Invariant Subcones of a Linear Completely Continuous Operator Leaving a Cone Fixed in a Banach Space

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Let $X$ be an infinite-dimensional Banach space with norm denoted by $\| \cdot \|$. By a cone $K \subset X$ we mean a subset of $X$ satisfying the following properties:

1. If $x \in K$, then $\lambda x \in K$, $\forall \lambda > 0$.
2. If $x, y \in K$, then $x + y \in K$.
3. If $x, y \in K$, $x \neq \theta$, then $x + y \neq \theta$, where $\theta$ is the null element of $X$.

In what follows we assume that $K$ is closed and its interior $\hat{K}$ is non-empty. A subcone $K_1 \subset K$ is said to be a proper subcone of $K$ if $K_1$ is a closed cone and $K_1 \neq \{\theta\}$, $K_1 \subset K$, $K_1 \subset K$.

Here, by a completely continuous operator $A : X \to X$ we mean a continuous operator, which maps bounded subsets into relatively compact subsets.

The result of this note is given by

**Theorem 1.** Let $K \subset X$ be a closed cone with non-empty interior. If $A : X \to X$ is a linear completely continuous operator such that $AK \subset K$ (being different from null operator from $K$ into itself), then there is a proper subcone $K_1 \subset K$ of $K$, such that each linear continuous operator $T : K \to K$, commuting with $A$ (i.e., $TA = AT$), leaves $K_1$ invariant (i.e., $TK_1 \subset K_1$).

**Remark 1.** The reasoning of the result given by Theorem 1 is the following one: in general, a linear completely continuous operator $A : K \to K$ has no eigenvectors $x \in K$ (corresponding to positive eigenvalues) but has necessarily invariant proper subcones $K_1 \subset K$. Several additional conditions under which $A$ admits eigenvectors $x \in K$ are given in [3]. For example, if in addition to the hypothesis of Theorem 1 we assume that for each $x \neq \theta$ which lies on the boundary of $K$, there is a natural number $n = n(x)$ such that $A^n x \in \hat{K}$, then there is $\lambda > 0$ and $x \in K$, such that $Ax = \lambda x$.  
The space $X$ need not to be complex (as in Theorem 2 of Lomonosov).

Theorem 1 was suggestes by the well-known theorem of Lomonosov, restated below as Theorem 2.

**Theorem 2 (Lomonosov [4]).** If $X$ is an infinite-dimensional complex Banach space and $A: X \to X$ is a linear completely continuous operator (different from null operator), then there is a closed subspace $X_1 \subset X$. $X_1 \neq \{\theta\}, X_1 \neq X$, invariant with respect to each linear continuous operator $T$ commuting with $A$ (i.e., if $TA = AT$, then $TX_1 \subset X_1$).

The proof of Theorem 1 is closely related to the proof of Lomonosov's theorem and makes use of a result of Krein and Rutman [3] (on the existence of eigenvectors of an one-to-one linear operator leaving invariant a closed cone in a finite dimensional space). The first result on the existence of invariant subspaces of a linear completely continuous operator on a Banach space is due to Aronszajn and Smith [1].

Several generalizations of Lomonosov theorem are given by Pearcy [5]. For other results see Ky Fan [2].

2. **Proof of Theorem 1**

Assume by contradiction that the conclusion of Theorem 1 is not true. First of all this assumption implies that $A$ is one-to-one therefore,

$$N_A = \{x \in K, Ax = \theta\} = \{\theta\} \quad (1)$$

and that $A$ does not admit eigenvectors $x \in K - \{\theta\}$ (i.e., there is no $\lambda > 0$ and $x \in K - \{\theta\}$, such that $Ax = \lambda x$).

Let $x_0 \in K$ be such that $\|x_0\| > 1$ and $\|Ax_0\| = 1 + d$ with $d > 0$.

Now let $0 < r < 1$ be such that $B(x_0, r) \subset K$ and

$$\|Ax - Ax_0\| \leq 1, \quad x \in B(x_0, r), \quad (2)$$

where $B = B(x_0, r)$ denotes the ball of radius $r$ about $x_0$.

Clearly, $\theta \in B(x_0, r)$ since $\|x_0\| > 1$. Moreover, $\theta \in AB(x_0, r)$ since we have

$$\|Ax\| \geq \|Ax_0\| - \|Ax - Ax_0\| \geq d, \quad x \in B(x_0, r). \quad (3)$$

Denote by $F$ the set of all linear continuous operators $T: K \to K$, commuting with $A$.

For each $y \in AB(x_0, r)$, set

$$K_y = \{Ty, T \in F\}. \quad (4)$$
Obviously, $K_y$ is a subcone of $K$, $Ay \in K_y$ and since $y \neq \theta$ we have $Ay \neq \theta$. Thus, $K_y \neq \{\theta\}$. On the other hand, $TK_y \subset K_y$, $\forall T \in F$, therefore $TK_y \subset K_y$. Taking into account that $K_y$ is a cone too, we must have $K_y = K$. Since $\overline{AB}(x_0, r) \subset K$, $x_0 \in K$, it follows that for each $y \in \overline{AB}(x_0, r)$, there is $T = T_y \in F$ such that

$$\|T_y y - x_0\| < r. \quad (5)$$

In view of the continuity of $T_y$, inequality (5) holds in a neighborhood $V_y$ of $y$, i.e.,

$$\|T_y z - x_0\| < r, \quad \forall z \in V_y \quad (6)$$

Inasmuch as $\overline{AB}$ is a compact subset of $K$, it follows that there is a finite number of elements $y_1, \ldots, y_n \in \overline{AB}$ such that

$$\overline{AB} \subset \bigcup_{i=1}^{n} \{z \in K. \|T_y z - x_0\| < r\}, \quad (7)$$

where $\overline{AB} = \overline{AB}(x_0, r)$.

Let us denote $T_i = T_{y_i}$,

$$a_i(z) = \max\{r - \|T_i z - x_0\|, 0\},$$

$$b_i(z) = \frac{a_i(z)}{\sum_{i=1}^{n} a_i(z)}, \quad i = 1, 2, \ldots, n.$$

In view of (7) it follows that $b_i$ is defined on $\overline{AB}$. Obviously, $a_i$ and $b_i$ are continuous functions.

Introducing the function $f$ by

$$f(x) = \sum_{i=1}^{n} b_i(Ax) T_i Ax, \quad x \in B(x_0, r), \quad (8)$$

we see that $f$ is continuous on $B = B(x_0, r)$.

Denote by $C$ the convex hull of $\bigcup_{i=1}^{n} T_i(\overline{AB}(x_0, r))$ and by $K_0 = \overline{C} \cap B(x_0, r)$ (which is a compact subset of $K$).

First of all we see that $f : B \to K_0$. Indeed, it is obvious that $f(x) \in C$, $\forall x \in B$.

Also, we have

$$\|f(x) - x_0\| = \left\| \sum_{i=1}^{n} b_i(Ax) T_i Ax - \sum_{i=1}^{n} b_i(Ax) x_0 \right\|$$

$$\leq \sum_{i=1}^{n} b_i(Ax) \|T_i Ax - x_0\| < r, \quad \forall x \in B$$

so $f(x) \in B$. 

Thus \( f: K_0 \to K_0 \), so by the classical fixed point theorem of Schauder-Tychonoff, there is \( x_1 \in K_0 \) such that \( f(x_1) = x_1 \). Since \( x_1 \in B(x_0, r) \), it follows that \( x_1 \neq \theta \). This fact implies that the closed cone

\[
M = \{ x \in K, Qx = x \},
\]

where

\[
Q = \sum_{i=1}^{n} b_i (Ax_i) T_i A
\]

is strictly included in \( K \) (since \( Q \) is a linear completely continuous operator, and the identity operator on \( K \) is not completely continuous) \((X\) is infinite dimensional). Moreover, \( x_1 \in M \), and \( M \) is a cone of a finite-dimensional space, namely.

\[
M = \{ x \in X, Qx = x \} = X_1.
\]

Clearly, \( A: M \to M \), therefore, by a well-known result of Krein and Rutman [3] it follows that \( A \) has at least an eigenvector \( x_2 \in X, x_2 \neq \theta \) (i.e., \( Ax_2 = \lambda x_2 \), with \( \lambda > 0 \)).

In view of our assumption, we have reached a contradiction, so the theorem is proved.

**Remark 2.** Perhaps one can obtain generalizations of Theorem 1 in the sense of Pearcy [5].

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**REFERENCES**