DISCRETE MATHEMATICS

Edge colorings of K_{2n} with a prescribed condition — I

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Abstract

A graph L is called a *lantern* if it has two adjacent vertices u, v such that all the other vertices of L are adjacent to both u and v, and L has no other edges. Let L be a lantern of order $2n \geqslant 8$. We prove that any edge-coloring of L using $2n-1$ colors can be extended to a proper edge-coloring of K_{2n} using the same set of colors. This result is used in some of our other papers on edge colorings of K_{2n}. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper all graphs we deal with are finite, simple and undirected. For a graph G, we use $V(G)$, $E(G)$, $\bar{G}$, and $|G|$ to denote its vertex set, edge set, complement, and order, respectively. We write $uv \in E(G)$ if $u, v \in V(G)$ are adjacent in G. If $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of G induced by U. Suppose that $E' \subseteq E(G)$ (resp. $V' \subset V(G)$). Then $G - E'$ (resp. $G - V'$) is the graph obtained from G by deleting E' (resp. V'). Similarly, if $E' \subseteq E(\bar{G})$, then $G + E'$ is the graph obtained from G by adding E'. We use K_n to denote a complete graph of order n, S_k to denote a star of size k, and $S = S_{n_1} \cup \cdots \cup S_{n_k}$ to denote the vertex-disjoint union of k stars $S_{n_1}, \ldots, S_{n_k}$. We use $G(X, Y)$ to denote a complete bipartite graph having bipartition (X, Y). A complete bipartite graph $G(X, Y)$ with $|X| = |Y| = n$ is denoted by $K_{n,n}$.

A tree is called *bad* if it is the vertex-disjoint union of two stars plus an edge joining the center of the first star to an end-vertex of the second star. A tree is *good* if it is not bad.

A *partial* latin square Q of side n is an $n \times n$ matrix defined on n symbols in which some cells may be empty and each of the non-empty cells contains exactly one

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symbol such that no symbol occurs more than once in any row or in any column. It is a (complete) latin square if there are no empty cells. A partial latin square Q is called *symmetric* if whenever a cell (i,j) contains a symbol σ, the cell (j,i) also contains σ.

In this paper, very often (for short) we shall say a coloring of a graph G instead of an edge-coloring of G. All the colorings we deal with are proper.

Some results on colorings of K_{2n} with prescribed conditions have been obtained. Dugdale and Hilton [2] proved the following theorem.

Theorem 1.1. *Let P be a Hamilton path of K_{2n}, where $n \geqslant 4$. Then there exists a $(2n-1)$-edge-coloring of K_{2n} such that all the edges of P receive distinct colors.*

The present authors [5] have proved the following theorem.

Theorem 1.2. *Let $S = S_{n_1} \cup \cdots \cup S_{n_k} \neq S_{2n-3} \cup S_1$ or $2S_2$ (if $n=3$) be a spanning star-forest of K_{2n}. Then there exists a $(2n-1)$-edge-coloring of K_{2n} such that all the edges of S receive distinct colors.*

Recently we have proved that for any good spanning tree T of K_{2n}, where $n \geqslant 4$, there exists a $(2n-1)$-edge-coloring of K_{2n} such that all the edges of T receive distinct colors. This result generalizes both Theorems 1.1 and 1.2. In proving this result, we used the following theorem.

Theorem 1.3. *Let $K_{n,n}$ be a complete bipartite graph having bipartition (X,Y). Let F be a 1-factor of $K_{n,n}$. Suppose $S = S_{m_1} \cup \cdots \cup S_{m_k} \neq S_{n-2} \cup S_1$, where $m_1 + \cdots + m_k = n-1 \geqslant 4$, is a subgraph of $K_{n,n} - F$. If all the centers of the star-components of S are in Y, then any edge-coloring of S using $n-1$ distinct colors can be extended to a proper edge-coloring of $K_{n,n} - F$ using the same set of colors.*

However, to prove Theorem 1.3, we need to apply the main theorem of this paper. In short, the main theorem of this paper is very basic in the study of extensions of partial colorings of K_{2n} to full colorings of K_{2n}.

A graph L is called a *lantern* (because it looks like a Chinese lantern if it is drawn nicely) if it has two adjacent vertices u, v such that all the other vertices of L are adjacent to both u and v, and L has no other edges. The two vertices u and v are called the *apexes* of the lantern L (Fig. 1).

Theorem 1.4 (Main theorem). *Let L be a lantern of order $2n$, where $n \geqslant 4$. Then any edge-coloring of L using $2n-1$ colors can be extended to a proper edge-coloring of K_{2n} using the same set of colors.*

From Theorem 1.4 we deduce the following corollary by deleting a vertex from K_{2n}.

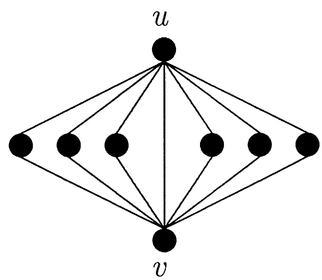

Fig. 1. A lantern of order 8.

Corollary 1.5. *Let L be a lantern of order* $2n-1$, *where* $n \geqslant 4$. *Then any edge-coloring of L using exactly* $2n-1$ *colors can be extended to a proper edge-coloring of* K_{2n-1} *using the same set of colors.*

2. Preliminary results

Let L be a lantern with vertex set $V(L) = \{v_1, v_2, u_1, \ldots, u_{2n-2}\}$, where v_1 and v_2 are the apexes of L. Let ϕ be an edge-coloring of L using $2n-1$ colors $1, 2, \ldots, 2n-1$. By permutation of vertices of L, if necessary, suppose that $\phi(v_1 u_j) = i_j$, $\phi(v_2 u_j) = i_{j+1}$, $j = 1, 2, \ldots, k-1$, $\phi(v_1 u_k) = i_k$ and $\phi(v_2 u_k) = i_1$. Then we say that $(u_1, u_2, \ldots, u_k)$ form a *ϕ-cycle* of length k. A ϕ-cycle is said to be *odd* (*even*) if k is odd (even). We use $o(\phi)$ to denote the number of odd ϕ-cycles in L. Since L is of even order, $o(\phi)$ is always even for any $(2n-1)$-edge-coloring ϕ of L. Suppose L has m ϕ-cycles of length $k_1, k_2, \ldots, k_m$, where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_m$, then we call $(k_1, k_2, \ldots, k_m)$ the *cycle decomposition sequence* of ϕ.

Let ϕ_1 and ϕ_2 be two colorings of a lantern L using the same set of $2n-1$ colors $\{1, 2, , \ldots, 2n-1\}$. We say that ϕ_1 and ϕ_2 are *equivalent* if ϕ_2 can be obtained from ϕ_1 by permutations of colors/permutations of vertices of L. Clearly ϕ_1 and ϕ_2 are equivalent if and only if they have the same cycle decomposition sequence. For example, a lantern with 10 vertices has seven distinct colorings corresponding to the following seven cycle decomposition sequences $(2,2,2,2), (2,2,4), (2,3,3), (2,6), (3,5)$,$(4,4)$ and (8).

In the proof of the main theorem, we shall partition $G = K_{2n}$ into three parts: We first split $V(G)$ into disjoint union of $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. We call $G[X]$ *the upper* K_n and $G[Y]$ *the lower* K_n, which are denoted by G_1 and G_2, respectively, and we denote the complete balanced bipartite graph $G(X, Y)$ by H. For any subgraph T of G, let T_1 and T_2 denote the subgraph of T induced by $V(T) \cap X$ and $V(T) \cap Y$, respectively, and let T_H denote $T - E(T_1 \cup T_2)$.

We shall now describe how to split a lantern L into L_1, L_2 and L_H: We first put $x_1 = v_1$ and $y_1 = v_2$. For any even ϕ-cycle $(u_1, u_2, \ldots, u_k)$, we put $u_i \in X$ if i is odd and $u_i \in Y$ if i is even. Then for each color i_j on $(u_1, u_2, \ldots, u_k)$, either i_j occurs twice in

H, or once in G_1 and once in G_2. Next, since $\mathrm{o}(\phi)$ is always even, we first pair off the $\mathrm{o}(\phi)$ odd ϕ-cycles. For each pair of odd ϕ-cycles $(u_1, u_2, \ldots, u_p)$ and $(w_1, w_2, \ldots, w_q)$, we put

$u_i \in X$ if i is odd and $u_i \in Y$ if i is even;

$w_i \in Y$ if i is odd and $w_i \in X$ if i is even.

Then it is not difficult to check that for the colors on $(u_1, u_2, \ldots, u_p)$ and $(w_1, w_2, \ldots, w_q)$, we have the following two possibilities:

(P_1) Exactly one color occurs in both G_1 and H, and exactly one color occurs in both G_2 and H;

(P_2) each of the other colors occurs either twice in H or once in G_1 and once in G_2.

The above split of L is called the *natural split* of L.

In the proof of our main theorem, we have to recolor some edges of L, if necessary, so that we can obtain a required coloring of K_{2n}. In doing so, we need to use some known results on completing a partial latin square of side n. These results are closely related to extensions of partial colorings of $K_{n,n}$:

Let $K_{n,n}$ be a complete bipartite graph having bipartition (X, Y), where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. There exists a 1-1 correspondence between n-colorings of $K_{n,n}$ using colors $1, 2, \ldots, n$ and latin squares of side n on symbols $1, 2, \ldots, n$: Each n-coloring φ of $K_{n,n}$ gives rise to a latin square $L = (\varphi(y_i x_j))$ of side n; conversely, for any latin square $L = (l_{ij})$ of side n, we can construct an n-coloring φ of $K_{n,n}$ by putting $\varphi(y_i x_j) = l_{ij}$. Thus completing a partial latin square of side n is equivalent to extending a partial coloring of $K_{n,n}$ to an n-coloring of $K_{n,n}$. (By the same way, there also exists a 1-1 correspondence between $(2n-1)$-edge-colorings of K_{2n} using colors $1, 2, \ldots, 2n-1$ and symmetric latin squares of side $2n$ on symbols $1, \ldots, 2n-1, 2n$, whose diagonal cells are all occupied by $2n$.)

We shall now apply the following two lemmas to prove Theorems 2.3 and 2.4 which in turn will be used to prove the main theorem.

Lemma 2.1 (Hall [4]). *A partial latin square such that all the cells in the first r rows are occupied and all the other cells being unoccupied, can be completed to form a latin square.*

(For a proof of Lemma 2.1 see also [1].)

Lemma 2.2 (Häggkist [3]). *Let G be a regular bipartite graph of degree d with $2n$ vertices. Let B_1 be a set of b_1 independent edges of G, and let B_2 be a set of b_2 edges of G disjoint from B_1. If $d - b_1 \geqslant \frac{1}{2}(n-1)$ and $b_2 \leqslant d - b_1 - 1$, then G contains a 1-factor F such that $B_1 \subseteq F$ and $F \cap B_2 = \emptyset$.*

Let P be a partial latin square. The *weight* of a row R_i (resp. column C_j) of P is the number of occupied cells in R_i (resp. C_j) and is denoted by $w(R_i)$ (resp. $w(C_j)$).

We call a row or a column *empty* if it has no preassigned cells, and *full* if it has no empty cells.

Theorem 2.3. *Let P be a partial latin square of side $2n \geqslant 4$ such that the first column and the first row are full. Suppose that $w(R_2) \leqslant n$ and all the other cells from the third row onwards are empty. Then P can be completed to form a latin square of side $2n$.*

Proof. Let P' be a partial latin square such that the first two rows of P' are the same as those of P and the other rows are empty. Since $w(R_2) \leqslant n$, the second row of P' has at least $n \geqslant 2$ empty cells. Let $(2, i_1), (2, i_2), \ldots, (2, i_k)$ be the empty cells of the second row, $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the symbols in the cells $(1, i_1), (1, i_2), \ldots, (1, i_k)$, respectively, and B be the set of symbols not preassigned in the second row of P'. For any symbol α_j, where $j = 1, \ldots, k$, if $\alpha_i \in B$, then we fill α_j in the cell $(2, i_{j+1})$ (we set $i_{k+1} = i_1$). Then we arbitrarily fill the remaining symbols of B in the remaining empty cells of the second row. We have thus completed the first two rows of P'. By Lemma 2.1 we can complete P' to a latin square of side $2n$. By permuting some rows of this latin square, if necessary, we obtain a required latin square of side $2n$ which contains P. $\square$

Theorem 2.4. *Let P be a partial latin square of side $2m + 1 \geqslant 7$ on symbols $\sigma_1, \ldots, \sigma_{2m+1}$ such that*

(1) *the first column and the first row of P are full*;

(2) $w(R_2) \leqslant m$, $w(R_3) = \cdots = w(R_m) \leqslant 2$, *and* $w(R_{m+1}) = \cdots = w(R_{2m+1}) = 1$.

Then P can be completed to form a latin square L of side $2m + 1$.

Proof. Let P' be a partial latin square of side $2m+1$ such that the first m rows of P' are the same as those of P, and the other rows are empty. If we can complete P' to form a latin square L' of side $2m+1$, then by permuting some rows of L', if necessary, we can obtain a required latin square L which contains P.

Let $\Sigma = \{\sigma_1, \ldots, \sigma_{2m+1}\}$. Let C_i be the ith column of P' and $C = \{C_1, \ldots, C_{2m+1}\}$. We identify P' with a subgraph G of the complete bipartite graph $K_{2m+1,2m+1}$ with bipartition (Σ, C) and $E(G) = E_1 \cup E_2 \cup \cdots \cup E_{2m+1}$, where $E_i = \{e \in E(K_{2m+1,2m+1}) \mid e = \{\sigma_j, C_k\}$ and σ_j is preassigned in cell (i, k) of $P'\}$. For a fix i, since all the preoccupied cells of the ith row of P' are filled in distinct symbols, E_i consists of independent edges. We observe that extending E_i to a 1-factor of $K_{2m+1,2m+1}$ corresponds to completing row i of P' to a full row. Since the first row of P' is full, $E_1 = F_1$ is already a 1-factor of $K_{2m+1,2m+1}$. We also have $|E_2| = w(R_2) \leqslant m$, $|E_3| = \cdots = |E_m| \leqslant 2$, and $|E_{m+1}| = \cdots = |E_{2m+1}| = 0$. We shall extend E_2, and then E_3, and so on, to a 1-factor of $K_{2m+1,2m+1}$.

Let $G_0 = K_{2m+1,2m+1}$ and $G_1 = G_0 - F_1$. We shall first find a 1-factor F_2 of G_1 containing E_2 and disjoint from $E_3, \ldots, E_m$. Since the first cell of each of the first m rows is preoccupied, for $i = 1, \ldots, m$, E_i contains an edge, say e_i, incident with the same vertex C_1. Thus any 1-factor of G_1 containing E_2 does not contain $e_3,\dots,e_m$. So we only need to find a 1-factor F_2 of G_1 containing E_2 and disjoint from $E_3' = E_3\backslash\{e_3\}$, $\dots, E_m' = E_n\backslash\{e_m\}$.

We apply Lemma 2.2 with $G = G_1$, $B_1 = E_2$, $B_2 = E_3' \cup \cdots \cup E_m'$ and $d = 2m$. We have $b_1 = |B_1| \leqslant m$ and $b_2 = |B_2| \leqslant m - 2$. So

$$b_1 + b_2 \leqslant m + (m-2) = 2m - 2 \leqslant 2m - 1 = d - 1$$

and

$$d - b_1 \geqslant 2m - m = m \geqslant \tfrac{1}{2}(2m + 1 - 1).$$

Hence, by Lemma 2.2, a required F_2 exists.

Let $G_2 = G_1 - F_2$. Now suppose that we have a sequence of graphs $G_0, G_1, \dots, G_p$, where $p < m$, and, for $r \geqslant 1$, $G_r = G_{r-1} - F_r$, F_r is a 1-factor of G_{r-1} containing E_r and disjoint from $E_{r+1},\dots,E_{2m+1}$. We want to extend this sequence by finding a 1-factor F_{p+1} of G_p containing E_{p+1} and disjoint from $E_{p+2},\dots,E_{2m+1}$.

We apply Lemma 2.2 again with $G = G_p$, $B_1 = E_{p+1}$, $B_2 = E_{p+2} \cup \cdots \cup E_{2m+1}$ and $d = 2m + 1 - p$. We have $b_1 \leqslant 2$ and $b_2 \leqslant 2(m - p - 1)$. So

$$b_1 + b_2 \leqslant 2 + 2(m - p - 1) = 2m - 2p \leqslant d - 1 = 2m - p$$

and

$$d - b_1 \geqslant 2m + 1 - p - 2 = 2m - p - 1 \geqslant 2m - m = m \geqslant \tfrac{1}{2}(2m + 1 - 1).$$

Again, by Lemma 2.2, we can find a required F_{p+1}. Hence the first m rows of P' can be completed. Since the other rows of P' are empty, by Lemma 2.1, we can complete P' to form a latin square of side $2m + 1$. $\square$

3. Proof of the main theorem

We will show in Section 4 that the main theorem holds for $n = 4, 5, 6$. Now we suppose that $n \geqslant 7$. We shall prove the main theorem by induction on n.

We consider the natural split of L into G_1, G_2 and H. Suppose that $o(\phi) = 2k > 0$. For each pair of odd ϕ-cycles, by (P_1), there are exactly one color occuring in both G_1 and H and exactly one color occuring in both G_2 and H. Without loss of generality, we assume that $\phi(x_1x_{2i}) = \phi(y_1x_{2i+1}) = \sigma_{2i-1}$ and $\phi(y_1y_{2i}) = \phi(x_1y_{2i+1}) = \sigma_{2i}$, where $i = 1,\dots,k$. By (P_2), each of the other colors occurs either only in H or once in G_1 and once in G_2. Let

$$\begin{aligned} L' = L &+ y_3y_{2k+2} + x_2y_{2k+2} + \cdots + y_{2i+1}y_{2k+2} + x_{2i}y_{2k+2} \\ &+ \cdots + y_{2k+1}y_{2k+2} + x_{2k}y_{2k+2}, \end{aligned}$$

and ϕ' be a coloring of L' such that $\phi'(x_1x_{2i}) = \phi'(y_{2i+1}y_{2k+2}) = \sigma_{2i}$ and $\phi'(x_1y_{2i+1}) = \phi'(x_{2i}y_{2k+2}) = \sigma_{2i-1}$, where $i = 1,\dots,k$, and for any other edge e of L', $\phi'(e) = \phi(e)$ (see Fig. 2). If we can extend ϕ' to a $(2n-1)$-coloring of K_{2n}, then by permutating the colour σ_{2i-1} with σ_{2i} in the cycle $x_1x_{2i}y_{2k+2}y_{2i+1}x_1$, for each $i = 1,\dots,k$, we can obtain a required coloring of K_{2n} which extends ϕ.

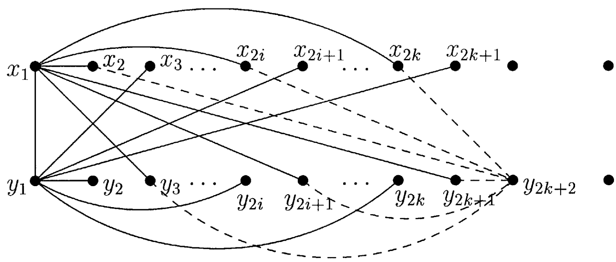

Fig. 2. The graph L'.

We consider two cases separately.

Case 1: $n \geqslant 8$ *is even*. Suppose that $o(\phi) = 0$. Then in the natural splits of L, each color occurs either only in H or once in G_1 and once in G_2. Let $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_H$ be the set of colors used in L_1, L_2 and L_H, respectively. Then $|\mathscr{C}_1| = |\mathscr{C}_2| = n-1, |\mathscr{C}_H| = n, \mathscr{C}_1 = \mathscr{C}_2$ and $\mathscr{C}_1 \cap \mathscr{C}_H = \emptyset$. Since L_1 (resp. L_2) is a star, it is obvious that we can extend the coloring of L_1 (resp. L_2) to an $(n-1)$-coloring of G_1 (resp. G_2) using the same set of colors $\mathscr{C}_1$. As mentioned in Section 2, we can identify the coloring of L_H with a partial latin square P of side n such that the first column and the first row of P are full and all the other cells are empty. By Theorem 2.3 we can complete P to form a latin square of side n. Hence we can extend the coloring of L_H to an n-coloring of H using the same set of colors $\mathscr{C}_H$. Finally, by combining these colorings we obtain a required $(2n-1)$-coloring of K_{2n}.

Now suppose that $o(\phi) = 2k > 0$. As stated at the beginning of this section, we only need to extend ϕ' to a $(2n-1)$-coloring of K_{2n}. We denote the three parts of L' in G_1, G_2 and G_H by L'_1, L'_2 and L'_H, respectively, and denote the sets of colors used in L'_1, L'_2 and L'_H by $\mathscr{C}'_1, \mathscr{C}'_2$ and $\mathscr{C}'_H$, respectively. Then we have $|\mathscr{C}'_1| = |\mathscr{C}'_2| = n-1$, $|\mathscr{C}'_H| = n$, $\mathscr{C}'_1 = \mathscr{C}'_2$ and $\mathscr{C}'_1 \cap \mathscr{C}'_H = \emptyset$. Observe that L'_1 is a star of size $n-1$. It is obvious that the coloring of L'_1 can be extended to an $(n-1)$-coloring of G_1 using the set of colors $\mathscr{C}'_1$. We also observe L'_2 is a subgraph of the lantern of order n with y_1 and y_{2k+2} as its apexes. Thus, by induction, we can extend the coloring of L'_2 to an $(n-1)$-coloring of G_2 using the same set of colors $\mathscr{C}'_2$. Since there is a 1-1 correspondence between n-colorings of $K_{n,n}$ and latin squares of side n, we can identify the coloring of L'_H with a partial latin square P such that the first row and the first column of P are full, $w(R_i) = 1$, where $i \neq 1,\ 2k+2$, and

$$w(R_{2k+2}) = 1 + \frac{o(\phi)}{2} \leqslant 1 + \left\lfloor \frac{|L| - 2}{6} \right\rfloor \leqslant \frac{n}{2}.$$

Hence by Theorem 2.3 (we can permute R_2 with R_{2k+2}), P can be completed to form a latin square of side n. Thus we can extend the coloring of L'_H to a proper coloring of H using the set of colors $\mathscr{C}'_H$. Finally, by combining these colorings, we obtain a required coloring of K_{2n}.

Case 2: $n \geqslant 7$ *is odd*. Suppose that $o(\phi) = 0$. Let $\phi(x_1 y_1) = \sigma$. Then each color occurs either only in H or once in G_1 and once in G_2. Let $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_H$ be the sets of

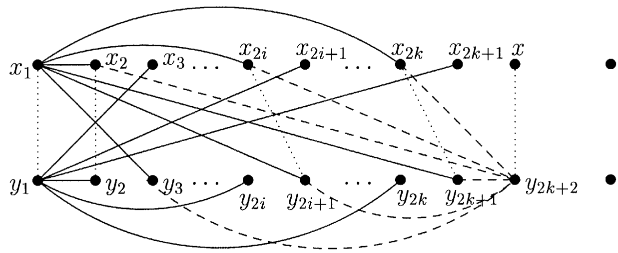

Fig. 3. The graph L^*.

colors used in L_1, L_2 and L_H, respectively. Then $\mathscr{C}_1 = \mathscr{C}_2$, $\mathscr{C}_i \cap \mathscr{C}_H = \emptyset, |\mathscr{C}_i| = n-1$ and $|\mathscr{C}_H| = n$. By Theorem 2.4, we can extend the coloring of L_H to an n-coloring ψ of H. Let $\{x_1', x_2', \dots, x_n'\}$ be a permutation of $V(G_1)$ such that $\psi(x_i' y_i) = \sigma$, for $i = 1, 2, \dots, n$. Let $\phi(x_1 x_i') = \alpha_i$ and $\phi(y_1 y_i) = \beta_i$, for $i = 2, \dots, n$. Since $n \geqslant 7$, we can set $\{\gamma_2, \dots, \gamma_n\} = \mathscr{C}_1 = \{\alpha_2, \dots, \alpha_n\} = \mathscr{C}_2 = \{\beta_2, \dots, \beta_n\}$ such that $\gamma_i \neq \alpha_i, \beta_i$ for any $i = 2, \dots, n$. Now we add a vertex x_0 to $V(G_1)$ and join x_0 to each vertex of $V(G_1)$. Then we obtain a lantern of order $n+1$ with x_0 and x_1 as its apexes. We color $x_0 x_1$ with σ and color $x_0 x_i'$ with γ_i, for $i = 2, \dots, n$. Combining this coloring with the coloring of L_1, we obtain an n-coloring of this lantern. By induction, we can extend this coloring to an n-coloring of K_{n+1} having vertex set $\{x_0, x_1, \dots, x_n\}$ with the set of colors $\mathscr{C}_1 \cup \{\sigma\}$. By deleting x_0, we obtain an n-coloring of G_1 such that color σ is missing at x_1, and color γ_i is missing at x_i', for $i = 2, \dots, n$. Similarly, we obtain an n-coloring of G_2 with the set of colors $\mathscr{C}_2 \cup \{\sigma\} (= \mathscr{C}_1 \cup \{\sigma\})$ such that σ is missing at y_1, and color γ_i is missing at y_i, for $i = 2, \dots, n$. Consequently, by recoloring $x_i' y_i$ with γ_i, where $i = 2, \dots, n$, we obtain a proper $(2n-1)$-coloring which extends ϕ.

Now suppose that $o(\phi) = 2k > 0$. Again, let $\phi(x_1 y_1) = \sigma$. As stated at the beginning of our proof, we need only to extend ϕ' of L' to a $(2n-1)$-coloring of K_{2n}. Let x be the vertex of $V(G_1)$ such that $\phi'(x_1 x) = \phi'(y_1 y_{2k+2})$. Then x is different from $x_2, x_4, \dots, x_{2k}$ because $\phi'(x_1 x_{2i}) = \phi'(y_1 y_{2i})$ for $i = 1, \dots, k$. Now we add the edges $x y_{2k+2}$, $x_2 y_2$ and $x_{2i} y_{2i+1}$, where $i = 2, \dots, k$, to L' to form a new graph L^*. We color these edges with σ and let ϕ^* be this coloring of L^* (see Fig. 3). We shall prove that we can extend ϕ^* to a $(2n-1)$-coloring of K_{2n}.

Let $\mathscr{C}_1^*, \mathscr{C}_2^*$ and $\mathscr{C}_H^*$ be the sets of colors used in L_1^*, L_2^* and L_H^*, resepectively. Then $\mathscr{C}_1^* = \mathscr{C}_2^*, \mathscr{C}_i^* \cap \mathscr{C}_H^* = \emptyset, |\mathscr{C}_i^*| = n-1$ and $|\mathscr{C}_H^*| = n$. We shall first extend the coloring of L_H^* to an n-coloring of H. As stated in Section 2, we identify the coloring of L_H^* with a partial latin square P of side n such that the first column and the first row of P are full, $w(R_2) = 2$, $w(R_{2i+1}) = 2$, for $i = 2, \dots, k$, $w(R_{2k+2}) = 2 + (o(\phi)/2)$, and each of the other rows has only one preassigned symbol. For $n = 9$, we have $o(\phi) \leqslant 4 = (9-1)/2$. For $n \geqslant 11$, we have

$$w(R_{2k+2}) = 2 + \frac{o(\phi)}{2} \leqslant 2 + \frac{1}{2}\left\lfloor \frac{|L|-2}{3} \right\rfloor = 2 + \frac{1}{2}\left\lfloor \frac{2n-2}{3} \right\rfloor \leqslant \frac{n-1}{2}. \quad (1)$$

Clearly when $n=7$ and $o(\phi)=2$, (1) also holds. Now for the case when $n=7$ and $o(\phi)=4$, (1) does not hold, but ϕ has only one cycle decomposition sequence, namely $(3,3,3,3)$. We shall settle this special case in Section 5. Hence we may assume that $w(R_{2k+2})\leqslant(n-1)/2$. Then by Theorem 2.4 (in fact we need to permute some rows of P to get the kind of partial latin square given in the statement of Theorem 2.4), P can be completed to form a latin square of side n. Hence the coloring of L_H^* can be completed to an n-coloring ψ of H.

Let $\{x_1',\ldots,x_n'\}$ be a permutation of $V(G)$ such that $\psi(x_i'y_i)=\sigma$ for $i=1,\ldots,n$. We have $x_1'=x_1$, $x_2'=x_2$, $x_{2k+2}'=x$, and $x_{2i+1}'=x_{2i}$ for $i=2,\ldots,k$, because previously we have colored x_2y_2, xy_{2k+2} and $x_{2i}y_{2i+1}$ with σ.

We shall next adjoin a new vertex x_0 (resp. y_0) to G_1 (resp. G_2) and use induction on the complete graph G_1^* (resp. G_2^*) having vertex set $\{x_0,x_1,\ldots,x_n\}$ (resp. $\{y_0,y_1,\ldots,y_n\}$). The set of colors we use in both G_1^* and G_2^* is $\mathscr{C}_1^*\cup\{\sigma\}=\mathscr{C}_2^*\cup\{\sigma\}$. If we can make the coloring of G_1^* and G_2^* in such a way that, after deleting x_0 and y_0, the missing colors at vertices x_i' and y_i $(i=1,2,\ldots,n)$ are the same, then we can replace the color σ by the missing color on the edge $x_i'y_i$ $(i=1,2,\ldots,n)$. Thus we can obtain a $(2n-1)$-coloring of K_{2n} which extends ϕ^*.

We first extend the coloring ϕ^* of L_2^* to an n-coloring ψ_2 of G_2^* using the set of colors $\mathscr{C}_2^*\cup\{\sigma\}$ such that $\psi_2(y_0y_i)\neq\phi^*(x_1x_i')$ for $i=2,\ldots,n$. In order to achieve this aim, we set $\psi_2(y_0y_1)=\sigma$, $\psi_2(y_0y_{2k+2})=\phi^*(x_1x_3')$, and then construct a lantern $L(2)$ of order $n+1$ in G_2^* with apexes y_1 and y_{2k+2}, and color $L(2)$ with the set of colors $\mathscr{C}_1^*\cup\{\sigma\}$ such that $\phi^*(x_1x_i')$ is used to color y_1y_i or $y_{2k+2}y_i$ (which will be specified later), where $i\neq3$. We partition $V(G_1)\backslash\{x_1,x_3',x\}$ into three classes V_1,V_2 and V_3 and color the remaining uncolored edges of $L(2)$ as follow:

(i) $V_1=\{x_5',x_7',\ldots,x_{2k+1}'\}$.
(We put $\psi_2(y_{2k+2}y_{2i+1})=\phi^*(y_{2k+2}y_{2i+1})=\phi'(y_{2k+2}y_{2i+1})=\sigma_{2i}=\phi'(x_1x_{2i})=\phi^*(x_1x_{2i+1}')$, for $i=2,\ldots,k$.)

(ii) $V_2=\{x_i'\in V(G_1)\backslash\{x_1,x_3',x\}\mid x_i'\notin V_1$ and $\phi^*(y_1y_i)\neq\phi^*(x_1x_i')\}$.
(We put $\psi_2(y_{2k+2}y_i)=\phi^*(x_1x_i')$, if $x_i'\in V_2$.)

(iii) $V_3=\{x_i'\in V(G_1)\backslash\{x_1,x_3',x\}\mid x_i'\notin V_1$ and $\phi^*(y_1y_i)=\phi^*(x_1x_i')\}$.
(For any $x_i'\in V_3$, the edge $y_{2k+2}y_i$ has not been colored before, therefore we call $y_{2k+2}y_i$ a *free edge*. We call color $c\in C_2^*\cup\sigma$ a *free color* if c has not been used to color any edge incident with y_{2k+2} in G_2^* before. Clearly, σ is a free color. Note that $x_2'\in V_3$ and $\phi^*(x_1x_2')$ was used to color $y_{2k+2}y_3$. So $\phi^*(x_1x_2')$ is not a free color. However, for any other $x_i'\in V_3$, the color $\phi^*(x_1x_i')$ is a free color since it has not been used to color any edge incident with y_{2k+2} in G_2^*. Let $\{y_{2k+2}y_1^*,\ldots,y_{2k+2}y_j^*\}$ be the set of all free edges and $\{c_1,\ldots,c_j\}$ be the set of all free colors. Let $\{c_1',\ldots,c_j'\}$ be a permutation of $\{c_1,\ldots,c_j\}$ such that $c_i'\neq\phi(y_{2k+2}y_i^*)$ for $i=1,\ldots,j$. Then put $\psi_2(y_{2k+2}y_i^*)=c_i'$.)

By the induction hypothesis, we can extend the coloring ψ_2 of $L(2)$ to an n-coloring of G_2^*. Deleting y_0 from G_2^* we get an n-coloring of G_2. Let α_i be the color missing at y_i. Then $\alpha_i=\psi_2(y_0y_i)\neq\phi^*(x_1x_i')$ because $\phi^*(x_1x_i')$ was used to color y_1y_i or $y_{2k+2}y_i$.

Next we adjoin a vertex x_0 to $V(G_1)$ and add an edge joining x_0 to x_i' for all $i=1,2,\ldots,n$. We obtain a lantern $L(1)$ of order $n+1$ with x_0 and x_1 as its apexes. We color x_0x_i' with α_i, for $i=1,2,\ldots,n$. By the induction hypothesis, we can extend this coloring of $L(1)$ to an n-coloring of G_1^*. Deleting x_0 from G_1^*, we obtain an n-coloring of G_1 such that α_i is missing at x_i'.

Finally we recolor the edge $x_i'y_i$ with α_i, for $i=1,\ldots,n$. We then obtain a $(2n-1)$-coloring of K_{2n} which extends ϕ^*. Hence, ϕ can be extended to a $(2n-1)$-coloring of K_{2n}. $\square$

4. Proof of the main theorem for the cases $n = 4, 5, 6$

When $n=4,6$, we have $o(\phi)\leqslant\lfloor(|L|-2)/3\rfloor\leqslant 3$. Thus $o(\phi)=0$ or 2. The proof is the same as Case 1 of the proof of the main theorem, except when $o(\phi)=2$ and $L_2'=S_{n-1}+y_2y_4$, in which case we cannot use induction because $n\leqslant 6$. However, in this case, it is obvious that we can first extend the coloring ϕ', restricted to S_{n-1}, to a proper coloring of G_2 using the same set of colors. Then by renaming some colors and reordering some vertices, if necessary, we obtain a required edge-coloring of G_2 which extends the coloring of $L_2'=S_{n-1}+y_2y_4$.

When $n=5$, there are seven cycle decomposition sequences, namely, (2,2,2,2), (2,2,4), (2,3,3), (2,6), (3,5), (4,4) and (8). We list below the symmetric matrices corresponding to the colorings of K_{2n} which extend the colorings of the lanterns. Let $V(K_{2n})=\{v_1,\ldots,v_{2n}\}$. The edge v_iv_j is colored with the symbol in the cell (i,j) in the corresponding symmetric matrix. The symbols in the shielded cells correspond to the colors on the edges of the lantern. These colorings are found by using a computer (Fig. 4).

5. A special case in the proof of the main theorem

In the proof of the main theorem, when $n=7$ and $o(\phi)=4$, the inequality (1) does not hold. In this case there is only one cycle decomposition sequence, namely $(3,3,3,3)$. We now give an example showing that a required coloring of K_{14} exists. This coloring is also found by using a computer (Fig. 5).

6. Applications

A *total coloring* of a graph G is a map $\phi: E(G)\cup V(G)\to\mathscr{C}$, where $\mathscr{C}$ is a set of colors such that no two incident or adjacent elements of $E(G)\cup V(G)$ receive the same color. The *total chromatic number* $\chi_{\mathrm{T}}(G)$ of G is the minimum value of $|\mathscr{C}|$ for which G has a total coloring. It is well known that $\chi_{\mathrm{T}}(K_{2n-1})=2n-1$. The following theorem on total colorings with prescribed conditions can be easily induced from the main theorem.

	2	3	4	5	6	7	8	9	1
2		1	6	7	8	9	4	5	3
3	1		7	8	9	4	5	6	2
4	6	7		9	1	8	2	3	5
5	7	8	9		2	3	6	1	4
6	8	9	1	2		5	3	4	7
7	9	4	8	3	5		1	2	6
8	4	5	2	6	3	1		7	9
9	5	6	3	1	4	2	7		8
1	3	2	5	4	7	6	9	8	

$(2,2,2,2)$

	2	3	4	5	6	7	8	9	1
2		1	6	4	5	9	7	8	3
3	1		7	8	9	2	6	5	4
4	6	7		9	8	1	2	3	5
5	4	8	9		1	6	3	7	2
6	5	9	8	1		3	4	2	7
7	9	2	1	6	3		5	4	8
8	7	6	2	3	4	5		1	9
9	8	5	3	7	2	4	1		6
1	3	4	5	2	7	8	9	6	

$(4,4)$

	2	3	4	5	6	7	8	9	1
2		1	6	4	5	9	7	8	3
3	1		7	8	9	4	5	6	2
4	6	7		9	8	1	2	3	5
5	4	8	9		7	2	3	1	6
6	5	9	8	7		3	1	2	4
7	9	4	1	2	3		6	5	8
8	7	5	2	3	1	6		4	9
9	8	6	3	1	2	5	4		7
1	3	2	5	6	4	8	9	7	

$(2,3,3)$

	2	3	4	5	6	7	8	9	1
2		1	6	4	5	9	7	8	3
3	1		7	8	9	4	5	6	2
4	6	7		9	8	1	2	3	5
5	4	8	9		1	2	3	7	6
6	5	9	8	1		3	4	2	7
7	9	4	1	2	3		6	5	8
8	7	5	2	3	4	6		1	9
9	8	6	3	7	2	5	1		4
1	3	2	5	6	7	8	9	4	

$(2,6)$

	2	3	4	5	6	7	8	9	1
2		1	6	7	4	9	5	8	3
3	1		7	8	9	5	6	4	2
4	6	7		9	8	1	2	3	5
5	7	8	9		2	6	3	1	4
6	4	9	8	2		3	1	5	7
7	9	5	1	6	3		4	2	8
8	5	6	2	3	1	4		7	9
9	8	4	3	1	5	2	7		6
1	3	2	5	4	7	8	9	6	

$(2,2,4)$

	2	3	4	5	6	7	8	9	1
2		1	5	4	8	9	6	7	3
3	1		7	8	9	2	5	6	4
4	5	7		9	1	6	3	8	2
5	4	8	9		2	3	7	1	6
6	8	9	1	2		5	4	3	7
7	9	2	6	3	5		1	4	8
8	6	5	3	7	4	1		2	9
9	7	6	8	1	3	4	2		5
1	3	4	2	6	7	8	9	5	

$(3,5)$

	2	3	4	5	6	7	8	9	1
2		1	6	4	5	9	7	8	3
3	1		7	8	9	2	5	6	4
4	6	7		9	8	1	2	3	5
5	4	8	9		2	3	1	7	6
6	5	9	8	2		4	3	1	7
7	9	2	1	3	4		6	5	8
8	7	5	2	1	3	6		4	9
9	8	6	3	7	1	5	4		2
1	3	4	5	6	7	8	9	2	

(8)

Fig. 4. The matrices corresponding to the colorings of the lanterns of order 10.

	2	3	4	5	6	7	8	9	10	11	12	13	1
2		1	5	4	8	6	7	11	9	13	10	12	3
3	1		6	2	5	8	10	12	13	7	11	9	4
4	5	6		1	9	10	11	13	12	3	7	8	2
5	4	2	1		11	12	13	3	7	8	9	10	6
6	8	5	9	11		13	12	1	2	10	3	4	7
7	6	8	10	12	13		1	2	11	9	4	3	5
8	7	10	11	13	12	1		4	3	2	5	6	9
9	11	12	13	3	1	2	4		5	6	8	7	10
10	9	13	12	7	2	11	3	5		4	6	1	8
11	13	7	3	8	10	9	2	6	4		1	5	12
12	10	11	7	9	3	4	5	8	6	1		2	13
13	12	9	8	10	4	3	6	7	1	5	2		11
1	3	4	2	6	7	5	9	10	8	12	13	11	

$(3, 3, 3, 3)$

Fig. 5. The matrix corresponding to a coloring of a lantern of order 14.

Theorem 6.1. *Let S be a spanning star of K_{2n-1}, where $n \geqslant 4$, and let ϕ be a partial total coloring of K_{2n-1} such that all the edges of S receive distinct colors and all the vertices of S receive distinct colors. Then ϕ can be extended to a total coloring of K_{2n-1} with the same set of $2n-1$ colors.*

We can also apply the main theorem to complete a kind of partial symmetric latin squares. The following theorem can be easily induced from the 1–1 correspondence between $(2n-1)$-colorings of K_{2n} using colors $1, 2, \ldots, 2n-1$ and symmetric latin squares of side $2n$ on symbols $1, \ldots, 2n-1, 2n$, whose diagonal cells are all occupied by $2n$, as mentioned in Section 2.

Theorem 6.2. *Let P be a partial symmetric latin square of order $2n \geqslant 8$ such that two rows and two columns of P are full and all the other cells are empty. If the two preassigned symbols in the main diagonal of P are the same, then P can be completed to form a symmetric latin square of order $2n$ with constant diagonal.*

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