THE COMPLEXITY OF PARTIAL DERIVATIVES

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Abstract. Let $L$ denote the nonscalar complexity in $k(x_1,\ldots,x_n)$. We prove $L(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n) \approx 3L(f)$. Using this we determine the complexity of single power sums, single elementary symmetric functions, the resultant and the discriminant as root functions, up to order of magnitude. Also we linearly reduce matrix inversion to computing the determinant.

1. Introduction

Let $k$ be an infinite field, $x_1,\ldots,x_n$ be indeterminates over $k$. Given $f_1,\ldots,f_q \in k(x)$, let $L(f_1,\ldots,f_q)$ be the minimal number of nonscalar multiplications/divisions sufficient to compute $f_1,\ldots,f_q$ from $x_1,\ldots,x_n$ allowing additions/subtractions and multiplications by arbitrary scalars from $k$ for free. $L(f_1,\ldots,f_q)$ is called the complexity of $f_1,\ldots,f_q$. (For details see e.g. Borodin and Munro [1], Strassen [6]).

One way to obtain lower bounds for the complexity of a set $f_1,\ldots,f_q$ of polynomials (=rational functions) is by the degree method (Strassen [7]). Unfortunately in the case of single polynomials one gets only trivial results. An interesting recent paper of Schnorr [4] deals with this problem and extends the method to yield nontrivial lower bounds for certain single functions. In the present paper we reduce the complexity of a single polynomial to that of several polynomials by means of the following simple but surprising inequality

$$L\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \leq 3L(f),$$

proved in a completely elementary way. Combining (1) with the degree bound in its original form we obtain rather sharp complexity bounds, such as

$$L\left(\sum_{i=1}^{n} x_i^m\right) \simeq n \log m,$$

$$L(\sigma_q) \simeq n \log \min(q, n-q).$$
where $\sigma_q$ is the $q$th elementary symmetric function in $n$ variables,

$$L\left(\prod_{i \neq j} (x_i - x_j)\right) \asymp n \log n.$$ 

Here $\asymp$ means equality of order of magnitude.

Of course (1) is useful not only for proving lower bounds. It easily implies, e.g. that computing the inverse of a matrix is not much harder than computing its determinant. In this connection we remark that inequalities similar to (1) hold for other cost measures (e.g. when counting all operations).

Throughout the paper $\log$ means $\log_2$. We apply Bezout’s theorem in the form of Bezout’s inequality for affine space (see Heintz [2], Schnorr [4]).

2. Main result

**Theorem 1.** Let $f \in k(x)$. Then

$$L\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \asymp 3L(f).$$

For the proof we need the following

**Lemma.** Let $K$ be a field, $y_i \in K$ nonzero ($1 \leq i \leq s$), $\alpha_{ij} \in K$ ($1 \leq j < i \leq s$) and $z_1, \ldots, z_s$ indeterminates over $K$. Define $h_1, \ldots, h_s$ by

$$h_i = y_i \left(\sum_{j=1}^{i-1} \alpha_{ij}h_j + z_i\right), \quad i \leq s. \tag{2}$$

Write

$$h_s = \sum_{\alpha \neq 1} d_\alpha z_\alpha$$

with $d_\alpha \in K$. Then

$$d_s = y_s,$$

$$d_i = \left(\sum_{\alpha \neq 1} d_\alpha \alpha_{ii}\right)y_i \quad \text{for} \ i < s. \tag{3}$$

**Proof.** Let $h_i = \sum_{\alpha \neq 1} d_\alpha z_\alpha$. Then from (2)

$$d_{is} = y_i \left(\sum_{j=1}^{i-1} \alpha_{ij}d_{jr} + \delta_{ir}\right). \tag{4}$$
We introduce \((s \times s)\)-matrices

\[
D = (d_{ij}) = \begin{pmatrix} d_{11} & 0 \\ \vdots & \ddots \\ d_{s1} & \cdots & d_{ss} \end{pmatrix},
\]

\[
A = \begin{pmatrix} 0 \\ \vdots \\ \alpha_{s1} & \cdots & \alpha_{ss-1} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ 0 \end{pmatrix}.
\]

Then (4) is equivalent to

\[
D = Y(AD + 1).
\]  

(5)

Since \(d_{ii} = y_i \neq 0\), \(D\) and \(Y\) are invertible. Multiplying (5) from the left by \(Y^{-1}\) and from the right by \(D^{-1}\) we get

\[
Y^{-1} = A + D^{-1}.
\]

Multiplication from the left by \(D\) and from the right by \(Y\) yields

\[
D = (DA + 1)Y.
\]

This means

\[
d_{ij} = \left( \sum_{i=1}^{s} d_{ii} \alpha_{rij} + \delta_{ij} \right) y_i.
\]

For \(i = s\) this is (3). 

**Proof of Theorem 1.** Let \(L(f) \leq r\). Then there is a sequence \(g_1, \ldots, g_r \in k(x)\) such that for all \(i \leq r\) we have \(g_i = u_i + \nu_i\), where

\[
u_i = \sum_{i=1}^{r} \gamma_{ii} g_i + q_i
\]

(6)

for some \(\beta_{ii}, \gamma_{ii} \in k, p_i, q_i \in k + \sum_{r=1}^{n} kx_i\), and such that

\[
f = \sum_{i=1}^{r} \alpha_{ii} g_i + m
\]

for some \(\alpha_{ii} \in k, m \in k + \sum_{r=1}^{n} kx_i\). In addition we may assume that all \(u, \nu_i\) are nonzero.

The proof now proceeds as follows: It will first be shown that any partial derivative \(\partial f / \partial x_{\nu} (\nu \leq n)\) is of the form

\[
\frac{\partial f}{\partial x_{\nu}} = h_s(\zeta_{r+1}, \ldots, \zeta_{r+s})
\]

(7)
where \( h_t(z_1, \ldots, z_s) \) is defined as in the lemma (relative to suitable parameters \( s, \alpha_{i,t}, y_t \)) and the \( \zeta_{3i} \) are elements from the ground field \( k \). Since scalar multiplications are free it will then be sufficient to show that the coefficients \( d_t \) of \( h_t \) can be computed from \( g_1, \ldots, g_s \) with at most \( 2r \) multiplications/divisions. This will be done according to the recursion (3).

Now fix \( \nu \leq n \). Using (6) and Leibniz’s formula we obtain for all \( i \leq r \)

\[
\frac{\partial g_i}{\partial x_\nu} = v_i \left( \sum_{j=1}^{i-1} \beta_{ij} \frac{\partial g_i}{\partial x_j} + \frac{\partial g_i}{\partial x_j} + \frac{\partial q_i}{\partial x_j} \right) + u_i \left( \sum_{j=1}^{i-1} \gamma_{ij} \frac{\partial g_i}{\partial x_j} + \frac{\partial q_i}{\partial x_j} \right)
\]

if \( g_i = u_i \cdot v_i \) and

\[
\frac{\partial g_i}{\partial x_\nu} = \frac{1}{v_i} \left( \left( \sum_{j=1}^{i-1} \beta_{ij} \frac{\partial g_i}{\partial x_j} + \frac{\partial g_i}{\partial x_j} \right) + \left( -\frac{u_i}{v_i} \right) \left( \sum_{j=1}^{i-1} \gamma_{ij} \frac{\partial g_i}{\partial x_j} + \frac{\partial q_i}{\partial x_j} \right) \right)
\]

if \( g_i = u_i / v_i \). Moreover

\[
\frac{\partial f}{\partial x_\nu} = \sum_{i=1}^{r} \alpha_{i,j} \frac{\partial g_i}{\partial x_j} + \frac{\partial m}{\partial x_\nu}
\]

In order to get into the situation of the lemma put \( s = 3r + 1 \), and define \( \alpha_{i,l} \) for \( 1 \leq l \leq r \) by

\[
\alpha_{3i-2,l} = \begin{cases} 
\beta_{ij} & \text{if } l = 3j, \\
0 & \text{if } 3 \nmid l,
\end{cases}
\]

\[
\alpha_{3i-1,l} = \begin{cases} 
\gamma_{ij} & \text{if } l = 3j, \\
0 & \text{if } 3 \nmid l,
\end{cases}
\]

\[
\alpha_{3i,l} = \begin{cases} 
1 & \text{if } l = 3i - 2, \\
1 & \text{if } l = 3i - 1, g_i = u_i \cdot v_i, \\
-1 & \text{if } l = 3i - 1, g_i = u_i / v_i, \\
0 & \text{otherwise},
\end{cases}
\]

for \( i \leq r \), and

\[
\alpha_{3i+1,l} = \begin{cases} 
\alpha_{ij} & \text{if } l = 3j, \\
0 & \text{if } 3 \nmid l.
\end{cases}
\]

Furthermore, put \( y_* = 1 \) and for \( i \leq r \) put

\[
y_{3i-2} = v_i, \quad y_{3i-1} = u_i, \quad y_{3i} = 1
\]

in case \( g_i = u_i \cdot v_i \), and

\[
y_{3i-2} = 1, \quad y_{3i-1} = \frac{u_i}{v_i}, \quad y_{3i} = \frac{1}{v_i}
\]

in case \( g_i = u_i / v_i \). Note that all \( y_* \neq 0 \).
Finally put

$$\zeta_{3i-2} = \frac{\partial p_i}{\partial x_\nu}, \quad \zeta_{3i-1} = \frac{\partial q_i}{\partial x_\nu}, \quad \zeta_{3i} = 0 \quad (i \leq r), \quad (14)$$

and

$$\zeta_s = \frac{\partial m}{\partial x}.$$

Let $h_1, \ldots, h_s$ be defined as in the lemma.

**Claim.** For all $i \leq r$ we have

$$\frac{\partial g_i}{\partial x_\nu} = h_{3i}(\zeta_1, \ldots, \zeta_s) = h_{3i}(\zeta) \quad \text{and} \quad \frac{\partial f}{\partial x_\nu} = h_s(\zeta).$$

The first assertion is proved by induction on $l \leq r$. We treat the case $g_i = u_i/v_i$, leaving the case $g_i = u_i * v_i$ and the second assertion to the reader.

$$\frac{\partial g_i}{\partial x_\nu} = \frac{1}{v_i} \left[ \left( \sum_{j=1}^{l-1} \beta_{ij} \frac{\partial g_i}{\partial x_j} + \frac{\partial p_i}{\partial x_\nu} \right) + \left( \frac{\partial q_i}{\partial x_\nu} - \frac{u_i}{v_i} \right) \left( \sum_{j=1}^{l-1} \gamma_{ij} \frac{\partial q_i}{\partial x_j} + \frac{\partial p_i}{\partial x_\nu} \right) \right] \quad \text{(by (9))}$$

$$= \frac{1}{v_i} \left[ y_{3i-2} \left( \sum_{j=1}^{3i-3} \alpha_{3i-2,j} h_j(\zeta) + \zeta_{3i-2} \right) + y_{3i-1} \left( \sum_{j=1}^{3i-2} \alpha_{3i-1,j} h_j(\zeta) + \zeta_{3i-1} \right) \right]$$

(by induction hypothesis, (11), (13), (14))

$$= \frac{1}{v_i} \left[ h_{3i-2}(\zeta) + h_{3i-1}(\zeta) \right] \quad \text{(by (2))}$$

$$= y_{3i} \left( \sum_{j=1}^{3i-1} \alpha_{3i,j} h_j(\zeta) + \zeta_{3i} \right) \quad \text{(by (11), (13), (14))}$$

$$= h_{3i}(\zeta) \quad \text{(by (2)).}$$

It follows from our claim that any partial derivative $\partial f/\partial x_\nu$ ($\nu \leq n$) is of the desired form (7). Therefore

$$L \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right| g_1, \ldots, g_r \right) \leq L(d_1, \ldots, d_s | g_1, \ldots, g_r) \quad (15)$$

where $d_1, \ldots, d_s$ are the coefficients of $h_s$. By (3) each $d_i$ ($j < s$) is obtained from $d_{j+1}, \ldots, d_s$ using just one (nonscalar) multiplication, one factor being $y_j$. By (12) and (13) multiplication by $y_j$ means either multiplication or division by some element from $k + \sum_{\nu=1}^n k x_\nu + \sum_{\nu=1}^{r'} k g_\nu$. Furthermore, at least $r$ of these multiplications are
multiplications by 1, and also $y_s = 1$. Hence

$$L(d_1, \ldots, d_s | g_1, \ldots, g_r) \leq (s-1) - r = 2r.$$ 

This and (15) imply the theorem. \qed

To a reader who would like to improve the constant 3 in Theorem 1 we suggest to look at the example

$$f = \sum_{i=1}^{n} \frac{x_i}{y_i} \in k(x_1, \ldots, y_n).$$

3. Applications

**Corollary 1.** Assume $\text{char } k \not\equiv m$. Then

$$\frac{3}{4} n \log(m-1) \leq L \left( \sum_{i=1}^{n} x_i^m \right) \leq n l(m)$$

where $l(m)$ is the length of a shortest addition chain for $m$ (see Knuth [3]; $l(m) \leq 2 \log m$ always, $l(m) \sim \log m$).

**Proof.** It suffices to show that the left inequality. The theorem yields

$$L \left( \sum_{i=1}^{n} x_i^m \right) \geq \frac{3}{4} L(x_1^{m-1}, \ldots, x_n^{m-1}).$$

Now we apply the degree bound (Strassen [7]). W.l.o.g. $k$ algebraically closed. In case $\text{char } k \not\equiv m - 1$

$$\deg \text{graph}(x_1^{m-1}, \ldots, x_n^{m-1})$$

$$\leq \# \{(\xi_1, \ldots, \xi_n) \in k^n | \xi_1^{m-1} = \cdots = \xi_n^{m-1} = 1\}$$

$$= (m-1)^n,$$

hence $L(x_1^{m-1}, \ldots, x_n^{m-1}) \geq n \log(m-1)$. In case $k \mid m - 1$

$$\deg \text{graph}(x_1^{m-1} + x_1, \ldots, x_n^{m-1} + x_n)$$

$$\leq \# \{(\xi_1, \ldots, \xi_n) | \xi_1^{m-1} + \xi_1 = \cdots = \xi_n^{m-1} + \xi_n = 0\}$$

$$= (m-1)^n,$$

hence $L(x_1^{m-1}, \ldots, x_n^{m-1}) = L(x_1^{m-1} + x_1, \ldots, x_n^{m-1} + x_1) \geq n \log(m-1)$. \qed
Corollary 2. Let \( \sigma_1, \ldots, \sigma_n \) be the elementary symmetric functions in \( x_1, \ldots, x_n, \)
\( q \leq \frac{1}{2} n. \) Then

\[
\frac{3}{8} (n - q + 1) \log(q - 1) \leq L(u_q) \leq n \log q + 2n,
\]
and

\[
|I.(\sigma_{n-q}) - I.(\sigma_q)| \leq \frac{2n}{x}.\]

Proof. Left inequality of (16): W.l.o.g. \( k \) algebraically closed and of infinite degree of transcendency over its prime field. Denote by \( \sigma_q^{(m)} \) the \( q \)th elementary symmetric function in \( m \) variables. Obviously

\[
\frac{\partial \sigma_q^{(m)}}{\partial x_i} = \sigma_{q-1}^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\]
and

\[
\sigma_i^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sigma_i^{(m)}(x_1, \ldots, x_n) - x_i \sigma_i^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]
The last recursion yields

\[
\sigma_{q-1}^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sigma_{q-1}^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - x_i \sigma_{q-2}^{(m)} + x_i^2 \sigma_{q-3}^{(m)} - \cdots + (-1)^{q-1} x_i^{q-1}.
\]
This together with (18) gives

\[
\frac{\partial \sigma_q}{\partial x_i} = \sigma_{q-1} - x_i \sigma_{q-2} + \cdots + (-1)^{q-1} x_i^{q-1}.
\]
The theorem tells us

\[
L(\sigma_q) \geq \frac{1}{3} L(\sigma_q, \ldots, \sigma_q).
\]
Again we apply the degree bound to the right-hand side of this inequality, getting

\[
L(\sigma_q, \ldots, \sigma_q) \geq \log \deg W,
\]
where \( W \subseteq k^{2n} \) is the graph of the polynomial map given by the equations

\[
y_1 = \frac{\partial \sigma_q}{\partial x_1}(x_1, \ldots, x_n) \\
\vdots \\
y_n = \frac{\partial \sigma_q}{\partial x_n}(x_1, \ldots, x_n).
\]
Choose $\lambda_1, \ldots, \lambda_{q-1}, \mu_{q-1}, \ldots, \mu_n \in k$ algebraically independent and intersect $W$ with the hypersurfaces

$$\sigma_1(x_1, \ldots, x_n) = \lambda_1$$

$$\vdots$$

$$\sigma_{q-1}(x_1, \ldots, x_n) = \lambda_{q-1}$$

and with the hyperplanes

$$y_q = \mu_{q-1}, \ldots, y_n = \mu_n.$$  

For the intersection $W_0$ we obtain by Bezout’s inequality

$$\deg W_0 \leq \deg W \cdot (q-1)!.$$  

By (20) and (21) it suffices to show that $W_0$ is finite, and

$$\# W \leq (q-1)^n - (q-1)!.$$  

For $(\xi, \eta) \in k^{2n}$ we have $(\xi, \eta) \in W_0$ if and only if (23), (24) and (25), where

$$\eta_i = \frac{\partial \eta_i}{\partial x_j}(\xi), \quad 1 \leq i \leq n,$$  

$$\sigma_i(\xi) = \lambda_i, \quad 1 \leq i < q,$$  

$$\xi^n_i - \lambda_i \xi_i^{q-2} + \cdots + (-1)^{q-1} \mu_i = (-1)^{q-1} \mu_i, \quad 0 \leq i \leq n.$$  

the last group of equations coming from (19). Since $\lambda_1, \ldots, \lambda_{q-1}, \mu_{q-1}, \ldots, \mu_n$ are algebraically independent there are precisely $(q-1)^n$ vectors $(\xi_0, \ldots, \xi_n)$ satisfying (25). Hence it suffices to show that any such vector has precisely $(q-1)!$ extensions to a vector $(\xi_1, \ldots, \xi_n)$ satisfying (24).

Now fix $(\xi_0, \ldots, \xi_n)$ satisfying (25). Since obviously

$$\sum_{i=0}^n (-1)^i \sigma_i(\xi) \xi_i^{q-i} = 0,$$  

(24) and (25) imply

$$\sum_{i=0}^n (-1)^i \sigma_i(\xi) \xi_i^{q-i} = (-1)^q \mu_i \xi_i^{q-1}, \quad 0 \leq i \leq n.$$  

Since the components of $(\xi_0, \ldots, \xi_n)$ are pairwise different, the system of linear equations

$$\sum_{i=0}^n z_i (-1)^i \xi_i^{q-i} = (-1)^q \mu_i \xi_i^{q-1}, \quad 0 \leq i \leq n$$

has a unique solution $\xi_0, \ldots, \xi_n$. Therefore, using (26), an extension $(\xi_1, \ldots, \xi_n)$ of $(\xi_0, \ldots, \xi_n)$ satisfies (24) if and only if

$$\sigma_i(\xi) = \lambda_i, \quad 1 \leq i < q,$$  

$$\sigma_i(\xi) = \xi_i, \quad q \leq j \leq n.$$  

(27)
Introducing
\[
f(t) = t^n - \lambda_1 t^{n-1} + \cdots + (-1)^{q-1} \lambda_{q-1} t^{n-q+1} \\
+ (-1)^q \xi_q t^{n-q} + \cdots + (-1)^n \xi_n.
\]

(27) is equivalent to
\[
f(t) = \prod_{i=1}^{n} (t - \xi_i). \quad \text{(28)}
\]

Since by the definition of the \(\xi_i\) and by (25), \(\xi_{q}, \ldots, \xi_{n}\) are roots of \(f(t)\) there is at least one solution \((\xi_1, \ldots, \xi_n)\) (extending \((\xi_q, \ldots, \xi_n)\)) of equation (28). Choosing any such solution we get all solutions by permuting the first \(q - 1\) components \(\xi_1, \ldots, \xi_{q-1}\). So it remains to show that \(\xi_1, \ldots, \xi_{q-1}\) are pairwise different. But in fact \(\xi_1, \ldots, \xi_{n}\) are algebraically independent since by (24) and (25) \(\lambda_1, \ldots, \lambda_{q-1}, \mu_q, \ldots, \mu_n\) are rational functions of them. Thus we have shown
\[
L(\sigma_q) \geq \frac{1}{2}(n - q + 1) \log(q - 1).
\]

Right inequality of (16). Let \(m = \lceil n/q \rceil\) and \(p = n - mq\). For any \(i = 0, \ldots, m - 1\) compute all the elementary symmetric functions \(\sigma_{i,1}, \ldots, \sigma_{i,o}\) in \(x_{i0+1}, \ldots, x_{i+1}\), and all elementary symmetric functions \(\sigma_{m,1}, \ldots, \sigma_{m,p}\) in \(x_{mq+1}, \ldots, x_m\). This can be done with cost \(mq \log q + p \log p \leq n \log q\) (see e.g. Strassen [7, p. 243]). We introduce the polynomials
\[
Q_i = 1 - \sigma_{i,1} t + \cdots + (-1)^q \sigma_{i,q} t^q, \\
Q_m = 1 - \sigma_{m,1} t + \cdots + (-1)^p \sigma_{m,p} t^p.
\]

Then
\[
\prod_{i=0}^{m} Q_i = 1 - \sigma_1 t + \cdots + (-1)^q \sigma_q t^q \pmod{t^{q+1}}.
\]

Since symbolic multiplication \(\pmod{t^{q+1}}\) of two polynomials with constant term 1 can be done with \(2q\) nonscalar operations, we can compute \(\sigma_1, \ldots, \sigma_q\) from the \(\sigma_{i,j}\) in time \(2qm \leq 2n\). Thus
\[
L(\sigma_q) \leq L(\sigma_1, \ldots, \sigma_q) \leq n \log q + 2n.
\]

(17) follows from the equation
\[
\sigma_{n-q} = \sigma_q \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right) \cdot x_1 \cdots x_n,
\]
valid for \(1 \leq q \leq n\). \(\Box\)
An interesting consequence of Corollary 2 is the following: There is a polynomial

\[ f(x_1, \ldots, x_{2n}, t) = \sum_{i=0}^{2n} f_i(x_1, \ldots, x_{2n}) t^i \]

such that

\[ L(f) = 2n - 1, \quad L(f_n) \geq \frac{1}{2} n \log(n - 1). \]

(Take \( f = (t - x_1) \cdot \ldots \cdot (t - x_{2n}) \). Compare this also with Valiant [9;§4].)

**Corollary 3** (resultant). Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \) be indeterminates over \( k \). Then

\[ \frac{1}{2} n \log n \leq L\left( \prod_{i,j} (x_i - y_j) \right) < n(\log n + 1). \]

**Proof.** Left inequality: It suffices to show that

\[ L\left( \prod_{i,j} (x_i - y_j) \right) \geq \frac{1}{2} n \log n, \]

where \( \eta_1, \ldots, \eta_n \in k \) are algebraically independent over the prime field. (Just adjoin the \( y_i \) to the ground field \( k \).) Put

\[ f = \prod_{i,j} (x_i - y_j), \quad g = \prod_i (t - \eta_i), \]

where \( t \) is a new indeterminate. Then

\[ f = \prod_{i=1}^n g(x_i), \tag{29} \]

\[ \frac{\partial f}{\partial x_i} = \frac{g'(x_i)}{g(x_i)}. \tag{30} \]

The theorem and the degree bound yield

\[ L(f) \geq \frac{1}{2} L\left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \geq \frac{1}{2} \deg \text{graph } \phi. \]

where \( \phi \) is the polynomial map defined by \( f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n \). We intersect graph \( \phi \), which lives in \( k^{2n+1} \) with coordinate variables \( x_1, \ldots, x_n, z_1, \ldots, z_n \) (say), with the equations

\[ z_n = z_1 = \ldots = z_n. \]

By (29) and (30) the intersection is

\[ V = \{ (\xi, \phi(\xi)) | \text{either } f(\xi) = 0 \text{ or } \forall \xi \ g(\xi) - g'(\xi) = 0 \}. \]

By Bezout's inequality,

\[ \deg \text{graph } \phi = \deg V. \]
Now \( g \) and therefore \( g - g' \) have algebraically independent coefficients (except for the highest which is 1). Thus \( g - g' \) has \( n \) distinct roots, i.e.

\[
\# \{ (\xi, \phi(\xi)) | \forall i \; g(\xi_i) - g'(\xi_i) = 0 \} = n^n.
\]

Now (29) implies that the above \( n^n \) points satisfy \( f(\xi) \neq 0 \). This means that \( V \) contains \( n^n \) isolated points, so \( \deg V \geq n^n \).

**Right inequality:** First compute the elementary symmetric functions in \( y_1, \ldots, y_n \) in time \( n \log n \), next evaluate the polynomial \( t^n - \sigma_1(y)t^{n-1} + \cdots + (-1)^n \sigma_n(y) \) at the points \( x_1, \ldots, x_n \) in time \( 8n \log n \) and finally multiply the values. (Compare Borodin and Munro [1]).

We do not know whether our method allows to prove a nonlinear lower bound for the complexity of the resultant of two polynomials as a function of their coefficients.

**Corollary 4** (discriminant). Let \( x_1, \ldots, x_n \) be indeterminates over \( k \). Then

\[
\frac{1}{e} n \log n - \frac{2}{3} n < L \left( \prod_{i \neq j} (x_i - x_j) \right) < n(\log n + 1).
\]

**Proof. Upper bound** (suggested by J. Stoss): First compute the coefficients of \( A(t) = \prod_{i=1}^n (t - x_i) \) with cost \( n \log n \), then the coefficients of \( \frac{dA}{dt} = \sum_{i=1}^n \Pi_{1 \leq j \leq i} (t - x_j) \) without additional cost. Now evaluate \( \frac{dA}{dt} \) at \( x_1, \ldots, x_n \) and multiply the values. This can be done with cost \( 8n \log n + n - 1 \) (actually \( 7n \log n + n - 1 \) is sufficient, using byproducts of step 1).

**Lower bound:** Let \( f = \prod_{1 \leq i < j \leq n} (x_i - x_j) \), \( p = \lfloor n/2 \rfloor \), \( q = \lfloor n/2 \rfloor \). For clarity replace \( x_1, \ldots, x_n \) by \( x_1, \ldots, x_p, y_1, \ldots, y_q \). Then

\[
f = g(x) \cdot \prod_{i,j} (x_j - y_j)^2 \cdot h(y),
\]

and therefore

\[
\frac{\partial f}{\partial y_i} = f \cdot \left( - \sum_{l=1}^p \frac{2}{x_l - y_i} + \frac{\partial h}{\partial y_i} (y) \cdot \frac{1}{h(y)} \right).
\]

The theorem yields

\[
L(f, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_q}) \leq 3L(f).
\]

Adjoining the \( y_j \) to the ground field and calling them \( \eta_l \) we get

\[
L \left( f(x, \eta), f(x, \eta) \sum_{l=1}^p \frac{1}{x_l - \eta_1}, \ldots, f(x, \eta) \sum_{l=1}^p \frac{1}{x_l - \eta_q} \right) \leq 3L(f).
\]
Dividing $f(x, \eta)$ by the other $q$ terms under the bracket we obtain

$$L \left( \frac{\prod_{i=1}^{p} (x_i - \eta_i)}{\sum_{i=1}^{p} \prod_{i \neq j} (x_i - x_j)}, \ldots, \frac{\prod_{i=1}^{p} (x_i - \eta_q)}{\sum_{i=1}^{p} \prod_{i \neq j} (x_i - x_j)} \right) \leq 3L(f) + q.$$ 

If $\phi$ is the rational map defined by the above $q$ (reduced) quolynomials, the degree bound yields

$$3L(f) + q \geq \log \deg \text{graph} \phi.$$ 

Now

$$\deg \text{graph} \phi \geq \# \text{ of components of } \phi^{-1}(0) - \# \text{ of components of } \{ xi \in k^p | \forall j \exists i \xi_i = \eta_j \}$$

$$= p(p-1) \cdot \ldots \cdot (p-q+1) = p!,$$

because the $\eta_j$ are pairwise distinct. Therefore

$$3L(f) \geq \log p! - q \geq \frac{n}{2} \log \frac{n}{2e} - \frac{n}{2}$$

$$> \frac{n}{2} \log n - 2n. \quad \square$$

Corollary 4 implies that the complexity of the Vandermonde determinant $\Pi_{i,j} (x_i - x_j)$ is at least $\lambda n \log n - n$. (Its square is the discriminant.)

**Corollary 5.** Let $a_{ij}$ $(1 \leq i, j \leq n)$ be undeterminates over $k$, $(b_{ij}) = (a_{ij})^{-1}$ as matrices. Then

$$L(\{b_{ij} | 1 \leq i, j \leq n\}) \leq 3L(\det(a_{ij})) + n^2.$$ 

**Proof.** By Cramer's rule

$$\frac{\partial}{\partial a_{ij}} \det(a) = b_{ij} \det(a).$$

Thus

$$L(\{b_{ij} | 1 \leq i, j \leq n\}) \cdot L(\{\det(a) \cdot \left( \frac{\partial}{\partial a_{ij}} \det(a) \cdot 1 \leq i, j \leq n \right) \}) + n^2$$

$$= 3L(\det(a)) + n^2. \quad \square$$

Corollary 5, in conjunction with e.g. Strassen [5, 8], shows that the determinant has roughly the same complexity as matrix multiplication or inversion. It would be interesting to have a similar result for solving a system of linear equations.
Corollary 6. Let \( f = \sum_{i,j,l} \tau_{ij} x_i y_j z_l \) be a trilinear form. Then
\[
L(f) \geq \frac{1}{6} \text{rank}(\tau_{ij}).
\]

Proof. Differentiate with respect to the \( x_i \) and apply Korollar 3 and Lemma 6 of Strassen [8]. \( \square \)

4. An extension

Theorem 2. Let \( f \in k(x_1, \ldots, x_n) \) be computable from \( \{x_1, \ldots, x_n\} \cup k \) using \( A \) additions/subtractions, \( S \) scalar multiplications (i.e. multiplications by elements from \( k \)) and \( M \) further multiplications/divisions. Then \( \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\} \) can be computed from \( \{x_1, \ldots, x_n\} \cup k \) using \( 2A + 4M \) additions/subtractions, \( 2S \) scalar multiplications and \( 3M \) further multiplications/divisions.

In particular, let \( L_1 \) denote the complexity when all operations have unit cost, \( L_2 \) the complexity when additions/subtractions are free but all multiplications/divisions (including scalar multiplications) count. Then
\[
\begin{align*}
L_1 \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) & \leq 4L_1(f), \\
L_2 \left( f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) & \leq 3L_2(f).
\end{align*}
\]

Sketch of proof of Theorem 2. We may assume that the given computation sequence is of the form \( g_1, \ldots, g_{n+m+r} \), where \( g_i = x_i \) for \( 1 \leq i \leq n \), \( g_i \in k \) for \( n < i < n + m \), and any \( g_i \) with \( n + m < i < n + m + r \) is obtained by adding/subtracting or multiplying/dividing two previous \( g \)’s or by multiplying a previous \( g \) by some element from \( k \).

Proceed as in the proof of Theorem 1 with the following provisions:

1. Each addition/subtraction or multiplication/division in the given computation yields three rows of the matrix \((\alpha_{ij})\) as before. Each scalar multiplication as well as each of the \( n + m \) initial steps give rise to only one row of \( (\alpha_{ij}) \).

2. All \( \alpha_{ij} \in \{0, 1, -1\} \).

Since we may assume that in the original computation any intermediate result except the last one is being referred to, all columns of the matrix \((\alpha_{ij})\) except the last one are nonzero.

Now the lemma can be applied and (7) holds with \( \xi_{vi} = \delta_{vi} \). It is clear that the given computation together with the new one provided by the lemma use \( 2S \) scalar multiplications and \( 3M \) further multiplications/divisions. It remains to estimate the total number of additions/subtractions. If we content ourselves first with computing the \( d_i \) up to sign only, the number \( B \) of additions/subtractions used can be made equal to \( A \) plus the number of nonzero \( \alpha_{ij} \) minus the number of nonzero
columns of the matrix \((a_{ij})\), i.e.

\[
B = A + (4A + S + 4M) - (n + m + 3A + S + 3M - 1) = 2A + M - (n + m - 1).
\]

Now we can adjust the signs of \(d_1, \ldots, d_{n+m}\) (we are interested only in \(d_1, \ldots, d_n\)) using at most \(n + m - 1\) additional subtractions, since not all of \(d_1, \ldots, d_{n+m}\) have the wrong sign. (To see this observe that the first \(i\) such that our procedure yields \(d_i\) with the correct sign cannot exceed \(n + m\).) \(\square\)

References