On a nonlinear nonlocal ODE arising in magnetic recording

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Abstract

In this paper we study the uniqueness of the solution for a nonlinear ODE with nonlocal terms. We consider a limit case of a one-dimensional equation arising in magnetic recording. The equation models the tape deflection where the magnetic head profile, with trenches to control the tape position, is a known function.

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1. Introduction

Different kinds of nonlocal terms appear in a great number of partial differential equations of elliptic type. In this work we will consider a particular case where the unknown $u$ appears evaluated at a distinguished point $x_0$ of the domain. The simplest example of an elliptic problem with this type of term is the following:

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The solution of the problem depends on the value of lambda: for \( \lambda \neq 8 \) the only solution is \( u = 0 \), whereas for \( \lambda = 8 \) infinitely many solutions appear. These are given by \( u = cx(x - 1) \) for any \( c \neq 0 \). Notice that this is not an eigenvalue problem and therefore the question of uniqueness is significant.

In the next section we present a problem arising in magnetic recording that we will study in Section 3.

### 2. The magnetic tape

A magnetic tape is driven with constant velocity over the magnetic head and its position \( u \) is given as the solution of the ODE

\[
\begin{align*}
-\frac{d^2 u}{dx^2} &= k \left( \frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]} \quad 0 < x < L, \\
&\quad u(0) = u(L) = 0
\end{align*}
\]  

(2.1)

where \( 0 < L_1 < L_2 < L \), \( \chi_{[L_1, L_2]} \) is the characteristic function of the interval \( [L_1, L_2] \), \( \delta \) is the head profile, \( k \) is a positive constant and \( u \) satisfies

\[
u(x) > \delta(x) \quad \text{if} \quad L_1 \leq x \leq L_2.
\]  

(2.2)

(2.1) is the limit case of a system where the pressure \( p \) of the air is modelled by the compressible Reynolds equation and the position of the tape \( u \) is modelled by the beam equation (see [1–3]). Problem (2.1) has been analyzed in [4] and [5].

In [4] existence and uniqueness is proved using a shooting method under the assumption

\[
\delta \in C^2 \quad \text{and} \quad \delta''(x) < 0, \quad L_1 \leq x \leq L_2.
\]  

(2.3)

This assumption is very restrictive, mathematically and physically, because magnetic heads do not usually satisfy the concavity condition (2.3) and are generally discontinuous (see [2,3,5]).

In [5] the existence of solutions is proved using a sub- and super-solution method under more general assumptions:

\[
\delta \quad \text{is piecewise continuous with jump discontinuous at} \quad \xi_1, \ldots, \xi_s \quad \text{where}
\]

\[
\xi_0 = L_1 < \xi_1 \ldots \xi_s < L_2 = \xi_{s+1}, \quad \text{and} \quad \delta \in C^1[\xi_i, \xi_{i+1}] \quad \text{for} \quad 0 \leq i \leq s,
\]  

(2.4)

and

\[
\delta(L_1) < \delta'(L_1)L_1, \quad \delta(L_2) < (L_2 - L)\delta'(L_2).
\]  

(2.5)

Uniqueness was proved in case (2.3), but not for the general case (2.4), (2.5). The question of uniqueness is not just a mere mathematical issue. Its analysis is also necessary for simulating the solution with a numerical approach.

The main result of this paper is enclosed in the following theorem.

**Theorem 2.1.** Assume that (2.4) and (2.5) are satisfied. Then there exists a unique solution \( u \) to (2.1) satisfying (2.2).
Note that the inequality \( \delta(L_1) < \delta'(L_1)L_1 \) means that the tangent to the head at \( x = L_1 \) intersects the \( x \)-axis in the interval \((0, L_1)\). Similarly, the second inequality in (2.5) means that the tangent to the head at \( x = L_2 \) intersects the \( x \)-axis in the interval \((L_2, L)\).

3. Proof of the Theorem 2.1

By [5, Theorem 2.1] we know that any solution \( u \) to (2.1) satisfies

\[
u \in W^{2,\infty}(0, L).
\]

We assume without loss of generality that

\[
\delta(L_1) = 0, \quad \delta(x) \geq 0, \quad \delta \geq \frac{k}{2}(x - L_1)^2 \text{ if } x \in [L_1, L_2].
\]

Remark 3.1. Notice that if \( \delta \) does not satisfy (3.2) we can introduce the change

\[
\tilde{u} = u - \delta(L_1) + \gamma(x - L_1), \quad \tilde{\delta} = \delta - \delta(L_1) + \gamma(x - L_1)
\]

where \( \gamma \), defined by

\[
\gamma = \max \left\{ 0, -\min_{x \in (L_1, L_2)} \left\{ \frac{\delta(x) - \delta(L_1)}{x - L_1} \right\} \right\} + k(L_2 - L_1)
\]

is bounded by (2.4). Then \( \tilde{\delta} \) satisfies (3.2) and \( \tilde{u} \) satisfies

\[
\left\{ \begin{array}{l}
-\frac{\partial^2 \tilde{u}}{\partial x^2} = k \left( \frac{\tilde{u}(L_1)}{\tilde{u}(x) - \tilde{\delta}(x)} - 1 \right) \chi_{[L_1, L_2]}, \\
\tilde{u}(0) = -\delta(L_1) - \gamma L_1, \\
\tilde{u}(L) = -\delta(L_1) + \gamma(L - L_1), \\
\tilde{u}(x) - \tilde{\delta}(x) > 0,
\end{array} \right. \quad 0 < x < L,
\]

\[
\left\{ \begin{array}{l}
-\frac{\partial^2 u}{\partial x^2} = k \left( \frac{\lambda}{u(\lambda) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \\
u(\lambda, 0) = \gamma_1, \\
u(\lambda, L) = \gamma_2, \\
u(x) - \delta(x) > 0
\end{array} \right. \quad 0 < x < L,
\]

(3.3)

As in [5], we consider the unique solution \( u(\lambda) \) of the problem

\[
\left\{ \begin{array}{l}
-\frac{\partial^2 u}{\partial x^2} = k \left( \frac{\lambda}{u(\lambda) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \\
u(\lambda, 0) = \gamma_1, \\
u(\lambda, L) = \gamma_2, \\
u(x) - \delta(x) > 0
\end{array} \right. \quad 0 < x < L,
\]

(3.4)

for \( \gamma_1 > -L_1\delta'(L_1) \) and \( \gamma_2 > (L - L_2)\delta'(L_2) + \delta(L_2) \). By [5, Lemma 2.1] we know that for any \( \lambda > \delta(L_1) = 0 \) there exists a unique solution \( u(\lambda) > \delta \) to (3.4).

Lemma 3.1. If \( \lambda_1 > \lambda_2 \) then \( u(\lambda_1) \geq u(\lambda_2) \) in \([0, L]\).

Proof. Consider \( u(\lambda_1) - u(\lambda_2) \) which satisfies

\[
-\frac{\partial^2}{\partial x^2} (u(\lambda_1) - u(\lambda_2)) - \left( \frac{\lambda_1}{u(\lambda_1) - \delta(x)} - \frac{\lambda_1}{u(\lambda_2) - \delta(x)} \right) \chi_{[L_1, L_2]} = (\lambda_1 - \lambda_2) \frac{1}{u(\lambda_2) - \delta(x)} \chi_{[L_1, L_2]} \geq 0.
\]

(3.5)
Let us consider the continuous and Lipschitz function $\phi$ defined by $\phi(s) = s$ if $s < 0$ and 0 otherwise. Let us take $\phi(u(\lambda_1) - u(\lambda_2))$ as a test function in (3.5); we obtain
\[
\int_0^L \left[(u_x(\lambda_1) - u_x(\lambda_2))^2\right] \phi'(u(\lambda_1) - u(\lambda_2)) \, dx + \int_{L_1}^{L_2} \left(\frac{\lambda_1}{u(\lambda_1) - \delta} - \frac{\lambda_1}{u(\lambda_2) - \delta}\right) \phi(u(\lambda_1) - u(\lambda_2)) \, dx = \lambda_1 - \lambda_2 \int_{L_1}^{L_2} \frac{1}{u(\lambda_2) - \delta} \phi(u(\lambda_1) - u(\lambda_2)) \, dx \leq 0.
\]

Since $\frac{1}{u - \delta}$ is decreasing (as a function of $u$) for $u > \delta$, we obtain
\[
\left(\frac{\lambda_1}{u(\lambda_1) - \delta} - \frac{\lambda_1}{u(\lambda_2) - \delta}\right) \phi(u(\lambda_1) - u(\lambda_2)) \leq 0
\]
and then
\[
\int_0^L \left[(u_x(\lambda_1, x) - u_x(\lambda_2, x))^2\right] \phi'(u(\lambda_1) - u(\lambda_2)) \, dx \leq 0.
\]

By definition of $\phi$ we deduce the desired result. □

Let us argue by contradiction and consider that there exist two different solutions, $u_1$ and $u_2$, to (3.3) such that $u_1(L_1) = \lambda_1$, $u_2(L_1) = \lambda_2$ and
\[
\lambda_1 > \lambda_2.
\]
Then, $u_i$ (for $i = 1, 2$) satisfies
\[
-\frac{\partial^2 u_i}{\partial x^2} = k \left(\frac{\lambda_i}{u_i(x) - \delta(x)} - 1\right) \chi(L_1, L_2) \quad 0 < x < L,
\]
\[
u_i(0) = -\delta(L_1) - \gamma L_1, \quad u_i(L) = -\delta(L_1) + \gamma(L - L_1).
\]

Consider the new unknown $v$ and $w$ defined by
\[
v = u_1 - u_2, \quad w = \lambda_2 u_1 - \lambda_1 u_2 \quad \text{in} \ [L_1, L_2].
\]
Then $v$ satisfies
\[
-\frac{\partial^2 v}{\partial x^2} = k \left(\frac{\lambda_1}{u_1(x) - \delta(x)} - \frac{\lambda_2}{u_2(x) - \delta(x)}\right) \quad L_1 < x < L_2,
\]
\[
v(L_1) = \lambda_1 - \lambda_2, \quad v_x(L_1) = \frac{\lambda_1 - \lambda_2}{L_1}
\]
and $w$ satisfies
\[
-\frac{\partial^2 w}{\partial x^2} = k \left(\frac{\lambda_2 u_1}{u_1(x) - \delta(x)} - \frac{\lambda_1 u_2}{u_2(x) - \delta(x)} - (\lambda_2 - \lambda_1)\right) \quad 0 < x < L,
\]
\[
w(L_1) = w_x(L_1) = 0.
\]

Since
\[
\frac{\lambda_1}{u_1(x) - \delta(x)} - \frac{\lambda_2}{u_2(x) - \delta(x)} = \frac{\lambda_1 u_2(x) - \lambda_2 u_1(x) - (\lambda_1 - \lambda_2)\delta(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))} = \frac{-w - (\lambda_1 - \lambda_2)\delta(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))}
\]
we obtain by (3.6) and (3.2) \(- (\lambda_1 - \lambda_2)\delta(x) \leq 0\). Then, writing
\[
f(x) = \frac{1}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))} > 0
\]
we obtain
\[
v_{xx} = kf(x)(w + (\lambda_1 - \lambda_2)\delta), \quad x \in (L_1, L_2).
\]
In the same way,
\[
\frac{1}{u_1(x) - \delta(x)} - \frac{1}{u_2(x) - \delta(x)} = \frac{u_2(x) - u_1(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))} = - f(x)v
\]
and then
\[
w_{xx} = k\lambda_1\lambda_2 f v - k(\lambda_1 - \lambda_2), \quad x \in (L_1, L_2).
\]

**Lemma 3.2.** \( w \geq -\frac{k}{2}(\lambda_1 - \lambda_2)(x - L_1)^2 \) in \([L_1, L_2]\).

By Lemma 3.1 we deduce that \( v \geq 0 \) and by (3.14) we get
\[
w_{xx} \geq -k(\lambda_1 - \lambda_2) \quad \text{if } (L_1, L_2).
\]
Integrating (3.15) twice over \((L_1, x)\), as a result of (3.12) we obtain the desired result. \(\square\)

**End of the Proof of the Theorem.** By the previous lemma and from (3.13) we deduce
\[
w + (\lambda_1 - \lambda_2)\delta \geq (\lambda_1 - \lambda_2) \left( -\frac{k}{2}(x - L_1)^2 + \delta \right).
\]
By (3.2) it results that \((\lambda_1 - \lambda_2)(-\frac{k}{2}(x - L_1)^2 + \delta) \geq 0\) and substituting this in (3.13) we get
\[
v_{xx} \geq 0 \quad \text{in } (L_1, L_2).
\]
Since \( v(L_1) > 0, v_x(L_1) > 0 \) (see (3.10)) and from (3.16), \( v(L_2) \) satisfies
\[
0 < v(L_2) = u_1(L_2) - u_2(L_2)
\]
and \( v_x(L_2) \)
\[
0 < v_x(L_2) = u_{1x}(L_2) - u_{2x}(L_2).
\]
Integrating (3.7) in the interval \((L_2, L)\) we obtain
\[
u_1(L) = u_1(L_2) + (L - L_2)u_{1x}(L_2),
\]
and
\[
u_2(L) = u_2(L_2) + (L - L_2)u_{2x}(L_2).
\]
Subtracting the above expressions we get
\[
u_1(L) - u_2(L) = u_1(L_2) - u_2(L_2) + (L - L_2)(u_{1x}(L_2) - u_{2x}(L_2))
\]
and by (3.17) and (3.18) it results that \( u_1(L) - u_2(L) > 0 \) which contradicts (3.8). \(\square\)

**Remark 3.2.** The typical head profile satisfies
\[
\delta(x) - \delta(L_1) \leq \delta'(L_1)(x - L_1) \quad \text{in } [L_1, L_2].
\]
Then (2.5) is a necessary assumption.
If \( \delta'(L_1) \leq \frac{\delta(L_1)}{L_1} \) and \( \delta \) satisfies (3.19), then \( u'(L_1) \geq \frac{\delta(L_1)}{L_1} > 0 \) and \( \frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} \) is decreasing (as a function of \( x \)). We obtain \( u(L_2) > u(L_1) > 0 \) and \( u_x(L_2) > u_x(L_1) > 0 \) and then \( u(L) > 0 \), which contradicts (2.1).

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References