ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 311 (2007) 606–618

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Bezoutians and Tate resolutions

David A. Cox

Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002-5000, USA

Received 19 June 2006

Available online 3 January 2007

Communicated by Luchezar L. Avramov

Abstract

This paper gives an explicit construction of the Tate resolution of sheaves arising from the d-fold Veronese embedding of $\mathbb{P}^n$. Our description involves the Bezoutian of $n+1$ homogeneous forms of degree d in $n+1$ variables. We give applications to duality theorems, including Koszul duality.

Keywords: Bezoutian; Tate resolution

1. Introduction

Given a finite dimensional vector space W over a field k with dual V, a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$ gives a *Tate resolution* $T^\bullet(\mathcal{F})$, which is a minimal bi-infinite exact sequence of free graded $E = \bigwedge V$-modules

$$\cdots \longrightarrow T^{-2}(\mathcal{F}) \longrightarrow T^{-1}(\mathcal{F}) \longrightarrow T^{0}(\mathcal{F}) \longrightarrow T^{1}(\mathcal{F}) \longrightarrow T^{2}(\mathcal{F}) \longrightarrow \cdots.$$

These resolutions were introduced by Gel'fand [8] in 1984 and are part of the BGG correspondence [2] from 1978.

The paper [4] gives an explicit formula for $T^\bullet(\mathcal{F})$, namely

$$T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes_k H^i\big(\mathbb{P}(W), \mathcal{F}(p-i)\big), \tag{1.1}$$

E-mail address: dac@cs.amherst.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.11.029

where $\widehat{E} = \mathrm{Hom}_k(E, k) = \bigwedge W$ as an E-module. Also note that $\deg(W) = 1$ since $\deg(V) = -1$ and that $\widehat{E} \simeq E(-\dim(W))$ (noncanonically).

The maps $T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ are less well understood. For the ith summand of $T^p(\mathcal{F})$, the map to $T^{p+1}(\mathcal{F})$ looks like

$$
\begin{array}{ccc}
\widehat{E}(i-p) \otimes_k H^i(\mathcal{F}(p-i)) & \longrightarrow & \widehat{E}(i-p-1) \otimes_k H^i(\mathcal{F}(p+1-i)) \\
 & & \bigoplus \\
\vdots & \searrow & \widehat{E}(i-p-2) \otimes_k H^{i-1}(\mathcal{F}(p+2-i)) \\
 & & \bigoplus \\
 & & \vdots \\
 & \searrow & \bigoplus \\
 & & \widehat{E}(-p) \otimes_k H^0(\mathcal{F}(p)),
\end{array}
$$

where for simplicity we have omitted "$\mathbb{P}(W)$" in the cohomology groups. The horizontal map in this diagram is known from [4], while the diagonal maps are more mysterious. Examples of these diagonal maps can be found [4,5], and explicit descriptions of certain diagonal maps in the toric context were given by Khetan in his work [10,11] on sparse determinantal formulas in dimensions 2 and 3.

In this paper, we will use Bezoutians to describe the diagonal maps in the Tate resolution for a particular choice of $\mathcal{F}$. Let $S = k[x_0, \dots, x_n]$ have the standard grading and let $W = S_d$ be the graded piece in degree $d \geqslant 1$. Thus $\dim(W) = \binom{n+d}{d}$. Given any $\ell \in \mathbb{Z}$, the d-fold Veronese embedding

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}(W)$$

gives the coherent sheaf

$$\mathcal{F} = \nu_{d*}\mathcal{O}_{\mathbb{P}^n}(\ell)$$

on $\mathbb{P}(W)$. We will give an explicit construction of the Tate resolution $T^\bullet(\mathcal{F})$.

Since $\mathcal{O}_{\mathbb{P}(W)}(1)|_{\nu_d(\mathbb{P}^n)} = \nu_{d*}\mathcal{O}_{\mathbb{P}^n}(d)$, we have

$$H^i\big(\mathbb{P}(W), \mathcal{F}(j)\big) = H^i\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell + jd)\big).$$

This cohomology group will be denoted $H^i(\ell + jd)$. Using Serre duality and standard vanishing theorems for line bundles on $\mathbb{P}^n$, we also have

$$H^i(\ell + jd) = \begin{cases} S_{\ell+jd} & i = 0, \\ S^*_{-n-1-(\ell+jd)} & i = n, \\ 0 & \text{otherwise,} \end{cases}$$

where S_m is the graded piece of $S = k[x_0, \dots, x_n]$ in degree m.

In the Tate resolution, it follows that

$$\begin{aligned} T^p(\mathcal{F}) &= \widehat{E}(-p) \otimes_k H^0(\ell + pd) \bigoplus \widehat{E}(n-p) \otimes_k H^n\big(\ell + (p-n)d\big) \\ &= \widehat{E}(-p) \otimes_k S_{\ell+pd} \bigoplus \widehat{E}(n-p) \otimes_k S^*_{-n-1-(\ell+(p-n)d)}. \end{aligned}$$

To simplify the subscripts, we set $a = \ell + (p+1)d$ and $\rho = (n+1)(d-1)$. Then the description of $T^p(\mathcal{F})$ becomes

$$T^p(\mathcal{F}) = \widehat{E}(-p) \otimes_k S_{a-d} \bigoplus \widehat{E}(n-p) \otimes_k S^*_{\rho-a},$$

and the map $T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ has the following form:

$$\begin{array}{ccc} \widehat{E}(n-p) \otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \widehat{E}(n-p-1) \otimes_k S^*_{\rho-a-d} \\ & \searrow^{\delta_p} & \\ \bigoplus & & \bigoplus \\ \widehat{E}(-p) \otimes_k S_{a-d} & \xrightarrow{\beta_p} & \widehat{E}(-p-1) \otimes_k S_a. \end{array} \tag{1.2}$$

By [4], the map

$$\beta_p \in \operatorname{Hom}_E\big(\widehat{E}(-p) \otimes_k S_{a-d}, \widehat{E}(-p-1) \otimes_k S_a\big)_0 \simeq \operatorname{Hom}_k(W \otimes_k S_{a-d}, S_a)$$

(the subscript "0" means graded E-module homomorphisms of degree 0) corresponds to multiplication $W \otimes_k S_{a-d} = S_d \otimes_k S_{a-d} \to S_a$, and α_p similarly corresponds to the natural map $W \otimes_k S^*_{\rho-a} \to S^*_{\rho-a-d}$ induced by multiplication.

The diagonal map δ_p in (1.2) lies in

$$\operatorname{Hom}_E\big(\widehat{E}(n-p) \otimes_k S^*_{\rho-a}, \widehat{E}(-p-1) \otimes_k S_a\big)_0 \simeq \operatorname{Hom}_k\Big(\bigwedge^{n+1} W, S_{\rho-a} \otimes_k S_a\Big). \tag{1.3}$$

The map δ_p is not unique; hence our main result (Theorem 1.3 below) will give one possible choice for this map.

We next recall the definition of the Bezoutian.

Definition 1.1. Consider the polynomial ring $k[x_0, \ldots, x_n, y_0, \ldots, y_n]$.

(1) For $f \in k[x_0, \ldots, x_n]$ and $0 \leqslant j \leqslant n$, define $\Delta_j(f)$ to be the polynomial

$$\frac{f(y_0, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_n) - f(y_0, \ldots, y_{j-1}, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}.$$

(2) The *Bezoutian* of homogeneous polynomials $f_0, \ldots, f_n \in k[x_0, \ldots, x_n]$ of degree d is the $(n+1) \times (n+1)$ determinant

$$\Delta = \det \Delta_j(f_i).$$

Remark 1.2. Here are some observations about the Bezoutian of $f_0, \ldots, f_n$.

(1) Each $\Delta_j(f_i)$ is homogeneous of degree $d-1$ in $x_0, \ldots, x_n, y_0, \ldots, y_n$, so the Bezoutian is homogeneous of degree $\rho = (n+1)(d-1)$ in these variables.
(2) Writing Δ as a polynomial in the y_is with coefficients in $k[x_0, \ldots, x_n]$, we obtain

$$\Delta = \sum_{|\alpha| \leqslant \rho} \Delta_\alpha(x) y^\alpha,$$

where $\Delta_\alpha(x) \in S = k[x_0, \ldots, x_n]$ has degree $\rho - |\alpha|$.
(3) Under the natural bigrading of $k[x_0, \ldots, x_n, y_0, \ldots, x_n]$, the graded piece of Δ of bidegree $(\rho - a, a)$ is

$$\Delta_{\rho-a,a} = \sum_{|\alpha|=a} \Delta_\alpha(x) y^\alpha.$$

(4) Recall the isomorphism $k[x_0, \ldots, x_n, y_0, \ldots, x_n] \simeq S \otimes_k S$ given by $x_i \mapsto x_i \otimes 1$, $y_i \mapsto 1 \otimes x_i$. Since Δ is multilinear and alternating in $f_0, \ldots, f_n$, the Bezoutian construction gives a linear map

$$\bigwedge^{n+1} S_d \longrightarrow (S \otimes_k S)_\rho = \bigoplus_{a=0}^{\rho} S_{\rho-a} \otimes_k S_a.$$

Bezoutians can be defined in greater generality (see [1,12]), but the case considered in Definition 1.1 is the only one we need for our main result.

By Remark 1.2, the Bezoutian in degree $(\rho - a, a)$ gives a linear map

$$\bigwedge^{n+1} W = \bigwedge^{n+1} S_d \longrightarrow S_{\rho-a} \otimes_k S_a,$$

which by (1.3) corresponds to an E-module homomorphism

$$B_p : \widehat{E}(n-p) \otimes_k S^*_{\rho-a} \longrightarrow \widehat{E}(-p-1) \otimes_k S_a. \tag{1.4}$$

Theorem 1.3. *The sheaf* $\mathcal{F} = \nu_{d*}(\mathcal{O}_{\mathbb{P}^n}(\ell))$ *has a Tate resolution with*

$$T^p(\mathcal{F}) = \widehat{E}(-p) \otimes_k S_{a-d} \bigoplus \widehat{E}(n-p) \otimes_k S^*_{\rho-a}, \quad a = \ell + (p+1)d,$$

and the differential $d_p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ *is given by*

$$\begin{array}{ccc}
\widehat{E}(n-p) \otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \widehat{E}(n-p-1) \otimes_k S^*_{\rho-a-d} \\
\bigoplus & \searrow^{(-1)^p B_p} & \bigoplus \\
\widehat{E}(-p) \otimes_k S_{a-d} & \xrightarrow{\beta_p} & \widehat{E}(-p-1) \otimes_k S_a,
\end{array}$$

where B_p *is the Bezoutian map from* (1.4) *and* α_p, β_p *are as in* (1.2).

2. Proof of the main result

We begin with two lemmas needed for the proof of Theorem 1.3. The notation will be the same as for the previous section. First observe that the graded pieces of B_p from (1.4) induce linear maps

$$\bigwedge^{n+1+m} W \otimes_k S^*_{\rho-a} \longrightarrow \bigwedge^{m} W \otimes_k S_a$$

for any integer m. This follows from $\widehat{E}(n-p)_{p+1+m} = \bigwedge^{n+1+m} W$. These maps will be called B_p by abuse of notation. Then one of the graded pieces of the differentials d_p from Theorem 1.3 gives the diagram

$$\begin{array}{ccccc} \bigwedge^{n+2} W \otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \bigwedge^{n+1} W \otimes_k S^*_{\rho-a-d} & & \\ & \searrow^{(-1)^p B_p} & \bigoplus & \searrow^{(-1)^{p+1} B_{p+1}} & \\ & & W \otimes_k S_a & \xrightarrow{\beta_{p+1}} & S_{a+d}. \end{array}$$

Lemma 2.1. $(-1)^{p+1} B_{p+1} \circ \alpha_p + \beta_{p+1} \circ (-1)^p B_p = 0$ *in the above diagram.*

Proof. Given $f_0, \ldots, f_{n+1} \in W = S_d$, the polynomials $\Delta_j(f_i)$ from Definition 1.1 satisfy the identity

$$\sum_{j=0}^{n} \Delta_j(f_i)(x_i - y_i) = f_i(x) - f_i(y), \quad 0 \leqslant i \leqslant n+1,$$

by a telescoping sum argument. Here we write $f_i(x)$ for $f_i(x_0, \ldots, x_n)$, and similarly for $f_i(y)$. It follows that in the $(n+2) \times (n+2)$ matrix

$$\begin{pmatrix} f_0(x) - f_0(y) & f_1(x) - f_1(y) & \cdots & f_{n+1}(x) - f_{n+1}(y) \\ \Delta_0(f_0) & \Delta_0(f_1) & \cdots & \Delta_0(f_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_n(f_0) & \Delta_n(f_1) & \cdots & \Delta_n(f_{n+1}) \end{pmatrix},$$

the first row is a linear combination (in $k[x, y]$) of the remaining rows. Hence the determinant is zero. Now expand by minors along the first row and observe that the $(n+1) \times (n+1)$ minors of the last $n+1$ rows are Bezoutians. Hence we get an identity

$$\sum_{i=0}^{n+1} (-1)^i \Delta^i(x, y) f_i(x) = \sum_{i=0}^{n+1} (-1)^i \Delta^i(x, y) f_i(y),$$

where $\Delta^i(x, y)$ is the Bezoutian of $f_0, \ldots, \widehat{f_i}, \ldots, f_{n+1}$. Each side is homogeneous of degree $\rho + d$ in $k[x, y]$, where $\rho = (n+1)(d-1)$.

If we write $\Delta^i(x,y) = \sum_{|\alpha|\leqslant\rho} \Delta^i_\alpha(x) y^\alpha$, then we can write the identity as

$$\sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|\leqslant\rho} \Delta^i_\alpha(x) f_i(x) y^\alpha = \sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|\leqslant\rho} \Delta^i_\alpha(x) f_i(y) y^\alpha.$$

Using $k[x,y] \simeq S \otimes_k S$ and taking the graded piece of bidegree $(\rho - a, a + d)$ gives

$$\sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|=a+d} \Delta^i_\alpha(x) f_i(x) \otimes x^\alpha = \sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|=a} \Delta^i_\alpha(x) \otimes f_i(x) x^\alpha. \tag{2.1}$$

This is an identity in $S_{\rho-a} \otimes_k S_{a+d}$.

Now pick $\varphi \in S^*_{\rho-a}$. If we apply $\varphi \otimes 1$ to (2.1), we obtain the identity

$$\sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|=a+d} \varphi\big(\Delta^i_\alpha(x) f_i(x)\big) x^\alpha = \sum_{i=0}^{n+1}(-1)^i \sum_{|\alpha|=a} \varphi\big(\Delta^i_\alpha(x)\big) f_i(x) x^\alpha \tag{2.2}$$

in S_{a+d}. The left-hand side of (2.2) is $B_{p+1} \circ \alpha_p$ evaluated at $f_0 \wedge \cdots \wedge f_{n+1} \otimes \varphi$, while the right-hand side is $\beta_{p+1} \circ B_p$ evaluated at the same element. This shows that $B_{p+1} \circ \alpha_p - \beta_{p+1} \circ B_p = 0$, from which the lemma follows immediately. □

To prepare for the second lemma, let $N = \dim(W) = \binom{n+d}{d}$ and assume that $0 \leqslant \rho - a < d$, so that $S^*_{\rho-a-d} = 0$. Then one of the graded pieces of the differential d_p from Theorem 1.3 gives the diagram

$$\begin{array}{ccc} \bigwedge^N W \otimes_k S^*_{\rho-a} & & \\ & \searrow^{(-1)^p B_p} & \\ \bigoplus & & \\ \bigwedge^{N-n} W \otimes_k S_{a-d} & \xrightarrow{\beta_p} & \bigwedge^{N-n-1} W \otimes_k S_a. \end{array} \tag{2.3}$$

Lemma 2.2. *If* $0 \leqslant \rho - a < d$*, then the maps* B_p *and* β_p *in* (2.3) *have the following two properties*:

(1) B_p *is injective.*
(2) $\mathrm{Im}(B_p) \cap \mathrm{Im}(\beta_p) = \{0\}$.

Proof. The Bezoutian of $x_0^d, \ldots, x_n^d$ is easily seen to be

$$\Delta = \sum_{\beta \leqslant \beta_{d-1}} x^\beta y^{\beta_{d-1}-\beta},$$

where $\beta_{d-1} = (d-1, \ldots, d-1) \in \mathbb{Z}^n$ and $\beta \leqslant \beta_{d-1}$ means that every component of β is $\leqslant d-1$. This Bezoutian is also computed in [1].

The monomial basis of $W = S_d$ induces a basis of $\bigwedge^i W$ for every i. When $i = N$, the space has dimension one, and we write its basis element as

$$x_0^d \wedge \cdots \wedge x_n^d \wedge \omega \in \bigwedge^N W,$$

where ω is the wedge product of the remaining monomials of degree d. Given $\varphi \in S^*_{\rho-a}$, we obtain

$$B_p\big(x_0^d \wedge \cdots \wedge x_n^d \wedge \omega \otimes \varphi\big) = \omega \otimes \left(\sum_\beta \varphi\big(x^\beta\big)x^{\beta_{d-1}-\beta}\right) + \cdots, \tag{2.4}$$

where the sum inside the parentheses is over all β of degree $\rho - a$ satisfying $\beta \leqslant \beta_{d-1}$, and the omitted terms involve basis elements of $\bigwedge^{N-n-1} W$ different from ω.

Let φ be in the kernel of B_p. It follows that $\varphi(x^\beta) = 0$ for all x^β appearing in the above sum. But our hypothesis that $\rho - a < d$ guarantees that this sum includes *all* monomials of degree $\rho - a$. These monomials form a basis of $S_{\rho-a}$, so that φ must vanish. This proves that B_p is injective, as claimed.

For the second part of the lemma, let $A = \sum_i \omega_i \otimes p_i \in \bigwedge^{N-n} W \otimes_k S_{a-d}$, where $\{\omega_i\}_i$ is the basis of $\bigwedge^{N-n} W$ coming from monomials. We can assume that the basis includes $\omega_i = \omega \wedge x_i^d$ for $i = 0, \ldots, n$, where ω is as above. Then

$$\beta_p(A) = \omega \otimes \left(\sum_{i=0}^n x_i^d p_i\right) + \cdots,$$

where the omitted terms involve basis elements of $\bigwedge^{N-n-1} W$ different from ω. The monomials appearing in $\sum_{i=0}^n x_i^d p_i$ all have some x_i with an exponent $\geqslant d$, yet in the ω-term of (2.4), every x_i has exponent $\leqslant d - 1$. Hence, if $\beta_p(A) = B_p(x_0^d \wedge \cdots \wedge x_n^d \wedge \omega \otimes \varphi)$, then their ω-terms in $\bigwedge^{N-n-1} W \otimes_k S_a$ must vanish, which as above implies that $\varphi = 0$. Hence $\operatorname{Im}(B_p) \cap \operatorname{Im}(\beta_p) = \{0\}$. □

We can now prove our main result.

Proof of Theorem 1.3. We first show that the differential $d_p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ defined in Theorem 1.3 satisfies $d_{p+1} \circ d_p = 0$, i.e., $(T^\bullet(\mathcal{F}), d_\bullet)$ is a complex.

We know that $\alpha_{p+1} \circ \alpha_p = 0$ and $\beta_{p+1} \circ \beta_p = 0$. It remains to show that the map

$$\widehat{E}(n-p) \otimes S^*_{\rho-a} \longrightarrow \widehat{E}(-p-2) \otimes S_{a+d}$$

given by $(-1)^{p+1} B_{p+1} \circ \alpha_p + \beta_{p+1} \circ (-1)^p B_p$ is zero. Since

$$\operatorname{Hom}_E\big(\widehat{E}(n-p) \otimes S^*_{\rho-a}, \widehat{E}(-p-2) \otimes S_{a+d}\big)_0 \simeq \operatorname{Hom}_k\Big(\bigwedge^{n+2} W \otimes S^*_{\rho-a}, S_{a+d}\Big),$$

this follows immediately from Lemma 2.1.

Next we need to show that for each p, d_p is determined by the minimal generators of the kernel of d_{p+1}. This is where we use the power of the formula for $T^p(\mathcal{F})$ given in (1.1): it tells us the degrees of the minimal generators of $\mathrm{Ker}(d_{p+1})$ and the number of minimal generators in these degrees. Furthermore, $d_{p+1} \circ d_p = 0$ implies that d_p maps into the kernel. So we need to study how d_p behaves in the degrees of the minimal generators.

Recall that $a = \ell + (p+1)d$, so that $\rho - a < 0$ for large p. We will look closely at the case when $0 \leqslant \rho - a < d$. Here, $d_{p+1} = \beta_{p+1}$ and the complex looks like

$$\begin{array}{ccccc} \widehat{E}(n-p)\otimes_k S^*_{\rho-a} & & & & \\ & \searrow^{(-1)^p B_p} & & & \\ \bigoplus & & & & \\ \widehat{E}(-p)\otimes_k S_{a-d} & \xrightarrow{\beta_p} & \widehat{E}(-p-1)\otimes_k S_a & \xrightarrow{\beta_{p+1}} & \widehat{E}(-p-2)\otimes_k S_{a+d}. \end{array}$$

This is the first place where a nonzero diagonal map appears in the Tate resolution. Since $\widehat{E} \simeq E(-N)$ (this is the notation of Lemma 2.2), there are $\dim(S_{a-d})$ minimal generators of degree $N+p$ and $\dim(S^*_{\rho-a})$ minimal generators of degree $N-n+p$. The former are taken care of by the known formula for β_p. For the latter, notice that the above diagram in degree $N-n+p$ is precisely (2.3), and then Lemma 2.2 implies that $(-1)^p B_p$ maps injectively onto the minimal generators in this degree. Hence we have the desired behavior when $\rho - a < d$.

We now proceed by decreasing induction on p. Suppose that $\rho - a \geqslant d$ and that everything is fine for larger p. As above, there are $\dim(S_{a-d})$ minimal generators of degree $N+p$ and $\dim(S^*_{\rho-a})$ minimal generators of degree $N-n+p$, where the former are taken care of by β_p. But now in degree $N-n+p$, the differential d_p is given by

$$\begin{array}{ccc} \bigwedge^{N} W\otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \bigwedge^{N-1} W\otimes_k S^*_{\rho-a-d} \\ & \searrow^{(-1)^p B_p} & \\ \bigoplus & & \bigoplus \\ \bigwedge^{N-n} W\otimes_k S_{a-d} & \xrightarrow{\beta_p} & \bigwedge^{N-n-1} W\otimes_k S_a. \end{array}$$

The key observation is that the α_p in this diagram is dual to the multiplication map $W \otimes S_{\rho-a-d} \to S_{\rho-a}$, which is surjective since $\rho - a \geqslant d$. This implies that in the degree of the minimal generators, α_p is injective. It follows that $\alpha_p \oplus (-1)^p B_p$ is injective in this degree and its image intersects the image of β_p in $\{0\}$. This shows that d_p has the desired property and completes the proof of the theorem. $\square$

Remark 2.3. Here are two observations due to Evgeny Materov.

(1) The Tate resolution of Theorem 1.3 can be expressed as a mapping cone. Let $\mathcal{D}^\bullet$ denote the part of the Tate resolution in cohomological degree 0 (i.e., the part of (1.1) involving H^0). Thus $\mathcal{D}^\bullet$ is given by

$$\cdots \longrightarrow \mathcal{D}^p = \widehat{E}(-p)\otimes_k S_{a-d} \xrightarrow{\beta_p} \mathcal{D}^{p+1} = \widehat{E}(-p-1)\otimes_k S_a \longrightarrow \cdots.$$

Similarly, let $\mathcal{C}^\bullet$ denote the part of the Tate resolution in cohomological degree n, shifted by -1. Thus $\mathcal{C}^\bullet$ is given by

$$\cdots \longrightarrow \mathcal{C}^p = \widehat{E}(n-p+1)\otimes_k S^*_{\rho-a+d} \xrightarrow{\alpha_{p-1}} \mathcal{C}^{p+1} = \widehat{E}(n-p)\otimes_k S^*_{\rho-a} \longrightarrow \cdots.$$

The proofs of Lemma 2.1 and Theorem 1.3 give a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \widehat{E}(n-p+1)\otimes_k S^*_{\rho-a+d} & \xrightarrow{\alpha_{p-1}} & \widehat{E}(n-p)\otimes_k S^*_{\rho-a} & \longrightarrow \cdots \\
& \downarrow B_{p-1} & & \downarrow B_p & \\
\cdots \longrightarrow & \widehat{E}(-p)\otimes_k S_{a-d} & \xrightarrow{\beta_p} & \widehat{E}(-p-1)\otimes_k S_a & \longrightarrow \cdots,
\end{array}$$

so that the Bezoutians $\{B_{p-1}\}$ give a map of complexes $\mathcal{C}^\bullet \to \mathcal{D}^\bullet$. Then Theorem 1.3 implies that the Tate resolution is the mapping cone of this map of complexes. This explains the signs $(-1)^p$ and $(-1)^{p+1}$ appearing in the statement of the theorem.

(2) For a fixed degree, the Tate resolution of Theorem 1.3 is the Weyman complex discussed in [7, 13.1.C] and [14, 9.2]. These references describe everything except the diagonal maps. In [7, p. 432], the authors say that "No nice explicit expression ... is known" for these maps.

3. Application to duality

We conclude by exploring the relation between duality, Bezoutians, and the Tate resolution. We first recall how to extract information from the Tate resolution. Stated briefly, the key idea is to look at $T^\bullet(\mathcal{F})$ in a specific degree, but only *after* replacing W with a suitable subspace $U \subset W$. This is the functor $\mathbf{U}_l$ from [5], which is equivalent to the projection formula from [6, Section 1.2].

To make this precise, let $U \subset W$ be a subspace. Since $\mathbb{P}(W) = (W^* - \{0\})/k^*$, the linear subspace $\mathbb{P}(W/U) \subset \mathbb{P}(W)$ is the center of the projection $\pi : \mathbb{P}(W) \dashrightarrow \mathbb{P}(U)$. If $\mathbb{P}(W/U)$ is disjoint from the support of $\mathcal{F}$, then [5] and [6] show that

$$T^\bullet_U(\mathcal{F}) = \operatorname{Hom}_E\Bigl(\bigwedge U^*, T^\bullet(\mathcal{F})\Bigr)$$

is a Tate resolution of $\pi_*\mathcal{F}$ on $\mathbb{P}(U)$. Note also that $\mathcal{F}$ and $\pi_*\mathcal{F}$ have the same cohomology since $\pi : \mathbb{P}(W) \setminus \mathbb{P}(W/U) \to \mathbb{P}(U)$ is affine.

In the situation of Theorem 1.3, we have $W = S_d$, so that a subspace $U \subset W$ satisfies

$$\mathbb{P}(W/U) \cap \operatorname{Supp}(\mathcal{F}) = \emptyset$$

if and only if the homogeneous polynomials in U have no common zeros in $\mathbb{P}^n$. When this happens, the above paragraph and Theorem 1.3 give a minimal exact sequence of free graded E_U-modules $T^\bullet_U(\mathcal{F})$, where $T^p_U(\mathcal{F}) \to T^{p+1}_U(\mathcal{F})$ is

$$
\begin{array}{ccc}
\widehat{E}_U(n-p)\otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \widehat{E}_U(n-p-1)\otimes_k S^*_{\rho-a-d} \\
\oplus & \searrow^{(-1)^p B_p} & \oplus \\
\widehat{E}_U(-p)\otimes_k S_{a-d} & \xrightarrow{\beta_p} & \widehat{E}_U(-p-1)\otimes_k S_a.
\end{array}
\tag{3.1}
$$

Here, $E_U = \bigwedge U^*$ and $\widehat{E}_U = \bigwedge U$. As we will see, looking at this complex in specific degrees for specific choices of U will give some interesting duality theorems.

Example 3.1. First let $U = \mathrm{Span}(f_0, \dots, f_n) \subset W = S_d$, where $f_0, \dots, f_n$ have no common zeros on $\mathbb{P}^n$. As is well known, this happens $\Leftrightarrow$ $f_0, \dots, f_n$ is a regular sequence $\Leftrightarrow$ the Koszul complex of $f_0, \dots, f_n$ is exact.

Let $I = \langle f_0, \dots, f_n\rangle \subset S$ and $R = S/I$. Then consider $T^\bullet_U(\mathcal{F})$ in degree $p+1$. Using (3.1), we obtain the following exact sequence of vector spaces:

$$
\begin{array}{ccccc}
 & & \bigwedge^{n+1} U\otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \bigwedge^{n} U\otimes_k S^*_{\rho-a-d} \longrightarrow \cdots \\
 & & \oplus & \searrow^{(-1)^p B_p} & \oplus \\
\cdots & \longrightarrow & U\otimes_k S_{a-d} & \xrightarrow{\beta_p} & S_a.
\end{array}
$$

It follows that $(-1)^p B_p$ induces an isomorphism

$$\mathrm{Ker}(\alpha_p) \simeq \mathrm{Coker}(\beta_p).$$

Since $\mathrm{Ker}(\alpha_p) = R^*_{\rho-a}$ and $\mathrm{Coker}(\beta_p) = R_a$, we recover the known duality

$$R^*_{\rho-a} \simeq R_a.$$

Furthermore, $\bigwedge^{n+1} U$ has basis element $f_0 \wedge \cdots \wedge f_n$, so that if

$$\Delta = \sum_{|\alpha|\leqslant\rho} \Delta_\alpha(x) y^\alpha$$

is the Bezoutian of $f_0, \dots, f_n$, then the above isomorphism $R^*_{\rho-a} \simeq R_a$ is given by

$$\varphi \in R^*_{\rho-a} \longmapsto \sum_{|\alpha|=a} \varphi\big(\big[\Delta_\alpha(x)\big]\big)\big[x^\alpha\big] \in R_a, \tag{3.2}$$

where $[g] \in R$ denotes the coset of the polynomial $g \in S$.

Remark 3.2. Here are some comments about Example 3.1.

(1) It is known that the duality $R^*_{\rho-a} \simeq R_a$ can be computed by (3.2). Proofs can be found in [1,12] in the case when the f_i are homogeneous of degree d_i, as opposed to the equal degree

case considered here. Our contribution is to show that the Tate resolution gives a new proof of this explicit duality in the equal degree case.

(2) The proof given in [1] that (3.2) induces $R^*_{\rho-a} \simeq R_a$ uses the Bezoutian of $x_0^d, \ldots, x_n^d$. This is the same Bezoutian used in the proof of Lemma 2.2.

Example 3.3. Now suppose that $U = \mathrm{Span}(f_0, \ldots, f_n, f_{n+1}) \subset W$, where the polynomials $f_0, \ldots, f_n, f_{n+1}$ are linearly independent and have no common zeros in $\mathbb{P}^n$. We have one more polynomial than we had in Example 3.1. As we will see, this leads to a slightly different form of duality.

As in the previous example, let $I = \langle f_0, \ldots, f_n, f_{n+1}\rangle \subset S$ and $R = S/I$, and consider $T_U^\bullet(\mathcal{F})$ in degree $p+2$. Using (3.1), we obtain the following exact sequence of vector spaces:

$$\begin{array}{ccccccc}
 & \bigwedge^{n+2} U \otimes_k S^*_{\rho-a} & \xrightarrow{\alpha_p} & \bigwedge^{n+1} U \otimes_k S^*_{\rho-a-d} & \xrightarrow{\alpha_{p+1}} & \bigwedge^{n} U \otimes_k S^*_{\rho-a-2d} & \longrightarrow \cdots \\
 & & \searrow^{(-1)^p B_p} & & \searrow^{(-1)^{p+1} B_{p+1}} & & \\
 & \oplus & & \oplus & & \oplus & \\
\cdots \longrightarrow & \bigwedge^{2} U \otimes_k S_{a-d} & \xrightarrow{\beta_p} & U \otimes_k S_a & \xrightarrow{\beta_{p+1}} & S_{a+d}. &
\end{array}$$

It follows that $(-1)^p B_p$ induces an isomorphism

$$\mathrm{Ker}(\alpha_p) \simeq \mathrm{Ker}(\beta_{p+1}) / \mathrm{Im}(\beta_p). \tag{3.3}$$

Note that $\mathrm{Ker}(\alpha_p) = R^*_{\rho-a}$ and that the bottom row of the above diagram comes from the Koszul complex of $f_0, \ldots, f_{n+1}$. Hence

$$\mathrm{Ker}(\beta_{p+1}) = \mathrm{Syz}(f_0, \ldots, f_{n+1})_{a+d},$$

where a syzygy $(A_0, \ldots, A_{n+1})$ is said to have degree $a+d$ if $\sum_{i=0}^{n+1} A_i f_i = 0$ in S_{a+d}. Furthermore, the image of $\beta_p : \bigwedge^2 U \otimes_k S_{a-d} \to U \otimes_k S_a$ is the submodule of $\mathrm{Syz}(f_0, \ldots, f_{n+1})_{a+d}$ consisting of Koszul syzygies. Hence we set

$$\mathrm{Kosz}_{a+d} = \mathrm{Im}(\beta_p).$$

Then the duality (3.3) becomes

$$R^*_{\rho-a} \simeq \mathrm{Syz}(f_0, \ldots, f_{n+1})_{a+d} / \mathrm{Kosz}_{a+d}. \tag{3.4}$$

Notice also that B_p gives an explicit description of this duality since elements of $R^*_{\rho-a}$ can be regarded as linear functionals φ on $S_{\rho-a}$ that vanish on $I_{\rho-a}$. Then the left-hand side of (2.2) vanishes, so that (2.2) becomes

$$\sum_{i=0}^{n+1} (-1)^i \sum_{|\alpha|=a} \varphi\big(\Delta^i_\alpha\big) x^\alpha f_i = 0. \tag{3.5}$$

As noted in the proof of Lemma 2.2, this is β_{p+1} applied to $B_p(f_0 \wedge \cdots \wedge f_{n+1} \otimes \varphi)$. Thus

$$\left(\sum_{|\alpha|=a} \varphi\bigl(\Delta_\alpha^0\bigr)x^\alpha, -\sum_{|\alpha|=a} \varphi\bigl(\Delta_\alpha^1\bigr)x^\alpha, \ldots, (-1)^{n+1} \sum_{|\alpha|=a} \varphi\bigl(\Delta_\alpha^{n+1}\bigr)x^\alpha \right)$$

is an element of $\mathrm{Syz}(f_0, \ldots, f_{n+1})_{a+d}$ coming from B_p. We call this a *Bezout syzygy*. It follows that the duality (3.4) is computed in terms of Bezout syzygies.

Remark 3.4. Here are further comments on the duality of Example 3.3.

(1) If $K_\bullet$ is the Koszul complex of $f_0, \ldots, f_{n+1}$, then our hypothesis that the f_i do not vanish simultaneously on $\mathbb{P}^n$ implies that $K_\bullet$ is almost exact. In fact, the only place exactness fails is at K_1:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_2 & \xrightarrow{d_1} & K_1 & \xrightarrow{d_0} & S & \longrightarrow & R & \longrightarrow 0. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \text{ok} & & \text{ok} & & \textbf{no} & & \text{ok} & & \text{ok} & \end{array}$$

(This observation is used in [3].) The graded pieces of $\mathrm{Ker}(d_0)/\mathrm{Im}(d_1)$ are the $\mathrm{Syz}(f_0, \ldots, f_{n+1})_{a+d}/\mathrm{Kosz}_{a+d}$ appearing in (3.4). Thus size of

$$R = S/I = k[x_0, \ldots, x_n]/\langle f_0, \ldots, f_{n+1}\rangle$$

gives a precise measure of the failure of an arbitrary syzygy to be Koszul.

(2) One corollary of the duality (3.4) is that the syzygy module of $f_0, \ldots, f_{n+1}$ is generated by Koszul syzygies and Bezout syzygies.

(3) We can write the duality (3.4) more conceptually as follows. Set $\sigma = \sum_{i=0}^{n+1} \deg(f_i) - (n+1) = \rho + d$ and $b = a + d$. Then (3.4) becomes

$$R_{\sigma-b}^* \simeq \mathrm{Syz}(f_0, \ldots, f_{n+1})_b/\mathrm{Kosz}_b\,.$$

Furthermore, if $H_i(K_\bullet)$ is the ith homology of the Koszul complex, then this duality can be written as

$$H_0(K_\bullet)_{\sigma-b}^* \simeq H_1(K_\bullet)_b.$$

We also note that R is an almost complete intersection in this case. By [13], the Koszul homology $H_1(K_\bullet)_b$ is related to the symmetric algebra $\mathrm{Sym}(I/I^2)$.

(4) More generally, suppose that $f_0, \ldots, f_m \in S_d$ are linearly independent and do not vanish simultaneously on $\mathbb{P}^n$. Note that $m \geqslant n$ and that Examples 3.1 and 3.3 correspond to $m = n$ and $m = n + 1$, respectively. Let $K_\bullet$ be the Koszul complex of $f_0, \ldots, f_m$ and set $\sigma = \sum_{i=0}^m \deg(f_i) - (n+1)$. Then Examples 3.1 and 3.3 easily generalize to give a *Koszul duality*

$$H_i(K_\bullet)_{\sigma-a}^* \simeq H_{m-n-i}(K_\bullet)_a, \quad 0 \leqslant i \leqslant m-n,$$

that is computed by Bezoutians.

(5) The Koszul duality just stated applies more generally to homogeneous polynomials in S of arbitrary degrees (not necessarily equal) that do not vanish simultaneously on $\mathbb{P}^n$. The proof that some isomorphism exists is an easy spectral sequence argument; the fact that it is given by Bezoutians takes more work—this has been proved by Jouanolou [9]. So again, the Tate resolution gives a quick proof of the equal degree case of an explicit duality theorem.

A final comment is that the duality theorems of Examples 3.1 and 3.3 and Remark 3.4 come from the *same* Tate resolution. Once we describe the Tate resolution in terms of Bezoutians, we get immediate Bezoutian descriptions of *all* of these duality results. This indicates the deep relation between duality, Bezoutians, and the Tate resolution.

Acknowledgments

I thank Jessica Sidman, Ivan Soprounov and especially Evgeny Materov for their useful observations and careful reading of the paper. Thanks also go to Laurent Busé, Marc Chardin, David Eisenbud and Jean-Pierre Jouanolou for helpful conversations.

References

[1] E. Becker, J.-P. Cardinal, M.-F. Roy, Z. Szafraniec, Multivariate Bezoutians, Kronecker symbol and Eisenbud–Levine formula, in: L. González-Vega, T. Recio (Eds.), Algorithms in Algebraic Geometry and Applications, in: Progr. Math., vol. 143, Birkhäuser, Boston, 1996, pp. 79–104.
[2] I.N. Bernšteĭn, I.M. Gel'fand, S.I. Gel'fand, Algebraic bundles over $\mathbf{P}^r$ and problems of linear algebra, Funct. Anal. Appl. 12 (1978) 212–214; English translation from Funktsional. Anal. i Prilozhen. 12 (1978) 66–67.
[3] D. Cox, H. Schenck, Local complete intersections in $\mathbb{P}^2$ and Koszul syzygies, Proc. Amer. Math. Soc. 131 (2003) 2007–2014.
[4] D. Eisenbud, G. Fløystad, F.-O. Schreyer, Sheaf cohomology and free resolutions over exterior algebras, Trans. Amer. Math. Soc. 355 (2003) 4397–4426.
[5] D. Eisenbud, F.-O. Schreyer, Resultants and Chow forms via exterior syzygies, with an appendix by J. Weyman, J. Amer. Math. Soc. 16 (2003) 537–579.
[6] G. Fløystad, Exterior algebra resolutions arising from homogeneous bundles, Math. Scand. 94 (2004) 191–201.
[7] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.
[8] S.I. Gel'fand, Sheaves on $\mathbf{P}^n$ and problems in linear algebra, appendix to the Russian edition of C. Okonek, M. Schneider, H. Spindler (Eds.), Vector Bundles on Complex Projective Spaces, Mir, Moscow, 1984, pp. 278–305. Originally published by Birkhäuser, Boston, 1980.
[9] J.-P. Jouanolou, An explicit duality for quasi-homogeneous ideals, preprint, available at arXiv: math.AC/0607626, 2006.
[10] A. Khetan, Exact matrix formula for the unmixed resultant in three variables, J. Pure Appl. Algebra 198 (2005) 237–256.
[11] A. Khetan, The resultant of an unmixed bivariate system, in: International Symposium on Symbolic and Algebraic Computation, ISSAC 2002, Lille, J. Symbolic Comput. 36 (2003) 425–442.
[12] E. Kunz, Kähler Differentials, Adv. Lectures Math., Vieweg, Wiesbaden, 1986.
[13] A. Simis, W. Vasconcelos, The syzygies of the conormal bundle, Amer. J. Math. 103 (1981) 203–224.
[14] J. Weyman, Cohomology of Vector Bundles and Syzygies, Cambridge Univ. Press, Cambridge, 2003.