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Netlike partial cubes II. Retracts and netlike subgraphs

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ABSTRACT

First we show that the class of netlike partial cubes is closed under retracts. Then we prove, for a subgraph G of a netlike partial cube H, the equivalence of the assertions: G is a netlike subgraph of H; G is a hom-retract of H; G is a retract of H. Finally we show that a non-trivial netlike partial cube G, which is a retract of some bipartite graph H, is also a hom-retract of H if and only if G contains at most one convex cycle of length greater than 4.

1. Introduction

The class of netlike partial cubes was introduced in Part I [7] of this series of papers as a special class of partial cubes (isometric subgraphs of hypercubes) with median graphs, even cycles, benzenoid graphs and cellular bipartite graphs as particular elements.

This paper is entirely devoted to the study of retracts and hom-retracts of netlike partial cubes. A *retraction* (resp. *hom-retraction*) of a graph G is an idempotent nonexpansive (resp. edge-preserving) self-mapping of G. The retract construction has been a flourishing topic in graph theory since Pavol Hell's Ph.D. thesis [4]. Retracts have been one of the basic ingredients of metric graph theory; for example in the study of absolute retracts, the one with varieties of graphs – that is classes of graphs closed under retracts and products – and also to obtain fixed subgraph theorems in diverse classes of metric graphs, which will actually be the case for the class of netlike partial cubes (see [8]). In addition to the properties which will be used in [8], the main results of this paper deal with the links between retraction and hom-retraction.

Although a hom-retract is obviously a retract, the converse is not true in general. However there are graphs for which these two concepts coincide, an example of which are median graphs. In fact median graphs and certain bipartite graphs G satisfy the following two properties:

1. *Hom-Retract Property:*
   Any retract with at least two vertices of G is a hom-retract of this graph.
2. If G has at least two vertices and is a retract of a bipartite graph H, then G is a hom-retract of H.

The first property, which is weaker than the second, is a consequence in the case of median graphs of two results of Bandelt [2]. This property, which is also clearly satisfied by even cycles, is not a common property of partial cubes (see Section 7). The question of determining which partial cubes have the Hom-Retract Property arises naturally. In this paper we settle this question for netlike partial cubes.

Having proved in Section 3 that the class of netlike partial cubes is closed under retracts, we introduce in Section 4 the concept of netlike subgraphs of a netlike partial cube in order to extend a result of Bandelt [2, Theorem 1] on median graphs.

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It turns out that the definition of a netlike subgraph is compatible with the one of a median subgraph, that is any netlike subgraph of a median graph is a median subgraph of this graph.

The main result (Theorem 4.5) of this paper is that a netlike subgraph of a netlike partial cube H is a hom-retract of H. Because any retract of H is a netlike subgraph of H, the Hom-Retract Property follows immediately. To prove this result we show (Theorem 4.6) that, for any netlike subgraph G of H, there exists a minimal netlike extension G^* of G—that is a netlike subgraph of H which properly contains G and which is minimal with respect to the subgraph relation—such that G is a hom-retract of G^*. We also prove an analogous result (Proposition 6.1) by considering convex subgraphs instead of any netlike subgraphs.

The results of Section 6 are essential to prove the last result (Theorem 7.4) of this paper which deals with Property 2. As was noticed in [7], the class of netlike partial cubes is not closed under cartesian product. As we shall see, the Hom-Retract Property – which was a necessary condition in the choice of the class of netlike partial cubes – does not hold for cartesian product of netlike partial cubes. This observation is linked to the concept of prism-retractable graphs, a concept which was introduced by Sabidussi [10] in order to generalize Property 2. Contrary to the Hom-Retract Property, this property is not extendable to all netlike partial cubes. In Theorem 7.4 we characterize those that have this property.

2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G, $N_G[x] := \{x\} \cup N_G(x)$ is the *closed neighborhood* of x in G and $\delta_G(x) := |N_G(x)|$ is the *degree* of x in G. For a set X of vertices of a graph G we put $N_G[X] := \bigcup_{x \in X} N_G[x]$ and $N_G(X) := N_G[X] - X$, and we denote by $\partial_G(X)$ the *edge-boundary* of X in G, that is the set of all edges of G having exactly one endvertex in X. Moreover, we denote by $G[X]$ the subgraph of G induced by X, and we set $G - X := G[V(G) - X]$.

A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$. A path $P = \langle x_0, \ldots, x_n \rangle$ is called an (x_0, x_n)-*path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *inner* vertices, $n = |E(P)|$ is the *length* of P. If x and y are two vertices of a path P, then we denote by $P[x, y]$ the subpath of P whose endvertices are x and y.

A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, will be denoted by $\langle x_1, \ldots, x_n, x_0 \rangle$. The non-negative integer $n = |E(C)|$ is the *length* of C, and a cycle of length n is called a n-*cycle* and is often denoted by C_n.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of an (x, y)-*geodesic* (=shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure*. See van de Vel [12] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the *geodesic convexity*, that is the convexity on $V(G)$ which is induced by the geodesic interval operator I_G. In this convexity, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $\mathrm{co}_G(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. We will denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $\mathrm{co}_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we will say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $\mathrm{co}_G(H)$ of a subgraph H of G we will mean the smallest convex subgraph of G containing H as a subgraph, that is

$$\mathrm{co}_G(H) := G[\mathrm{co}_G(V(H))].$$

2.3. Netlike partial cubes

First we will recall some properties of *partial cubes*, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

$$W_{ab}^G := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$
$$U_{ab}^G := W_{ab}^G \cap N_G(W_{ba}^G).$$

Where no confusion is likely, we will simply denote W_{ab}^G and U_{ab}^G by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if G is bipartite and connected.

Two edges xy and uv are in the Djoković–Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If G is bipartite, the edges xy and uv are in relation Θ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation Θ is clearly reflexive and symmetric.

Theorem 2.1 (*Djoković [3, Theorem 1] and Winkler [13]*). *A connected bipartite graph G is a partial cube if and only if it has one of the following properties:*
(i) *For every edge ab of G, the sets W_{ab} and W_{ba} are convex (and thus half-spaces).*
(ii) *The relation Θ is transitive.*

Note that every interval and every polytope of a partial cube are finite. We will now recall the concept of pre-hull number of a partial cube. This concept was more generally defined for any convexities in [9].

Definition 2.2. Let G be a partial cube. The least non-negative integer n (if it exists) such that $\mathrm{co}_G(U_{ab}) = \mathcal{I}^n_G(U_{ab} \cup \{x\})$ for each edge ab of G and each $x \in \mathrm{co}_G(U_{ab})$ is called the *pre-hull number* of G, and is denoted by $ph(G)$. If no such n exists we put $ph(G) := \infty$.

As we will only deal with partial cubes whose pre-hull number is at most 1, except with a few counterexamples, it is useful to recall a simple characterization of these graphs.

Definition 2.3. We say that a set A of vertices of a graph G is *ph-stable* if, for all $u, v \in \mathcal{I}_G(A)$, $v \in I_G(u, w)$ for some $w \in A$.

We obtain immediately:

Lemma 2.4 (*Polat [7, Proposition 2.4]*). *If a set A of vertices of a graph G is ph-stable, then, for all $u, v \in \mathcal{I}_G(A)$, $I_G(u, v) \subseteq I_G(a, b)$ for some $a, b \in A$. In particular, each edge of $G[\mathcal{I}_G(A)]$ belongs to an (a, b)-geodesic for some $a, b \in A$, and moreover $\mathrm{co}_G(A) = \mathcal{I}_G(A)$.*

Proposition 2.5 (*Polat and Sabidussi [9, Theorem 7.5]*). *Let G be a partial cube. Then $ph(G) \leq 1$ if and only if U_{ab} and U_{ba} are ph-stable for every edge ab of G.*

We denote by $CV(G)$ (resp. $3V(G)$) the set of vertices of a graph G which belong to a cycle of G (resp. whose degree is at least 3). We say that a set $A \subseteq V(G)$ is *$\mathcal{C}$-convex* (resp. (3)-*convex*) if $CV(G[\mathcal{I}_G(A)]) \subseteq A$ (resp. $3V(G[\mathcal{I}_G(A)]) \subseteq A$). The set of $\mathcal{C}$-convex subsets of $V(G)$ and the one of (3)-convex subsets of $V(G)$ are convexities on $V(G)$ which are finer than the geodesic convexity.

Lemma 2.6 (*Polat [7, Proposition 3.5]*). *Any $\mathcal{C}$-convex set of a connected graph is ph-stable.*

Corollary 2.7. *If A is a $\mathcal{C}$-convex set of a connected graph G, then $\mathcal{I}_G(A)$ is convex.*

This is a consequence of Lemmas 2.4 and 2.6.

Lemma 2.8 (*Polat [7, Proposition 3.7]*). *Let A be a ph-stable set of vertices of a graph G. Then A is $\mathcal{C}$-convex if it is* (3)*-convex.*

Definition 2.9. We will say that a partial cube G is *netlike* if U_{ab} and U_{ba} are $\mathcal{C}$-convex for each edge ab.

In particular median graphs and even cycles are netlike partial cubes, and any convex subgraph of a netlike partial cube is a netlike partial cube.

Proposition 2.10 (*Polat [7, Proposition 2.6]*). *The pre-hull number of a netlike partial cube is at most* 1.

We have the following characterization of netlike partial cubes:

Proposition 2.11 (*Polat [7, Theorem 3.8]*). *A partial cube G is netlike if and only if U_{ab} and U_{ba} are ph-stable and* (3)*-convex for each edge ab.*

From another characterization [7, Theorem 3.10] of netlike partial cubes, we have the following property:

Proposition 2.12. *Each isometric cycle of a netlike partial cube is convex or its convex hull is a hypercube.*

A netlike partial cube G such that, for each edge ab, $\mathcal{I}_G(U_{ab})$ and $\mathcal{I}_G(U_{ba})$ induce trees, is called a *linear partial cube*.

Lemma 2.13 (*Polat [7, Theorem 7.4]*). *Let G be a partial cube. The following assertions are equivalent:*
(i) *G is linear.*
(ii) *G is a netlike partial cube which contains no hypercube of dimension greater than* 2.
(iii) *G is a netlike partial cube whose isometric cycles are convex.*

Lemma 2.14 (*Polat [7, Lemma 6.1]*). *Let ab be an edge of a netlike partial cube G. Then any convex cycle of $G[U_{ab}]$ is a* 4*-cycle.*

Lemma 2.15 (*Polat [7, Corollary 7.2]*). *A netlike partial cube is a median graph if and only if any of its convex cycles is a* 4*-cycle.*

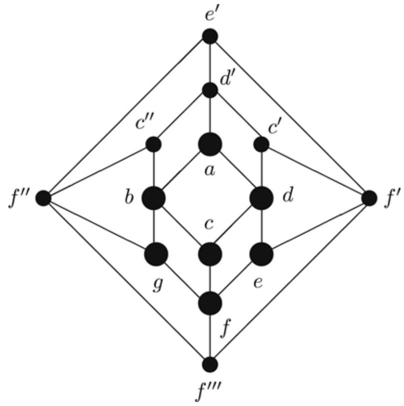

Fig. 1.

3. Retracts

We recall that, if G and H are two graphs, a map $f : V(G) \to V(H)$ is a *contraction* (*weak homomorphism* in [5]) if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Notice that a contraction $f : G \to H$ is a nonexpansive map between the metric spaces $(V(G), d_G)$ and $(V(H), d_H)$, i.e., $d_H(f(x), f(y)) \leq d_G(x, y)$ for all $x, y \in V(G)$. A contraction f of G onto an induced subgraph H of G is a *retraction*, and H is a *retract* (*weak retract* in [5]) of G, if its restriction to $V(H)$ is the identity. A retract of a graph G is a particular isometric subgraph of G.

Because a median graph is a retract of a hypercube, it follows that a retract of a median graph is also a median graph. We will see that this property also holds for netlike partial cubes.

Theorem 3.1. *The class of netlike partial cubes is closed under retracts.*

Proof. Let H be a retract of a netlike partial cube G, and let f be a retraction of G onto H. Let ab be an edge of H. We will prove that U^H_{ba} is $\mathcal{C}$-convex. Because H is an isometric subgraph of G, it follows that $W^H_{ab} = W^G_{ab} \cap V(H)$, $W^H_{ba} = W^G_{ba} \cap V(H)$ and $\mathcal{I}_H(U^H_{ba}) \subseteq \mathcal{I}_G(U^H_{ba}) \cap V(H)$.

We will show that $U^H_{ba} = U^G_{ba} \cap \mathcal{I}_H(U^H_{ba})$. Clearly $U^H_{ba} \subseteq U^G_{ba} \cap \mathcal{I}_H(U^H_{ba})$. Conversely, let u, v, w be three vertices of U^G_{ba} such that $v \in I_G(u, w)$. Let u', v', w' be the neighbors in U^G_{ab} of u, v, w, respectively. Then, by the Djoković–Winkler relation, v' is the only neighbor of v which belongs to $I_G(u', w')$. Therefore, if $u, w \in U^H_{ba}$ with $v \in I_H(u, w)$, then $f(v') \in I_H(u', v') \subseteq I_G(u', v')$ since H is an isometric subgraph of G. Hence $f(v') = v'$, and thus $v \in U^H_{ba}$. Consequently $U^G_{ba} \cap \mathcal{I}_H(U^H_{ba}) \subseteq U^H_{ba}$.

It follows that:

$$\begin{aligned} CV(H[U^H_{ba}]) &= CV(G[U^G_{ba}]) \cap \mathcal{I}_H(U^H_{ba}) \\ &\subseteq U^G_{ba} \cap \mathcal{I}_H(U^H_{ba}) \text{ since } U^G_{ba} \text{ is } \mathcal{C}\text{-convex} \\ &= U^H_{ba}. \end{aligned}$$

Therefore, U^H_{ba} is $\mathcal{C}$-convex. Analogously U^H_{ab} is also $\mathcal{C}$-convex. This proves that H is a netlike partial cube. □

Due to the fact that any isometric cycle of a retract of a graph G is also an isometric cycle of G, Lemma 2.13 and Theorem 3.1 immediately imply:

Corollary 3.2. *The class of linear partial cubes is closed under retracts.*

Remark 3.3. If the class of netlike partial cubes is closed under retracts, this is however not true for the class $\mathbb{PC}_1$ of all partial cubes whose pre-hull number is at most 1, as is shown by the following example. Let G be the graph in Fig. 1. Then the function which maps c' and c'' to c, d' to d, e' to e and f', f'' and f''' to f is clearly a retraction of G onto the subgraph H induced by the set $\{a, b, c, d, e, f, g\}$. It is easy to check that G, which is an isometric subgraph of the 4-cube Q_4, has a pre-hull number equal to 1, while H, which is the partial cube Q_3^- (the 3-cube minus a vertex), has a pre-hull number equal to 2. Also note that H is a convex subgraph of G. This proves that the class $\mathbb{PC}_1$ is not closed under convex subgraphs.

4. Hom-retracts and netlike subgraphs

A contraction $f : G \to H$ which preserves the edges is called a *homomorphism* of G into H. If a retraction $f : G \to H$ is a homomorphism, then we will say that f is a *hom-retraction* and that H is a *hom-retract*. We will extend to netlike partial cubes the following result of Bandelt:

Proposition 4.1 (*Bandelt [2, Theorem 1]*). *Let G be a median graph. Then the hom-retracts of G are the non-trivial (i.e., with at least two vertices) median subgraphs of G.*

We will first state a simple property of netlike partial cubes that will be frequently used.

Lemma 4.2. *For each edge ab of a netlike partial cube G, there exists a unique isometry (i.e. a distance-preserving bijection) ϕ_{ab} of $\mathcal{I}_G(U_{ab})$ onto $\mathcal{I}_G(U_{ba})$ such that, for every vertex $x \in U_{ab}$, $\phi_{ab}(x)$ is the neighbor of x in U_{ba}.*

ϕ_{ab} will be called the *canonical isometry of $\mathcal{I}_G(U_{ab})$ onto $\mathcal{I}_G(U_{ba})$*.

Proof. This is clear if $\mathcal{I}_G(U_{ab}) = U_{ab}$. Suppose that there there is a vertex $x \in \mathcal{I}_G(U_{ab}) - U_{ab}$. Then, because U_{ab} is $\mathcal{C}$-convex, there exists $u, v \in U_{ab}$ and a (u, v)-geodesic P containing x such that each inner vertex of P has degree 2 in $G[\mathcal{I}_G(U_{ab})]$ by Proposition 2.11. It follows that $I_G(u, v) = V(P)$ since U_{ab} is $\mathcal{C}$-convex. Let u' and v' be the neighbors of u and v in U_{ba}, respectively. Let P' be a (u', v')-geodesic. Then $\langle u', u\rangle \cup P \cup \langle v, v'\rangle \cup P'$ is an isometric cycle of G, and thus a convex cycle of G by Proposition 2.12. Therefore, $I_G(u', v') = V(P')$. Therefore, because $d_G(u, v) = d_G(u', v')$, the map f such that $f(u) = u'$ and $f(v) = v'$ has a unique extension of $I_G(u, v)$ to $I_G(u', v')$. □

Let G be a netlike partial cube. If a triple (x, y, z) of vertices of G has a median, then this median is unique. We will denote it by $m_G(x, y, z)$. If G is not a median graph, then some triple of vertices have no median. Hence the median operation m_G is partial. We will say that a subgraph G of a netlike partial cube H is *stable* under m_H if each triple (x, y, z) of vertices of G which has a median in H has also a median in G and $m_G(x, y, z) = m_H(x, y, z)$.

We will now define what we mean by a netlike subgraph.

Definition 4.3. A subgraph G of a netlike partial cube H is called a *netlike subgraph* of H if G is isometric in H and stable under m_H.

We can easily notice that the netlike subgraphs of a median graph are the median subgraphs of this graph. Clearly any convex subgraph and thus in particular any interval, and moreover any retract of a netlike partial cube H is a netlike subgraph of H. Furthermore, for any edge ab of H, the subgraphs $H[W_{ab}]$ and $H[W_{ba}]$ are also netlike subgraphs of H.

Proposition 4.4. *Let G be a netlike subgraph of a netlike partial cube H. Then G is a netlike partial cube such that $U^G_{ab} = U^H_{ab} \cap \mathcal{I}_G(U^G_{ab})$ for each edge ab of G.*

Proof. G is a partial cube since it is isometric in H. Let $ab \in E(G)$. Then $W^G_{ab} = W^H_{ab} \cap V(G)$ and $W^G_{ba} = W^H_{ba} \cap V(G)$ because G is isometric in H. Moreover $U^G_{ab} \subseteq U^H_{ab} \cap V(G)$, that implies $U^G_{ab} \subseteq U^H_{ab} \cap \mathcal{I}_G(U^G_{ab})$ since $U^G_{ab} \subseteq \mathcal{I}_G(U^G_{ab})$. Let $x \in U^H_{ab} \cap \mathcal{I}_G(U^G_{ab})$. Then $x \in I_G(u, v)$ for some $u, v \in U^G_{ab}$. Suppose that $x \neq a$. Then $\phi_{ab}(x) = m_H(\phi_{ab}(u), \phi_{ab}(v), x) = m_G(\phi_{ab}(u), \phi_{ab}(v), x)$ because G is stable under m_H. Hence $\phi_{ab}(x) \in V(G)$, and thus $x \in U^G_{ab}$.

It follows immediately that U^G_{ab} is $\mathcal{C}$-convex since U^H_{ab} is $\mathcal{C}$-convex by assumption. Consequently G is a netlike partial cube. □

Note that an isometric subgraph of a netlike partial cube H which is netlike in its own right is not necessarily a netlike subgraph of H, as is shown by the example of a 6-cycle in a 3-cube.

We will now state the following extension of Proposition 4.1.

Theorem 4.5. *Let G be a non-trivial subgraph of a netlike partial cube H. Then the following assertions are equivalent:*
(i) *G is a netlike subgraph of H.*
(ii) *G is a hom-retract of H.*
(iii) *G is a retract of H.*

We first prove another important result which will be the cornerstone of the proof of Theorem 4.5. If G is a netlike subgraph of a netlike partial cube H, then a *minimal netlike extension* of G in H is a netlike subgraph of H which properly contains G as a subgraph and which is minimal with respect to the subgraph relation.

Theorem 4.6. *Let G be a proper netlike subgraph of a netlike partial cube H. There exists a minimal netlike extension G^* of G in H such that G is a hom-retract of G^* whenever G is non-trivial.*

The proof of this theorem is very long, and we will deal with it in the next section. We will first give the proof of Theorem 4.5.

Proof of Theorem 4.5. The implications (ii) ⇒ (iii) and (iii) ⇒ (i) are obvious.

(i) ⇒ (ii): Let G be a non-trivial netlike subgraph of H. For each ordinal α, we inductively construct the subgraph G_α as follows:

- $G_0 := G$;
- $G_{\alpha+1}$ is a minimal netlike extension of G_α;
- if α is a limit ordinal, then $G_\alpha := \bigcup_{\beta<\alpha} G_\beta$.

Note that G_α is also a netlike subgraph of H if α is a limit ordinal because the set $\{G_\beta : \beta < \alpha\}$ is totally ordered by inclusion, and any geodesic is a finite graph. Let γ be the least ordinal such that $G_\gamma = H$.

Now, for each ordinal $\alpha \leq \gamma$, we will construct a hom-retraction f_α of G_α onto G_0. Let f_0 be the identity map on $V(G_0)$. Let $\alpha \geq 0$. Suppose that f_β has already been constructed for every $\beta < \alpha$. If $\alpha = \beta + 1$ for some ordinal β, then $f_\alpha := f_\beta \circ f_{G_\alpha}$ where f_{G_α} is a hom-retraction of G_α onto G_β induced by Theorem 4.6. Then f_α is obviously a hom-retraction of G_α onto G_0.

Suppose that α is a limit ordinal. Let $f_\alpha := \bigcup_{\beta<\alpha} f_\beta$, i.e. f_α is the map of G_α onto G_0 such that, for each vertex x of G_α, $f_\alpha(x) := f_\beta(x)$, where β is the least ordinal such that $x \in V(G_\beta)$. In particular $f_\alpha(x) = x$ if $x \in V(G_0)$. It remains to prove that f_α is a homomorphism. Let x, y be two adjacent vertices of G_α. Then there is an ordinal $\beta < \alpha$ such that $x, y \in V(G_\beta)$. Therefore, $f_\alpha(x) = f_\beta(x)$ and $f_\alpha(y) = f_\beta(y)$. It follows that $f_\alpha(x)$ and $f_\alpha(y)$ are adjacent because f_β is a homomorphism by the induction hypothesis. Consequently f_α is a hom-retraction of G_α onto G_0.

Finally f_γ is then the desired hom-retraction of H onto G. □

5. Proof of Theorem 4.6

We need two lemmas.

Lemma 5.1. *Let G be a netlike subgraph of a netlike partial cube H. Then any vertex of $H - G$ is adjacent to at most two vertices of G.*

Proof. Suppose that a vertex x of $H - G$ is adjacent to three vertices u, v, w of G. Since G is an isometric subgraph of H, the distance in G between any two of these neighbors is 2. Then we have two cases.

If u, v, w have a common neighbor y in G, then $\{x, y, u, v, w\}$ induces a $K_{2,3}$, contrary to the fact that H is a partial cube.

Therefore u, v, w are vertices of 6-cycle $\langle u, a, v, b, w, c, u\rangle$ of G. Then the edges $\{u, c\}$, $\{v, b\}$ and $\{x, w\}$ are in relation Θ in H, and thus the first two of these edges are in relation Θ in G because G is isometric in H. Hence $w \in I_G(c, b) \cap U^H_{cu}$, and thus $w \in U^G_{cu}$ by Proposition 4.4. It follows that x must belong to U^G_{uc}, contrary to the fact that $x \notin V(G)$. □

Lemma 5.2. *Let H be a netlike partial cube, and $\langle x, a, y, b, x\rangle$ a 4-cycle of H. Let*

$$X^H_{aybx} := U_{xa} \cap U_{xb}, \qquad Y^H_{aybx} := U_{ya} \cap U_{yb},$$
$$A^H_{aybx} := U_{ay} \cap N_H(Y), \qquad B^H_{aybx} := U_{by} \cap N_H(Y).$$

(i) *The restrictions of $\phi_{xb} \circ \phi_{xa}$ and of $\phi_{xa} \circ \phi_{xb}$ to X^H_{aybx} are equal, and this map, denoted by μ_H, is an isomorphism of $H[X^H_{aybx}]$ onto $H[Y^H_{aybx}]$.*

(ii) *$A^H_{aybx} = U_{ax} \cap N_H(X^H_{aybx}) = U_{ax} \cap U_{ay}$ and $B^H_{aybx} = U_{bx} \cap N_H(X^H_{aybx}) = U_{bx} \cap U_{by}$.*

(iii) *The sets $A^H_{aybx}, B^H_{aybx}, X^H_{aybx}, Y^H_{aybx}$ are convex.*

(iv) *For each pair of vertices $(\alpha, \beta) \in A^H_{aybx} \times B^H_{aybx}$, the triples of vertices (x, α, β) and (y, α, β) have a median, and*

$$m_H(y, \alpha, \beta) = \mu_H(m_H(x, \alpha, \beta)).$$

(v) *$X^H_{aybx} = \{m_H(x, \alpha, \beta) : (\alpha, \beta) \in A^H_{aybx} \times B^H_{aybx}\}$ and $Y^H_{aybx} = \{m_H(y, \alpha, \beta) : (\alpha, \beta) \in A^H_{aybx} \times B^H_{aybx}\}$.*

(vi) *Each triple (p, q, r) of vertices in X^H_{aybx} has a median, and*

$$\mu_H(m_H(p, q, r)) = m_H(\mu_H(p), \mu_H(q), \mu_H(r)).$$

(vii) *Let $p \in X^H_{aybx}$, $q \in W_{ax} \cap W_{xb}$ and $r \in W_{xa} \cap W_{bx}$ be such that the triple (p, q, r) has a median m. Then $m \in X^H_{aybx}$ and $\mu_H(m) = m_H(\mu_H(p), q, r)$.*

Proof. Because no confusion is likely, in the following we delete the symbols H and $aybx$ in the notation of the four different sets $A^H_{aybx}, B^H_{aybx}, X^H_{aybx}$ and Y^H_{aybx}. See Fig. 2. Note that these sets are not empty since they contain the vertices a, b, x and y, respectively.

(i) Let $u \in Y$. The edges $u\phi_{ya}(u)$ and ya are in relation Θ. Hence $\phi_{ya}(u) \in I_G(u, a)$. Then $\phi_{ya}(u)$ belongs to a cycle of $H[\mathcal{I}_H(U_{yb})]$ since $u \in Y$. Therefore, $\phi_{ya}(u) \in U_{yb}$ because H is netlike. In the same way $\phi_{yb}(u) \in U_{ya}$.

Clearly $\phi_{yb}(\phi_{ya}(u)) = \phi_{ya}(\phi_{yb}(u)) \in X$. Analogously, for each $m \in X$, $\phi_{xa}(\phi_{xb}(m)) = \phi_{xb}(\phi_{xa}(m)) \in Y$. It follows that the restriction μ_H of $\phi_{xb} \circ \phi_{xa}$ to X is a bijection of X onto Y, and moreover it is clearly an isomorphism of $G[X]$ onto $G[Y]$.

(ii) Let $\alpha \in A$. Then $\alpha = \phi_{xa}(\mu_H^{-1}(\phi_{ay}(\alpha)))$. Then $A = U_{ax} \cap N_H(X)$. Analogously $B = U_{bx} \cap N_H(X)$.

Then, in particular, $A \subseteq U_{ax} \cap U_{ay}$. On the other hand, for any vertex $\alpha' \in U_{ax} \cap U_{ay}$ we can prove, as in (i), that $\phi_{ay}(\alpha') \in U_{yb}$. It follows that $A = U_{ax} \cap U_{ay}$, and analogously $B = U_{bx} \cap U_{by}$.

(iii) Let $u, u' \in Y$, and let P be any (u, u')-geodesic. Then $P \cup \langle u, \phi_{yb}(u)\rangle \cup \phi_{yb}(P) \cup \langle \phi_{yb}(u'), u'\rangle$ is a cycle of $H[\mathcal{I}_H(U_{ya})]$, and thus the vertex set of this cycle is contained in U_{ya} because H is netlike. Similarly $P \cup \langle u, \phi_{ya}(u)\rangle \cup \phi_{ya}(P) \cup \langle \phi_{ya}(u'), u'\rangle$ is a cycle of $H[\mathcal{I}_H(U_{yb})]$, and thus the vertex set of this cycle is contained in U_{yb}. Hence $V(P) \subseteq U_{ya} \cap U_{yb} = Y$, and thus Y is convex. Analogously A, B and X are convex.

(iv) By Lemma 2.14, any convex cycle of $H[X]$ and of $H[Y]$ is a 4-cycle. Hence, by Lemma 2.15, $H[X]$ and of $H[Y]$ are median graphs. Therefore every triple of vertices of these graphs has a median.

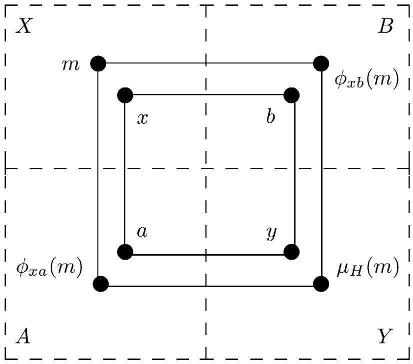

Fig. 2.

Let $(\alpha, \beta) \in A \times B$. Then the triple $(x, \phi_{ax}(\alpha), \phi_{bx}(\beta))$ has a median in $H[X]$, which is its median in H since X is convex by (iii), and this median is clearly the median of (x, α, β). Analogously $m_H(y, \alpha, \beta) = m_H(y, \phi_{xb}(\alpha), \phi_{by}(\beta)) \in Y$. Then, by the uniqueness of the median: $m_H(y, \alpha, \beta) = \mu_H(m_H(x, \alpha, \beta))$.

(v) By (iv), for each $(\alpha, \beta) \in A \times B$, $m_H(x, \alpha, \beta) \in X$ and $m_H(y, \alpha, \beta) \in Y$. Conversely, for each $m \in X$, $m = m_H(x, \phi_{xa}(m), \phi_{xb}(m))$; and, for each $u \in Y$, $u = m_H(y, \phi_{ya}(u), \phi_{yb}(u))$. Hence

$$X = \{m_H(x, \alpha, \beta) : (\alpha, \beta) \in A \times B\} \quad \text{and} \quad Y = \{m_H(y, \alpha, \beta) : (\alpha, \beta) \in A \times B\}.$$

(vi) is clear because $H[X]$ and $H[Y]$ are median graphs, X and Y are convex in H, and μ_H is an isomorphism of $H[X]$ onto $H[Y]$.

(vii) Let $p \in X$, $q \in W_{ax} \cap W_{xb}$ and $r \in W_{xa} \cap W_{bx}$ be such that the triple (p, q, r) has a median m. Then $p = m_H(x, \phi_{xa}(p), \phi_{xb}(p))$, and the triple $(\phi_{xa}(p), \phi_{xb}(p), q)$ has clearly a median in H:

$$\begin{aligned} m_H(\phi_{xa}(p), \phi_{xb}(p), q) &= m_H(\phi_{xa}(p), \mu_H(p), q) \quad \text{because } q \in W_{xb} \\ &= \phi_{xa}(p) \quad \text{because } q \in W_{ax}. \end{aligned}$$

Then, by the associativity of the median operation:

$$\begin{aligned} m &= m_H(p, q, r) = m_H(x, m_H(\phi_{xa}(p), \phi_{xb}(p), q), r) \\ &= m_H(x, \phi_{xa}(p), r) = m_H(x, p, r) \quad \text{since } r \in W_{xa} \cap W_{bx}. \end{aligned}$$

Hence $m \in I_H(x, p) \subseteq X$ since X is convex by (iii). Because $q \in W_{ax} \cap W_{xb}$ and $r \in W_{xa} \cap W_{bx}$, there exists a (q, r)-geodesic P which contains $\langle \phi_{xa}(m), m, \phi_{xb}(m) \rangle$ as a subpath. Then $P[q, \phi_{xa}(m)] \cup \langle \phi_{xa}(m), \mu_H(m), \phi_{xb}(m) \rangle \cup P[\phi_{xb}(m), r]$ is a (q, r)-geodesic. Therefore, $\mu_H(m) \in I_H(q, r)$.

Furthermore $\phi_{xa}(m) \in I_H(q, \phi_{xa}(p))$ because $m \in I_H(q, p)$ and $q \in W_{ax}$. Then $\mu_H(m) \in I_H(q, \mu_H(p))$ since $q \in W_{xb}$. Analogously $\mu_H(m) \in I_H(r, \mu_H(p))$. It follows that $\mu_H(m) = m_H(\mu_H(p), q, r)$. □

We will distinguish three cases for the proof of Theorem 4.6.

Case 1. *There exists a vertex* $x \in N_H(V(G))$ *which has two neighbors in* G.

Let a and b be the neighbors of x in G, and let y be the common neighbor of a and b in G. In the following we will denote the sets A^H_{aybx}, B^H_{aybx}, X^H_{aybx} and Y^H_{aybx} introduced in the statement of Lemma 5.2 by A^H, B^H, X^H and Y^H, respectively.

We introduce the following notations:

$$\begin{aligned} &Y := U^G_{ya} \cap U^G_{yb}, \\ &A := A^H \cap V(G), \\ &B := B^H \cap V(G), \\ &X := \{m_H(x, \alpha, \beta) : (\alpha, \beta) \in A \times B\}, \\ &G_x := H[V(G) \cup X]. \end{aligned}$$

Lemma 5.3. *G_x is a minimal netlike extension of G in H, and G is a hom-retract of G_x.*

Proof. *Claim* 1. $Y = Y^H \cap \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb})$.

By Lemma 5.2 and Proposition 4.4 since G is a netlike subgraph of H,

$$\begin{aligned} Y &= U^G_{ya} \cap U^G_{yb} = U^H_{ya} \cap \mathcal{I}_G(U^G_{ya}) \cap U^H_{yb} \cap \mathcal{I}_G(U^G_{yb}) \\ &= Y^H \cap \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb}). \end{aligned}$$

Claim 2. *$G[Y]$, $G[A]$ and $G[B]$ are isometric in H.*

By Claim 1, $Y = Y^H \cap \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb})$. The set Y^H is convex by Lemma 5.2(iii). Furthermore $\mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb})$ is convex in G by Corollary 2.7 since G is netlike by Proposition 4.4. Therefore, Y is convex because G is isometric in H and $G[Y]$ is isometric in H.

$G[A]$ and $G[B]$ are isometric in H since A^H and B^H are convex by Lemma 5.2(iii), and because G is isometric in H.

Claim 3. *$m_H(y, \alpha, \beta) \in Y$ for all $\alpha \in W^G_{ay} \cap W^G_{yb}$ and $\beta \in W^G_{ya} \cap W^G_{by}$ such that (y, α, β) has a median in H.*

By Lemma 5.2(i) and (vii), $m := m_H(y, \alpha, \beta) \in Y^H$. Moreover $m = m_G(y, \alpha, \beta) \in V(G)$ since G is stable under m_H. Because G is isometric in H, it follows that $m \in W^G_{ya} \cap W^G_{yb}$. Furthermore $m \in I_G(\alpha, y) \cap I_G(\beta, y)$. Then there are two vertices $\alpha' \in U^G_{ya} \cap I_G(\alpha, y)$ and $\beta' \in U^G_{yb} \cap I_G(\beta, y)$ such that $m \in I_G(\alpha', y) \cap I_G(\beta', y)$. Thus $m \in \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb})$, and therefore, $m \in Y^H \cap \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb}) = Y$ by Claim 1.

Claim 4. *$\mu := \mu_H|_X$ is a bijection of X onto Y.*

Let $m \in X$. Then $m = m_H(x, \alpha, \beta)$ for some $(\alpha, \beta) \in A \times B$. Hence, by Lemma 5.2(iv), $\mu(m) = m_H(y, \alpha, \beta) = \phi_{xa}(\phi_{xb}(m))$; and moreover $m_H(y, \alpha, \beta) \in Y$ by Claim 3 since $A \subseteq W^G_{ay} \cap W^G_{yb}$ and $B \subseteq W^G_{ya} \cap W^G_{by}$.

Furthermore, by Lemma 5.2(i), $u = \mu(m_H(x, \phi_{ya}(u), \phi_{yb}(u)))$ for every $u \in Y$. Therefore, μ is a bijection of X onto Y.

Claim 5. *$\phi_{xa}(X) \subseteq A$, $\phi_{xb}(X) \subseteq B$ and $X \subseteq X^H \cap N_H(A) \cap N_H(B)$.*

Let $m \in X$. Then $m \in X^H$ by Lemma 5.2(vii). Then, by Claim 4, $\phi_{xa}(m) = \phi_{ya}(\mu(m)) \in U^G_{ay} \cap N_G(Y) \subseteq A^H \cap V(G) = A$. Analogously $\phi_{xb}(m) \in B$. Therefore, $m \in X^H \cap N_H(A) \cap N_H(B)$.

It follows from Claim 5 and Lemma 5.1, that any $m \in X$ has exactly two neighbors in G: $\phi_{xa}(m)$ and $\phi_{xb}(m)$.

Claim 6. *$G_x[X]$ is isometric in H, and μ is an isomorphism of $G_x[X]$ onto $G_x[Y]$.*

Let $m, m' \in X$. Then

$$\begin{aligned} d_H(m, m') &= d_H(\phi_{xa}(m), \phi_{xa}(m')) \\ &= d_G(\phi_{xa}(m), \phi_{xa}(m')) \quad \text{by Claims 2 and 5} \\ &= d_{G_x}(m, m') \quad \text{by Claim 5.} \end{aligned}$$

The rest of the claim is a consequence of Lemma 5.2(i).

Claim 7. *$W^G_{ay} \cap W^G_{by} = \emptyset$.*

Suppose that this is not true, and let $p \in W^G_{ay} \cap W^G_{by}$. Because $p \in W^G_{ay}$, there is a (p, y)-geodesic P that passes through a. Since $p \in W^G_{by}$, there is exactly one edge cd of P which is in relation Θ with yb. Then $a \in I_G(c, y)$, and thus $x \in I_G(d, b)$, contrary to the hypothesis that $x \notin V(G)$.

Claim 8. *The subgraph G_x is isometric in H.*

By Claim 6, we only have to prove that $d_{G_x}(m, p) = d_H(m, p)$ for all $m \in X$ and $p \in V(G)$. If $p \in W^G_{ya} \cap W^G_{yb}$, then

$$\begin{aligned} d_H(m, p) &= d_H(\mu(m), p) + 2 \\ &= d_G(\mu(m), p) + 2 \quad \text{by Claim 4 and since } G \text{ is isometric in } H \\ &= d_{G_x}(\mu(m), p) + 2 \\ &= d_{G_x}(m, p). \end{aligned}$$

If $p \notin W^G_{ya} \cap W^G_{yb}$, then, by Claim 7, $p \in W^G_{ay} \Delta W^G_{by}$, where Δ denotes the symmetric difference of sets. Suppose that $p \in W^G_{ay}$. Then $p \in W^G_{yb}$. Hence

$$\begin{aligned} d_H(m, p) &= d_H(\phi_{by}(m), p) + 1 \\ &= d_G(\phi_{by}(m), p) + 1 \quad \text{because } \phi_{by}(m) \in A \text{ by Claim 5} \\ &= d_{G_x}(\phi_{by}(m), p) + 1 \\ &= d_{G_x}(m, p). \end{aligned}$$

Claim 9. *The subgraph G_x is stable under m_H.*

Let (p, q, r) be a triple of vertices of G_x which has a median m in H. Because G_x is isometric in H by Claim 8, it is sufficient to prove that $m \in V(G_x)$ to show that $m = m_{G_x}(p, q, r)$. We distinguish five cases.

(a) If $p, q, r \in V(G)$, then $m = m_G(p, q, r)$ because G is stable under m_H.

(b) If $p, q, r \in X$, then $m \in X^H$ since X^H is convex. Hence $\mu_H(m) \in Y^H$ by Lemma 5.2(i). Moreover, because $\mu(p), \mu(q), \mu(r) \in Y$ by Claim 4, it follows that

$$\begin{aligned} \mu_H(m) &= m_H(\mu(p), \mu(q), \mu(r)) \\ &= m_G(\mu(p), \mu(q), \mu(r)) \quad \text{since } G \text{ is stable under } m_H \\ &\in \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb}). \end{aligned}$$

Therefore, $\mu_H(m) \in Y^H \cap \mathcal{I}_G(U^G_{ya}) \cap \mathcal{I}_G(U^G_{yb}) = Y$ by Claim 1. Hence $m \in X$.

(c) $p, q \in X$ and $r \in V(G)$.

Then $m \in I_H(p, q) \subseteq X^H$ since X^H is convex. On the other hand, by Claim 7, $r \in W^G_{ax} \cup W^G_{bx}$. Suppose that $r \in W^G_{ax}$. Then

$$\begin{aligned}\phi_{xa}(m) &= m_H(\phi_{xa}(p), \phi_{xa}(q), r)\\ &= m_G(\phi_{xa}(p), \phi_{xa}(q), r) \quad \text{because } G \text{ is stable under } m_H.\end{aligned}$$

Hence $\phi_{xa}(m) \in A$, and thus $m \in X$.

(d) $p \in X$ and $q, r \in W^G_{ya}$ or $q, r \in W^G_{yb}$.

Suppose that $q, r \in W^G_{yb}$. Then $m \in W^H_{ax}$. It follows that $\phi_{xa}(p) \in I_H(p, m)$. Hence $m = m_H(\phi_{xa}(p), q, r) = m_G(\phi_{xa}(p), q, r)$ because G is stable under m_H, and thus $m \in V(G)$.

(e) $p \in X$, $q \in W^G_{ay} \cap W^G_{yb}$ and $r \in W^G_{ya} \cap W^G_{by}$.

By Lemma 5.2(vii), the fact that G is stable under m_H and Claim 3, $\mu_H(m) = m_H(\mu(p), q, r) = m_G(\mu(p), q, r) \in Y$. Therefore, $m \in X$ by Claim 4.

Claim 10. *G_x is a minimal netlike extension of G in H.*

By Claims 8 and 9, G_x is a netlike extension of G in H. Now, let $x' \in X$ and let Y', A', B' and X' be the subsets of $V(H)$ which are associated to x' in the same way that Y, A, B and X were associated to x. Then

$$Y' = U^G_{\mu(x')\phi_{xa}(x')} \cap U^G_{\mu(x')\phi_{xb}(x')} = U^G_{ya} \cap U^G_{yb} = Y.$$

It follows that $X' = \mu_H^{-1}(Y) = X$. Therefore, $G_{x'} = G_x$. Consequently G_x is minimal.

Claim 11. *G is a hom-retract of G_x.*

Let $f : V(G_x) \to V(G)$ be such that $f(u) = u$ if $u \in V(G)$, and $f(u) = \mu(u)$ if $u \in X$. We have to prove that f is a homomorphism. Let u, u' be two adjacent vertices of G_x. We have to show that $f(u)$ and $f(u')$ are adjacent. We are done if $u, u' \in X$ by Claim 4 and Lemma 5.2(i), and if $u, u' \in V(G)$.

Suppose that $u \in X$ and $u' \in V(G)$. Then $u' = \phi_{xa}(u)$ or $u' = \phi_{xb}(u)$, say $u' = \phi_{xb}(u) = f(u')$. Then $f(u) = \phi_{xa}(\phi_{xb}(u)) = \phi_{xa}(u') = \phi_{xa}(f(u'))$, and thus $f(u)$ and $f(u')$ are adjacent. Therefore, f is a hom-retraction of G_x onto G.

This completes the proof of Lemma 5.3. □

Case 2. *Every vertex in $N_H(V(G))$ has exactly one neighbor in G, and there is an edge bc of G which is in relation Θ with some edge in $\partial_H(V(G))$.*

Choose a vertex x in $N_H(V(G))$ and the edge bc of G in such a way that, if a is the neighbor of x in $V(G)$, then xa and bc are in relation Θ and $I_H(x, b) \cap U^H_{xa} = \{x, b\}$. Let

$$G_x := H[V(G) \cup \phi_{ax}(\mathcal{I}_G(U^G_{cb} \cup \{a\}))].$$

Lemma 5.4. *G_x is a minimal netlike extension of G in H, and G is a hom-retract of G_x.*

Proof. Because $I_G(x, b) \cap U^H_{xa} = \{x, b\}$, it follows, because H is netlike, that no vertex in $I_H(x, b)$ belongs to a cycle of $H[\mathcal{I}_H(U^H_{xa})]$, and no vertex in $I_H(a, c)$ belongs to a cycle of $H[\mathcal{I}_H(U^H_{ax})]$. Then there exists exactly one (x, b)-geodesic in H, say P, and exactly one (a, c)-geodesic in H, and thus in G, say Q.

Suppose that there is an edge $b'c'$ of G distinct from bc such that $b'c'$ and bc are in relation Θ and $b \notin I_H(x, b')$. Then, since G is isometric in H, there would exist a (b, b')-geodesic in G, and thus in H. Hence each vertex of $I_H(x, b)$ would belong to a cycle of $H[\mathcal{I}_H(U^H_{xa})]$, contrary to the above. Therefore

$$G_x = G \cup P \cup \langle x, a \rangle.$$

Claim 1. *G_x is isometric in H.*

Because $V(P) - \{x, b\} \subseteq \mathcal{I}_H(U^H_{xa}) - U^H_{xa}$, it suffices to show that $d_{G_x}(x, y) = d_H(x, y)$ for every $y \in V(G)$. Suppose that this is not true. Then there is $y \in V(G) \cap W^H_{xa} = W^G_{xa}$ such that $b \notin I_H(x, y)$. Hence $a \in I_H(x, y)$. Since $a \in W^H_{ax}$, for any (a, y)-geodesic R in G there exists an edge $b'c'$ in R which is in relation Θ with bc. Then $b \in I_H(x, b')$ by the above. Hence $b \in I_H(a, b') \subseteq I_H(a, y)$ since R is also an (a, y)-geodesic in H, because G is isometric in H. Therefore, $b \in I_H(x, y)$, contrary to the fact that $b \notin I_H(x, y)$ by the definition of y.

Claim 2. *G_x is stable under m_H.*

Let $u_0, u_1, u_2 \in V(G_x)$ such that the triple (u_0, u_1, u_2) has a median m in H. As in Case 1 we have to prove that $m \in V(G_x)$. We distinguish three cases.

(a) If $u_i \in V(G)$ for $i = 0, 1, 2$, then $m_H(u_0, u_1, u_2) = m_G(u_0, u_1, u_2)$ because G is stable under m_H, and thus $m \in V(G)$.

(b) If $u_i, u_j \notin V(G)$ for some $i \neq j$, then $m \in V(P[u_i, u_j])$, and thus $m \in V(G_x)$.

(c) Suppose that there is exactly one $i \in \{0, 1, 2\}$ such that $u_i \notin V(G)$, say $i = 0$. Then, for every $y \in V(G)$, a or b belongs to $I_G(y, u_0)$. Hence, clearly,

$$m = m_H(\phi_{xa}(u_0), u_1, u_2) = m_G(\phi_{xa}(u_0), u_1, u_2)$$

because G is stable under m_H, and thus $m \in V(G)$.

Claim 3. *G_x is a minimal netlike extension of G in H.*

By Claims 1 and 2, G_x is a netlike extension of G in H. Let $P = \langle x_0, \dots, x_n \rangle$ with $x_0 = x$ and $x_n = b$, and let $Q = \langle a_0, \dots, a_n \rangle$ with $a_0 = a$ and $a_n = c$. Then x_{n-1} is the only other vertex of G_x which belongs to $N_H(V(G))$. The edges $x_{n-1}x_n$ and a_0a_1 are in relation Θ. Hence clearly $G_{x_{n-1}} = G_x$. Therefore, G_x is minimal.

Claim 4. *G is a hom-retract of G_x.*

The map $f : V(G_x) \to V(G)$ such that $f(u) = u$ if $u \in V(G)$ and $f(x_i) = a_{i+1}$ for $i < n-1$, is clearly a hom-retraction of G_x onto G. □

Case 3. *Every vertex in $N_H(V(G))$ has exactly one neighbor in G, and no edge of G is in relation Θ with some edge in $\partial_H(V(G))$.*

Let x be any vertex in $N_H(V(G))$, and let

$$G_x := H[V(G) \cup \{x\}].$$

Lemma 5.5. *G_x is a minimal netlike extension of G in H, and G is a hom-retract of G_x whenever G is non-trivial.*

Proof. Let a be the neighbor of x in G. Suppose that G_x is not isometric in H. Then there is a vertex y of G such that $a \notin I_H(x, y)$ and $I_H(x, y) \cap V(G) = \{y\}$. Let $z \in N_G(y) \cap I_G(y, a)$. Because G is isometric in H and $a \notin I_H(x, y)$, it follows that $d_G(a, y) = d_H(a, y) = d_H(x, y) + 1$. Hence $d_H(a, z) = d_H(x, y)$, and moreover $d_H(a, y) = d_H(x, z)$ since $I_H(x, y) \cap V(G) = \{y\}$. It follows that the edges yz and xa are in relation Θ in H, contrary to the assumption. Therefore, G_x is isometric in H.

Suppose that, for some $p, q \in V(G)$, the triple (x, p, q) has a median m in H. Then

$$\begin{aligned} m &= m_H(a, p, q) \quad \text{because } G \text{ is isometric in } H \\ &= m_G(a, p, q) \quad \text{since } G \text{ is stable under } m_H. \end{aligned}$$

Hence $m \in V(G)$, and thus $m = m_{G_x}(x, p, q)$.

Then G_x is a minimal netlike extension of G in H. If G is non-trivial, then the map $f : V(G_x) \to V(G)$ such that $f(u) = u$ if $u \in V(G)$ and such that $f(x)$ is any neighbor of a in G, is a hom-retraction of G_x onto G. □

This completes the proof of Theorem 4.6.

6. Minimal convex extensions and mooring

Let G be a convex subgraph of a netlike partial cube H. Clearly a minimal netlike extension of G is not necessarily convex. However, as we will see, there is always an extension of G which is convex and which is minimal with respect to the convex subgraph relation. This is a consequence of the following result where, if C is a convex set, then by a *minimal convex extension* of C we mean a convex set C' which properly contains C and which is minimal with respect to inclusion.

Proposition 6.1. *Let G be a netlike partial cube, and C a non-empty convex set of G. We have the following properties:*

(i) *A set C' is a minimal convex extension of C if and only if $C' = \mathcal{I}_G(\{u\} \cup C)$ for some vertex $u \in N_G(C)$.*

(ii) *If C' is a minimal convex extension of C, then $C' \cap N_G(C)$ is ph-stable and (3)-convex (and thus $\mathcal{C}$-convex by Lemma 2.8).*

(iii) *If C' is a minimal convex extension of C, then, for any edge uv of G with $v \in C$ and $u \in C' - C$, $C = W_{vu}^{G[C']}$, $C' - C = W_{uv}^{G[C']}$ and $U_{uv}^{G[C']}$ is $\mathcal{C}$-convex in $G[C']$.*

Proof. Let $u \in N_G(C)$ and let v be the neighbor of u in C. This neighbor is unique because C is convex and G is bipartite.

Claim 1. $\mathcal{I}_G(U_{uv} \cap N_G(C)) \cap U_{uv} \subseteq N_G(C)$.

Let $x \in \mathcal{I}_G(U_{uv} \cap N_G(C)) \cap U_{uv}$. Then $x \in I_G(a, b)$ for some vertices $a, b \in U_{uv} \cap N_G(C)$. Let a', b' and x' be the neighbors in U_{vu} of a, b and x, respectively. Then $a', b' \in C$ and $x' \in I_g(a', b')$ since G is a partial cube. Therefore, $x' \in C$ by convexity. Hence $x \in N_G(C)$.

Claim 2. *The set $U_{uv} \cap N_G(C)$ is(3)-convex and $\mathcal{C}$-convex.*

$$\begin{aligned} V_{(3)}(G[\mathcal{I}_G(U_{uv} \cap N_G(C))]) &\subseteq \mathcal{I}_G(U_{uv} \cap N_G(C)) \cap V_{(3)}(G[\mathcal{I}_G(U_{uv})]) \\ &\subseteq \mathcal{I}_G(U_{uv} \cap N_G(C)) \cap U_{uv} \quad \text{since } U_{uv} \text{ is (3)-convex} \\ &\subseteq U_{uv} \cap N_G(C) \quad \text{by Claim 1.} \end{aligned}$$

Hence $U_{uv} \cap N_G(C)$ is (3)-convex, and analogously it is $\mathcal{C}$-convex.

Claim 3. *The set $U_{uv} \cap N_G(C)$ is ph-stable.*

By Claim 2, $U_{uv} \cap N_G(C)$ is $\mathcal{C}$-convex. Hence it is ph-stable by Lemma 2.6.

Claim 4. $\mathcal{I}_G(\{u\} \cup C) = \mathcal{I}_G(U_{uv} \cap N_G(C)) \cup C$.

Clearly $\mathcal{I}_G(U_{uv} \cap N_G(C)) \cup C \subseteq \mathcal{I}_G(\{u\} \cup C)$. Conversely let $x \in C$, and let $P = \langle u_0, \dots, u_n \rangle$ be a (u, x)-geodesic with $u_0 = u$ and $u_n = x$. Without loss of generality we can suppose that $u_{n-1} \notin C$. Then $x \in W_{vu}$. It follows that $v \in I_G(u, x)$. Hence $d_G(v, x) = d_g(u, u_{n-1})$ and $d_G(u, x) = d_G(v, u_{n-1})$. Therefore, the edges $u_{n-1}x$ and uv are in relation Θ, and thus $u_{n-1} \in U_{uv} \cap N_G(C)$. Consequently $V(P) \in \mathcal{I}_G(U_{uv} \cap N_G(C)) \cup C$. More generally $\mathcal{I}_G(\{u\} \cup C) \subseteq \mathcal{I}_G(U_{uv} \cap N_G(C)) \cup C$.

Claim 5. *$\mathcal{I}_G(\{u\} \cup C)$ is convex and is equal to $\mathcal{I}_G(\{a\} \cup C)$ for each vertex $a \in \mathcal{I}_G(\{u\} \cup C) - C$.*

By Claim 4, $\mathcal{I}_G(\{u\} \cup C) - C = \mathcal{I}_G(U_{uv} \cap N_G(C))$. By Claim 3, the set $U_{uv} \cap N_G(C)$ is ph-stable, and thus $\mathcal{I}_G(U_{uv} \cap N_G(C))$ is convex by Lemma 2.4. Therefore $\mathcal{I}_G(\{u\} \cup C)$ is convex. Now let $a \in \mathcal{I}_G(\{u\} \cup C) - C = \mathcal{I}_G(U_{uv} \cap N_G(C))$. Clearly $\mathcal{I}_G(\{a\} \cup C) \subseteq \mathcal{I}_G(\{u\} \cup C)$. On the other hand, since $U_{uv} \cap N_G(C)$ is ph-stable, it follows that $\mathcal{I}_G(U_{uv} \cap N_G(C)) \cup C \subseteq \mathcal{I}_G(\{a\} \cup C)$, therefore, $\mathcal{I}_G(\{a\} \cup C) = \mathcal{I}_G(\{u\} \cup C)$.

From Claim 5, it follows immediately that $\mathcal{I}_G(\{u\} \cup C)$ is a minimal convex extension of C. Conversely, let C' be a minimal convex extension of C. Let $u \in N_G(C) \cap C'$. By the above $\mathcal{I}_G(\{u\} \cup C)$ is a minimal convex extension of C, and moreover which is contained in C' since $u \in C'$. Hence $C' = \mathcal{I}_G(\{u\} \cup C)$. This proves the assertion (i). The assertion (ii) is a consequence of (i) and of Claims 2 and 3. Because $G[C']$ is a netlike partial cube, as a convex subgraph, the assertion (iii) is then a consequence of (i) and (ii). □

An induced subgraph H (or its vertex set) of a graph G is said to be *gated* if, for each $x \in V(G)$, there exists a vertex y (the *gate* of x) in H such that $y \in I_G(x, z)$ for every $z \in V(H)$.

The following concept, essentially due to Tardif [11], was initially defined for median graphs. Let H be a gated subgraph of a netlike partial cube G. We denote by $g_H(x)$ the gate of x in H. A self-contraction φ of G is a *mooring* of G onto H if $\varphi(u) = u$ for all $u \in V(H)$ and $u\varphi(u)$ is an edge of $G[I_G(u, g_H(u))]$ for all $u \notin V(H)$. We recall that any convex cycle of a netlike partial cube is gated [7, Corollary 6.4]. The following result will be useful in the next section.

Proposition 6.2. *If a netlike partial cube G contains a unique convex cycle C of length greater than 4, then there is a mooring of G onto C.*

Proof. Let H be a convex subgraph of G containing the convex cycle C, and let H' be a minimal convex extension of H. By Proposition 6.1(iii), if uv is an edge of G with $v \in V(H)$ and $u \in V(H' - H)$, then the set $U_{uv}^{H'}$ is $\mathcal{C}$-convex in H', and thus convex because the only convex cycle of G of length greater than 4 is contained in H. It follows that each vertex in $V(H') \cap N_G(V(H))$ has only one neighbor in H.

For each ordinal α, we construct the subgraph G_α as follows:

- $G_0 := C$;
- $G_{\alpha+1}$ is a minimal convex extension of G_α;
- if α is a limit ordinal, then $G_\alpha := \bigcup_{\beta<\alpha} G_\beta$.

Note that G_α is also a convex subgraph of G if α is a limit ordinal because the set $\{G_\beta : \beta < \alpha\}$ is totally ordered by inclusion. For each $x \in V(G)$ we denote by $\alpha(x)$ the smallest ordinal α such that $x \in V(G_\alpha)$.

Define the self-map φ of $V(G)$ such that $\varphi(x)$ is x if $\alpha(x) = 0$ and is the only neighbor of x in $G_{\alpha(x)-1}$ if $\alpha(x) > 0$. It suffices to prove that φ is a contraction to show that it is a mooring of G onto C. Let x and y be two adjacent vertices of G with $\alpha(x) \leq \alpha(y)$, we have to show that $\varphi(x)$ and $\varphi(y)$ are equal or adjacent. We are done if $\alpha(x) = \alpha(y) = 0$. If $\alpha(x) = \alpha(y) \neq 0$, then $\varphi(x)$ and $\varphi(y)$ are adjacent because $\varphi(y) \in U_{\varphi(x)x}^{G_{\alpha(x)}}$. If $\alpha(x) < \alpha(y)$, then $x = \varphi(y)$ by the definition of φ, and thus $\varphi(x)$ and $\varphi(y)$ are equal or adjacent according to whether $\alpha(x)$ is or is not equal to 0. □

7. Prism-retractable netlike partial cubes

Proposition 4.1 is very important in the study of median graphs because it is the cornerstone of the proof of Bandelt [2, Theorem 2] that median graphs are the hom-retracts of hypercubes. Independently of the concepts of subgraphs, the Hom-Retract Property is far from being a general property of partial cubes, and not even of the elements of $\mathbb{PC}_1$. Actually this property is not even satisfied by most of the cartesian products of netlike partial cubes. Take for example the cartesian products H of K_2 with the benzenoid graph G which is the union of two distinct 6-cycles having an edge in common. Then each image of G in H is a retract of H, but it clearly cannot be a hom-retract of H because of the existence of two distinct convex cycles of length greater than 4 in G. On the contrary the cartesian product of any even cycle by K_2 clearly has the Hom-Retract Property. More generally we will see, by studying a related problem, that the cartesian product of a netlike partial cube G by K_2 has the Hom-Retract Property if and only if G contains at most one convex cycle of length greater than 4.

The following definitions and results, which are essentially due to Sabidussi [10], were introduced in order to prove and generalize a property of median graph (Proposition 7.3). In this section we will suppose that the vertex set of K_2 is $\{0, 1\}$, i.e. $K_2 = \langle 0, 1\rangle$. For a graph G, the cartesian product $G\Box K_2$ is called the *prism over* G. For $i = 0, 1$, we denote by $G\Box\langle i\rangle$ the G-fiber of $G\Box K_2$ induced by $V(G) \times \{i\}$.

Definition 7.1. A graph G is called *prism-retractable* if $G\Box K_2$ can be hom-retracted onto any one of its G-fibers. In other words if $G\Box\langle 0\rangle$ (and $G\Box\langle 1\rangle$) is a hom-retract of $G\Box K_2$.

Proposition 7.2. *A non-trivial graph G is prism-retractable if and only if it is a hom-retract of a bipartite graph H whenever it is a retract of H.*

Proposition 7.3. *Any non-trivial median graph is prism-retractable.*

We will give a simple proof of this result distinct from the one given by Sabidussi.

Proof. Let G be a non-trivial median graph. Then $G\Box K_2$ is also a median graph, and moreover the G-fiber $G\Box\langle 0\rangle$ is a convex subgraph of $G\Box K_2$, and thus a median subgraph of $G\Box K_2$. Hence, by Proposition 4.1, $G\Box\langle 0\rangle$ is a hom-retract of $G\Box K_2$. □

Median graphs are not the only prism-retractable graphs. For example, cycles, complete graphs, unicyclic graphs, cartesian products of any graphs by K_2, are prism-retractable. Moreover, the class of all prism-retractable graphs is closed under hom-retracts and cartesian products.

We will now state the main result of this section which extends the last proposition to netlike partial cubes.

Theorem 7.4. *A non-trivial netlike partial cube is prism-retractable if and only if it contains at most one convex cycle of length greater that* 4*.*

We need a lemma, in which we use the following notation. For a graph G, a vertex x of G and a cycle C of G, and for and $i = 0, 1$, we denote by x^i the vertex (x, i) of $G\Box K_2$, and by C_i the subgraph $C\Box\langle i\rangle$ of $G\Box\langle i\rangle$.

Lemma 7.5. *Let C be a convex cycle of length greater than* 4 *of a prism-retractable netlike partial cube G, and let f be a hom-retraction of $G\Box K_2$ onto G_0. Then $f(C_1) = C_0$.*

Proof. Let $C = \langle x_1, \ldots, x_{2n}, x_1\rangle$ with $n > 2$.

(a) We will first show that $f(x_i^1) \neq f(x_j^1)$ if $i \neq j$. This is clear if $|i - j| = 1$ because f preserves the edges. Assume that $|i - j| \geq 2$, and without loss of generality that $i < j$.

Suppose that $f(x_i^1) = f(x_j^1)$. Then $\langle x_i^0, f(x_i^1), x_j^0\rangle$ is a geodesic. Hence $j = i + 2$ (the subscripts being modulo $2n$) and $f(x_i^1) = f(x_j^1) = x_{i+1}^0$ because C_0 is convex. It follows that $f(x_{i-1}^1)$, which is adjacent to x_{i-1}^0 and to x_{i+1}^0, is equal to x_i^0 since C_0 is convex. Analogously, we have that $f(x_{i+3}^1) = x_{i+2}^0$. Therefore, we successively obtain $f(x_{i-r-1}^1) = x_{i-r}^0$ and $f(x_{i+r+1}^1) = x_{i+r}^0$, $r = 0, \ldots, n - 1$. Hence, in particular for $r = n - 1$, we have that

$$x_{i-n+1}^0 = f(x_{i-n}^1) = f(x_{i+n}^1) = x_{i+n-1}^0$$

since $x_{i-n}^1 = x_{i+n}^1$, which is impossible because $x_{i-n+1}^0 \neq x_{i+n-1}^0$.

Consequently $f(C_1)$ is a cycle of G_0 of length $2n$.

(b) Suppose that $f(C_1)$ and C_0 are disjoint. Then the edges $x_1^0x_2^0$, $x_{2+n}^0x_{1+n}^0$, $f(x_1^1)f(x_2^1)$ and $f(x_{2+n}^1)f(x_{1+n}^1)$ are in relation Θ in G_0. It follows that $x_3^0, f(x_3^1) \in \mathcal{I}_{G_0}(U^{C_0}_{x_1^0x_2^0})$, therefore, $\langle x_2^0, x_3^0, f(x_3^1), f(x_2^1), x_2^0\rangle$ is a cycle of $G_0[\mathcal{I}_{G_0}(U^{C_0}_{x_1^0x_2^0})]$, with $x_3^0 \notin U^{C_0}_{x_1^0x_2^0}$ because C_0 is convex. This proves that $U^{C_0}_{x_1^0x_2^0}$ is not $\mathcal{C}$-convex, contrary to the fact that G_0 is netlike.

(c) Then there is an i such that $f(x_i^1) \in V(C_0)$. Then $f(x_i^1)$ is either x_{i-1}^0 or x_{i+1}^0. Suppose without loss of generality that $i = 1$ and $f(x_1^1) = x_{2n}^0$. Then $f(x_2^1)$ is adjacent to both x_{2n}^0 and x_2^0. Hence $f(x_2^1) = x_1^0$ because C_0 is convex. Then, we can successively prove that $f(x_{i+1}^1) = x_i^0$ for $i = 1, \ldots, 2n$. Consequently $f(C_1) = C_0$. □

Proof of Theorem 7.4. Let G be a non-trivial prism-retractable netlike partial cube. Suppose that G contains two distinct convex cycles $C = \langle x_1, \ldots, x_{2n}, x_1\rangle$ and $D = \langle y_1, \ldots, y_{2p}, y_1\rangle$ with n and p greater that 2. Let f be a hom-retraction of $H := G\Box K_2$ onto G_0. Then, by Lemma 7.5, $f(C_1) = C_0$ and $f(D_1) = D_0$. Without loss of generality we can suppose that $f(x_i^1) = x_{i+1}^0$ and $f(y_j^1) = y_{j+1}^0$, and that x_0 and y_0 are such that

$$r := d_G(x_0, y_0) = \min\{d_G(x_i, y_j) : i = 1, \ldots, 2n \text{ and } j = 1, \ldots, 2p\}.$$

We prove by induction that $d_G(x_i, y_i) = r$ for $i = 1, \ldots, 2n$, and thus that $n = p$. This is true by definition if $i = 0$. Suppose that this holds for some positive integer $i < 2n$. Then $d_H(x_i^1, y_i^1) = r$. Therefore

$$\begin{aligned} r \geq d_H(f(x_i^1), f(y_i^1)) &= d_H(x_{i+1}^0, y_{i+1}^0) \quad \text{by the choice of } f \\ &= d_G(x_{i+1}, y_{i+1}) \geq r. \end{aligned}$$

Hence $d_G(x_{i+1}, y_{i+1}) = r$.

Then, with a proof similar to that of part (b) in the proof of Lemma 7.5, we can show that the set $U_{x_1x_2}$ is not $\mathcal{C}$-convex, contrary to the fact that G is netlike.

Conversely let G be a netlike partial cube containing at most one convex cycle of length greater that 4. If G is a median graph, then it is prism-retractable by Proposition 7.3.

Suppose that it is not median, and let $C = \langle x_1, \ldots, x_{2n}, x_1\rangle$ be its unique convex cycle of length $2n > 4$. By Proposition 6.2, there is a mooring φ of G onto C. Let $f : V(G\Box K_2) \to V(G_0)$ be such that $f(x_i^1) = x_{i+1}^0$ for $i = 1, \ldots, 2n$, $f(x^1) = \varphi(x)^0$ for any $x \in V(G - C)$, and $f(x^0) = x^0$ for any $x \in V(G)$. Then, by the definition of a mooring, it follows that f is a homomorphism and thus a hom-retraction of $G\Box K_2$ onto G_0. □

By Proposition 7.2 and Theorem 7.4 we immediately obtain:

Corollary 7.6. *A non-trivial netlike partial cube G is a hom-retract of a bipartite graph H whenever it is a retract of H if and only if it contains at most one convex cycle of length greater than* 4*.*

Note that a netlike partial cube having a unique convex cycle of length greater than 4 is very close to a median graph in the sense that it suffices to "fill" the "hole" due to this cycle by an adequate hypercube to get a median graph. More precisely:

Proposition 7.7. *Let G be netlike partial cube containing a unique convex cycle C of length $2n > 4$. Let H be an n-cube such that C is a maximal isometric cycle of H and such that $V(H) \cap V(G) = V(C)$. Then $G^+ := G \cup H$ is a median graph.*

Proof. Let $C = \langle x_1, \ldots, x_{2n}, x_1\rangle$. Clearly $\text{co}_{G^+}(C) = H$. We will show that any isometric cycle of G^+ is a cycle of H or of G.

Suppose that there exists an isometric cycle Γ of G^+ that contains a vertex of $H - G$ and a vertex of $G - C$. Then $\Gamma \cap C$ has exactly two vertices x_i and x_{i+r} with $r \leq 2$. Without loss of generality we suppose that $i = 1$ and that $\Gamma = \langle y_1, \ldots, y_{2p}, y_1\rangle$ with $y_1 = x_1$ and $y_j = x_r$ for some j. Because Γ is isometric and C is convex, we have $j = r$ and $p > r$.

Suppose that the cycle $\Gamma' = \langle x_1, \ldots, x_r, y_{r+1}, \ldots, y_{2p}, x_1\rangle$ is isometric in G. Then it cannot be convex since C is the unique convex cycle of G of length greater that 4. Therefore, by Proposition 2.12, its convex hull in G is a hypercube. Hence, there is a 4-cycle, of this hypercube which has two edges in common with C, contrary to the fact that C is convex.

Therefore, Γ' is not isometric in G, and thus there is an (x_i, y_j)-geodesic P for some i, j with $1 < i < r$ and $r < j \leq 2p$, and that we can choose the notation so that i is maximum with respect to these properties. Because P is a geodesic, and by the choice of i, we have that $y_j \in W^G_{x_i x_{i+1}}$. Because C is convex, $y_r = x_r \in W^G_{x_{i+1} x_i}$ and because $\langle y_r, \ldots, y_j\rangle$ is a geodesic, it follows that there is an edge $y_{k+1}y_k$ of $\langle y_r, \ldots, y_{j-1}\rangle$ which is in relation Θ with $x_i x_{i+1}$. Then, because $x_1 \in \mathcal{I}_G(U^G_{x_i x_{i+1}})$, and $\langle y_{k+1}, \ldots, y_1\rangle$ is a geodesic since Γ is isometric in G^+, we have that $y_1 \in \mathcal{I}_G(U^G_{x_i x_{i+1}})$. Hence $\langle x_1, \ldots, x_i\rangle \cup P \cup \langle y_j, \ldots, y_{2p}, y_1\rangle$ is a cycle of $G[\mathcal{I}_G(U^G_{x_i x_{i+1}})]$ with $x_1 \notin U^G_{x_i x_{i+1}}$ since $i > 1$, therefore, $U^G_{x_i x_{i+1}}$ is not $\mathcal{C}$-convex, contrary to the fact that G is netlike.

Consequently, every isometric cycle of G^+ is a cycle of H or of G, and hence, by Proposition 2.12, its convex hull is a hypercube. Therefore, by a result of Bandelt [1] (see also [6, Theorem 5]), G^+ is a median graph. $\square$

In general this construction does not work if the graph contains more than one convex cycle of length greater that 4. For example take the benzenoid graph G which is the union of three distinct 6-cycles having pairwise an edge in common. Then the "filling" of one of the "holes" gives a graph that is not netlike, and the "filling" of the three "holes" gives a graph that not only is not median but that is not even an element of $\mathbb{PC}_1$ because it contains Q_3^- as an induced subgraph.

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