-1}$$

which sends the sequence $(p_\gamma)_{\gamma\in\Gamma}$ to $\sum_{\gamma\in\Gamma} p_\gamma x^\gamma$, and

$$\psi^d : {S_{n-1}}^{\Gamma_0} \rightarrow S_{(d+1)n-1},$$

its restriction. Let MQ^d and MS^d be the matrices of Ψ^d and ψ^d in the monomial bases, respectively. MS^d has size $\frac{((d+1)n+1)(d+1)n}{2} \times \frac{(d+1)^2 n(n+1)}{2}$. To get a square submatrix of it, remove all the columns corresponding to

$$s^i\, t^j\, u^{n-i-j}\, x^\gamma, \qquad (i,j) \in \mathcal{I}, \qquad \gamma_4 = 1,$$

and define $MS^d_{\mathcal{I}}$ to be the remaining submatrix. It is associated with a linear map

$$\psi^d_{\mathcal{I}} : S^{\Gamma_0\setminus\Gamma_4}_{n-1,\mathcal{I}} \oplus {S_{n-1}}^{\Gamma_4} \rightarrow S_{(d+1)n-1} \tag{33}$$

where $S_{n-1,\mathcal{I}}$ is the $\mathbb{K}$-vector space generated by all monomials whose exponents are not in $\mathcal{I}$, and Γ_4 is the set of all multiindices $\gamma \in \Gamma_0$ such that $\gamma_4 = 0$. Also, $MS^d_{\mathcal{I}}$ may be regarded as a maximal square submatrix of the coefficient matrix of the moving d-surfaces of degree $n-1$ that follow the rational surface.

Let $\mathrm{Res}_n(f_1, f_2, f_3)$, be the multivariate resultant associated with a sequence of three generic polynomials of degree n (cf. Dixon, 1908; Gelfand *et al.*, 1994; Cox *et al.*, 1998): it is an irreducible polynomial in the coefficients of f_i, which vanishes after a specialization of the coefficients in an algebraically closed field $\overline{k}$ if and only if the specialized system $f_i = 0$ has a solution in $\mathbb{P}^2(\overline{k})$.

In this situation, a similar result holds:

Theorem 5.1.

$$|MS^d_{\mathcal{I}}| = \pm|MP_{\mathcal{I}}|^{(d+1)d(d-1)/6}\,(\mathrm{Res}_n(x_1, x_2, x_3))^{(d+1)d/2}.$$

Proof. The proof follows applying *mutatis mutandis* all the tools developed in Section 3. Consider the following Koszul complex:

$$0 \longrightarrow {S_{n-1}}^2 \oplus S_{(d-1)n-1} \xrightarrow{\psi_0} {S_{dn-1}}^2 \oplus S_{2n-1} \xrightarrow{\psi_1} S_{(d+1)n-1} \longrightarrow 0. \tag{34}$$

Here, ψ_1 and ψ_0 are defined by (15) and (16) respectively. A similar version of Proposition 3.1 holds, applying the same trick used there to the formulation of the multivariate resultant as the determinant of a Koszul complex (cf. Demazure, 1984; Chardin, 1993).

Proposition 5.1. *The complex* (34) *is exact, and after a specialization of the coefficients* $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$*, it will be exact if and only if* $\mathrm{Res}_n(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ *does not vanish. The determinant of the complex with respect to the monomial bases is equal to* $\pm\mathrm{Res}_n(x_1, x_2, x_3)^{d-1}$.

The linear map $\psi^d_{\mathcal{I}}$ may be factored as follows:

$$S^{\Gamma_0\setminus\Gamma_4}_{n-1,\mathcal{I}} \oplus {S_{n-1}}^{\Gamma_4} \xrightarrow{\widetilde{\psi}_2} {S_{dn-1}}^2 \oplus S_{2n-1} \xrightarrow{\psi_1} S_{(d+1)n-1}, \tag{35}$$

where $\widetilde{\psi}_2$ is defined as in (26).

In order to apply to this situation the proof of Theorem 3.2, the "triangular" version of Lemma 3.1 is needed. Let

$$\rho^d : {S_{n-1}}^{\Gamma_1} \oplus S_{dn-1} \rightarrow S_{(d+1)n-1}$$

be the linear map defined as in (9), and set $MT^d_{\mathcal{I}}$ the matrix in the monomial bases corresponding to the restriction of ρ^d to

$$(S^{\Gamma_1\setminus\Gamma_4}_{n-1,\mathcal{I}} \oplus {S_{n-1}}^{\Gamma_1\cap\Gamma_4}) \oplus S_{dn-1}.$$

It is a square matrix of the same size as $MS^d_{\mathcal{I}}$. The following equality follows straightforwardly:

Lemma 5.1.

$$\pm|MT^d_{\mathcal{I}}| = |MP_{\mathcal{I}}|^d\,\mathrm{Res}_n(x_1, x_2, x_3)^{d(d-1)/2}.$$

Now, the same proof performed in Theorem (3.2) for the tensor product case, may be applied for triangular polynomials, giving the desired result. □

Corollary 5.1. (Conjecture 6.2 in Cox *et al.*, 2000)

$$|MS^2_{\mathcal{I}}| = |MP_{\mathcal{I}}|^3 \operatorname{Res}_n(x_1, x_2, x_3).$$

Corollary 5.2. (General version of Theorem 5.1 in Cox *et al.*, 2000) *If* $\operatorname{Res}_n(x_1, x_2, x_3) \neq 0$*, then* $|MS^d_{\mathcal{I}}| = 0$ *implies* $|MP_{\mathcal{I}}| = 0$.

Theorem 5.1 has the following interpretation in the language of moving surfaces: *If* $\operatorname{Res}_n(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \neq 0$*, and there are exactly* n *linearly independent moving planes of degree* $n-1$ *that follow the rational surface, then the dimension of the* $\mathbb{C}$*-vector space of* d*-surfaces of degree* $n-1$ *that follow the rational surface is equal to* $\frac{n(d+1)d(dn+d+5-n)}{12}$.

Proof. The fact that there are n linearly independent moving planes of degree $n-1$, implies that MP has maximal rank. Lemma 5.2 in Cox *et al.* (2000) implies that there exists in index set $\mathcal{I}$, $|\mathcal{I}| = n$, such that $MP_{\mathcal{I}}$ is non-singular. □

5.1. the method of moving quadrics when the surface is not properly parametrized

We are going to give here the analogue of Section 4 for triangular surfaces. The main result will be an improvement of Theorem 5.2 in Cox *et al.* (2000), extending the validity of the method to the case when the parametrization is not generically one-to-one. Some extra care must be taken, because the method combines moving planes and moving quadrics.

As before, set $d = 2$, and suppose that $\mathcal{I}$ is fixed. If $|MS^2_{\mathcal{I}}| \neq 0$, the kernel of MQ^2, i.e. the space of moving quadrics which follow the surface, will have dimension equal to $\frac{n^2+7n}{2}$.

As in Section 4, suppose that $MQ^2 = [MS^2_{\mathcal{I}}, R_{\mathcal{I}}]$, where $R_{\mathcal{I}}$ has $\frac{n(n+1)}{2} + 3n$ columns. In Cox *et al.* (2000) (proof of Theorem 5.2), $(n^2-n)/2$ linearly independent vectors are chosen from the basis of the kernel of MQ^2 by considering a matrix T of $(5n+5n^2) \times (n^2-n)/2$ such that $MQ^2 \cdot T = 0$ as follows:

$$T := \begin{bmatrix} \bar{T} \\ 0 \\ \mathbb{I} \end{bmatrix}.$$

Here, 0 is a block of $4n$ rows indexed by $s^i\, t^j\, x_i^a x_4^b$, $(i,j) \in \mathcal{I}$, $b \geq 1$, and $\mathbb{I}$ denotes the identity matrix of order $(n^2-n)/2$ whose rows are indexed by $s^i\, t^j\, x_4^2$ $(i,j) \notin \mathcal{I}$.

With this structure, one can check that

$$T = \begin{bmatrix} -{MS^2_{\mathcal{I}}}^{-1} \cdot \widetilde{R_{\mathcal{I}}} \\ 0 \\ \mathbb{I} \end{bmatrix},$$

where $\widetilde{R_{\mathcal{I}}}$ denotes a block of $R_{\mathcal{I}}$.

Because of Theorem 5.1, the fact that $|MS^2_{\mathcal{I}}| \neq 0$ implies that $MP^2_{\mathcal{I}}$ is not singular, hence one can also find $n-1$ linearly independent moving planes that follow the surface by solving the system $MP \cdot T' = 0$, where $MP = [MP_{\mathcal{I}}\,,\, R']$ and

$$T' = \begin{bmatrix} -MP_{\mathcal{I}}{}^{-1} \cdot \widetilde{R'} \\ \mathbb{I} \end{bmatrix}.$$

As in the previous section, let $\mathbb{T}$ be the matrix made from T by ordering its rows with the order defined in (30).

Consider also the following order:

$$\begin{gathered} x_1,\ x_2,\ x_3,\ x_4, \\ s(x_1,\ x_2,\ x_3,\ x_4), \\ s\,t(x_1,\ x_2,\ x_3,\ x_4), \\ s\,t^2(x_1,\ x_2,\ x_3,\ x_4), \\ \ldots \end{gathered}$$

and let $\mathbb{T}'$ be the matrix made from T' by ordering its rows with it.

Let C be the vector defined in (31) and set $C' := (X_1, X_2, X_3, X_4)$. Consider also $M \in \mathbb{Z}[X_1, X_2, X_3, X_4]^{n(n+1)/2 \times 10n(n+1)/2}$ and $M' \in \mathbb{Z}[X_1, X_2, X_3, X_4]^{n(n+1)/2 \times 4n(n+1)/2}$ as follows:

$$M := \begin{bmatrix} C & C & C & \ldots & C \\ 0 & C & C & \ldots & C \\ 0 & 0 & C & \ldots & C \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & C \end{bmatrix},$$

$$M' := \begin{bmatrix} C' & C' & \ldots & C' \\ 0 & C' & \ldots & C' \\ & \ldots & & \\ 0 & 0 & \ldots & C' \end{bmatrix}.$$

Let $\widetilde{T}$ be the square matrix of size $n(n+1)/2$ constructed in the proof of Theorem 5.2 in Cox *et al.* (2000) by collecting all the coefficients of these moving planes and moving quadrics. The following proposition may be proven mutatis mutandis the arguments given in Proposition 4.1.

Proposition 5.2.

$$|\widetilde{T}| = \pm |[M \cdot \mathbb{T}\,,\, M' \cdot \mathbb{T}']|.$$

Similarly, one can formulate and prove versions of Proposition 4.2, Corollary 4.1 and Theorem 4.1 for the triangular case. This leads to the following corollary:

Corollary 5.3. *Suppose that the surface*

$$\left(\frac{\tilde{x}_1}{\tilde{x}_4}, \frac{\tilde{x}_2}{\tilde{x}_4}, \frac{\tilde{x}_3}{\tilde{x}_4} \right)$$

has no base points, and that there are exactly n moving planes of degree $n-1$ following the surface. Then, the method of moving quadrics always compute a power of the implicit equation of the surface.

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