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On the quasi-heredity and the semi-simplicity of cellular algebras

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Abstract

Some simpler homological characterizations of quasi-hereditary algebras inside the class of cellular algebras are presented in terms of cell modules. Moreover, some new criteria for the semi-simplicity of cellular algebras are given by using the cohomology groups of cell modules and simple modules.

Keywords: Cellular algebra; Quasi-hereditary algebra; Semisimple algebra; Cell module

1. Introduction

Cellular algebras have been introduced by Graham and Lehrer [4] in order to investigate, in an axiomatic framework, the modular representations of Hecke algebras and related algebras with geometric connections like Brauer algebras and Temperley–Lieb algebras. One of the important features of cellular algebras is that from the theoretical point of view, the problem of determining a parameter set for all irreducible representations is reduced to questions in linear algebra.

There are close connections between cellular algebras and quasi-hereditary algebras. In fact, the class of cellular algebras has a large intersection with the class of quasi-hereditary algebras. In [6] it was shown that a cellular algebra A is quasi-hereditary if and only if A has Cartan determinant one, and this is equivalent to that the decomposition matrix of A is square. In this way, one can obtain many quasi-hereditary algebras from

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cellular algebras, for instance, from Brauer algebras [4,6] and Birman–Wenzl algebras [9]. Recently, Xi [10] gave a homological characterization of the quasi-heredity of cellular algebras in terms of cell modules. Unlike the characterization of Cartan determinant, the homological characterization does not need any information on simple modules.

A special case of quasi-heredity is the semi-simplicity. It is well-known that all split semisimple algebras which are naturally included in the class of quasi-hereditary algebras are cellular. However, cellular algebras are not always semisimple. The problem of determining semi-simplicity was theoretically reduced to the computation of the discriminants of bilinear forms defined on cell modules in [4], which is a local solution. It is shown in [11] that a cellular algebra A is semisimple if and only if all eigenvalues of the Cartan matrix of A are rational numbers and the Cartan determinant equals one.

The purpose of this paper is to give much simpler homological characterizations of the quasi-heredity of cellular algebras and some new criteria for their semi-simplicity by using the cohomology groups of cell modules and simple modules. Our main results can be stated as follows.

Theorem 1.1. *Let K be a field and A a cellular K-algebra with involution i and cell chain $0 = J_{m+1} \subset J_m \subset J_{m-1} \subset \cdots \subset J_1 = A$. Denote by $W(\lambda)$ the cell module associated to the cell ideal $J_\lambda/J_{\lambda+1}$, $1 \leqslant \lambda \leqslant m$. Then the following statements are equivalent*:

(a) *The algebra A is quasi-hereditary.*
(b) $\mathrm{Ext}^1_A(W(\lambda), W(\lambda)^*) = 0$ *for each* $1 \leqslant \lambda \leqslant m$.
(c) $\mathrm{Ext}^2_{A/J_\mu}(W(\lambda), W(\lambda)^*) = 0$ *for each* $1 \leqslant \lambda \leqslant m$ *and* $\lambda + 1 \leqslant \mu \leqslant m + 1$.

In the above we denote by $W(\lambda)^*$ the module $\mathrm{Hom}_K(i(W(\lambda)), K)$.

The above condition (b) improves a result obtained by Xi in [10], where the vanishing of all Ext^1-groups between cell modules and dual cell modules was required. We note that the approach there does not work in our case.

The following theorem provides some homological characterizations of the semi-simplicity of cellular algebras.

Theorem 1.2. *Let K be a field and A a cellular K-algebra with respect to an involution i and a poset $(\Lambda, \geqslant)$. Denote by $W(\lambda)$ the cell module associated to $\lambda \in \Lambda$. Let Λ_0 be the subset of Λ, which parametrizes the isomorphism classes of simple A-modules. Then the following conditions are equivalent*:

(a) *The algebra A is semisimple.*
(b) $\mathrm{Ext}^1_A(W(\lambda), S(\mu)) = 0$ *for any* $\lambda, \mu \in \Lambda_0$ *satisfying* $\mu \geqslant \lambda$, *where simple A-module* $S(\mu) \simeq W(\mu)/\mathrm{rad}(W(\mu))$.
(c) $\mathrm{Ext}^1_A(W(\lambda), W(\mu)) = 0$ *for any* $\lambda, \mu \in \Lambda_0$ *satisfying* $\mu \geqslant \lambda$.
(c′) $\mathrm{Ext}^1_A(W(\lambda), W(\mu)) = 0$ *for any* $\lambda, \mu \in \Lambda$ *satisfying* $\mu \geqslant \lambda$.
(c″) $\mathrm{Ext}^1_A(W(\lambda), W(\mu)) = 0$ *for all* $\lambda, \mu \in \Lambda$.

The contents of this paper are as follows. In Section 2 we recall the definitions of cellular and quasi-hereditary algebras and then we assemble a few necessary facts which are often used in the paper. The proof of Theorem 1.1 is given in Section 3 after establishing several key lemmas. The last section is devoted to the proof of Theorem 1.2. Nevertheless, these criteria for the semi-simplicity cannot be generalized to the case of the second cohomology groups.

2. Preliminaries

In this section we shall recall the two equivalent definitions of cellular algebras and the definition of quasi-hereditary algebras. We also collect several facts which will be used freely in later sections.

For simplicity we assume that K is a field. Throughout the paper, A denotes a finite-dimensional associative K-algebra with the identity 1, and A-mod denotes the category of all finitely generated left A-modules. By a module we mean a left module, unless otherwise specified.

Definition 2.1 (Graham and Lehrer [4]). A K-algebra A is called a *cellular algebra* with cell datum (Λ, M, C, i) if the following conditions are satisfied:

(C1) The finite set Λ is partially ordered and for each $\lambda \in \Lambda$ there is a finite indexing set $M(\lambda)$. The algebra A has a K-basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

(C2) The map i is a K-linear anti-automorphism of A which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$ for all $\lambda \in \Lambda$ and all S and T in $M(\lambda)$.

(C3) For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{S' \in M(\lambda)} r_a(S', S)C^{\lambda}_{S',T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly larger than λ, and where the coefficients $r_a(S', S) \in K$ are independent of T.

In the following, a K-linear anti-automorphism i of A with $i^2 = \mathrm{id}$ will be called an *involution*. We now recall the equivalent definition of cellular algebras, which is more handy for our theoretical and structural purposes because it does not depend on a choice of basis.

Definition 2.2 (König and Xi [5]). Let A be a K-algebra with an involution i. A two-sided ideal J of A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that there is an isomorphism of A-bimodules $\alpha : J \simeq W \otimes_K i(W)$ (where $i(W) \subset J$ is the i-image of W) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W \otimes_K i(W) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x\otimes y \mapsto i(y)\otimes i(x)} \\ J & \xrightarrow{\alpha} & W \otimes_K i(W) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J'_m \oplus J'_{m-1} \oplus \cdots \oplus J'_1$ (for some m) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=j}^{m} J'_l$ gives a chain of two-sided ideals of $A: 0 = J_{m+1} \subset J_m \subset J_{m-1} \subset \cdots \subset J_1 = A$ (each of them fixed by i) and for each j ($j = m, m-1, \ldots, 1$) the quotient J_j/J_{j+1} is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j+1}.

The above chain in A is called a *cell chain*, and the modules $W(j)$, $1 \leqslant j \leqslant m$, which are obtained from the sections J_j/J_{j+1} of the cell chain, are called *cell modules* of the cellular algebra A. It is proved in [5] that a cell ideal J is either $J^2 = 0$ or a heredity ideal (see Definition 2.3 below). Moreover, there is a natural bijection between isomorphism classes of simple A-modules and the set $\Lambda_0 := \{\lambda \mid 1 \leqslant \lambda \leqslant m \text{ such that } J_\lambda^2 \not\subseteq J_{\lambda+1}\}$. The inverse of this bijection is given by sending such a λ to the top of the cell module $W(\lambda)$ (see [4,5]).

Assume that the cardinality of Λ_0 is n, which equals the number of non-isomorphic simple A-modules. For the convenience in the proofs later on, we relabel the original cell chain as follows:

$$\begin{aligned} 0 &= J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(i+1,0)} = J_{(i,s(i)+1)} \subset J_{(i,s(i))} \subset \cdots \\ &\subset J_{(i,1)} \subset J_{(i,0)} = J_{(i-1,s(i-1)+1)} \subset \cdots \subset J_{(1,0)} = A, \end{aligned}$$

where the ideals $J_{(i,0)}$, $1 \leqslant i \leqslant n$, are just those ideals J_λ in the original cell chain with $\lambda \in \Lambda_0$, and $s(i)$ denotes the number of ideals J_μ in the original cell chain satisfying $J_{(i+1,0)} \subsetneq J_\mu \subsetneq J_{(i,0)}$. Thus $s(i) \geqslant 0$. Moreover, if $s(i) > 0$, then $J^2_{(i,k)} \subset J_{(i,k+1)}$ for each $1 \leqslant k \leqslant s(i)$.

The cell module associated to cell ideal $J_{(i,j)}/J_{(i,j+1)}$, in which, $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant s(i)$, will be denoted by $W(i,j)$. For simplicity we shall always write $W(i)$ for $W(i,0)$ in the rest of this paper, except where otherwise stated. Such a notation precisely indicates that the cell module corresponds to an idempotent cell ideal. Note that for each idempotent cell ideal $J_{(i,0)}/J_{(i,1)}$, there is a primitive idempotent e_i of A such that $J_{(i,0)} = Ae_iA + J_{(i,1)}$ and, moreover, $W(i) \simeq Ae_i/J_{(i,1)}e_i$. The latter has a simple top, which is denoted by $S(i)$. Thus, $S(1), \ldots, S(n)$ form a complete set of non-isomorphic simple A-modules. Let Λ be the index set $\{(i,j) \mid 1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant s(i)\}$ endowed with lexicographic ordering, and Λ_0 be the subset $\{(i,0) \mid 1 \leqslant i \leqslant n\}$ inheriting the ordering of Λ. Clearly, we may identify Λ_0 with the index set $\{1, 2, \ldots, n\}$ with its natural ordering. We shall always fix the ordering for labelling the simple A-modules.

For each i, let $P(i)$ be the projective cover of $S(i)$, and denote by $\Delta(i)$ the maximal factor module of $P(i)$ with composition factors of the form $S(j)$, $j \leqslant i$, called a *standard module*. Dually, let $I(i)$ be the injective envelope of $S(i)$ and denote by $\nabla(i)$ the maximal submodule of $I(i)$ with composition factors of the form $S(j)$, $j \leqslant i$, called a *costandard module*. It should be pointed out that only in some special cases, standard modules coincide with cell modules.

Let us also recall the definition of quasi-hereditary algebras arising in the representation theory of complex Lie algebras and algebraic groups.

Definition 2.3 (Cline, Parshall, and Scott [1]). Let A be a K-algebra. An ideal J of A is called a *heredity ideal* if J is idempotent, $J(\operatorname{rad} A)J = 0$ and J is a projective left (or right) A-module. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_{n+1} \subset J_n \subset \cdots \subset J_1 = A$ of ideals in A such that J_j/J_{j+1} is a heredity ideal of A/J_{j+1} for all j. Such a chain is then called a *heredity chain* of the quasi-hereditary algebra A.

It is known [5,6] that a cell chain of the cellular algebra A is a heredity chain if and only if there is no nilpotent cell ideal arising from the cell chain, and this is equivalent to $\Lambda = \Lambda_0$.

We also need that the notation $[X : S(k)]$ denotes the Jordan–Hölder multiplicity of $S(k)$ in any A-module X. Obviously, $[X : S(k)] = \dim_K \operatorname{Hom}_A(P(k), X)$ if K is a splitting field for A. For a cellular algebra A, define $d_{(i,j)k} = [W(i,j) : S(k)]$ for all $(i,j) \in \Lambda$ and $k \in \Lambda_0$, thus give rise to a matrix $D = (d_{(i,j)k})$, which is the so-called *decomposition matrix* of A.

The following lemma collects some known facts from [4] on cellular algebras which we shall need in the sequel.

Lemma 2.4. *Let A be a cellular K-algebra with involution i and cell chain* $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. *Then we have the following*:

(a) *The decomposition matrix D is lower unitriangular, namely, $d_{(i,j)k} = 0$ unless $i \geqslant k$, and $d_{ii} := d_{(i,0)i} = 1$, where $(i,j) \in \Lambda$ and $k \in \Lambda_0$. In particular,* $\operatorname{Hom}_A(W(i), W(k)) = 0$ *unless $k \geqslant i$, and* $\operatorname{End}_A(W(i)) \simeq K$. *Moreover, K is a splitting field for A.*

(b) *Let $P = Ae_k$, $1 \leqslant k \leqslant n$. Then P has an A-module filtration $0 = J_{(n+1,0)}e_k \subset J_{(n,s(n))}e_k \subset \cdots \subset J_{(1,0)}e_k = P$ such that the factor modules $J_{(i,j)}e_k/J_{(i,j+1)}e_k$ are isomorphic to the modules $\bigoplus_{d_{(i,j)k}} W(i,j)$, in which we put $J_{(i,s(i)+1)} = J_{(i+1,0)}$.*

We remark that the factor module $J_{(i,j)}e_k/J_{(i,j+1)}e_k$ appearing in the above lemma may be zero, and this occurs if and only if $d_{(i,j)k}$ is also zero.

Let A be a cellular algebra with respect to an involution i and X an A-module. Following [5,10], we define the *dual* X^* of X to be the A-module $\operatorname{Hom}_K(i(X), K)$, where $i(X)$ is equal to X as a vector space, but with the right A-module structure given by $x \cdot a = i(a)x$ for all $x \in X$ and $a \in A$.

Observe that the functor $*$ is a self-dual functor, and furthermore, it has the following easily verified properties.

Lemma 2.5. *Let A be a cellular K-algebra with involution i. Then we have the following*:

(a) *For any simple A-module $S(k)$ and any $M \in A$-mod, we have that $S(k)^* \simeq S(k)$, $P(k)^* \simeq I(k)$, $\operatorname{top}(M) \simeq \operatorname{soc}(M^*)$ and $[M : S(k)] = [M^* : S(k)]$.*

(b) $\dim_K \operatorname{Ext}_A^j(X,Y) = \dim_K \operatorname{Ext}_A^j(Y^*, X^*)$ *for any $j \geqslant 0$ and any $X, Y \in A$-mod.*

(c) *Let $\lambda, \mu \in \Lambda_0$. Then $\operatorname{Hom}_A(W(\lambda), W(\mu)^*) \neq 0$ if and only if $\lambda = \mu$. Moreover,* $\dim_K \operatorname{Hom}_A(W(\lambda), W(\lambda)^*) = 1$.

Proof. The assertion (a) is an easy consequence of dual functor and the known fact $Ae_k \simeq Ai(e_k)$ (see [4]). The assertion (c) follows readily from (a), (b), and Lemma 2.4. So it only needs to give a proof of (b). Use induction on j, the case $j = 0$ being trivial since the functor $*$ is self-dual.

Let $j \geqslant 1$, and suppose that (b) is true for $j - 1$. Let $0 \to Z \to P \to X \to 0$ be an exact sequence in A-mod with P a projective cover of X. Then $0 \to X^* \to P^* \to Z^* \to 0$ is an exact sequence with P^* an injective A-module. It follows from $\mathrm{Ext}_A^j(X, Y) \simeq \mathrm{Ext}_A^{j-1}(Z, Y)$ and $\mathrm{Ext}_A^{j-1}(Y^*, Z^*) \simeq \mathrm{Ext}_A^j(Y^*, X^*)$ that $\dim_K \mathrm{Ext}_A^j(X, Y) = \dim_K \mathrm{Ext}_A^{j-1}(Z, Y) = \dim_K \mathrm{Ext}_A^{j-1}(Y^*, Z^*) = \dim_K \mathrm{Ext}_A^j(Y^*, X^*)$, which is our desired result. □

3. Quasi-heredity of cellular algebras

In this section we present some homological characterizations of quasi-hereditary algebras inside the class of cellular algebras by means of cell modules. We shall prove the stronger statements Theorem 3.3 and Theorem 3.4 which have Theorem 1.1 as a corollary.

From now on we fix a cellular algebra A with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. However, this does not prevent us from discussing the quasi-heredity of the cellular algebra A since it has been shown in [6] that A is quasi-hereditary with respect to an involution i and a cell chain if and only if any cell chain of A with respect to any involution is a heredity chain.

Before beginning with the following lemma, we need one more notation. Let A be a cellular algebra with cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. For any $1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant n-k$, define $Q_{(k,i)} := Ae_k/J_{(k+i,1)}e_k$, which is a projective $A/J_{(k+i,1)}$-module. The modules $Q_{(k,i)}$ play a prominent role in our study.

The following lemma can help us determine the composition factors of some cell modules.

Lemma 3.1. *Let A be a cellular algebra with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. If $\mathrm{Ext}_A^1(Q_{(k,i)}, Q_{(k,i)}^*) = 0$ for some $1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant n - k$, then*:

(a) $Q_{(k,i)} \simeq Ae_k/J_{(k+i+1,0)}e_k$.
(b) *If $s(k+i) \geqslant 1$, then $[W(k+i, j) : S(k)] = 0$ for all $1 \leqslant j \leqslant s(k+i)$.*

Proof. Observe that (a) evidently holds when $s(k+i) = 0$. For $s(k+i) \geqslant 1$, the two assertions will be proved by using induction on j. In the case of $j = 1$, we have the exact sequence of A-modules

$$0 \to J_{(k+i,1)}e_k/J_{(k+i,2)}e_k \to Ae_k/J_{(k+i,2)}e_k \to Q_{(k,i)} \to 0,$$

namely, the sequence

$$0 \to \bigoplus_{d_{(k+i,1)k}} W(k+i, 1) \to Ae_k/J_{(k+i,2)}e_k \to Q_{(k,i)} \to 0$$

is exact, which induces an exact sequence

$$\begin{aligned} 0 \to\ & \mathrm{Hom}_A\big(Q_{(k,i)}, Q^*_{(k,i)}\big) \to \mathrm{Hom}_A\big(Ae_k/J_{(k+i,2)}e_k, Q^*_{(k,i)}\big) \\ \to\ & \mathrm{Hom}_A\Bigg(\bigoplus_{d_{(k+i,1)k}} W(k+i,1), Q^*_{(k,i)}\Bigg) \to \mathrm{Ext}^1_A\big(Q_{(k,i)}, Q^*_{(k,i)}\big). \end{aligned}$$

The last term is zero by the hypothesis of the lemma. Meanwhile, we have

$$\begin{aligned} \dim_K \mathrm{Hom}_A\big(Q_{(k,i)}, Q^*_{(k,i)}\big) &= \dim_K \mathrm{Hom}_{A/J_{(k+i,1)}}\big(Q_{(k,i)}, Q^*_{(k,i)}\big) = \big[Q^*_{(k,i)} : S(k)\big] \\ &= \dim_K \mathrm{Hom}_{A/J_{(k+i,2)}}\big(Ae_k/J_{(k+i,2)}e_k, Q^*_{(k,i)}\big) \\ &= \dim_K \mathrm{Hom}_A\big(Ae_k/J_{(k+i,2)}e_k, Q^*_{(k,i)}\big) \end{aligned}$$

since $Q_{(k,i)}$ is a projective $A/J_{(k+i,1)}$-module and $Ae_k/J_{(k+i,2)}e_k$ is a projective $A/J_{(k+i,2)}$-module. As a result, $\mathrm{Hom}_A(\bigoplus_{d_{(k+i,1)k}} W(k+i,1), Q^*_{(k,i)}) = 0$. Suppose that $d_{(k+i,1)k} \neq 0$. Then $\mathrm{Hom}_A(W(k+i,1), Q^*_{(k,i)}) = 0$. Note that $W(k+i,1) \subset J_{(k+i,1)}/J_{(k+i,2)}$ and the latter is a nilpotent cell ideal of $A/J_{(k+i,2)}$, thus $W(k+i,1)$ can be viewed as an $A/J_{(k+i,1)}$-module since it is annihilated by $J_{(k+i,1)}/J_{(k+i,2)}$. But $Q^*_{(k,i)}$ is an injective $A/J_{(k+i,1)}$-module, so we obtain that $d_{(k+i,1)k} = [W(k+i,1) : S(k)] = \dim_K \mathrm{Hom}_{A/J_{(k+i,1)}}(W(k+i,1), Q^*_{(k,i)}) = \dim_K \mathrm{Hom}_A(W(k+i,1), Q^*_{(k,i)}) = 0$, which is absurd. Hence, $[W(k+i,1) : S(k)] = d_{(k+i,1)k} = 0$, which implies that $J_{(k+i,1)}e_k = J_{(k+i,2)}e_k$ and $Q_{(k,i)} \simeq Ae_k/J_{(k+i,2)}e_k$.

Assume now that $[W(k+i,l) : S(k)] = 0$ for all $1 \leqslant l \leqslant j-1 (< s(k+i))$. We show that $[W(k+i,j) : S(k)] = 0$. The induction hypothesis means that $J_{(k+i,1)}e_k = \cdots = J_{(k+i,j)}e_k$ and $Q_{(k,i)} \simeq Ae_k/J_{(k+i,j)}e_k$. Thus, there is an exact sequence of A-modules

$$0 \to J_{(k+i,j)}e_k/J_{(k+i,j+1)}e_k \to Ae_k/J_{(k+i,j+1)}e_k \to Q_{(k,i)} \to 0,$$

which yields the exact sequence

$$\begin{aligned} 0 \to\ & \mathrm{Hom}_A\big(Q_{(k,i)}, Q^*_{(k,i)}\big) \to \mathrm{Hom}_A\big(Ae_k/J_{(k+i,j+1)}e_k, Q^*_{(k,i)}\big) \\ \to\ & \mathrm{Hom}_A\big(J_{(k+i,j)}e_k/J_{(k+i,j+1)}e_k, Q^*_{(k,i)}\big) \to 0. \end{aligned}$$

Comparing the K-dimensions of the first two terms, we get

$$\mathrm{Hom}_A\Bigg(\bigoplus_{d_{(k+i,j)k}} W(k+i,j), Q^*_{(k,i)}\Bigg) = \mathrm{Hom}_A\big(J_{(k+i,j)}e_k/J_{(k+i,j+1)}e_k, Q^*_{(k,i)}\big) = 0.$$

Note also that $W(k+i,j)$ is contained in the nilpotent cell ideal $J_{(k+i,j)}/J_{(k+i,j+1)}$ of $A/J_{(k+i,j+1)}$. So $W(k+i,j)$ can be viewed as an $A/J_{(k+i,j)}$-module, and then $\mathrm{Hom}_{A/J_{(k+i,j)}}(\bigoplus_{d_{(k+i,j)k}} W(k+i,j), Q^*_{(k,i)}) = 0$. This forces that $[W(k+i,j) : S(k)] = 0$ since $Q^*_{(k,i)} \simeq (Ae_k/J_{(k+i,j)}e_k)^*$, which is an injective $A/J_{(k+i,j)}$-module. We also obtain that $J_{(k+i,j)}e_k = J_{(k+i,j+1)}e_k$ and $Q_{(k,i)} \simeq Ae_k/J_{(k+i,j+1)}e_k$. In particular, we get $Q_{(k,i)} \simeq Ae_k/J_{(k+i,s(k+i)+1)}e_k = Ae_k/J_{(k+i+1,0)}e_k$ by setting $j = s(k+i)$, as desired. □

In order to apply Lemma 3.1 to the proof of our theorem, we also need the following lemma.

Lemma 3.2. *Let A be a cellular algebra with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$ and let $1 \leqslant k \leqslant n$. If $\mathrm{Ext}^1_A(W(s), W(s)^*) = 0$ for any $k \leqslant s \leqslant n$, then for all $k \leqslant t \leqslant n$ and $0 \leqslant i \leqslant n-t$, we have the following*:

(a) $\mathrm{Ext}^1_A(Q_{(t,i)}, Q^*_{(t,i)}) = 0$.
(b) $\mathrm{Ext}^1_A(Q_{(t,i)}, W(\mu)^*) = 0$ *for any* $\mu \geqslant t$.
(c) $\mathrm{Ext}^1_A(W(t), W(j)^*) = 0$ *for any* $1 \leqslant j \leqslant n$.

Proof. We first prove the case of $k = n$. In this situation, $t = n$ and $i = 0$. By the definition of $Q_{(n,0)}$, we know that $Q_{(n,0)} \simeq W(n)$. Thus $\mathrm{Ext}^1_A(Q_{(n,0)}, Q^*_{(n,0)}) = 0$. Applying Lemma 3.1, we obtain that $Q_{(n,0)} \simeq Ae_n/J_{(n+1,0)}e_n = Ae_n$, which is a projective A-module. The results follow at once.

For the case $k < n$, we prove the lemma by (downward) induction on t. The initial step $t = n$ has already been shown as above.

Now assume that the results are true for $t \geqslant l+1 (> k)$, that is,

(a′) $\mathrm{Ext}^1_A(Q_{(t,i)}, Q^*_{(t,i)}) = 0$ for $l+1 \leqslant t \leqslant n$ and $0 \leqslant i \leqslant n-t$;
(b′) $\mathrm{Ext}^1_A(Q_{(t,i)}, W(\mu)^*) = 0$ for $l+1 \leqslant t \leqslant n,\ 0 \leqslant i \leqslant n-t$ and $\mu \geqslant t$;
(c′) $\mathrm{Ext}^1_A(W(t), W(j)^*) = 0$ for $l+1 \leqslant t \leqslant n$ and $1 \leqslant j \leqslant n$.

For the induction step $t = l$, we have that $0 \leqslant i \leqslant n-l$. We first prove the assertions (a) and (b) by a second induction on i. In the subcase of $i = 0$, we have that $Q_{(l,0)} \simeq Ae_l/J_{(l,1)}e_l$, which is isomorphic to $W(l)$. Thus $\mathrm{Ext}^1_A(Q_{(l,0)}, Q^*_{(l,0)}) = 0$ by the condition. Using the induction assumption (c′), we see that $\dim_K \mathrm{Ext}^1_A(W(l), W(\mu)^*) = \dim_K \mathrm{Ext}^1_A(W(\mu), W(l)^*) = 0$ for any $\mu \geqslant l+1$, that is, $\mathrm{Ext}^1_A(W(l), W(\mu)^*) = 0$ for any $\mu \geqslant l+1$. Combining this with the condition that $\mathrm{Ext}^1_A(W(l), W(l)^*) = 0$, we have $\mathrm{Ext}^1_A(Q_{(l,0)}, W(\mu)^*) = 0$ for any $\mu \geqslant l$. Now, assume that the subcase of $i = \lambda - 1$ $(< n-l)$ has already been shown, namely, $\mathrm{Ext}^1_A(Q_{(l,\lambda-1)}, Q^*_{(l,\lambda-1)}) = 0$ and $\mathrm{Ext}^1_A(Q_{(l,\lambda-1)}, W(\mu)^*) = 0$ for any $\mu \geqslant l$. Let us prove the subcase $i = \lambda$. Using the induction hypothesis $\mathrm{Ext}^1_A(Q_{(l,\lambda-1)}, Q^*_{(l,\lambda-1)}) = 0$ and Lemma 3.1, we see that $Q_{(l,\lambda-1)} \simeq Ae_l/J_{(l+\lambda,0)}e_l$. Thereby, we have the following short exact sequence of A-modules

$$0 \to J_{(l+\lambda,0)}e_l/J_{(l+\lambda,1)}e_l \to Q_{(l,\lambda)} \to Q_{(l,\lambda-1)} \to 0,$$

that is,

$$0 \to \bigoplus_{d_{l+\lambda,l}} W(l+\lambda) \to Q_{(l,\lambda)} \to Q_{(l,\lambda-1)} \to 0, \tag{1}$$

which induces an exact sequence

$$\operatorname{Ext}_A^1\big(Q_{(l,\lambda-1)}, Q_{(l,\lambda-1)}^*\big) \to \operatorname{Ext}_A^1\big(Q_{(l,\lambda)}, Q_{(l,\lambda-1)}^*\big) \to \operatorname{Ext}_A^1\Bigg(\bigoplus_{d_{l+\lambda,l}} W(l+\lambda), Q_{(l,\lambda-1)}^*\Bigg).$$

Thanks to the induction hypotheses on the subcase $i = \lambda - 1$, both end terms vanish, thus the middle term $\operatorname{Ext}_A^1(Q_{(l,\lambda)}, Q_{(l,\lambda-1)}^*) = 0$, too. Applying $\operatorname{Hom}_A(-, W(l+\lambda)^*)$ to (1) gives rise to the following exact sequence

$$\begin{aligned}\operatorname{Ext}_A^1\big(Q_{(l,\lambda-1)}, W(l+\lambda)^*\big) &\to \operatorname{Ext}_A^1\big(Q_{(l,\lambda)}, W(l+\lambda)^*\big)\\ &\to \operatorname{Ext}_A^1\Bigg(\bigoplus_{d_{l+\lambda,l}} W(l+\lambda), W(l+\lambda)^*\Bigg).\end{aligned}$$

Again by the induction hypothesis on $i = \lambda - 1$, the first term vanishes. The third term also vanishes by the conditions of the lemma. Whence, we get $\operatorname{Ext}_A^1(Q_{(l,\lambda)}, W(l+\lambda)^*) = 0$. Further, we may obtain an exact sequence from (1) as follows:

$$\operatorname{Ext}_A^1\big(Q_{(l,\lambda-1)}, Q_{(l,\lambda)}^*\big) \to \operatorname{Ext}_A^1\big(Q_{(l,\lambda)}, Q_{(l,\lambda)}^*\big) \to \operatorname{Ext}_A^1\Bigg(\bigoplus_{d_{l+\lambda,l}} W(l+\lambda), Q_{(l,\lambda)}^*\Bigg).$$

From the previous arguments, both end terms of the above sequence vanish, thus we see that $\operatorname{Ext}_A^1(Q_{(l,\lambda)}, Q_{(l,\lambda)}^*) = 0$. It follows that the assertion (a) holds.

Next, for any $\mu \geqslant l$, applying $\operatorname{Hom}_A(-, W(\mu)^*)$ to (1) yields the exact sequence

$$\operatorname{Ext}_A^1\big(Q_{(l,\lambda-1)}, W(\mu)^*\big) \to \operatorname{Ext}_A^1\big(Q_{(l,\lambda)}, W(\mu)^*\big) \to \operatorname{Ext}_A^1\Bigg(\bigoplus_{d_{l+\lambda,l}} W(l+\lambda), W(\mu)^*\Bigg).$$

By the induction hypothesis on the subcase $i = \lambda - 1$, the first term is zero. Besides, the last term is zero by the induction assumption (c′). Hence, $\operatorname{Ext}_A^1(Q_{(l,\lambda)}, W(\mu)^*) = 0$ for all $\mu \geqslant l$, which is the assertion (b).

It remains only to prove that $\operatorname{Ext}_A^1(W(l), W(j)^*) = 0$ for all $1 \leqslant j \leqslant n$. According to the preceding arguments, we have that $\operatorname{Ext}_A^1(Q_{(l,i)}, Q_{(l,i)}^*) = 0$ for all $0 \leqslant i \leqslant n-l$. By Lemma 3.1, this just means that $Q_{(l,i)} = Ae_l/J_{(l+i,1)}e_l \simeq Ae_l/J_{(l+i+1,0)}e_l$ for all $0 \leqslant i \leqslant n-l$. Therefore, for any $1 \leqslant i \leqslant n-l$, we get an exact sequence

$$0 \to \bigoplus_{d_{l+i,l}} W(l+i) \to Q_{(l,i)} \to Q_{(l,i-1)} \to 0,$$

which yields an exact sequence

$$\operatorname{Hom}_A\Bigg(\bigoplus_{d_{l+i,l}} W(l+i), W(j)^*\Bigg) \to \operatorname{Ext}_A^1\big(Q_{(l,i-1)}, W(j)^*\big) \to \operatorname{Ext}_A^1\big(Q_{(l,i)}, W(j)^*\big)$$

$$\to \operatorname{Ext}_A^1\left(\bigoplus_{d_{l+i,l}} W(l+i), W(j)^*\right)$$

for any $j \leqslant l$. From Lemma 2.5(c), we see that the first term is zero. The last term is also zero, thanks to the induction assumption (c′). This forces that the middle two terms are isomorphic, and then $\operatorname{Ext}_A^1(Q_{(l,0)}, W(j)^*) \simeq \operatorname{Ext}_A^1(Q_{(l,n-l)}, W(j)^*)$. Note that $Q_{(l,0)} \simeq W(l)$ and $Q_{(l,n-l)} \simeq Ae_l/J_{(n+1,0)}e_l = Ae_l$. So the latter is a projective A-module. Consequently, $\operatorname{Ext}_A^1(W(l), W(j)^*) = 0$ for any $j \leqslant l$. Observe that $\operatorname{Ext}_A^1(W(l), W(j)^*) = 0$ has already been verified for any $j \geqslant l+1$ in the proof of the subcase $i = 0$. Thus, the proof of Lemma 3.2 is finished. □

Recall that modules with a division ring as endomorphism ring are called *Schurian*. Now we can prove the first main result.

Theorem 3.3. *Let A be a cellular algebra with cell modules $W(i,j)$, $(i,j) \in \Lambda$, and standard modules $\{\Delta(i) \mid i \in \Lambda_0\}$. Then the following statements are equivalent*:

(a) *The algebra A is quasi-hereditary.*
(b) *All standard modules $\Delta(i)$ are Schurian, equivalently, $[\Delta(i):S(i)] = 1$ for each $i \in \Lambda_0$.*
(c) $\operatorname{Ext}_A^1(W(i), W(i)^*) = 0$ *for each $i \in \Lambda_0$.*

Proof. Obviously the condition (a) implies (b) (see [3] or [8]).

(b) ⇒ (c). For any $i \in \Lambda_0$, by the definition of standard modules, we always have an exact sequence of A-modules

$$0 \to Z \to P(i) \to \Delta(i) \to 0,$$

which induces an exact sequence

$$\operatorname{Hom}_A\big(Z, \Delta(i)^*\big) \to \operatorname{Ext}_A^1\big(\Delta(i), \Delta(i)^*\big) \to 0.$$

If $Z = 0$, then the first term of the above sequence is zero, and so is the second one. Otherwise, all the composition factors of $\operatorname{top}(Z)$ have index greater than i, thus $\operatorname{Hom}_A(Z, \Delta(i)^*) = 0$ since all composition factors of $\Delta(i)^*$ are of the form $S(k)$ with $k \leqslant i$. This also forces that $\operatorname{Ext}_A^1(\Delta(i), \Delta(i)^*) = 0$. Observe that there is also an exact sequence of A-modules

$$0 \to L \to \Delta(i) \to W(i) \to 0, \tag{2}$$

which yields the following exact sequence

$$\operatorname{Hom}_A\big(L, \Delta(i)^*\big) \to \operatorname{Ext}_A^1\big(W(i), \Delta(i)^*\big) \to \operatorname{Ext}_A^1\big(\Delta(i), \Delta(i)^*\big).$$

The last term vanishes by the above argument. From the condition $[\Delta(i):S(i)] = 1$ and the fact $[W(i):S(i)] = 1$, we know that $[L:S(i)] = 0$, which implies that $\operatorname{Hom}_A(L, \Delta(i)^*) = 0$

since $\mathrm{soc}(\Delta(i)^*) \simeq \mathrm{top}(\Delta(i)) = S(i)$. So we get $\mathrm{Ext}_A^1(W(i), \Delta(i)^*) = 0$, that is, $\mathrm{Ext}_A^1(\Delta(i), W(i)^*) = 0$. Applying $\mathrm{Hom}_A(-, W(i)^*)$ to (2), we again get an exact sequence

$$\mathrm{Hom}_A\big(L, W(i)^*\big) \to \mathrm{Ext}_A^1\big(W(i), W(i)^*\big) \to \mathrm{Ext}_A^1\big(\Delta(i), W(i)^*\big) = 0.$$

The first term equals zero since $\mathrm{soc}(W(i)^*) \simeq S(i)$ and $[L : S(i)] = 0$. Hence, $\mathrm{Ext}_A^1(W(i), W(i)^*) = 0$, which is the condition (c).

(c) ⇒ (a). Let $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$ be a cell chain which produces the cell modules $W(i, j)$, where $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant s(i)$. In order to prove that the cell chain is a heredity chain, it suffices to show that those cell modules $W(i, j)$ with $j \neq 0$ are zero, equivalently, to show that there does not exist $i \in \Lambda_0$ such that $s(i) \neq 0$. Suppose that $s(i) > 0$ for some $i \in \Lambda_0$. Then by Lemma 2.4(a), the composition factors of $W(i, 1)$ are of the form $S(k)$ with $k \leqslant i$. By the condition (c) and Lemma 3.2, we have that $\mathrm{Ext}_A^1(Q_{(k,i-k)}, Q^*_{(k,i-k)}) = 0$ for all $1 \leqslant k \leqslant i$. Furthermore, we get that $[W(i, 1) : S(k)] = 0$ for all $1 \leqslant k \leqslant i$ by Lemma 3.1. Thus, $W(i, 1) = 0$, which contradicts $s(i) > 0$. This completes the proof of the theorem. □

The remainder of this section is devoted to giving another criterion for a cellular algebra to be quasi-hereditary via the second cohomology groups of certain cell modules. We shall establish the following theorem.

Theorem 3.4. *Let A be a cellular algebra with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. Denote by $W(i, j)$ the cell module associated to $(i, j) \in \Lambda$. Then A is quasi-hereditary if and only if $\mathrm{Ext}^2_{A/J_{(p,q)}}(W(i), W(i)^*) = 0$ for each $i \in \Lambda_0$, $i \leqslant p \leqslant n$ and $1 \leqslant q \leqslant s(p) + 1$.*

For the proof, we need two key lemmas below. Let us continue to use the notations of the previous parts.

The following fact is similar to Lemma 3.1.

Lemma 3.5. *Let A be a cellular algebra with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$. Let $1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant n - k$. If $\mathrm{Ext}^2_{A/J_{(k+i,q)}}(Q_{(k,i)}, Q^*_{(k,i)}) = 0$ for each $1 \leqslant q \leqslant s(k + i) + 1$, then*:

(a) *$Q_{(k,i)} \simeq Ae_k/J_{(k+i+1,0)}e_k$ as A-modules.*
(b) *If $s(k + i) \geqslant 1$, then $[W(k + i, j) : S(k)] = 0$ and $J_{(k+i,j)}e_k = J_{(k+i+1,0)}e_k$ for all $1 \leqslant j \leqslant s(k + i)$.*

Proof. Observe that the conclusion (a) holds obviously if $s(k + i) = 0$. Now we consider the case of $s(k + i) > 0$. Note that there is an exact sequence of $A/J_{(k+i,2)}$-modules

$$0 \to J_{(k+i,1)}e_k/J_{(k+i,2)}e_k \to Ae_k/J_{(k+i,2)}e_k \to Q_{(k,i)} \to 0,$$

namely,

$$0 \to \bigoplus_{d_{(k+i,1)k}} W(k+i,1) \to Ae_k/J_{(k+i,2)}e_k \to Q_{(k,i)} \to 0, \tag{3}$$

which provides the following exact sequence

$$\begin{aligned} \mathrm{Ext}^1_{A/J_{(k+i,2)}}\big(Ae_k/J_{(k+i,2)}e_k, Q^*_{(k,i)}\big) \;&\to\; \mathrm{Ext}^1_{A/J_{(k+i,2)}}\Big(\bigoplus_{d_{(k+i,1)k}} W(k+i,1), Q^*_{(k,i)}\Big) \\ &\to\; \mathrm{Ext}^2_{A/J_{(k+i,2)}}\big(Q_{(k,i)}, Q^*_{(k,i)}\big) \end{aligned}$$

in $A/J_{(k+i,2)}$-mod. The first term vanishes just since $Ae_k/J_{(k+i,2)}e_k$ is a projective $A/J_{(k+i,2)}$-module. By the condition of the lemma, the last term vanishes as well. Assume that $d_{(k+i,1)k} \neq 0$. We thus get $\mathrm{Ext}^1_{A/J_{(k+i,2)}}(W(k+i,1), Q^*_{(k,i)}) = 0$. By applying $\mathrm{Hom}_{A/J_{(k+i,2)}}(-, W(k+i,1)^*)$ to (3), we get the long exact sequence

$$\begin{aligned} 0 \;&\to\; \mathrm{Hom}_{A/J_{(k+i,2)}}\big(Q_{(k,i)}, W(k+i,1)^*\big) \\ &\to\; \mathrm{Hom}_{A/J_{(k+i,2)}}\big(Ae_k/J_{(k+i,2)}e_k, W(k+i,1)^*\big) \\ &\to\; \mathrm{Hom}_{A/J_{(k+i,2)}}\Big(\bigoplus_{d_{(k+i,1)k}} W(k+i,1), W(k+i,1)^*\Big) \\ &\to\; \mathrm{Ext}^1_{A/J_{(k+i,2)}}\big(Q_{(k,i)}, W(k+i,1)^*\big). \end{aligned}$$

According to the above argument, the last term is equal to zero. Note also that the cell module $W(k+i,1)$ arises from the nilpotent cell ideal $J_{(k+i,1)}/J_{(k+i,2)}$ of $A/J_{(k+i,2)}$. Hence, $W(k+i,1)$, and also $W(k+i,1)^*$ can be seen as $A/J_{(k+i,1)}$-modules, and thus

$$\begin{aligned} \dim_K \mathrm{Hom}_{A/J_{(k+i,2)}}\big(Q_{(k,i)}, W(k+i,1)^*\big) \;&=\; \dim_K \mathrm{Hom}_{A/J_{(k+i,1)}}\big(Q_{(k,i)}, W(k+i,1)^*\big) \\ &=\; \big[W(k+i,1)^* : S(k)\big]. \end{aligned}$$

The last equality follows from that $Q_{(k,i)}$ is a projective $A/J_{(k+i,1)}$-module. However, $\dim_K \mathrm{Hom}_{A/J_{(k+i,2)}}(Ae_k/J_{(k+i,2)}e_k, W(k+i,1)^*)$ is also equal to $[W(k+i,1)^* : S(k)]$ just since $Ae_k/J_{(k+i,2)}e_k$ is a projective $A/J_{(k+i,2)}$-module. This means that we have $\mathrm{Hom}_{A/J_{(k+i,2)}}(W(k+i,1), W(k+i,1)^*) = 0$, which is absurd because it contains the non-zero homomorphism $W(k+i,1) \twoheadrightarrow \mathrm{top}(W(k+i,1)) \simeq \mathrm{soc}(W(k+i,1)^*) \hookrightarrow W(k+i,1)^*$. Thus we get $d_{(k+i,1)k} = 0$, that is, $[W(k+i,1) : S(k)] = 0$. Therefore $J_{(k+i,1)}e_k = J_{(k+i,2)}e_k$ and so $Q_{(k,i)} \simeq Ae_k/J_{(k+i,2)}e_k$ as A-modules. Continuing by induction, we obtain that $d_{(k+i,j)k} = 0$ for all $1 \leqslant j \leqslant s(k+i)$, namely $[W(k+i,j) : S(k)] = 0$ for all $1 \leqslant j \leqslant s(k+i)$. Thus, $J_{(k+i,j)}e_k = J_{(k+i,j+1)}e_k$, and so $Q_{(k,i)} \simeq Ae_k/J_{(k+i,s(k+i)+1)}e_k = Ae_k/J_{(k+i+1,0)}e_k$. Hence, both (a) and (b) hold when $s(k+i) \geqslant 1$, finishing the proof. □

The following lemma points out the relationship between the Ext^2-groups of cell modules, and the Ext^2-groups of the modules $Q_{(k,i)}$ in our context.

Lemma 3.6. *Let A be a cellular algebra with involution i and cell chain $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$ and let $1 \leqslant k \leqslant n$. If $\mathrm{Ext}^2_{A/J_{(p,q)}}(W(s), W(s)^*) = 0$ for any $k \leqslant s \leqslant n$, $s \leqslant p \leqslant n$, and $1 \leqslant q \leqslant s(p) + 1$, then $\mathrm{Ext}^2_{A/J_{(t+i,q)}}(Q_{(t,i)}, Q^*_{(t,i)}) = 0$ for all $k \leqslant t \leqslant n$, $0 \leqslant i \leqslant n - t$, and $1 \leqslant q \leqslant s(t + i) + 1$.*

Proof. The case $k = n$. In this case, $t = n$ and $i = 0$. Note that $Q_{(n,0)}$ is isomorphic to $W(n)$. Thus, for each $1 \leqslant q \leqslant s(n) + 1$, we have that $\mathrm{Ext}^2_{A/J_{(n,q)}}(Q_{(n,0)}, Q^*_{(n,0)}) = 0$ by the condition of the lemma.

In the case of $k < n$, we prove the lemma by (downward) induction on t. The above argument implies that the case $t = n$ is true.

We now assume that the assertion holds for $t \geqslant l + 1 (> k)$, namely, $\mathrm{Ext}^2_{A/J_{(t+i,q)}}(Q_{(t,i)}, Q^*_{(t,i)}) = 0$ for all $l + 1 \leqslant t \leqslant n$, $0 \leqslant i \leqslant n - t$, and $1 \leqslant q \leqslant s(t + i) + 1$. Considering the induction step $t = l$, we see that $0 \leqslant i \leqslant n - l$. Let us show the assertion by a second induction on i. In the subcase of $i = 0$, we have that $Q_{(l,0)} \simeq W(l)$. Thus, by the condition of the lemma we have that $\mathrm{Ext}^2_{A/J_{(l,q)}}(Q_{(l,0)}, Q^*_{(l,0)}) = 0$ for each $1 \leqslant q \leqslant s(l) + 1$. Suppose next that the subcases of $i \leqslant \lambda - 1 (< n - l)$ have already been shown, that is, $\mathrm{Ext}^2_{A/J_{(l+j,q)}}(Q_{(l,j)}, Q^*_{(l,j)}) = 0$ for all $0 \leqslant j \leqslant \lambda - 1$ and $1 \leqslant q \leqslant s(l + j) + 1$. We are now in the position to prove the subcase $i = \lambda$. To this end, it will take several steps. In the rest of our proof, one further bit of notation will be handy: if Θ is a class of A-modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of A-mod whose objects are the modules M which have a Θ-filtration, namely there is a finite chain $0 = M_{m+1} \subset M_m \subset M_{m-1} \subset \cdots \subset M_1 = M$ of submodules of M such that all factors M_j/M_{j+1}, $1 \leqslant j \leqslant m$, belong to Θ.

Step 1. *Let $0 \leqslant r \leqslant \lambda - 1$. Then $Q_{(l,r)} \in \mathcal{F}(W(l), W(l + 1), \ldots, W(l + r))$.*

Using the induction assumption that $\mathrm{Ext}^2_{A/J_{(l+j,q)}}(Q_{(l,j)}, Q^*_{(l,j)}) = 0$ for all $0 \leqslant j \leqslant r$ and $1 \leqslant q \leqslant s(l + j) + 1$, we see that $J_{(l+j,q)}e_l = J_{(l+j+1,0)}e_l$ by Lemma 3.5(b). In particular, $J_{(l+j,1)}e_l = J_{(l+j+1,0)}e_l$ holds for each $0 \leqslant j \leqslant r$. Considering the following filtration of $Q_{(l,r)}$

$$0 = M_{r+1} \subset M_r \subset \cdots \subset M_j \subset \cdots \subset M_0 = J_{(l,0)}e_l/J_{(l+r,1)}e_l = Ae_l/J_{(l+r,1)}e_l = Q_{(l,r)},$$

where $M_j = J_{(l+j,0)}e_l/J_{(l+r,1)}e_l$, we have that $M_j/M_{j+1} \simeq J_{(l+j,0)}e_l/J_{(l+j+1,0)}e_l = J_{(l+j,0)}e_l/J_{(l+j,1)}e_l \simeq \bigoplus_{d_{l+j,l}} W(l + j)$, where $0 \leqslant j \leqslant r$. It, therefore, follows that $Q_{(l,r)} \in \mathcal{F}(W(l), W(l + 1), \ldots, W(l + r))$.

Step 2. *Let $0 \leqslant r \leqslant \lambda - 1$ and $r + 1 \leqslant \nu \leqslant \lambda$. Then $\mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}(W(l + \nu), Q^*_{(l,r)}) = 0$ for each $1 \leqslant q \leqslant s(l + \lambda) + 1$.*

When $\nu = \lambda$. According to the induction hypothesis on the case of $l+\lambda$ and Lemma 3.5, we have that $Q_{(l+\lambda,0)} \simeq Ae_{l+\lambda}/J_{(l+\lambda,q)}e_{l+\lambda}$. Hence, $W(l + \lambda)$, which is isomorphic to $Q_{(l+\lambda,0)}$, is a projective $A/J_{(l+\lambda,q)}$-module. Thus, $\mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}(W(l + \lambda), Q^*_{(l,r)}) = 0$ being trivial.

For the case $\nu < \lambda$, we have the following exact sequence of $A/J_{(l+\lambda,q)}$-modules

$$0 \to K_1 \to Ae_{l+\nu}/J_{(l+\lambda,q)}e_{l+\nu} \to W(l + \nu) \to 0, \tag{4}$$

in which, $K_1 = J_{(l+\nu,1)}e_{l+\nu}/J_{(l+\lambda,q)}e_{l+\nu}$. If $K_1 = 0$, then $W(l+\nu)$ is a projective $A/J_{(l+\lambda,q)}$-module. So the assertion holds obviously. For $K_1 \neq 0$, the induction assumption on the case $l+\nu(\geqslant l+1)$ and Lemma 3.5 insure that $J_{(l+\lambda,q)}e_{l+\nu} = J_{(l+\lambda,1)}e_{l+\nu}$ and $J_{(l+\nu+j+1,0)}e_{l+\nu} = J_{(l+\nu+j,1)}e_{l+\nu}$, where $0 \leqslant j \leqslant \lambda-\nu$. Thus there is a chain of submodules of K_1

$$\begin{aligned} 0 &= M_{\lambda-\nu+1} \subset M_{\lambda-\nu} \subset \cdots \subset M_j = J_{(l+\nu+j,0)}e_{l+\nu}/J_{(l+\lambda,1)}e_{l+\nu} \subset \cdots \subset M_1 \\ &= J_{(l+\nu+1,0)}e_{l+\nu}/J_{(l+\lambda,1)}e_{l+\nu} = J_{(l+\nu,1)}e_{l+\nu}/J_{(l+\lambda,1)}e_{l+\nu} = K_1, \end{aligned}$$

which provides that $M_j/M_{j+1} \simeq J_{(l+\nu+j,0)}e_{l+\nu}/J_{(l+\nu+j+1,0)}e_{l+\nu} = J_{(l+\nu+j,0)}e_{l+\nu}/J_{(l+\nu+j,1)}e_{l+\nu} \simeq \bigoplus_{d_{l+\nu+j,l+\nu}} W(l+\nu+j)$. As a result, $K_1 \in \mathcal{F}(W(l+\nu+1), W(l+\nu+2), \ldots, W(l+\lambda))$. Now applying $\mathrm{Hom}_{A/J_{(l+\lambda,q)}}(-, Q^*_{(l,r)})$ to (4) gives rise to an exact sequence

$$\begin{aligned} \mathrm{Hom}_{A/J_{(l+\lambda,q)}}\big(K_1, Q^*_{(l,r)}\big) &\to \mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}\big(W(l+\nu), Q^*_{(l,r)}\big) \\ &\to \mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}\big(Ae_{l+\nu}/J_{(l+\lambda,q)}e_{l+\nu}, Q^*_{(l,r)}\big). \end{aligned}$$

It is clear that the last term equals zero. Noting that $Q_{(l,r)} \in \mathcal{F}(W(l), W(l+1), \ldots, W(l+r))$, as shown in Step 1, we get $Q^*_{(l,r)} \in \mathcal{F}(W(l)^*, W(l+1)^*, \ldots, W(l+r)^*)$. This, together with $K_1 \in \mathcal{F}(W(l+\nu+1), W(l+\nu+2), \ldots, W(l+\lambda))$, means that the first term in the above exact sequence is also zero by Lemma 2.5(c). Consequently, $\mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}(W(l+\nu), Q^*_{(l,r)}) = 0$, as desired.

Step 3. *Let* $0 \leqslant r \leqslant \lambda-1$ *and* $1 \leqslant q \leqslant s(l+\lambda)+1$. *Then* $\mathrm{Ext}^2_{A/J_{(l+\lambda,q)}}(W(l+r+1), Q^*_{(l,r)}) = 0$.

The case $r = \lambda-1$ is obvious since $W(l+\lambda)$ is a projective $A/J_{(l+\lambda,q)}$-module, as mentioned in Step 2.

For any $r < \lambda-1$, there is an exact sequence of $A/J_{(l+\lambda,q)}$-modules

$$0 \to K_2 \to Ae_{l+r+1}/J_{(l+\lambda,q)}e_{l+r+1} \to W(l+r+1) \to 0, \tag{5}$$

where $K_2 = J_{(l+r+1,1)}e_{l+r+1}/J_{(l+\lambda,q)}e_{l+r+1}$. Using the induction hypothesis on $l+r+1$ and Lemma 3.5, we have that $J_{(l+\lambda,q)}e_{l+r+1} = J_{(l+\lambda,1)}e_{l+r+1}$ and $J_{(l+r+j+2,0)}e_{l+r+1} = J_{(l+r+j+1,1)}e_{l+r+1}$, where $0 \leqslant j \leqslant \lambda-r-1$. This yields a filtration of K_2 as follows:

$$\begin{aligned} 0 &= M_{\lambda-r} \subset M_{\lambda-r-1} \subset \cdots \subset M_j = J_{(l+r+j+1,0)}e_{l+r+1}/J_{(l+\lambda,1)}e_{l+r+1} \subset \cdots \subset M_1 \\ &= J_{(l+r+2,0)}e_{l+r+1}/J_{(l+\lambda,1)}e_{l+r+1} = J_{(l+r+1,1)}e_{l+r+1}/J_{(l+\lambda,1)}e_{l+r+1} = K_2. \end{aligned}$$

We observe that $M_j/M_{j+1} \simeq J_{(l+r+j+1,0)}e_{l+r+1}/J_{(l+r+j+2,0)}e_{l+r+1} = J_{(l+r+j+1,0)} \times e_{l+r+1}/J_{(l+r+j+1,1)}e_{l+r+1} \simeq \bigoplus_{d_{l+r+j+1,l+r+1}} W(l+r+j+1)$, and thus $K_2 \in \mathcal{F}(W(l+r+2), W(l+r+3), \ldots, W(l+\lambda))$. By Step 2 we obtain that $\mathrm{Ext}^1_{A/J_{(l+\lambda,q)}}(K_2, Q^*_{(l,r)}) = 0$, which implies that $\mathrm{Ext}^2_{A/J_{(l+\lambda,q)}}(W(l+r+1), Q^*_{(l,r)}) = 0$ by dimension shifting in (5).

Step 4. $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,r)}, Q^*_{(l,r)}) = 0$ *for all* $0 \leqslant r \leqslant \lambda$.

Use induction on r, the case $r = 0$ being trivial, again by the condition of the lemma. Suppose that the assertion holds for $r-1$, namely $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,r-1)}, Q^*_{(l,r-1)}) = 0$. We now show the case of r. Note that we have already proved $\operatorname{Ext}^2_{A/J_{(l+j,q)}}(Q_{(l,j)}, Q^*_{(l,j)}) = 0$ for $0 \leqslant j \leqslant \lambda - 1$ and $1 \leqslant q \leqslant s(l + j) + 1$. Thus we know by Lemma 3.5 that $Q_{(l,r-1)} = Ae_l/J_{(l+r-1,1)}e_l \simeq Ae_l/J_{(l+r,0)}e_l$. Whence, there exists an exact sequence of $A/J_{(l+\lambda,q)}$-modules

$$0 \to \bigoplus_{d_{l+r,l}} W(l+r) \to Q_{(l,r)} \to Q_{(l,r-1)} \to 0, \tag{6}$$

which induces the exact sequence

$$\begin{aligned} \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r-1)}, Q^*_{(l,r-1)}\big) &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r)}, Q^*_{(l,r-1)}\big) \\ &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\Big(\bigoplus_{d_{l+r,l}} W(l+r), Q^*_{(l,r-1)}\Big). \end{aligned}$$

The first term is zero following from the induction hypothesis on $r - 1$. Thanks to Step 3, we see that the last term of the above sequence is also zero, and thus $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,r)}, Q^*_{(l,r-1)}) = 0$. Now applying $\operatorname{Hom}_{A/J_{(l+\lambda,q)}}(-, W(l+r)^*)$ to (6), we have the following exact sequence

$$\begin{aligned} \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r-1)}, W(l+r)^*\big) &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r)}, W(l+r)^*\big) \\ &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\Big(\bigoplus_{d_{l+r,l}} W(l+r), W(l+r)^*\Big). \end{aligned}$$

Again by Step 3, the first term vanishes. The last term also vanishes by the condition of the lemma. This forces that $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,r)}, W(l + r)^*) = 0$. Finally, applying $\operatorname{Hom}_{A/J_{(l+\lambda,q)}}(-, Q^*_{(l,r)})$ to (6) provides an exact sequence

$$\begin{aligned} \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r-1)}, Q^*_{(l,r)}\big) &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\big(Q_{(l,r)}, Q^*_{(l,r)}\big) \\ &\to \operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}\Big(\bigoplus_{d_{l+r,l}} W(l+r), Q^*_{(l,r)}\Big). \end{aligned}$$

Both end terms of the above sequence vanish according to the preceding arguments. This shows that $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,r)}, Q^*_{(l,r)}) = 0$. In particular, we have thus proved that $\operatorname{Ext}^2_{A/J_{(l+\lambda,q)}}(Q_{(l,\lambda)}, Q^*_{(l,\lambda)}) = 0$, thereby finishing the proof of the lemma. □

Now we are in the position to prove Theorems 3.4 and 1.1.

Proof of Theorem 3.4. For the 'if' part, the assumptions of Theorem 3.4 guarantee that Lemma 3.6 can be applied, which means that Lemma 3.5 can also be applied. By imitating the proof of (c) $\Rightarrow$ (a) in Theorem 3.3, it is not difficult for us to verify that the algebra A is quasi-hereditary. Conversely, we know that there is no nilpotent cell ideal appearing in the cell chain of A, and also that $W(i)$ coincides with $\Delta(i)$ for each $i \in \Lambda_0$. Hence, $\mathrm{Ext}_A^2(W(i), W(i)^*) = 0$ follows directly from the property of standard modules of quasi-hereditary algebras (see [3]). Using the known fact that $\mathrm{Ext}_{A/J}^j(M, N) \simeq \mathrm{Ext}_A^j(M, N)$ for any heredity ideal J of A and any A/J-modules M and N (see [2] or [7]), we can easily deduce the 'only if' part. $\square$

Proof of Theorem 1.1. Note that if A is quasi-hereditary, then $\Lambda = \Lambda_0$. It follows that the condition (a) implies (b) by Theorem 3.3. Conversely, the condition (b) means that $\mathrm{Ext}_A^1(W(\lambda), W(\lambda)^*) = 0$ for all $\lambda \in \Lambda_0$, thus A is quasi-hereditary according to Theorem 3.3. Similarly, we have that (a) and (c) are equivalent, using Theorem 3.4. $\square$

As a corollary of Theorem 1.1, we have the following result given in [10].

Corollary 3.7. *For a cellular algebra A the following are equivalent*:

(a) *The algebra A is quasi-hereditary.*
(b) $\mathrm{Ext}_A^1(W(\lambda), W(\mu)^*) = 0$ *for all* $\lambda, \mu \in \Lambda$.

Remark 3.8. In his paper [10], Xi also proved that a cellular algebra A is quasi-hereditary if and only if $\mathrm{Ext}_A^2(W(\lambda), W(\mu)^*) = 0$ for all $\lambda, \mu \in \Lambda$. When this is compared with the condition (b) of Theorem 1.1, a question arises naturally: for a cellular algebra A, if $\mathrm{Ext}_A^2(W(\lambda), W(\lambda)^*) = 0$ for each $\lambda \in \Lambda$, is A quasi-hereditary? The question will have a positive answer if one can deduce that $\mathrm{Ext}^2_{A/J_{(k+i,q)}}(Q_{(k,i)}, Q^*_{(k,i)}) = 0$ for all $1 \leqslant q \leqslant s(k+i)+1$, under the condition that $\mathrm{Ext}_A^2(Q_{(k,i)}, Q^*_{(k,i)}) = 0$.

4. Semi-simplicity of cellular algebras

In this section, we are going to deal with the semi-simplicity of cellular algebras by considering the first cohomology groups of some cell modules and simple modules. The issue of semi-simplicity reduces in [4] to the computation of the discriminants of bilinear forms associated to cell modules. We are interested in a homological characterization here.

Let us prove the following theorem which has Theorem 1.2 as an immediate consequence.

Theorem 4.1. *For a cellular algebra A the following conditions are equivalent*:

(a) *The algebra A is semisimple.*
(b) $\mathrm{Ext}_A^1(W(\lambda), S(\mu)) = 0$ *for any* $\lambda, \mu \in \Lambda_0$ *satisfying* $\mu \geqslant \lambda$.
(c) $\mathrm{Ext}_A^1(W(\lambda), W(\mu)) = 0$ *for any* $\lambda, \mu \in \Lambda_0$ *satisfying* $\mu \geqslant \lambda$.

Proof. Obviously, the condition (a) implies both (b) and (c) since $W(\lambda) \simeq S(\lambda)$ for all $\lambda \in \Lambda_0$ in this situation.

(b) $\Rightarrow$ (a). Observe that for any $\lambda \in \Lambda_0$, there always exists an exact sequence of A-modules

$$0 \to \mathrm{rad}\big(W(\lambda)\big) \to W(\lambda) \to S(\lambda) \to 0,$$

which provides the following exact sequence

$$\mathrm{Hom}_A\big(\mathrm{rad}\big(W(\lambda)\big), S(\mu)\big) \to \mathrm{Ext}_A^1\big(S(\lambda), S(\mu)\big) \to \mathrm{Ext}_A^1\big(W(\lambda), S(\mu)\big)$$

for any $\mu \geqslant \lambda$. Since $[\mathrm{rad}(W(\lambda)) : S(\mu)] = 0$ for any $\mu \geqslant \lambda$, the first term of the above sequence is zero. The last term is also zero by the condition. Thus we have that $\mathrm{Ext}_A^1(S(\lambda), S(\mu)) = 0$ for any $\mu \geqslant \lambda$. Thanks to Lemma 2.5, we also have $\mathrm{Ext}_A^1(S(\lambda), S(\mu)) = 0$ for any $\lambda \geqslant \mu$. Hence, $\mathrm{Ext}_A^1(S(\lambda), S(\mu)) = 0$ for any $\lambda, \mu \in \Lambda_0$, which implies that A is semisimple.

(c) $\Rightarrow$ (a). Let $0 = J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(1,0)} = A$ be a cell chain of the cellular algebra A. It is enough to prove that both $W(l) \simeq S(l)$ and $s(l) = 0$ hold for any $1 \leqslant l \leqslant n$.

Use induction on l, the case $l = 1$ just means that $W(1) \simeq S(1)$. We need to verify that $s(1) = 0$. Suppose that $s(1) > 0$. Then $W(1,1)$ is not zero. Note that $W(1,1)$ has only $S(1)$ as a composition factor. Therefore, the condition $\mathrm{Ext}_A^1(W(1), W(1)) = 0$ means that $\mathrm{Ext}_A^1(W(1), W(1,1)) = 0$. Thus, the following exact sequence

$$0 \to J_{(1,1)}e_1/J_{(1,2)}e_1 \to Ae_1/J_{(1,2)}e_1 \to W(1) \to 0$$

splits since $J_{(1,1)}e_1/J_{(1,2)}e_1 \simeq \bigoplus_{d_{(1,1)1}} W(1,1)$. Hence, we have that $Ae_1/J_{(1,2)}e_1 \simeq W(1) \oplus \bigoplus_{d_{(1,1)1}} W(1,1)$. Obviously, $d_{(1,1)1} \neq 0$, which forces that $Ae_1/J_{(1,2)}e_1$ is decomposable. This is absurd.

We now assume that both $W(l) \simeq S(l)$ and $s(l) = 0$ are true for any $1 \leqslant l \leqslant j-1 (< n)$. Since $s(l) = 0$ for all $1 \leqslant l \leqslant j-1$, it follows that the cell chain of A has the following form:

$$\begin{aligned} 0 = {} & J_{(n+1,0)} = J_{(n,s(n)+1)} \subset J_{(n,s(n))} \subset \cdots \subset J_{(j+1,0)} = J_{(j,s(j)+1)} \subset \cdots \subset J_{(j,1)} \\ & \subset J_{(j,0)} \subset J_{(j-1,0)} \subset \cdots \subset J_{(1,0)} = A. \end{aligned}$$

For any $1 \leqslant i \leqslant j-2$, the induction hypothesis implies that $[W(i+1) : S(i)] = \cdots = [W(j-1) : S(i)] = 0$. Therefore $J_{(i+1,0)}e_i/J_{(i+2,0)}e_i = \cdots = J_{(j-1,0)}e_i/J_{(j,0)}e_i$, that is, $J_{(i+1,0)}e_i = J_{(j,0)}e_i$, and then $W(i) \simeq Ae_i/J_{(i+1,0)}e_i = Ae_i/J_{(j,0)}e_i$ for any $1 \leqslant i \leqslant j-2$. Combining this with the fact $W(j-1) \simeq Ae_{j-1}/J_{(j,0)}e_{j-1}$, we have that $W(i) \simeq Ae_i/J_{(j,0)}e_i$ for any $1 \leqslant i \leqslant j-1$. Assume that $W(j)$ has a composition factor $S(i)$ with $1 \leqslant i \leqslant j-1$, namely, $d_{ji} \neq 0$. Then there exists an exact sequence of A-modules

$$0 \to J_{(j,0)}e_i/J_{(j,1)}e_i \to Ae_i/J_{(j,1)}e_i \to W(i) \to 0,$$

that is,

$$0 \to \bigoplus_{d_{ji}} W(j) \to Ae_i/J_{(j,1)}e_i \to W(i) \to 0,$$

which splits by the condition that $\mathrm{Ext}_A^1(W(i), W(j)) = 0$ for any $i \leqslant j$, a contradiction. This proves that $W(j)$ has no composition factors $S(i)$ with $i \leqslant j-1$. According to Lemma 2.4(a), we get $W(j) \simeq S(j)$.

It remains only to prove that $s(j)$ equals zero. From $[W(j):S(i)] = 0$ for all $1 \leqslant i \leqslant j-1$, we see that $J_{(j,0)}e_i/J_{(j,1)}e_i = 0$, namely, $J_{(j,0)}e_i = J_{(j,1)}e_i$ for all $1 \leqslant i \leqslant j-1$. Consequently, we get $W(i) \simeq Ae_i/J_{(j,1)}e_i$ for any $1 \leqslant i \leqslant j-1$. Combining this with the fact $W(j) \simeq Ae_j/J_{(j,1)}e_j$, we have that $W(i) \simeq Ae_i/J_{(j,1)}e_i$ for any $1 \leqslant i \leqslant j$. Suppose that $s(j) \neq 0$. Then $W(j,1)$ is not zero. Thus, there exists some $S(i)$ with $1 \leqslant i \leqslant j$, such that $[W(j,1):S(i)] \neq 0$, namely, $d_{(j,1)i} \neq 0$. Observe that there is an exact sequence

$$0 \to J_{(j,1)}e_i/J_{(j,2)}e_i \to Ae_i/J_{(j,2)}e_i \to W(i) \to 0$$

in A-mod, that is,

$$0 \to \bigoplus_{d_{(j,1)i}} W(j,1) \to Ae_i/J_{(j,2)}e_i \to W(i) \to 0$$

is exact. Since $\mathrm{Ext}_A^1(W(i), W(k)) = 0$ for any $i \leqslant k$ and $W(k) \simeq S(k)$ for any $k \leqslant j$, as proved, we obtain that $\mathrm{Ext}_A^1(W(i), S(k)) = 0$ for any $i \leqslant k \leqslant j$. For the case $k \leqslant i-1$, we have that $\dim_K \mathrm{Ext}_A^1(W(i), S(k)) = \dim_K \mathrm{Ext}_A^1(S(k), W(i)^*) = \dim_K \mathrm{Ext}_A^1(W(k), W(i)) = 0$, namely, $\mathrm{Ext}_A^1(W(i), S(k)) = 0$. As a result, $\mathrm{Ext}_A^1(W(i), S(k)) = 0$ for any $1 \leqslant k \leqslant j$, which implies that the last exact sequence splits. This is a contradiction, and thus the proof is completed. □

Remark 4.2. Of course, one would expect that Theorem 4.1 could be generalized to the case of the second cohomology groups. But such an attempt is usually futile. For example, let A be the quotient of the path algebra (over the field K) of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

modulo the ideal generated by $\beta\alpha$. An involution on A can be given by fixing the vertices, but interchanging the paths α and β. Let $J_{(2,0)}$ be the ideal generated by e_2, β, α, and $\alpha\beta$, and let $J_{(1,0)} = A$. One can check easily that the algebra A is cellular with the defined involution and the cell chain $0 \subset J_{(2,0)} \subset J_{(1,0)} = A$. Moreover, we see that $W(2) \simeq P(2)$ and $W(1) \simeq S(1)$. There is no difficulty to get that $\mathrm{Ext}_A^2(W(1), S(1)) = \mathrm{Ext}_A^2(W(1), S(2)) = \mathrm{Ext}_A^2(W(2), S(2)) = 0$ and $\mathrm{Ext}_A^2(W(1), W(1)) = \mathrm{Ext}_A^2(W(1), W(2)) = \mathrm{Ext}_A^2(W(2), W(2)) = 0$. However, the algebra A is not semisimple.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. That the conditions (a), (b), and (c) are equivalent follows from Theorem 4.1. When the algebra is semisimple, we see that $\Lambda = \Lambda_0$, and thus the conditions (c) and (c′) just say the same thing. Note that the condition (a) means that $W(\lambda) \simeq S(\lambda)$ for each $\lambda \in \Lambda_0 = \Lambda$ and thus (a) implies (c″). The implications (c″) ⇒ (c′) and (c′) ⇒ (c) are trivial. □

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