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Direct Sums and Direct Products of Finite-Dimensional Modules over Path Algebras

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The algebras in this paper are over the associative algebras R obtained from extended Coxeter–Dynkin quivers with no oriented cycles. The finite-dimensional indecomposable R-modules can, in principle, be described. Taking direct products and direct sums, respectively, of finite-dimensional R-modules over an infinite indexing set are two natural ways of getting infinite-dimensional R-modules. The latter are the infinite-dimensional pure-projective modules and direct summands of the former are the pure-injective modules. The focus in this paper is on these two classes of infinite-dimensional modules. Every module is a submodule of a pure-injective module and a quotient of a pure-projective module. When is an extension of a pure-injective module by a pure-injective module pure-injective? This question is answered in this paper. The answer is analogous to the answer of the corresponding question for pure-projective modules. The structures of some quotients of direct products of finite-dimensional modules are also obtained. © 1992 Academic Press, Inc.

1. INTRODUCTION

Throughout the paper K will be an algebraically closed field, even though an arbitrary field will do in many of the results. All algebras will be K-algebras and all modules will be unital right modules. Let R be the path algebra of an extended Coxeter–Dynkin quiver with no oriented cycle. As a K-vector space R is generated by the edges and vertices of the graph. Multiplication in R is given by path composition. For brevity we shall say that R is *tame* if it is the path algebra of an extended Coxeter–Dynkin quiver with no oriented cycle. It is also a hereditary finite-dimensional algebra. The seminal reference in the study of R-modules is [20]. The representations of the quiver with no oriented cycle give rise to R-modules and conversely, see [6], for example.

The representations of $\tilde{A}_{12}$ are called *Kronecker modules* following Ringel's terminology. The corresponding path algebra is called a *Kronecker algebra*. The paradigmatic nature of Kronecker algebras among all finite-

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dimensional hereditary tame algebras is captured by [6, Theorem 5.1], parts of which are quoted in Theorem 2.9. Armed with [20] and this theorem it becomes a formality to extend to arbitrary finite-dimensional hereditary tame algebras many results originally proved for Kronecker algebras. We shall come across several illustrations of this procedure in this paper, see also [18].

We now recall some definitions. A submodule L of a module N is said to be *pure* if it is a direct summand of any submodule M with $L \subseteq M \subseteq N$, and M/L is of finite length. Call L *pure-injective* if it is a direct summand of any module in which it is contained as a pure submodule. A module N is *pure-projective* if every exact sequence

$$0 \to L \to M \to N \to O$$

with L pure in M splits. A module is pure-projective precisely if it is a direct sum of finite-dimensional submodules, while direct summands of direct products of finite-dimensional modules are the pure-injective modules, see [16] and [14], respectively, or [19].

While the structure of pure-projective modules is more attractive than that of pure-injective modules, both classes of modules be described by cardinal invariants, see [18]. Also pure-injective modules reflect the complexity of the module theory of the algebra, see [19] for a more precise statement. We should also remark that every module is a pure submodule of a pure-injective module, see e.g., [19]. In view of these facts it is worthwhile to obtain as much information as we can on these classes of modules.

A module M is said to be *bounded* if it has only finitely many isomorphism classes of indecomposable submodules of finite length. Bounded modules provide a link between pure-injective modules and pure-projective modules, as the next proposition shows.

PROPOSITION 1.1 [18]. *A bounded module is both pure-projective and pure-injective.*

EXAMPLES 1.2. We recall that a *Kronecker module* M is best viewed as a pair (M_1, M_2) of K-vector spaces together with a K-bilinear map from $K^2 \times M_1$ to M_2, which may be described by two linear maps from M_1 to M_2. Call M_1 the *domain space* of M and M_2 the *range* or *target space* of M. The algebra, R', over which M is a module is called a *Kronecker* algebra. It is a certain subalgebra of the algebra of 3×3 matrices over K, see e.g., [16].

We now give some important examples of Kronecker modules. Let $K[\zeta]$ be the polynomial ring in one variable over K. For each positive integer n, let P_n denote the subspace of $K[\zeta]$ spanned by polynomials of degree strictly less than n. Let P_0 denote the zero subspace. Multiplication by ζ and inclusion give rise to two linear maps from P_{n-1} to P_n. In this way we get a family of representations, $P_{n-1} = (P_{n-1}, P_n)$, of $\tilde{A}_{12}$, i.e., Kronecker modules, whose isomorphism classes we denote by $\{\mathrm{III}^n\}_{n=1}^{\infty}$ or simply III. They are all submodules of $P = (K[\zeta], K[\zeta])$, where the pair of linear maps are given by multiplication by ζ and inclusion.

Let $[S]$ denote the K-subspace of $K[\zeta]$ spanned by the subset S. For any subset S of P_n, $(0, [S])$ is a subrepresentation of P_{n-1}. For $\theta \in K$, the isomorphism classes of $P_n/(0, [(\zeta - \theta)^n])$ and $P_n/(0, [1])$ are denoted respectively by II_θ^n and II_∞^n. They give rise to the family II. For each θ in K the ascending union of II_θ^n over all positive integers n is denoted by $\mathrm{II}_\theta^\infty$. The isomorphism class of $P_n/(0, [1, \zeta^n])$ is denoted by I^n. We denote $\{\mathrm{I}^n\}_{n=1}^{\infty}$ by I. The designation I, II, and III is due to Aronszajn and Fixman, see [1].

For types in III or I we can introduce a preorder $\leqslant$ as follows: $\pi_1 \leqslant \pi_2$ if there is a nonzero homomorphism from a representation of type π_1 to one of type π_2. In fact $\mathrm{Hom}(\mathrm{III}^m, \mathrm{III}^n) = 0$ if and only if $m > n$. This preorder is consistent with the partial orders in Sections 2 and 4 of [20] on preprojective and pre-injective modules, respectively. All other modules of finite length are said to be *regular*. When R is an arbitrary finite dimensional hereditary tame algebra the analogue of II_θ^n is denoted in [20] by S_i^n. A module with neither a pre-injective nor a preprojective direct summand is said to be *regular*.

We can now state the following improvement on Proposition 1.1. The hypothesis on the pre-injective direct summand is needed because the injective hull of every module, bounded or not, is pre-injective.

PROPOSITION 1.3[18]. *Let M be a module with no infinite-dimensional pre-injective direct summand. Then M is pure-projective and pure-injective if and only if it is bounded.*

The results in Sections 2 and 3 of the paper are of the following form: Let M and N be pure-projective (pure-injective) and $\mathrm{Ext}(N, M) \neq 0$, then in order that an extension of M by N be pure-projective (pure-injective) at least one of the modules must be bounded. In all but one instance, Proposition 3.3, it is shown that *boundedness* is indispensable. Section 4 can be viewed as a complement to [9], where it is shown that the quotient of a product of modules, over a countable indexing set, by the direct sum is pure-injective if and only if the field K is countable.

Remark 1.4. An anonymous reader of this paper has suggested the symbol *Kr* for a Kronecker algebra. There is also a need for a more euphonious replacement for the mouthful *finite-dimensional hereditary tame algebra.*

2. Extensions of Pure-Projective Modules by Pure-Projective Modules

In this section we summarize results from [16] because in the next section we shall see that analogous results hold for pure-injective modules. Some of the results in [16] were proved only for Kronecker modules. We remedy the situation here by using results in [20] to modify the proofs in [16]. In several instances the proofs look the same with references to [1] replaced by appropriate references to [20]. We shall say that a module is of type I, II, or III according as it is a direct sum of pre-injective, regular, or preprojective finite-dimensional indecomposable modules. The proofs of all results stated in this section without proof can be found in [16, Section 2], unless otherwise stated.

THEOREM 2.1. *Extensions of pure-projective modules by pure-projective modules are pure-projective in the following cases*:

(i) *Extensions of modules of type* I *by modules of type* I.

(ii) *Extensions of modules of type* I *by modules of type* II.

(iii) *Extensions of type* I *by modules of type* III.

(iv) *Extensions of modules of type* II *by modules of type* III.

PROPOSITION 2.2[5]. *Extensions of a module of type* II *by a bounded module of type* II *are pure-projective.*

Following [20] we say a finite-dimensional R-module is *torsion* if it has no nonzero preprojective direct summand. The submodule of an arbitrary module M generated by the finite-dimensional torsion submodules of M is called the *torsion submodule* of M and denoted by tM. The module M is said to be *torsion* if $tM = M$, *torsion-free* if $tM = 0$. In any case tM is always a pure submodule of M [20, Theorem 4.1]. An extension of a torsion module M by a torsion-free module splits if and only if M is the direct sum of a bounded module and a divisible module. Thanks to [20], the proof of this theorem in [13, Theorem 3.5] for Kronecker modules goes through, with only formal changes, for modules over any finite-dimensional hereditary tame algebra. The proof of Proposition 2.2 and an example showing the indispensability of *bounded* are in [5].

PROPOSITION 2.3 [16, Proposition 2.3]. *An extension of a bounded torsion module by a module of type* I *is pure-projective.*

By 4.5 of [20] every torsion module M is $\bigoplus_{\theta \in K \cup \{\infty\}} M_\theta$, where each M_θ is a module over a discrete valuation ring. This is the analogue of the primary decomposition of a torsion abelian group and is the key to the proof of Proposition 2.3. If each M_θ is bounded the proof of [16, Proposition 2.3] can be adapted to show that an extension of M by a pure-projective module of type I is pure-projective.

PROPOSITION 2.4. *An extension of a module of type* III *by one of type* I *is always pure-projective if and only if the module of type* III *is finite-dimensional.*

Proof. Let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of modules with L finite-dimensional of type III and N is of type I. Let $M = M_1 \dotplus M_2$ where M_1 is a maximal pure-projective submodule of M of type *I*. So M_2 has no preinjective direct summand. Since L is finite-dimensional, its image under the projection of M onto M_1 is finite-dimensional. So we may suppose that it is contained in $M_1' \dotplus M_2$, where M_1' is a finite-dimensional direct summand of M_1, hence of type I. So $M = M_1' \dotplus M_2' \dotplus M_2$ where $M_1 = M_1' \dotplus M_2'$. Since M_2' is pure-projective and $M_1 \subseteq M_1' \dotplus M_2$ with $N_1 = (M_1' \dotplus M_2)/L$ a direct summand of N, hence also of type I, we may suppose that we have the following exact sequence $0 \to L \to M_1' \dotplus M_2 \to N_1 \to 0$. We shall complete the proof of the pure-projectivity of M by showing that M_2 is finite-dimensional. Since M_1' is finite-dimensional its image is contained in a finite-dimensional direct summand of N_1 with direct complement N_1', say. The inverse image of N_1' is contained in M_2. Since L is of type III and N_1' is of type I, it follows from Theorems 3.6 and 3.7 of [20] that M_2 is finite-dimensional. Therefore, M is pure-projective. If L is not finite-dimensional we use the example on p. 281 of [16] and Theorem 2.9 to get an example of M that is not pure-injective. ∎

At the end of the section we shall show that in all cases we cannot dispense with the hypothesis of *boundedness*.

LEMMA 2.5. *An extension of a preprojective finite-dimensional module by a bounded module of type* II *is pure-projective.*

Proof. The proof of Lemma 2.5 in [16] is valid for arbitrary finite-dimensional hereditary tame algebras. Note that pure-projective on the penultimate line of p. 282 of [16] should read pure-injective. For the argument on p. 283 the reference to a table should be replaced by $\mathrm{Ext}(S_n^t, S_n^t) \neq 0$. ∎

Using Lemma 2.5 we can now prove Theorem 2.6 in the same way as [16, Theorem 2.6].

THEOREM 2.6 [16, Theorem 2.6]. *An extension of a pure-projective module of type* III *by a pure-projective module of type* II *is always pure-projective only in the following cases*:

(a) *The module of type* III *is finite-dimensional while that of type* II *is bounded.*

(b) *The module of type* II *is finite-dimensional.*

Let s be the number of indecomposable projective R-modules. The sequence $\{P_n\}_{n=1}^{\infty}$ of isomorphism classes of indecomposable preprojective modules with the preorder in [20, p. 350] alluded to in Example 1.2 can be shown to have the property stated in Lemma 2.7(a), (b), see e.g., [18, Lemma 1.2].

LEMMA 2.7. (a) $\operatorname{Ext}(P_{i+s}, P_i) \neq 0$ *for all positive integers, i.*

(b) *For any positive integer i,* $\operatorname{Ext}(P_j, P_i) = 0$, $j < i + s$.

(c) *Any extension* M *of a bounded torsion-free module* M_1 *by a bounded torsion-free module* M_2 *is bounded.*

(d) [8, *Proposition* 1, *p.* 84] *Let L and N be R-modules. An extension of L by N is equivalent to an extension with middle term* $L \oplus N$ *as a vector space. The module action is given by* $r(l, n) = (rl + f(r)(n), rn)$ *for some linear map f from R to* $\operatorname{Hom}_K(N, L)$.

Proof of (c). Bounded torsion-free modules are direct sums of preprojective indecomposable modules. Let m be the maximum of the bounds on M_1 and M_2. Using the partial order on p. 350 of [20] we can deduce that M is also bounded by m. ∎

The above lemmas are designed to make the proof of Proposition 2.8 identical to the proof of [16, Proposition 2.8].

PROPOSITION 2.8. *An extension of a module of type* III *by a bounded module of type* III *is pure-projective.*

Proof. Let M be an extension of a module L of type III by a bounded module N of type III. So N is a direct sum of submodules of type P_j's with a bound n (say) on the positive integers j. Let $L = L_1 \dotplus L_2$, where L_1 has no direct summand of type P_j with j exceeding n, while if P_j is a direct summand of L_2 then j strictly exceeds n. The embedding of L in M and the projection of L onto L_1 results, by pushout, in an extension M_1 of L_1 by N. By Lemma 2.7(c), M_1 is bounded. Moreover, if P_j is a direct summand of M_1 then $j \leqslant n$. This follows from the preorder on the P_j's on p. 350 of [20], see Example 1.2. The above-mentioned pushout makes M an extension of L_2 by M_1. By [20, Lemma 1E(b), (c)] and Lemma 2.7(b) this extension splits. Therefore, M is pure-projective as required. ∎

An alternative proof of Proposition 2.8 can be given using the fundamental Auslander–Reiten formula connecting Ext and Hom. We thank the anonymous reader referred to in Remark 1.4 for this point.

We conclude the section by showing that the hypothesis of *boundedness* cannot be removed in any of the places it occurs. Since this was done in [16] for Kronecker modules, the functor T below settles it for all finite-dimensional hereditary tame algebras below. We have included the properties of T that we shall need in Section 4. There is a unique infinite-dimensional R-module Q_R with properties analogous to those of the $K[\zeta]$-module, $K(\zeta)$; see [12]. When R is a Kronecker algebra $Q_R = (K(\zeta), K(\zeta))$ with the pair of linear transformations given by the identity map and multiplication by ζ.

THEOREM 2.9 [3, 6, 9, 2, 11, 12]. *For each finite-dimensional hereditary tame algebra R, there is a full and exact embedding T from the category of Kronecker modules to the category of R-modules.*

(a) *The embedding T commutes with direct limits, direct sums, and direct products.*

(b) *It is closed under pure submodules, pure extensions, and pure quotients.*

(c) $T(Q_{R'}) = Q_R$, *where R′ is the Kronecker algebra.*

(d) *T preserves the respective types* I, II, *and* III.

(e) *T preserves and reflects pure-injectivity.*

3. Extensions of Pure-Injective Modules by Pure-Injective Modules

In investigating extensions of pure-injective modules by pure-injective modules over finite-dimensional hereditary tame algebras there is the added complication that a direct summand of a direct product of indecomposable finite-dimensional modules is not always a direct product of finite-dimensional modules. We shall consider direct products of indecomposable modules of type I, II, or III, respectively. Let A be a set of isomorphism types of finite-dimensional indecomposable modules. By abuse of terminology we shall say that a pure-injective module is of type A, if it is a direct summand of modules whose isomorphism types are in A. Two cases are considered in most of the proofs in this section: the bounded case and the unbounded case. In the former case it will often be possible to refer to the corresponding result in Section 2. The results in this section are exact analogues of those in Section 2.

To make the comparison easy we retain the same numbering scheme, e.g., Proposition 3.1 is obtained from Proposition 2.1 by replacing *pure-projective* by *pure-injective*. Following [20] we say that a module M is divisible if $\operatorname{Ext}(S, M) = 0$ for all simple regular modules, M. Therefore extensions of divisible modules by divisible modules are divisible, as are direct products of divisible modules. In [20] it is shown that pre-injective modules are divisible and divisible modules are pure-injective. Therefore, pure-injective modules of type I are divisible. Theorem 5.4 of [20] tells us that every divisible module is a direct sum of indecomposable pre-injective modules, Prüfer modules, and copies of Q_R.

The proof of Proposition 3.1 is similar to that of Proposition 2.1 with some properties of divisible modules thrown in. Also extensions of pure-injective modules of type II by pure-injective modules of type II can be treated by the methods of [10] and Sections 38–40 of [7]. So we shall move on to Proposition 3.3. Recall that a bounded module is both pure-injective and pure-projective and a divisible module is pure-injective.

PROPOSITION 3.3. *An extension of a bounded torsion module M_2 by a pure-injective module, M_1 of type* I *is pure-injective.*

Proof. Let $0 \to M_2 \to M \to M_1 \to 0$ be an exact sequence. Let M_1' be the direct summand of M_1 consisting of indecomposable pre-injective modules and Prüfer modules. Then the inverse image M_2' of M_1' is an extension of M_2 by M_1'. Therefore, M_2' is a torsion module. For any torsion-free module module N, $\operatorname{Ext}(N, M_1') = 0 = \operatorname{Ext}(N, M_2)$ by Proposition 4.7 of [20] and Theorem 3.5 of [13]. Therefore, $\operatorname{Ext}(N, M_2') = 0$. So by Theorem 3.5 of [13], M_2' is a direct sum of a bounded module and a divisible module. Therefore M_2' is pure-injective. The direct complement, M_3', of M_1' in M_1 is a direct sum of copies of Q_R. (If M_1 is bounded it would have neither a Prüfer module nor Q_R as a direct summand and the proof would end here.) So the exact sequence

$$0 \to M_2' \to M \to M_3' \to 0$$

splits because M_2' is pure-injective. So M is pure-injective as claimed. ∎

If M is an extension of a pure-injective module of type II by a pure-injective module of type I, then $\operatorname{Ext}(N, M) = 0$ for all torsion-free modules N, i.e., M is *cotorsion*. In that case, $M = M_1 \oplus M_2 \oplus M_3$, where M_1 is divisible, M_2 has no nonzero torsion-free direct summand, and M_3 is torsion-free and has no nonzero divisible submodule, see Proposition 2 of [14]. As explained there, M_2 is the intractable component.

Let P be the Kronecker module $(K[\zeta], K[\zeta])$. The submodule $L = (0, K[\zeta])$ is a direct sum of modules of type III^1. Hence it is pure-injective. The quotient P/L is a direct sum of modules of type I^1. So, it too is pure-injective. However P is not pure-injective, see e.g., Section 13 of [19]. Therefore an extension of a bounded pure-injective module of type III by a bounded pure-injective module of type I need not be pure-injective. This example shows that it is necessary to assume that the module of type III in Proposition 3.4 is finite-dimensional. It also shows that we cannot delete *torsion-free* from the hypotheses in Lemma 2.7(c). However, an extension of a bounded module of type II by a bounded module of type II is bounded. This can be deduced from Proposition 2.2.

Since divisible modules are pure-injective, pure-projective modules of type I are pure-injective.

PROPOSITION 3.4. *An extension M of a finite-dimensional module of type* III *by a pure-projective module of type* I *is pure-injective.*

Proof. See Proposition 2.4 where M is shown to be a direct sum of a finite-dimensional module and a divisible module, hence pure-injective. ∎

We now give an example to show that an extension M of a finite-dimensional indecomposable pure-injective module M_1 *of type* III *by an unbounded pure-injective module M_2 of type* I *is not always pure-injective.* The example is a Kronecker module and is such that Theorem 2.9 transfers it to a corresponding example over any finite-dimensional hereditary tame algebra.

An unbounded pure-injective module of type I that is not pure-projective has $Q_{R'}$ or a Prüfer module as a direct summand, see e.g., [20, Theorem 5.4]. Let M_2' be the direct complement of $Q_{R'}$ in M_2. By Proposition 12 of [15] there is an indecomposable extension N of M_1 by $Q_{R'}$. By Lemma 2.7(d), this extension is given by a linear map l'. We obtain an extension M of M_1 by M_2 by letting l' act as 0 on M_2'. The construction makes N a direct summand of M. By Section 13 of [19] N, hence M, is not pure-injective. A Prüfer module is a quotient of P by a Kronecker module isomorphic to M_1, up to a change of basis of $K \oplus K$. Just as in the case $Q_{R'}$ we get the non-pure-injective module P as a direct summand of M. Hence M is not pure-injective.

LEMMA 3.5. *An extension of a torsion-free finite-dimensional module by a bounded module of type* II *is pure-injective.*

Proof. This follows from Lemma 2.5 because the pure-projective module there is bounded. ∎

There is an example, just before Lemma 2.5 in [16], of an indecomposable infinite-dimensional torsion-free module, M, that is an extension of a bounded Kronecker module of type III by a bounded Kronecker module of type II. Again by Section 13 of [19] M is not pure-injective.

A module M is said to be *separable* if every finite subset of M is contained in a finite-dimensional direct summand of M. For instance, a pure-projective module is separable. This can be used to give an easy proof, using Proposition 2.6 for instance, that an extension of a pure-projective module by a finite-dimensional module is pure-projective. The same proof can be used for separable pure-injective modules. Lemma 3.5 and the remarks following it are summarized in the next theorem.

THEOREM 3.6. *An extension of a pure-injective module of type* III *by a pure-injective module of type* II *is pure-injective in the follwing case*:

(i) *The module of type* III *is finite-dimensional while that of type* II *is bounded.*

(ii) *The module of type* III *is separable and the module of type* II *is finite-dimensional.*

The example on p. 282 of [16] that shows the necessity of *boundedness* in Theorem 2.6 cannot serve the same purpose in Theorem 3.6 because an unbounded pure-projective module of type II is not pure-injective. The example that does that is in the next proposition.

PROPOSITION 3.7. *Let N is an unbounded pure-injective module of type* II. *Then there is a non pure-injective extension of a finite-dimensional preprojective module by N.*

Proof. We shall give a proof using Kronecker modules and then use Theorem 2.9 to transfer the resulting modules to modules over an arbitrary finite-dimensional hereditary tame algebra.

The notation here is from Example 1.2. From Section 13 of [19] we get from the assumption that N is unbounded, that N contains either $\prod_{\theta \in K'} N_\theta$, or a p-adic module, where K' is an infinite subset of K and $N_\theta = ([x_\theta], [y_\theta])$ is of type II_θ^1. Denoting the identity map and multiplication by ζ map in P by a and b, respectively, we see that $(b - \theta a) x_\theta = 0$ and $a x_\theta = y_\theta$. We now show that in the first instance N contains a submodule N' isomorphic to P.

For each nonnegative integer i, N_1 contains the element $v_i = (\theta^i x_\theta)_{\theta \in K'}$. Replace x_θ by y_θ to get the corresponding element w_i of N_2. Let N_1' and N_2' be the subspaces spanned by the v_i's and w_i's, respectively. The isomorphism of N' with P is obtained by mappings that take v_i and w_i to ζ^i in both the domain and range spaces of P.

Let $L = (0, [w])$ be a Kronecker module of type III^1. We now construct an extension M of L by N which will be shown later not to be pure-injective. Let $s = (s_n)_{n=0}^{\infty}$ be a sequence of ones and zeros with an increasing but finite lengths of zeros. On a basis of N_1 that includes $(v_i)_{i=1}^{\infty}$ we define a linear functional l on N_1 as follows: $l(v_i) = s_i$, l is identically zero on all other elements of the chosen basis of N_1. Let (a, b) be a fixed basis of $K \oplus K$. Using this basis and the linear functional l we make M a Kronecker module as on p. 284 of [16]. In particular, $av_i = w_i + s_i w$ and $bv_i = w_{i+1}$. This implies that for each positive integer k, w is contained in a submodule of M with no direct summand of type III^t with $t \leq k$.

Suppose that M is pure-injective. Then it is a direct summand of a direct product of finite-dimensional indecomposable modules. If M has zero projection on any component we delete such a component. All other components are *irredundant*. We claim that one of these components must be of type III^m for some integer m: If all the irredundant components were either pre-injective or regular then $\text{Ext}(Q_{R'}, M)$ would be zero. (Recall from 2.9 that R' is the Kronecker algebra.) This follows from the properties of Ext and the fact that the torsion submodule of a module is pure and a finite-dimensional module is pure-injective. Now both $\text{Hom}(Q_{R'}, N)$ and $\text{Ext}(Q_{R'}, N)$ are zero because $Q_{R'}$ is divisible and torsion-free. So from the long exact sequence for Ext applied to $0 \to L \to M \to N \to 0$, we get that $\text{Ext}(Q_{R'}, L)$ is isomorphic to $\text{Ext}(Q_{R'}, M)$. The former is not zero by [15, Proposition 6]. Therefore there is an irredundant preprojective component, i.e., a component of type III^m for some positive integer m. So the projection onto such a component is not zero when restricted to M. Hence by Corollary 2.2 of [20], M has an indecomposable preprojective direct summand, M' say.

The projection of M onto M' is not zero on $[w]$. If it were then N would have a direct summand of type III^k for some integer k which is not the case, e.g., by the same Ext argument used above. However, we showed earlier in this proof that w is in the range space of a submodule M'' with no direct summand of type III^t with $t \leq k$. The projection π of M onto M' restricted to M'' is zero as pointed out in Example 1.2. So $\pi(w)$ is zero. This contradiction establishes that M is not pure-injective.

Suppose N contains a p-adic module. The latter in turn contains an isomorph, P', of P up to a change of basis of $K \oplus K$. Using P' in place of P in the above argument, we again arrive at a contradiction. ∎

After the last proposition it is a relief that the proof of the next proposition is similar to that of Proposition 2.8 once we bear in mind that a product of modules bounded by m is also bounded by m, see e.g., [17]. Therefore it is a pure-projective module.

PROPOSITION 3.8. *An extension M of a pure-injective module L of type* III *by a bounded pure-injective module N of type* III *is pure-injective.*

Proof. We first do the case when L is a direct product of modules of type P_j. Let n be the bound on N. The proof now proceeds as that of Proposition 2.8 with the reference to Lemma 1E (b) and (c) of [20] replaced by Lemma 1E (a) and (b) of [20].

Suppose $L \oplus L'$ is a direct summand of a direct product of modules of type P_j. The embedding of L in M and the inclusion of L in $L \oplus L'$ give by pushout an extension M' of $L \oplus L'$ by N. By the first part of the proof, M' is pure-injective. Also the above-mentioned pushout gives that M' is a split extension of M by L'. So M is a direct summand of a pure-injective module. Hence it is pure-injective. ∎

PROPOSITION 3.9. *Let N be an unbounded pure-injective module of type* III. *Then there is a non-pure-injective extension of a finite-dimensional preprojective module by N.*

Proof. Just as in Proposition 3.7 it is enough to prove the proposition for Kronecker modules. Every nonzero submodule of N has a finite-dimensional direct summand, by Proposition 2.7 of [20]. Since N is unbounded it follows from Theorem G of [20] and the fact that a bounded module is pure-injective that N contains an unbounded pure-projective module N'.

Let $N' = \bigoplus_{k=1}^{\infty} N_k$, where N_k is of type III^k with domain and range spaces spanned by the sets $v_k = (v_{21}, v_{22}, ..., v_{2,k-1})$ and $w_k = (w_{21}, w_{22}, ..., w_{2k})$. We now define a linear functional on a basis of N_1 that includes v_k for all positive integers k: For $3 \leqslant k$ odd let $l(v_{2,[k/2]}) = 1$, l is identically zero on all other elements in the basis. Here $[(k+1)/2]$ is the integer part of $k/2$. Using this linear functional and the basis (a, b) of $K \oplus K$ we obtain an extension M of $(0, [w])$ by N as in Proposition 3.7. The choice of l and the fact that there is no bound on the positive integers k ensure that for each positive integer s, w is contained in a submodule of M with no direct summand of type III^t with $t \leqslant s$. If M were pure-injective it would have to be a direct summand of a direct product of modules of type P_n for various positive integers n. Working with the above property of w we obtain a contradiction along similar lines as in the proof of Proposition 3.7. ∎

4. QUOTIENTS OF DIRECT PRODUCTS BY DIRECT SUMS OF FINITE-DIMENSIONAL MODULES

In this section it will be convenient to denote modules by their isomorphism types. Let I_n: $n = 1, 2, 3, ...,$ be the set of isomorphism classes of indecomposable pre-injective R-modules. As shown in [20], $\prod_{n=1}^{\infty} I_n$ is divisible. Therefore by Theorem 5.4 of [20] it is a direct sum of copies of indecomposable pre-injective modules, Prüfer modules, and Q_R. The first result in this section says how many copies of each of summand is involved. Here c denotes $2^{\aleph_0}$.

PROPOSITION 4.1. *The module* $M = \prod_{n=1}^{\infty} I_n$ *is a direct sum of*

(a) *a single copy of* I_n *for each positive integer n,*

(b) *c copies of* S_t^{∞} *for each t it in* $K \cup \{\infty\}$.

(c) *c copies of* Q_R.

Proof. (a) follows from Theorem 3.3 and Remark 3.4 of [20]. The functor T and cardinality considerations allow us to prove parts (b) and (c) for Kronecker modules only.

Let $M_n = (M_{n1}, M_{n2})$ be a Kronecker module of type I^n. It will be convenient to use the description of I^n in Section 2 of [1]. For each θ in K, there are bases $(x_{1n\theta}, x_{2n\theta}, ..., x_{nn\theta})$, $(y_{2n\theta}, y_{3n\theta}, ..., y_{nn\theta})$ of M_{n1}, M_{n2}, respectively. For a fixed basis of $K \oplus K$, the action of $K \oplus K$ on M_{n1} into M_{n2} is given by $b - \theta a x_{11\theta} = 0$, $b - \theta a x_{in\theta} = a x_{i-1, n\theta} = y_{in\theta}$, $i = 2, 3, ..., n$; $a x_{nn\theta} = 0$. When $\theta = \infty$ replace every occurrence of $b - \theta a$ by a and every occurrence of a by b in the above description of the action of $K \oplus K$. Let F be the two-element field. Let S be a set of representatives of a basis of $\prod_{\aleph_0} F / \bigoplus_{\aleph_0} F$. Elements of S are sequences of zeros and ones. Let $s = (s_n)_{n=1}^{\infty}$ be any element of S. For a fixed θ in $K \cup \{infty\}$ the sets $\{(s_n x_{jn\theta})_{n=1}^{\infty}\}_{j=1}^{\infty}$ and $\{(s_n y_{jn\theta})_{n=1}^{\infty}\}_{j=2}^{\infty}$ span the domain and range spaces of a submodule of M of type $\mathrm{II}_{\theta}^{\infty}$ (if $j > n$, set $x_{jn\theta}$ and $y_{jn\theta}$ equal to 0.) Since the cardinality of S is c and $\mathrm{Hom}(\mathrm{II}_{\theta}^{\infty}, \mathrm{II}_{\eta}^{\infty}) = 0$ if $\theta \neq \eta$, we have proved (b).

Starting from the element 1 we can recover all of $Q_{R'} = (K(\zeta), K(\zeta))$ by dividing it by $(\zeta - \theta)^n$ and multiplying it by ζ^n for all elements $\theta \in K$ and all positive integers n. To prove (c) it is enough to exhibit an element that will play the role of 1. The set S in the last paragraph is then used to get c copies of Q_R from this single element. Let $[n/2]$ be the integer part of $n/2$. The desired element is $(y_{[n/2]\, n\infty})_{n=1}^{\infty}$. ∎

The next result was suggested by analogous results in abelian groups due to Fuchs, Golema, and Hulanicki, see Section 42 of [7].

PROPOSITION 4.2 [11]. *Suppose the field K is countable. Let* M_n, $n = 1, 2, ...,$ *be R-modules. Then* $\prod_{n=1}^{\infty} M_n / \bigoplus_{n=1}^{\infty} M_n$ *is pure-injective.*

What are the invariants of the pure-injective modules in Proposition 4.2? When all the modules involved are torsion-regular i.e., have no pre-injective direct summand- we can fall back on [7]. When the modules are nonsingular, defined below, we can also borrow from abelian group theory.

The next proposition illustrates this case. If M is a $K[\zeta]$-module then (M, M) may be considered a Kronecker module with the identity map and multiplication by ζ providing a pair of linear maps on M. Kronecker modules that are isomorphic to modules obtained as above are said to be *nonsingular*, see p. 281 of [1]. The module P in Example 1.2 is nonsingular. By abuse of notation we identify M with (M, M). In the proposition below J_p is the completion of the localisation of $K[\zeta]$ at the prime p.

PROPOSITION 4.3. *Let K be a countable field. Let $M = \prod_{\aleph_0} P / \bigoplus_{\aleph_0} P$. ($P$ is the Kronecker module in Example 1.2) Then $M = \bigoplus_c Q_R \oplus \prod_p A_p$, where A_p is the p-adic completion of $\bigoplus_c J_p$.*

Proof. With Z in the place of P this is a theorem of S. Balcerzyk [4]. We can imitate the proof outlined in [7, Exercise 7, p. 177]. ∎

If K is uncoutable the proof of Proposition 2.1 in [9] shows that M in Proposition 4.3 is not pure-injective.

We conclude the paper by using the functor T to extend Proposition 2.1 of [9] to all finite-dimensional hereditary tame algebras. We should remark that our proof of Proposition 4.4 is implicit in [9, Proposition 3.1].

PROPOSITION 4.4. *Let $(P_n)_{n=1}^{\infty}$ be the sequence of preprojective indecomposable R-modules. If K is uncountable, then $M = \prod_n P_n / \bigoplus_n P_n$ is not pure-injective.*

Proof. Let $M' = \prod_m \mathrm{III}^m / \bigoplus_m \mathrm{III}^m$. It follows from Theorem 2.9 that $T(M')$ is a direct summand of M. If M is pure-injective, so would $T(M')$. By Theorem 2.9(e), this implies that M' is pure-injective, contradicting Proposition 2.1 of [9]. ∎

REFERENCES

1. N. ARONSZAJN AND U. FIXMAN, Algebraic spectral problems, *Studia Math.* **30** (1968), 273–338.
2. D. BAER, H. BRUNE, AND H. LENZING, A homological approach to representations of algebras, II. Tame hereditary algebras, *J. Pure Appl. Algebra* **26** (1982), 141–153.
3. D. BAER AND H. LENZING, A homological approach to representations of algebras, I. The wild case, *J. Pure Appl. Algebra* **26** (1982), 227–233.
4. S. BALCERZYK, On factor groups of some subgroups of a complete direct sum of infinite cyclic groups, *Bull. Acad. Polon. Sci.* **7** (1959), 141–142.
5. J. DIEUDONNE, Sur les p-groupes abeliens infinis, *Portugal. Math.* **11** (1952), 1–5.
6. V. DLAB AND C. M. RINGEL, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* **173** (1976).
7. L. FUCHS, "Infinite Abelian Groups, I," Academic Press, New York, London, 1973.
8. N. JACOBSON, Structure of rings, *in* "American Mathematical Society Colloquium Publications," Vol. 37, American Mathematical Society, Providence, RI, 1956.

9. U. JENSEN AND H. LENZING, Algebraic compactness of reduced products and applications to global dimension, *Comm. Algebra* **11** (1983), 305–325.
10. I. KAPLANSKY, "Infinite Abelian Groups," Univ. of Michigan Press, Ann Arbor, MI, 1954, 1969.
11. H. LENZING, Homological transfer from finitely presented to infinite modules, *in* "Lecture Notes in Mathematics" (R. Göbel, L. Lady, and A. Mader, Eds.), Vol. 1006, pp. 734–761, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.
12. H. LENZING, The pure-projective dimension of torsion-free divisible modules, *Comm. Algebra* **12** (1984), 649–662.
13. F. OKOH, A splitting criterion for pairs of linear transformations, *Illinois J. Math.* **22** (1978), 379–388.
14. F. OKOH, Cotorsion modules over tame finite-dimensional hereditary algebras, *in* "Lecture Notes in Mathematics" (M. Auslander and E. Luis, Eds.), Vol. 903, pp. 263–269, Springer-Verlag, Berlin/Heidelberg/New York, 1981.
15. F. OKOH, Indecomposable rank two modules over some artinian ring, *J. London Math. Soc.* **22** (1980), 411–422.
16. F. OKOH, On pure-projective modules over Artin algebras, *J. Algebra* **77** (1982), 275–285.
17. F. OKOH, Separable modules over finite-dimensional algebras, *J. Algebra* **116** (1988), 400–414.
18. F. OKOH, Pure-injective modules over path algebras, *J. Pure Appl. Algebra* **75** (1991), 75–83.
19. M. PREST, "Model Theory and Modules," London Mathematical Society Lecture Notes Series, Vol. 130, Cambridge Univ. Press, Cambridge, 1988.
20. C. M. RINGEL, Infinite-dimensional representations of finite-dimensional algebras, *Sympos. Mat. Inst. Alta Mat.* **23** (1979), 321–412.