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On a Class of Quasilinear Partial Integrodifferential Equations with Singular Kernels

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We prove local and global existence theorems for a model equation in nonlinear viscoelasticity. In contrast to previous studies, we allow the memory function to have a singularity. We approximate the equation by equations with regular kernels and use energy estimates to prove convergence of the approximate solutions.

1. INTRODUCTION

Many constitutive models for viscoelastic materials lead to equations of motion which have the form of a quasilinear hyperbolic PDE perturbed by a dissipative integral term of Volterra type. In the recent literature, a number of existence theorems have been proved for such equations [2–4, 10–14, 17, 21–22, 26]. These papers establish the existence of classical solutions locally in time and (in some cases) globally in time if the given data are suitably small. For large data, global existence does not hold in general and shocks are expected to develop [7, 9, 18, 19, 25].

Common to all the works referred to above is the assumption that the kernel in the integral term is sufficiently smooth on $[0, \infty)$. We are here interested in the possibility that this kernel is singular at 0. Kinetic theories for chain molecules [5, 24, 28] and some experimental data [15] suggest that this a realistic possibility, at least for some viscoelastic materials.

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Although some rheological properties of models with singular kernels have been investigated (see, e.g., [1]), there do not seem to be many studies from a fundamental mathematical point of view.

The only existence theorem for (nonlinear) models with singular kernels that we are aware of is a result of Londen [16] concerning the existence of weak solutions to an abstract integrodifferential equation. His existence theorem is applicable to the problem introduced below in the special case where $\psi \equiv \phi$. Londen's assumptions require the viscoelastic memory function to have a singularity which is stronger than logarithmic.

Renardy [23] has studied linear wave propagation. His results show that certain singular kernels do not permit propagation of singularities and have a smoothing effect. Hannsgen and Wheeler [8] show (for the constant coefficient linear problem on a bounded domain) that the evolution operator is compact for positive time if and only if the kernel is singular. This suggests that, if anything, models with singular kernels should have "nicer" existence properties than those with regular kernels. However, this also indicates that one cannot expect the methods of previous existence proofs to extend to singular kernels. These proofs rely on an iteration scheme that treats the hyperbolic part as the principal term and the integral as a perturbation. This, of course, works irrespective of the sign of the integral. If, however, singular kernels lead to smoothing, then the opposite sign of the integral must lead to instantaneous blow-up, and a local existence theorem cannot hold.

In this paper, we focus on the history value problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_{-\infty}^{t} a'(t-\tau)\,\psi(u_x(x,\tau))_x\,d\tau + f(x,t),$$

$$0 \leqslant x \leqslant 1, \quad -\infty < t < \infty, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad -\infty < t < \infty, \tag{1.2}$$

$$u(x,t) = v(x,t), \quad 0 \leqslant x \leqslant 1, \qquad -\infty < t \leqslant 0, \tag{1.3}$$

which was studied by Dafermos and Nohel [4]. (Closely related problems with regular kernels have also been studied by MacCamy [17], Dafermos and Nohel [3], Staffans [26], Hattori [9], and Hrusa and Nohel [13]. See [12] for a summary of these works.)

Like Dafermos and Nohel, we assume $\phi(0) = \psi(0) = 0$, $\phi' > 0$, $\psi' > 0$, $\phi' - a(0)\,\psi' > 0$. They require that the kernel a is strongly positive definite; for technical reasons we make the stronger assumption that a is positive, monotone decreasing, and convex. While they assume that $a, a', a'' \in L^1(0, \infty)$, we allow a' to have a singularity at 0, e.g., $a'(t) \sim -t^{-\alpha}$, $0 < \alpha < 1$, as $t \downarrow 0$.

For definiteness, we shall always consider (1.1) with Dirichlet boundary conditions (1.2). We emphasize, however, that our local existence proof can be applied without change for Neumann or mixed boundary conditions or for the all-space problem (i.e., x varies from $-\infty$ to ∞). We have purposely avoided the use of Poincaré inequalities in our estimates for this reason. The global result can also be generalized to other boundary conditions. For the case of Neumann conditions, we need a trivial modification in the statement of the theorem, due to the possibility of rigid motions which need not decay as $t \to \infty$. We do not know how to extend the global result to the all-space problem. Recent work on this problem by Hrusa and Nohel [13] makes very essential use of the assumption that the kernel is regular.

It is not easy to quantify the regularizing effect of a singular kernel in general terms. Roughly speaking, certain types of waves are smoothed, while others are not. For those waves that are smoothed, the precise degree of smoothing depends crucially on the nature of the singularity in the kernel. This is discussed in detail for linear problems in [29].

In our treatment, we assume that f is smooth on $[0, 1] \times (-\infty, \infty)$ and that the history v satisfies Eq. (1.1) and the boundary conditions (1.2) for $t \leqslant 0$. This ensures that the data (f and v) are compatible with the boundary conditions and that derivatives of v as $t \uparrow 0$ are compatible with derivatives of u as $t \downarrow 0$. It is possible to remove the assumption that v satisfies the equation (provided f and v are compatible with the boundary conditions), with the result that certain derivatives of u may be discontinuous across $t = 0$.

The paper is organized as follows: In Section 2, we prove some preliminary lemmas concerning the kernel. In Section 3, we prove an existence result for a linear problem with variable coefficients. This is done by approximating the problem by problems with regular kernels, for which existence is known. We then use energy estimate that hold uniformly as the kernel becomes singular to show that the solutions of these approximate problems converge to a limit. In Section 4, we establish local existence for the nonlinear problem by using the results of Section 3 and a contraction argument. Section 5 contains a brief discussion of global existence. We notice that once local existence is known, the assumption $a'' \in L^1$ is not essential for the global existence proof of Dafermos and Nohel and can be avoided by a minor modification.

Our global existence theorem requires the data to be small. It is conceivable that for certain singular kernels, global smooth solutions of (1.1), (1.2), (1.3) also exist for large data. However, we have been unable to verify this.

With the exception of Section 2, subscripts x and t indicate partial differentiation. A prime denotes the derivative of a function of a single variable, and we use the symbol $:=$ for an equality in which the left-hand side is defined by the right-hand side. All derivatives should be interpreted in the distributional sense.

2. Preliminaries

This section contains some preliminary results (concerning the kernel a) that will be used in the subsequent sections. Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. For each $b \in L^1(0, \infty)$, $T \in R$, and $u \in L^2((-\infty, T]; H)$, we set

$$Q(u, t, b) := \int_{-\infty}^{t} \langle u(s), \int_{-\infty}^{s} b(s-\tau)\, u(\tau)\, d\tau \rangle\, ds \qquad \forall\, t \in (-\infty, T]. \tag{2.1}$$

We use a hat to denote the Laplace transform evaluated along the imaginary axis, i.e.,

$$\hat{q}(\omega) := \int_0^\infty e^{-i\omega t} q(t)\, dt \qquad \forall\, \omega \in R, \tag{2.2}$$

for real and H-valued functions q. For $T \in R$, $h > 0$, $u \colon (-\infty, T] \to H$, and $t \in (-\infty, T]$, we employ the notations

$$\Delta_h u(t) := u(t) - u(t-h) \qquad \forall\, t \in (-\infty, T], \tag{2.3}$$

and

$$u_t(\tau) := u(t-\tau) \qquad \forall\, \tau \geqslant 0; \tag{2.4}$$

in particular,

$$\hat{u}_t(\omega) := \int_0^\infty e^{-i\omega\tau} u(t-\tau)\, d\tau \qquad \forall\, \omega \in R. \tag{2.5}$$

The concept of a strongly definite kernel will play a central role in our analysis. We recall that a real-valued function $b \in L^1_{\mathrm{loc}}[0, \infty)$ is said to be positive definite (or of positive type) if

$$\int_0^t w(s) \int_0^s b(s-\tau)\, w(\tau)\, d\tau\, ds \geqslant 0 \qquad \forall\, t \geqslant 0, \tag{2.6}$$

for every $w \in C[0, \infty)$; b is called strongly positive definite if there exists a constant $\lambda > 0$ such that the function defined by $b(t) - \lambda e^{-t}$, $t > 0$, is positive definite. As the terminology suggests, strongly positive definite implies positive definite.

Throughout this section, we assume that

$$a, a' \in L^1(0, \infty), \qquad a \text{ is strongly positive definite.} \tag{2.7}$$

It follows from (2.7) that $a \in AC[0, \infty)$, $a(0) > 0$, and[1]

$$\operatorname{Re} \hat{a}(\omega) \geqslant \frac{\lambda}{\omega^2 + 1} \qquad \forall \omega \in R, \tag{2.8}$$

for some constant $\lambda > 0$. Consequently, $\operatorname{Re} \hat{a}$ is integrable and $(1/2\pi) \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega)\, d\omega = (1/2)\, a(0)$. (See, for example, [20] for more information on strongly positive definite kernels.)

In our analysis of Eq. (1.1), terms of the form $\lim_{h \downarrow 0} (1/h^2)\, Q(\Delta_h u, t, a)$ will arise, where it is known a priori merely that $u \in L^2((-\infty, T]; H)$. Of course, this is not sufficient to guarantee that the limit in question exists. However, if we know from other considerations that the limit does exist, some rather useful conclusions can be drawn.

LEMMA 2.1. *Let $T \in R$ and $u \in L^2((-\infty, T]; H)$ be given. Assume that (2.7) holds and that $\lim_{h \downarrow 0} (1/h^2)\, Q(\Delta_h u, t, a)$ exists (and is finite) for a.e. $t \in (-\infty, T]$. Then, for a.e. $t \in (-\infty, T]$,*

$$\lim_{h \downarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a) = \frac{1}{2} a(0)\, \|u(t)\|^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 \operatorname{Re} \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2\, d\omega$$
$$- \frac{i}{\pi} \overline{\left\langle u(t), \int_{-\infty}^{\infty} \operatorname{Im}(\widehat{a'}(\omega))\, \hat{u}_t(\omega)\, d\omega \right\rangle}. \tag{2.9}$$

In particular, each term in (2.9) is well-defined for a.e. $t \in (-\infty, T]$.

Proof. For each $h > 0$, we have

$$\frac{1}{h^2} Q(\Delta_h u, t, a) = \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega)\, \|\hat{u}_t(\omega) - \hat{u}_{t-h}(\omega)\|^2\, d\omega, \tag{2.10}$$

by Parseval's identity. Next, we observe that

$$\begin{aligned} \hat{u}_{t-h}(\omega) &= \int_0^\infty u(t - h - \tau)\, e^{-i\omega\tau}\, d\tau \\ &= \int_h^\infty u(t - \sigma)\, e^{-i\omega\sigma}\, e^{i\omega h}\, d\sigma \\ &= e^{i\omega h} \hat{u}_t(\omega) - e^{i\omega h} \int_0^h u(t - \sigma)\, e^{-i\omega\sigma}\, d\sigma, \end{aligned} \tag{2.11}$$

[1] In fact, for $a \in L^1(0, \infty)$ to be strongly positive definite it is necessary and sufficient that (2.8) hold for some $\lambda > 0$.

and consequently

$$\frac{1}{h^2} Q(\Delta_h u, t, a) = \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega) \left\| (1 - e^{i\omega h}) \hat{u}_t(\omega) + e^{i\omega h} \int_0^h u(t-\sigma) e^{-i\omega\sigma} d\sigma \right\|^2 d\omega. \tag{2.12}$$

Using the fundamental theorem of calculus and the dominated convergence theorem, we find that

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{2\pi h^2} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega) \left\| e^{i\omega h} \int_0^h u(t-\sigma) e^{-i\omega\sigma} d\sigma \right\|^2 d\omega \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \hat{a}(\omega) \|u(t)\|^2 d\omega \\ = \frac{1}{2} a(0) \|u(t)\|^2. \end{aligned} \tag{2.13}$$

(In particular, the limit on the left-hand side of (2.13) exists for a.e. $t \in (-\infty, T]$.) The desired result now follows from the facts that $\lim_{h \downarrow 0} (1/h)(1 - e^{i\omega h}) = -i\omega$, $|(1/h)(1 - e^{i\omega h})| \leqslant |\omega|$ $\forall h > 0$, and $\operatorname{Im} \widehat{a'}(\omega) = \omega \operatorname{Re} \hat{a}(\omega)$. ∎

It is important to note that the first and second terms on the right-hand side of (2.9) are nonnegative. The next lemma provides a useful estimate for the last term in this expression.

LEMMA 2.2. *Assume that* (2.7) *holds and let* $\varepsilon > 0$ *be given. Then, there exists a constant* $C(\varepsilon)$ *such that*

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} (\operatorname{Im} \widehat{a'}(\omega)) \hat{u}_t(\omega) d\omega \right\|^2 \\ \leqslant \varepsilon \int_{-\infty}^{\infty} \omega^2 \operatorname{Re} \hat{a}(\omega) \|\hat{u}_t(\omega)\|^2 d\omega \\ + C(\varepsilon) \int_{-\infty}^{\infty} \|\hat{u}_t(\omega)\|^2 d\omega \qquad \text{a.e. } t \in (-\infty, T], \end{aligned} \tag{2.14}$$

for every $T \in R$ *and every* $u \in L^2((-\infty, T]; H)$. (*No claim is made that the integrals in* (2.14) *are all finite.*)

Proof. Observe that

$$|\operatorname{Im} \widehat{a'}(\omega)| = \sqrt{|\omega \operatorname{Im} \widehat{a'}(\omega)|} \cdot \sqrt{|\operatorname{Im} \widehat{a'}(\omega)/\omega|}, \qquad \omega \neq 0. \tag{2.15}$$

Using (2.15) and the Cauchy–Schwarz inequality, we find that for each $\alpha > 0$,

$$\left\| \int_{-\infty}^{\infty} (\operatorname{Im} \widehat{a'}(\omega))\, \hat{u}_t(\omega)\, d\omega \right\|^2$$

$$\leqslant 4\alpha \int_{-\alpha}^{\alpha} \|\hat{u}_t(\omega)\|^2\, d\omega \cdot \sup_{[-\alpha,\alpha]} |\operatorname{Im} \widehat{a'}(\omega)|^2$$

$$+ 2\left(\int_{A_\alpha} |\omega \operatorname{Im} \widehat{a'}(\omega)| \cdot \|u_t(\omega)\|^2\, d\omega\right) \cdot \left(\int_{A_\alpha} \left|\frac{\operatorname{Im} \widehat{a'}(\omega)}{\omega}\right| d\omega\right), \qquad (2.16)$$

where $A_\alpha := (-\infty, \alpha] \cup [\alpha, \infty)$. Recalling that $\operatorname{Im} \widehat{a'}(\omega) = \omega \operatorname{Re} \hat{a}(\omega)$ and that $\operatorname{Re} \hat{a}$ is integrable over $(-\infty, \infty)$, the lemma follows from (2.16) for a sufficiently large choice of α.

Combining Lemmas 2.1 and 2.2, and making use of the simple algebraic inequality $|AB| \leqslant \eta A^2 + B^2/4\eta$ for all $\eta > 0$, we easily establish

LEMMA 2.3. *Assume that* (2.7) *holds. Then, for each* $\varepsilon > 0$, *there exists a constant* $C(\varepsilon)$ *such that*

$$\lim_{h \downarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a) \geqslant \left(\frac{1}{2} a(0) - \varepsilon\right) \|u(t)\|^2$$

$$- C(\varepsilon) \int_{-\infty}^{t} \|u(s)\|^2\, ds \qquad a.e.\ t \in (-\infty, T], \qquad (2.17)$$

for every $T \in R$ *and every* $u \in L^2((-\infty, T], H)$ *for which* $\lim_{h \downarrow 0} (1/h^2)$ $Q(\Delta_h u, t, a)$ *exists a.e. in* $t \in (-\infty, T]$.

To discuss certain continuity properties of solutions of (1.1), it is important to know whether or not the mapping $t \mapsto \hat{u}_t(\cdot)$ is continuous from $(-\infty, T]$ to be weighted L^2-space $L^2(R; H/(1 + \omega^2 \operatorname{Re} \hat{a}(\omega))\, d\omega)$ with norm given by $|||g|||^2 := \int_{-\infty}^{\infty} (1 + \omega^2 \operatorname{Re} \hat{a}(\omega))\, \|g(\omega)\|^2\, d\omega$. Using the relationship between $\hat{}$ and the Fourier transform, the fact that the Fourier transform of a product is equal to the convolution of the Fourier transforms, and the formula for the Fourier transform of a step function, we find that for each $t \in (-\infty, T]$ and every real η such that $t + \eta \in (-\infty, T]$, $\hat{u}_t$ is given by

$$2\hat{u}_{t+\eta}(\omega) = e^{i\omega(T-t-\eta)} \hat{u}_T(\omega) - i\mathscr{H}[e^{i(T-t-\eta)\cdot}\, \hat{u}_T(\cdot)](\omega),$$

where $\mathscr{H}$ denotes the Hilbert transform. The question thus reduces to boundedness of the Hilbert transform on $L^2(R; H/(1 + \omega^2 \operatorname{Re} \hat{a}(\omega))\, d\omega)$. Using Theorem 6.2 of [6, p. 255], we find

LEMMA 2.4. *Let* $T \in R$ *and* $u \in L^2((-\infty, T]; H)$ *be given. Assume that* (2.7) *holds,* $\int_{-\infty}^{\infty} \omega^2 \operatorname{Re} \hat{a}(\omega)\, \|\hat{u}_T(\omega)\|^2\, d\omega$ *exists, and that the "(A_2)-condition"*

$$\sup_I \left(\frac{1}{|I|}\int_I (1+\omega^2 \operatorname{Re} \hat{a}(\omega))\, d\omega\right) \cdot \left(\frac{1}{|I|}\int_I \frac{d\omega}{1+\omega^2 \operatorname{Re} \hat{a}(\omega)}\right) < \infty \quad (2.18)$$

holds, where the sup *in* (2.18) *is taken over all intervals* $I \subset R$. *Then,* $\hat{u}_t(\cdot) \in L^2(R; H/(1+\omega^2 \operatorname{Re} \hat{a}(\omega))\, d\omega)$ *for all* $t \leqslant T$, *and the mapping* $t \mapsto \hat{u}_t(\cdot)$ *is continuous from* $(-\infty, T]$ *to* $L^2(R; H/(1+\omega^2 \operatorname{Re} \hat{a}(\omega))\, d\omega)$.

Remark 2.2. Condition (2.18) holds if $\widehat{a'}(\omega) \sim \omega^{-\alpha}$ as $\omega \to \infty$, with $0 < \alpha \leqslant 1$. This is essentially the case if $a'(t) \sim t^{\alpha - 1}$ as $t \to 0$. Such kernels are suggested by molecular theories [5, 24, 28]. In this case $(\int_{-\infty}^{\infty} (1+\omega^2 \operatorname{Re} \hat{a}(\omega))\, \|\hat{u}_t(\omega)\|^2\, d\omega)^{1/2}$ is equivalent to a fractional order Sobolev norm of u_t.

Our next lemma will be used to modify the global existence proof of Dafermos and Nohel [4].

LEMMA 2.5. *Assume that* (2.7) *holds. Then, for each* $\varepsilon > 0$, *there exists a constant* $C(\varepsilon)$ *such that*

$$\int_{-\infty}^{t} \left\| \int_{-\infty}^{s} a'(s-\tau)\, u(\tau)\, d\tau \right\|^2 ds \leqslant \varepsilon \int_{-\infty}^{t} \|u(\tau)\|^2\, d\tau + C(\varepsilon)\, Q(u, t, a)$$

$$\forall t \in (-\infty, T], \qquad (2.19)$$

for every $T \in R$ *and every* $u \in L^2((-\infty, T]; H)$.

Proof. Taking Laplace transforms, (2.19) reduces to

$$|\widehat{a'}(\omega)|^2 \leqslant \varepsilon + C(\varepsilon) \operatorname{Re} \hat{a}(\omega) \qquad \forall \omega \in R. \qquad (2.20)$$

This last inequality is immediate since $\operatorname{Re} \hat{a}(\omega) > 0$ and $\lim_{|\omega| \to \infty} |\widehat{a'}(\omega)| = 0$ (by the Riemann–Lebesgue lemma). ▮

Remark 2.3. If $a'' \in L^1(0, \infty)$, then (2.19) holds with $\varepsilon = 0$ and $C(0) < \infty$. This version of the lemma was used by Dafermos and Nohel [4].

We now discuss approximation of a by regular kernels. At this point,[2] we assume

$$a, a' \in L^1(0, \infty), \qquad (2.21)$$

$$\left.\begin{array}{l} a \geqslant 0, \quad a' \leqslant 0, \quad a'' \geqslant 0 \quad \text{(in the sense of measures);} \\ \text{the measure } a'' \text{ has a nontrivial absolutely} \\ \text{continuous component.} \end{array}\right\} \quad (2.22)$$

[2] The problem of approximating an arbitrary strongly positive definite kernel by "regularized" strongly positive definite kernels does not appear to be easy. We could base our existence argument on an approximation method other than approximating the kernel, e.g., finite differences. If this is done, (2.22) is not needed, but the proofs become much more complicated. Moreover, (2.22) is a natural assumption from the viewpoint of applications to viscoelasticity.

As is well known, this implies that a is strongly positive definite. (Corollary 2.2 of [20].) For each $\delta > 0$, we define the approximating kernel $a_\delta : [0, \infty) \to R$ by

$$a_\delta(t) := \int_{-\delta}^{\delta} \rho_\delta(\tau)\, a(t+\delta-\tau)\, d\tau \qquad \forall t \geqslant 0, \tag{2.23}$$

where ρ_δ is a standard mollifier with support contained in $[-\delta/2, \delta/2]$.

It follows from (2.21), (2.22), (2.23) that for every $\delta > 0$

$$a_\delta \in C^\infty[0, \infty), \qquad a_\delta \geqslant 0, \qquad a'_\delta \leqslant 0, \qquad a''_\delta \geqslant 0, \tag{2.24}$$

$$a_\delta,\, a'_\delta,\, a''_\delta \in L^1(0, \infty), \tag{2.25}$$

and

$$\|a_\delta\|_1 \leqslant \|a\|_1, \qquad \|a'_\delta\|_1 \leqslant a(0), \tag{2.26}$$

where $\|\cdot\|_1$ denotes the norm in $L^1(0, \infty)$. (Of course, $\|a''_\delta\|_1$ does not necessarily remain bounded as $\delta \downarrow 0$.) It also follows that a_δ is strongly positive definite for δ sufficiently small and that $a_\delta \to a$ pointwise (and in $L^1(0, \infty)$) as $\delta \downarrow 0$. Moreover, $\sup_{\omega \in R} |\widehat{a'_\delta}(\omega)| \leqslant a(0)$ for all $\delta > 0$, and $\operatorname{Re} \hat{a}_\delta \to \operatorname{Re} \hat{a}$ in $L^1(R)$ as $\delta \downarrow 0$. Therefore, a simple modification of the proof of Lemma 2.3 yields

LEMMA 2.6. *Assume that* (2.21), (2.22) *hold and let* $\varepsilon > 0$ *be given. Then, there exist constants* $C(\varepsilon)$, $\delta_0(\varepsilon) > 0$ *such that for every* $\delta \in (0, \delta_0(\varepsilon)]$

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h^2} Q(\Delta_h u, t, a_\delta) &\geqslant \left(\frac{1}{2} a(\delta) - \varepsilon\right) \|u(t)\|^2 \\ &\quad - C(\varepsilon) \int_{-\infty}^{t} \|u(s)\|^2\, ds \qquad \text{a.e. } t \in (-\infty, T], \end{aligned} \tag{2.27}$$

for every $T \in R$ *and every* $u \in L^2((-\infty, T]; H)$ *such that* $\lim_{h \downarrow 0} (1/h^2)\, Q(\Delta_h u, t, a_\delta)$ *exists a.e. in* $t \in (-\infty, T]$.

In our subsequent use of this material, we shall always take H to be (the complexification of) $L^2(0, 1)$.

3. LINEAR EQUATIONS

In this section, we study the linear history value problem

$$\begin{aligned} u_{tt}(x, t) &= \alpha(x, t)\, u_{xx}(x, t) \\ &\quad + \int_{-\infty}^{t} a'(t-\tau)\, \beta(x, \tau)\, u_{xx}(x, \tau)\, d\tau + f(x, t), \end{aligned}$$

$$x \in [0, 1], \quad t \in (-\infty, T], \tag{3.1}$$

$$u(0, t) = u(1, t) = 0, \qquad t \in (-\infty, T], \tag{3.2}$$

$$u(x, t) = v(x, t), \qquad x \in [0, 1], \quad t \in (-\infty, 0], \tag{3.3}$$

where T is a given positive number. We begin by stating an existence result for the case when the kernel does not have a singularity. There are many such existence theorems in the literature. (See, for example, [2, 10], and the references therein.) The particular one which we give here has been formulated with smoothness assumptions which are appropriate for our treatment of quasilinear equations in the next section.

We assume that the coefficients satisfy

$$\alpha, \alpha_x, \alpha_t, \alpha_{xx}, \alpha_{xt}, \alpha_{tt}, \beta, \beta_x, \beta_t, \beta_{xx}, \beta_{xt}, \beta_{tt} \in L^\infty((-\infty, T]; L^2(0, 1)), \tag{3.4}$$

$$\alpha(x, t) \geqslant \underline{\alpha} > 0 \qquad \forall x \in [0, 1], \quad t \in (-\infty, T]. \tag{3.5}$$

Of f and v we require

$$\begin{gathered} f, f_x, f_t \in L^\infty((-\infty, T]; L^2(0, 1)) \cap L^2((-\infty, T]; L^2(0, 1)), \\ f_{tt} \in L^2((-\infty, T]; L^2(0, 1)), \end{gathered} \tag{3.6}$$

$$\begin{aligned} &v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}, v_{xxx}, v_{xxt}, v_{xtt}, v_{ttt} \in L^\infty((-\infty, 0]; L^2(0, 1)) \\ &\quad \cap L^2((-\infty, 0]; L^2(0, 1)). \end{aligned} \tag{3.7}$$

In addition, we assume that v satisfies the equation and boundary conditions for $t \leqslant 0$, i.e.,

$$\begin{aligned} v_{tt}(x, t) &= \alpha(x, t)\, v_{xx}(x, t) \\ &\quad + \int_{-\infty}^{t} a'(t-\tau)\, \beta(x, \tau)\, v_{xx}(x, \tau)\, d\tau + f(x, t), \\ &\qquad x \in [0, 1], \quad t \in (-\infty, 0], \end{aligned} \tag{3.8}$$

$$v(0, t) = v(1, t) = 0, \qquad t \in (-\infty, 0]. \tag{3.9}$$

LEMMA 3.1. *Assume that* $a', a'' \in L^1(0, \infty)$, α *and* β *satisfy* (3.4), *and that* (3.5) *holds for some constant* $\underline{\alpha} > 0$. *Let* f *and* v *satisfying* (3.6) *through* (3.9) *be given. Then, the history value problem* (3.1), (3.2), (3.3) *has a unique solution* u *with*

$$u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in L^\infty((-\infty, T]; L^2(0, 1)). \tag{3.10}$$

If, in addition,

$$f_x \in C([0, T]; L^2(0, 1)), \tag{3.11}$$

then the solution has the additional regularity

$$u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in C([0, T]; L^2(0, 1)) \tag{3.12}$$

for positive time.

We have been unable to locate an existence theorem in the literature which has precisely the same smoothness conditions as Lemma 3.1. However, this type of result is standard and we omit the proof. For example, a minor modification of the proof of Theorem 2.1 of [4] can be used to establish Lemma 3.1.

We now prove an existence theorem which allows a' to have a singularity at 0. For this case, we must assume that the memory term satisfies the appropriate sign conditions, i.e., that (2.21), (2.22) hold and

$$\beta(x, t) \geqslant \underline{\beta} > 0 \qquad \forall x \in [0, 1], \quad t \in (-\infty, T]. \tag{3.13}$$

THEOREM 3.1. *Assume that* (2.21), (2.22), (3.4), (3.5), (3.13) *hold, and let f and v satisfying* (3.6) *through* (3.9) *be given. Assume further that* $v_{xxtt} \in L^2((-\infty, 0]; L^2(0, 1))$. *Then, the history value problem* (3.1), (3.2), (3.3) *has a unique solution u which satisfies* (3.10). *If, in addition,* (2.18) *and* (3.11) *hold, then u has the additional regularity* (3.12) *for positive time.*

Proof. Consider the family of approximating problems

$$u_{tt}^{(\delta)}(x, t) = \alpha(x, t)\, u_{xx}^{(\delta)}(x, t) + \int_{-\infty}^{t} a_\delta'(t-\tau)\, \beta(x, \tau)\, u_{xx}^{(\delta)}(x, \tau)\, d\tau$$
$$+ f^{(\delta)}(x, t), \qquad x \in [0, 1], \quad t \in (-\infty, T], \tag{3.14}$$

$$u^{(\delta)}(0, t) = u^{(\delta)}(1, t) = 0, \qquad t \in (-\infty, T], \tag{3.15}$$

$$u^{(\delta)}(x, t) = v(x, t), \quad x \in [0, 1], \qquad t \in (-\infty, 0], \tag{3.16}$$

for $\delta > 0$, where a_δ is defined by (2.33) and $f^{(\delta)}$ approximates f in such a way that v satisfies Eq. (3.14) for $t \leqslant 0$ and $f^{(\delta)}$, $f_x^{(\delta)}$, $f_t^{(\delta)} \to f, f_x, f_t$ in $L^\infty((-\infty, T]; \; L^2(0, 1)) \cap L^2((-\infty, T]; \; L^2(0, 1))$, $f_{tt}^{(\delta)} \to f_{tt}$ in $L^2((-\infty, T]; L^2(0, 1))$ as $\delta \downarrow 0$. (The existence of such an approximation to f follows from our assumptions on f and v and a straightforward extension theorem. It is here that the assumption $v_{xxtt} \in L^2((-\infty, 0]; L^2(0, 1))$ is used.) It follows from Lemma 3.1 that for each $\delta > 0$, (3.14), (3.15), (3.16) has a unique solution $u^{(\delta)}$ with $u^{(\delta)}$, $u_x^{(\delta)}$, $u_t^{(\delta)}$, $u_{xx}^{(\delta)}$, $u_{xt}^{(\delta)}$, $u_{tt}^{(\delta)}$, $u_{xxx}^{(\delta)}$, $u_{xxt}^{(\delta)}$, $u_{xtt}^{(\delta)}$, $u_{ttt}^{(\delta)} \in L^\infty((-\infty, T]; L^2(0, 1))$.

Our objective is to show that $u^{(\delta)}$ obeys certain a priori bounds, uniformly in δ, that imply the existence of a sequence $\{u^{(\delta_n)}\}_{n=1}^{\infty}$ which converges to a solution as $\delta_n \downarrow 0$. In order to simplify the notation, we suppress the superscripts on $u^{(\delta)}$ and $f^{(\delta)}$. For the purpose of deriving such bounds, we set

$$V := \operatorname*{ess\text{-}sup}_{s\in(-\infty,0]} \int_0^1 \{v_x^2 + v_{xx}^2 + v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2 + v_{ttt}^2\}(x,s)\,dx \\ + \int_{-\infty}^0 \int_0^1 \{v_{xx}^2 + v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2 + v_{ttt}^2\}(x,s)\,dx\,ds, \tag{3.17}$$

$$F := \operatorname*{ess\text{-}sup}_{s\in(-\infty,T]} \int_0^1 \{f_x^2 + f_t^2\}(x,s)\,dx \\ + \int_{-\infty}^T \int_0^1 \{f_x^2 + f_t^2 + f_{tt}^2\}(x,s)\,dx\,ds, \tag{3.18}$$

$$\tfrac{1}{2}\Gamma_0 := \operatorname*{ess\text{-}sup}_{s\in(-\infty,0]} \int_0^1 \{\alpha^2 + \alpha_x^2 + \alpha_t^2 + \alpha_{xx}^2 + \alpha_{xt}^2 + \alpha_{tt}^2 \\ + \beta^2 + \beta_x^2 + \beta_t^2 + \beta_{xx}^2 + \beta_{xt}^2 + \beta_{tt}^2\}(x,s)\,dx, \tag{3.19}$$

$$\tfrac{1}{2}\Gamma_1 := \operatorname*{ess\text{-}sup}_{s\in[0,T]} \int_0^1 \{\alpha^2 + \alpha_x^2 + \alpha_t^2 + \alpha_{xx}^2 + \alpha_{xt}^2 + \alpha_{tt}^2 + \beta^2 \\ + \beta_x^2 + \beta_t^2 + \beta_{xx}^2 + \beta_{xt}^2 + \beta_{tt}^2\}(x,s)\,dx, \tag{3.20}$$

and

$$E[u](t) := \operatorname*{ess\text{-}sup}_{s\in[0,t]} \int_0^1 \{u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)\,dx \\ \forall t \in [0,T], \tag{3.21}$$

and we observe that there exists a constant $\lambda > 0$ such that

$$\frac{\alpha(x,t)}{\beta(x,t)} \geqslant \lambda \qquad \forall x \in [0,1], \quad t \in (-\infty, T], \tag{3.22}$$

by virtue of (3.4), (3.5), (3.13).

An integration by parts in (3.14) yields

$$u_{tt} = \gamma^{(\delta)} u_{xx} + \int_{-\infty}^t a_\delta(t-\tau)[\beta u_{xx}]_t(x,\tau)\,d\tau + f, \tag{3.23}$$

where

$$\gamma^{(\delta)}(x, t) := \alpha(x, t) - a_\delta(0)\,\beta(x, t). \tag{3.24}$$

We apply the backward difference operator Δ_h (in the time variable) to (3.23), thus obtaining

$$\Delta_h u_{tt} = \Delta_h[\gamma^{(\delta)} u_{xx}] + \int_{-\infty}^{t} a_\delta(t-\tau)\,\Delta_h[(\beta u_{xx})_t](x, \tau)\,d\tau + \Delta_h f. \tag{3.25}$$

Then, we multiply (3.25) by $\Delta_h[(\beta u_{xx})_t]$ and integrate over $[0, 1] \times (-\infty, t]$, $t \in [0, T]$. After several integrations by parts, we divide by h^2 and let $h \downarrow 0$. The outcome of his tedious, but straightforward, computation is

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \{\beta\gamma^{(\delta)} u_{xxt}^2 + \beta u_{xtt}^2\}(x, t)\,dx + \lim_{h\downarrow 0}\frac{1}{h^2}\,Q(\Delta_h[(\beta u_{xx})_t], t, a_\delta)\\
&\quad + \int_0^1 \{\beta\gamma_t^{(\delta)} u_{xx} u_{xxt} + \beta f_t u_{xxt}\}(x, t)\,dx\\
&= \int_{-\infty}^{t}\int_0^1 \left\{\frac{3}{2}\beta\gamma_t^{(\delta)} u_{xxt}^2 - \frac{3}{2}\beta_t\gamma^{(\delta)} u_{xxt}^2 + \frac{1}{2}\beta_t u_{xtt}^2\right.\\
&\quad - \beta_x u_{xtt} u_{ttt} + \beta\gamma_{tt}^{(\delta)} u_{xx} u_{xxt} + 2\beta_t u_{xxt} u_{ttt} + \beta_{tt} u_{xx} u_{ttt}\\
&\quad - \beta_{tt}\gamma^{(\delta)} u_{xx} u_{xxt} - \beta_t\gamma_t^{(\delta)} u_{xx} u_{xxt} - \beta_{tt}\gamma_t^{(\delta)} u_{xx}^2\\
&\quad \left. + \beta f_{tt} u_{xxt} - \beta_t f_t u_{xxt} - \beta_{tt} f_t u_{xx}\right\}(x, s)\,dx\,ds \qquad \text{a.e. } t \in (-\infty, T],
\end{aligned} \tag{3.26}$$

where Q is defined by(2.1) with $H = L^2(0, 1)$. (We note that u_t, u_{tt}, $\Delta_h u$, $\Delta_h u_t$, and $\Delta_h u_{tt}$ all vanish at $x = 0, 1$ by virtue of (3.15). All of the spatial integrations by parts used in the derivation of (3.26) were carried out in such a way that the boundary terms (at $x = 0, 1$) vanish.)

It is not a priori evident that $\lim_{h\downarrow 0}(1/h^2)\,Q(\Delta_h[(\beta u_{xx})_t], t, a_\delta)$ exists for a.e. $t \in (-\infty, T]$. However, all of the other limits involved in the derivation of (3.26) exist for a.e. $t \in (-\infty, T]$, and consequently so does the limit in question.

Using (3.5), (3.13), (3.24), Lemma 2.6 (with ε sufficiently small relative to $\underline{\lambda}$), and the algebraic inequality $|AB| \leqslant \eta A^2 + (1/4\eta)\,B^2\ \forall \eta > 0$, we find that the left-hand side of (3.26) is bounded from below by

$$\int_0^1 \{\tfrac{1}{4}\underline{\lambda}\underline{\beta}^2 u_{xxt}^2 + \tfrac{1}{2}\underline{\beta} u_{xtt}^2\}(x, t)\, dx$$

$$- C \int_0^1 \{(\alpha_t^2 + \beta_t^2)\, u_{xx}^2 + f_t^2\}(x, t)\, dx$$

$$- C \int_{-\infty}^t \int_0^1 \{\beta^2 u_{xxt}^2 + \beta_t^2\, u_{xx}^2\}(x, s)\, dx\, ds$$

$$\forall t \in (-\infty, T], \quad \delta \in (0, \delta_0], \tag{3.27}$$

where C is a positive constant (which depends on $\underline{\lambda}$ and $\underline{\beta}$.)

Differentiating (3.14) with respect to t and x, and splitting the convolution integrals, we obtain

$$u_{ttt} = \alpha u_{xxt} + \alpha_t u_{xx} + f_t + \int_{-\infty}^0 a'_\delta(t-\tau)[\beta v_{xxt} + \beta_t v_{xx}](x, \tau)\, d\tau$$

$$+ \int_0^t a'_\delta(t-\tau)[\beta u_{xxt} + \beta_t u_{xx}](x, \tau)\, d\tau, \tag{3.28}$$

$$\alpha u_{xxx} + \int_0^t a'_\delta(t-\tau)[\beta u_{xxx}](x, \tau)\, d\tau = u_{xtt} - \alpha_x u_{xx} - f_x$$

$$- \int_{-\infty}^0 a'_\delta(t-\tau)[\beta v_{xxx} + \beta_x v_{xx}](x, \tau)\, d\tau$$

$$- \int_0^t a'_\delta(t-\tau)[\beta_x u_{xx}](x, \tau)\, d\tau. \tag{3.29}$$

It follows easily from (3.28) that

$$\int_0^1 u_{ttt}^2(x, t)\, dx \leqslant 5 \int_0^1 \{\alpha^2 u_{xxt}^2 + \alpha_t^2 u_{xx}^2 + f_t^2\}(x, t)\, dx$$

$$+ 10a(0)^2 \operatorname*{ess-sup}_{s \in [0, t]} \int_0^1 \{\beta^2 u_{xxt}^2 + \beta_t^2 u_{xx}^2\}(x, s)\, dx$$

$$+ 10a(0)^2 \operatorname*{ess-sup}_{s \in (-\infty, 0]} \int_0^1 \{\beta^2 v_{xxt}^2 + \beta_t^2 v_{xx}^2\}(x, s)\, dx$$

$$\text{a.e. } t \in [0, T]. \tag{3.30}$$

Using Gronwall's inequality in (3.29), we obtain, after a straightforward computation,

$$\int_0^1 [\alpha u_{xxx}]^2(x, t)\, dx$$

$$\leqslant 8 \exp[2a(0)\, \underline{\lambda}^{-1}] \operatorname*{ess-sup}_{s \in [0,t]}$$

$$\times \int_0^1 \{u_{xtt}^2 + \alpha_x^2 u_{xx}^2 + f_x^2 + a(0)^2 \beta_x^2 u_{xx}^2\}(x, s)\, ds$$

$$+ 4a(0)^2 \exp[2a(0)\, \underline{\lambda}^{-1}] \operatorname*{ess-sup}_{s \in (-\infty, 0]} \int_0^1 \{\beta^2 v_{xxx}^2 + \beta_x^2 v_{xx}^2\}(x, s)\, dx$$

$$\text{a.e. } t \in [0, T]. \qquad (3.31)$$

Combining (3.26), (3.30), and (3.31), and recalling the lower bound (3.27), we conclude that there exists a positive constant K such that

$$E[u](t) \leqslant K\{F + (1 + \Gamma_0 + \Gamma_1 T)\, V\}$$

$$+ K \cdot (1 + \Gamma_1) \cdot (1 + T^2) \int_0^t E[u](s)\, ds$$

$$\forall t \in [0, T], \quad \delta \in (0, \delta_0]. \qquad (3.32)$$

(The constant K depends on $\underline{\alpha}$, $\underline{\beta}$, $\underline{\lambda}$, and a, but is independent of F, V, Γ_0, Γ_1, T, and δ.) Gronwall's inequality and (3.32) yield

$$E[u](T) \leqslant K\{F + (1 + \Gamma_0 + \Gamma_1 T)\, V\} \exp[K \cdot (1 + \Gamma_1) \cdot (T + T^3)] \qquad (3.33)$$

for all $\delta \in (0, \delta_0]$.

To assist the reader in following the derivation of (3.32), we show the detailed estimation of a few typical terms. By the Sobolev embedding theorem, $\beta_x^2(x, t) \leqslant \Gamma_0$ for all $x \in [0, 1]$, $t \in (-\infty, 0]$, and $\beta_x^2(x, t) \leqslant \Gamma_1$ for all $x \in [0, 1]$, $t \in [0, T]$. Therefore,

$$\left| \int_{-\infty}^t \int_0^1 \beta_x u_{xtt} u_{ttt}(x, s)\, dx\, ds \right|$$

$$\leqslant \tfrac{1}{2} \int_{-\infty}^t \int_0^1 \{\beta_x^2 u_{xtt}^2 + u_{ttt}^2\}(x, s)\, dx\, ds$$

$$= \tfrac{1}{2} \int_{-\infty}^0 \int_0^1 \{\beta_x^2 v_{xtt}^2 + v_{ttt}^2\}(x, s)\, dx\, ds$$

$$+ \tfrac{1}{2} \int_0^t \int_0^1 \{\beta_x^2 u_{xtt}^2 + u_{ttt}^2\}(x, s)\, dx\, ds$$

$$\leqslant \tfrac{1}{2}(\Gamma_0 + 1)\, V + \tfrac{1}{2}(\Gamma_1 + 1) \int_0^t E[u](s)\, ds \qquad \forall t \in [0, T]. \quad (3.34)$$

Next, we observe that

$$\tfrac{1}{2} \max_{\xi \in [0,1]} v_{xx}^2(\xi, s) \leqslant \int_0^1 \{v_{xx}^2 + v_{xxx}^2\}(x, s)\, ds \qquad \forall s \in (-\infty, 0], \quad (3.35)$$

and consequently

$$\tfrac{1}{2} \int_{-\infty}^0 \max_{\xi \in [0,1]} v_{xx}^2(\xi, s)\, ds \leqslant V. \quad (3.36)$$

In addition, we note that

$$u_{xx}(x, t) = v_{xx}(x, 0) + \int_0^t u_{xxt}(x, s)\, ds \qquad \forall x \in [0, 1], \qquad t \in [0, T], \quad (3.37)$$

from which we easily deduce the estimates

$$\begin{aligned} \int_0^1 u_{xx}^2(x, t)\, dx &\leqslant 2 \int_0^1 v_{xx}^2(x, 0)\, dx + 2t \int_0^t \int_0^1 u_{xxt}^2(x, s)\, dx\, ds \\ &\leqslant 2V + 2T^2 E[u](t) \qquad \forall t \in [0, T], \end{aligned} \quad (3.38)$$

and

$$\tfrac{1}{2} \max_{x \in [0,1]} u_{xx}^2(x, t) \leqslant 2V + (1 + 2T^2)\, E[u](t) \qquad \forall t \in [0, T]. \quad (3.39)$$

Using (3.36) and (3.39), we find

$$\begin{aligned} &\left| \int_{-\infty}^t \int_0^1 \beta_{tt} u_{xx} u_{ttt}(x, s)\, dx\, ds \right| \\ &\quad \leqslant \tfrac{1}{2} \int_{-\infty}^t \int_0^1 \{\beta_{tt}^2 u_{xx}^2 + u_{ttt}^2\}(x, s)\, dx\, ds \\ &\quad \leqslant \tfrac{1}{2} \int_{-\infty}^t \max_{\xi \in [0,1]} u_{xx}^2(\xi, s) \int_0^1 \beta_{tt}^2(x, s)\, dx\, ds \\ &\qquad + \tfrac{1}{2} \int_{-\infty}^t \int_0^1 u_{ttt}^2(x, s)\, dx\, ds \\ &\quad \leqslant \tfrac{1}{2} \int_{-\infty}^0 \max_{\xi \in [0,1]} v_{xx}^2(\xi, s) \int_0^1 \beta_{tt}^2\, (x, s)\, dx\, ds \\ &\qquad + \tfrac{1}{2} \int_{-\infty}^0 \int_0^1 v_{ttt}^2(x, s)\, dx\, ds \end{aligned}$$

$$+ \tfrac{1}{2}\int_0^t \max_{\xi\in[0,1]} u_{xx}^2(\xi, s)\int_0^1 \beta_{tt}^2(x, s)\,dx\,ds$$

$$+ \tfrac{1}{2}\int_0^t\int_0^1 u_{ttt}^2(x, s)\,dx\,ds$$

$$\leqslant \tfrac{1}{2}(\Gamma_0+1)\,V+\Gamma_1 VT+\tfrac{1}{2}\Gamma_1\cdot(1+2T^2)\int_0^t E[u](s)\,ds$$

$$+ \tfrac{1}{2}\int_0^t E[u](s)\,ds \qquad \forall t\in[0, T]. \tag{3.40}$$

The other terms can all be handled in a similar manner.

We conclude from (3.33) that $u_{xxx}^{(\delta)}$, $u_{xxt}^{(\delta)}$, $u_{xtt}^{(\delta)}$, and $u_{ttt}^{(\delta)}$ are bounded in $L^\infty([0, T];\ L^2(0, 1))$ independently of $\delta\in(0, \delta_0]$. It follows from (3.38) (and similar inequalities for the other derivatives) that $u_{xx}^{(\delta)}$, $u_{xt}^{(\delta)}$, $u_{tt}^{(\delta)}$, $u_x^{(\delta)}$, $u_t^{(\delta)}$, and $u^{(\delta)}$ are also bounded in $L^\infty([0, T];\ L^2(0, 1))$ independently of $\delta\in(0, \delta_0]$. Therefore, there exists a function u: $[0, 1]\times(-\infty, T]\to R$, with $u=v$ on $[0, 1]\times(-\infty, 0]$, and a sequence $\{\delta_n\}_{n=1}^\infty$, with $\delta_n\downarrow 0$ as $n\to\infty$, such that

$$u^{(\delta_n)}, u_x^{(\delta_n)}, u_t^{(\delta_n)}, u_{xx}^{(\delta_n)}, u_{xt}^{(\delta_n)}, u_{tt}^{(\delta_n)}, u_{xxx}^{(\delta_n)}, u_{xxt}^{(\delta_n)}, u_{xtt}^{(\delta_n)}, u_{ttt}^{(\delta_n)}$$
$$\to u, u_x, u_t, \text{etc.} \qquad \text{weakly star in } L^\infty([0, T]; L^2(0, 1)) \tag{3.41}$$

as $n\to\infty$. Standard embedding theorems and (3.41) imply

$$u^{(\delta_n)}, u_x^{(\delta_n)}, u_t^{(\delta_n)}, u_{xx}^{(\delta_n)}, u_{xt}^{(\delta_n)}, u_{tt}^{(\delta_n)} \to u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}$$
$$\text{uniformly on } [0, 1]\times[0, T] \tag{3.42}$$

as $n\to\infty$. It thus follows easily that u satisfies (3.1), (3.2), (3.3).

Suppose that (2.18) and (3.11) hold. To show that the third-order derivatives of u belong to $C([0, T]; L^2(0, 1))$, we argue along the lines of Strauss [27]. We first note that Theorem 2.1 of [27] implies that u_{xxx}, u_{xxt}, u_{xtt}, and u_{ttt} are weakly continuous from $(-\infty, T]$ to $L^2(0, 1)$. Indeed, by (3.10), we have u_{xx}, u_{xt}, $u_{tt}\in C((-\infty, T];\ L^2(0, 1))$ and consequently u_{xxx}, u_{xxt}, $u_{xtt}\in C((-\infty, T]; H^{-1}(0, 1))$. Thus, Theorem 2.1 of [27] implies that u_{xxx}, u_{xxt}, and u_{xxt} are weakly continuous from $(-\infty, T]$ to $L^2(0, 1)$. The weak continuity of u_{ttt} then follows from differentiation of (3.1) with respect to t. Now, the basic idea is to show that a certain energy which acts like a 'variable norm" of third derivatives is continuous. This, in conjunction with the aforementioned weak continuity, will imply the desired strong continuity.

We apply the procedure used to derive (3.26) to (3.1), (3.2), (3.3). We thus conclude that for a.e. $t\in(-\infty, T]$, u satisfies (3.26) with a_δ replaced by a. Using Lemmas 2.1 and 2.4, and the fact that the right-hand side of (3.26) is continuous in t, we find that

$$H[u](t) := \tfrac{1}{2}\int_0^1 \{\alpha\beta u_{xxt}^2 + \beta u_{xtt}^2\}(x, t)\, dx \tag{3.43}$$

is continuous in t. (Observe that $H[u]$ is coercive in u_{xxt} and u_{xtt}, and that $f_t \in C((-\infty, T]; L^2(0, 1))$ by (3.6).) A minor modification of the proof of Theorem 4.2 of [27] yields

$$u_{xxt}, u_{xtt} \in C((-\infty, T]; L^2(0, 1)). \tag{3.44}$$

Differentiating (3.1) with respect to x and t, and using (3.44), we conclude that

$$u_{xxx} \in C([0, T]; L^2(0, 1)), \tag{3.45}$$

and

$$u_{ttt} \in C((-\infty, T]; L^2(0, 1)). \tag{3.46}$$

It is interesting to note that (3.44) and (3.46) hold even without the assumption (3.11). In particular if v satisfies (3.7), (3.8), and (3.9), it automatically satisfies $v_{xxt}, v_{xtt}, v_{ttt} \in C((-\infty, 0]; L^2(0, 1))$. Moreover, if f_x belongs to $C((-\infty, 0]; L^2(0, 1))$, then so does v_{xxx}. Finally, we note that the a priori bound (3.33) also holds for the "exact solution" u. ∎

4. Local Existence

We now apply the results of the preceding section to establish a local existence theorem for the quasilinear history value problem (1.1), (1.2), (1.3).

THEOREM 4.1. *Assume that* $\phi, \psi \in C^3(R)$, (2.21) *and* (2.22) *hold, and that*

$$\phi'(\xi) > 0, \qquad \psi'(\xi) > 0 \qquad \forall \xi \in R. \tag{4.1}$$

Assume further that f satisfies (3.6) *for every $T > 0$, v satisfies* (3.7), $v_{xxtt} \in L^2((-\infty, 0]; L^2(0, 1))$, *and that Eqs.* (1.1), (1.2) *hold* (*with* $u = v$) *for* $t \le 0$. *Then, the history value problem* (1.1), (1.2), (1.3) *has a unique solution u defined on a maximal time interval* $(-\infty, T_0)$, $T_0 > 0$, *which satisfies* (3.10) *for every* $T < T_0$. *If, in addition,* (2.18) *holds and*

$$f_x \in C([0, \infty); L^2(0, 1)), \tag{4.2}$$

then (3.12) *holds for every* $T \in (0, T_0)$. *Moreover, if*

$$\operatorname*{ess-sup}_{t \in (-\infty, T_0)} \int_0^1 \{u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x, t)\, dx < \infty, \tag{4.3}$$

then $T_0 = \infty$.

Proof. For each $M, T > 0$, let $Z(M, T)$ denote the set of all functions w: $[0, 1] \times (-\infty, T] \to R$ such that

$$w, w_x, w_t, w_{xx}, w_{xt}, w_{tt}, w_{xxx}, w_{xxt}, w_{xtt}, w_{ttt} \in L^\infty((-\infty, T]; L^2(0, 1)), \tag{4.4}$$

$$w(0, t) = w(1, t) = 0 \qquad \forall t \in (-\infty, T], \tag{4.5}$$

$$w(x, t) = v(x, t) \qquad \forall x \in [0, 1], \quad t \in (-\infty, 0], \tag{4.6}$$

and

$$\operatorname*{ess-sup}_{t \in [0, T]} \int_0^1 \{w_{xxx}^2 + w_{xxt}^2 + w_{xtt}^2 + w_{ttt}^2\}(x, t)\, dx \leqslant M. \tag{4.7}$$

We note that $Z(M, T)$ is nonempty for M sufficiently large. Henceforth, we tacitly make this assumption.

It follows from (4.1) that $\inf_{\xi \in R} [\phi'(\xi)/\psi'(\xi)] \geqslant 0$. We temporarily make the stronger assumption

$$\underline{\phi} := \inf_{\xi \in R} \phi'(\xi) > 0, \qquad \underline{\psi} := \inf_{\xi \in R} \psi'(\xi) > 0, \qquad \underline{\gamma} := \inf_{\xi \in R} \frac{\phi'(\xi)}{\psi'(\xi)} > 0, \tag{4.8}$$

which will be removed later. Identifying α with $\phi'(w_x)$ and β with $\psi'(w_x)$, it follows immediately from Theorem 3.1 that for $w \in Z(M, T)$, the history value problem

$$u_{tt}(x, t) = \phi'(w_x)\, u_{xx}(x, t) + \int_{-\infty}^{t} a'(t - \tau)\, \psi'(w_x) u_{xx}(x, \tau)\, d\tau + f(x, t),$$
$$x \in [0, 1], \quad t \in (-\infty, T], \tag{4.9}$$

(1.2), (1.3) has a unique solution u which satisfies (3.10). Moreover, the corresponding $\underline{\alpha}$, $\underline{\beta}$, and $\underline{\lambda}$ can be chosen independently of M and T.

Let S denote the mapping which carries w into the solution of (4.9), (1.2), (1.3). Our goal is to show that, for appropriately chosen M and T, S has a unique fixed point in $Z(M, T)$ which is obviously a solution of (1.1), (1.2), (1.3). For this purpose, we employ the contraction mapping principle and the complete[3] metric ρ given by

$$\rho(w, \bar{w})^2 := \max_{t \in [0, T]} \int_0^1 \{(w_{xx} - \bar{w}_{xx})^2 + (w_{xt} - \bar{w}_{xt})^2 + (w_{tt} - \bar{w}_{tt})^2\}(x, t)\, dx. \tag{4.10}$$

Observe that for $w \in Z(M, T)$, we have

$$w_{xx}(x, t) = v_{xx}(x, 0) + \int_0^t w_{xxt}(x, s)\, ds \qquad \forall x \in [0, 1], \qquad t \in [0, T]. \tag{4.11}$$

Therefore,

$$\begin{aligned} \int_0^1 w_{xx}^2\,(x, t)\, dx &\leqslant 2 \int_0^1 v_{xx}^2(x, 0)\, dx + 2t \int_0^t \int_0^1 w_{xxt}^2(x, s)\, ds \\ &\leqslant 2V + 2Mt^2 \qquad \forall t \in [0, T], \end{aligned} \tag{4.12}$$

where V is defined by (3.17), and so clearly

$$\sup_{t \in [0, T]} \int_0^1 w_{xx}^2\,(x, t)\, dx \leqslant 2V + 2MT^2 \qquad \forall w \in Z(M, T). \tag{4.13}$$

Similarly, the following inequalities hold for all $w \in Z(M, T)$:

$$\sup_{t \in [0, T]} \int_0^1 w_{xt}^2(x, t)\, dx \leqslant 2V + 2MT^2, \tag{4.14}$$

$$\tfrac{1}{2} \sup_{\substack{x \in [0, 1] \\ t \in [0, T]}} w_{xx}^2(x, t) \leqslant 2V + (1 + 2T^2)\, M, \tag{4.15}$$

$$\sup_{\substack{x \in [0, 1] \\ t \in [0, T]}} w_{xt}^2(x, t) \leqslant 2V + (1 + 2T^2)\, M, \tag{4.16}$$

$$\sup_{\substack{x \in [0, 1] \\ t \in [0, T]}} w_x^2(x, t) \leqslant 2V(1 + 2T^2) + (2T^2 + 4T^4)\, M. \tag{4.17}$$

The a priori estimate (3.33) and the above inequalities show that S maps $Z(M, T)$ into itself provided that T is sufficiently small relative to M. From now on, we assume that T is small enough so that S maps $Z(M, T)$ into $Z(M, T)$.

To show that S is a contraction, let $M, T > 0$ and $w, \bar{w} \in Z(M, T)$ be

[3] Completeness of ρ follows from Alaoglu's theorem and sequential weak star lower semi-continuity of the norm in $L^\infty([0, T]; L^2(0, 1))$.

given, and set $u := Sw$, $\bar{u} := S\bar{w}$, $W := w - \bar{w}$, $U := u - \bar{u}$. A simple computation shows that U satisfies

$$U_{tt} = \phi'(w_x)\, U_{xx} + \int_0^t a'(t-\tau)\, \psi'(w_{xx})\, U_{xx}(x, \tau)\, d\tau$$

$$+ [\phi'(w_x) - \phi'(\bar{w}_x)]\, \bar{u}_{xx}$$

$$+ \int_0^t a'(t-\tau)[\psi'(w_x) - \psi'(\bar{w}_x)]\, \bar{u}_{xx}(x, \tau)\, d\tau,$$

$$\forall x \in [0, 1], \quad t \in [0, T], \tag{4.18}$$

$$U(0, t) = U(1, t) = 0, \qquad \forall t \in [0, T] \tag{4.19}$$

$$U(x, t) = 0 \qquad \forall x \in [0, 1], \quad t \in (-\infty, 0]. \tag{4.20}$$

Integrating the first convolution term in (4.18) by parts, we obtain

$$U_{tt} = \chi'(w_x)\, U_{xx} + \int_0^t a(t-\tau)[\psi'(w_x)\, U_{xx}]_t(x, \tau)\, d\tau + [\phi'(w_x) - \phi'(\bar{w}_x)\,]\, \bar{u}_{xx}$$

$$+ \int_0^t a'(t-\tau)[\psi'(w_x) - \psi'(\bar{w}_x)]\, \bar{u}_{xx}(x, \tau)\, d\tau, \tag{4.21}$$

where

$$\chi(\xi) := \phi(\xi) - a(0)\, \psi(\xi) \qquad \forall \xi \in R. \tag{4.22}$$

We multiply (4.21) by $[\psi'(w_x)\, U_{xx}]_t$ and integrate over $[0, 1] \times [0, t]$, $t \in [0, T]$, performing various integrations by parts and exploiting (4.19), (4.20). This yields

$$\tfrac{1}{2} \int_0^1 \{\psi'(w_x)\, \chi'(w_x)\, U_{xx}^2 + \psi'(w_x)\, U_{xt}^2\}(x, t)\, dx + Q([\psi'(w_x)\, U_{xx}]_t, t, a)$$

$$= -\int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)]\, \psi'(w_x)\, \bar{u}_{xx}\, U_{xx}(x, t)\, dx$$

$$- \int_0^1 \psi'(w_x)\, U_{xx}(x, t) \int_0^t a'(t-\tau)[\psi'(w_x) - \psi'(\bar{w}_x)]\, \bar{u}_{xx}(x, \tau)\, d\tau\, dx$$

$$+ \int_0^t \int_0^1 \{\tfrac{1}{2}\psi''(w_x)\, w_{xt}\, U_{xt}^2 - \psi''(w_x)\, w_{xx}\, U_{xt}\, U_{tt} + \psi''(w_x)\, w_{xt}\, U_{xx}\, U_{tt}$$

$$+ \tfrac{1}{2}[\chi''(w_x)\, \psi'(w_x) - \chi'(w_x)\, \psi''(w_x)]\, w_{xt}\, U_{xx}^2$$

$$+ [\phi'(w_x) - \phi'(\bar{w}_x)]\, \psi'(w_x)\, \bar{u}_{xxt}\, U_{xx}$$

$$+ [\phi''(w_x) - \phi''(\bar{w}_x)]\, \psi'(w_x)\, \bar{u}_{xx} U_{xx} W_{xt}\}(x, s)\, dx\, ds$$

$$+ \int_0^t \int_0^1 \psi'(w_x)\, U_{xx}(x, s) \int_0^s a'(s-\tau)\{[\psi'(w_x) - \psi'(\bar{w}_x)]\, \bar{u}_{xxt}$$

$$+ [\psi''(w_x) - \psi''(\bar{w}_x)]\, \bar{u}_{xx} W_{xt}\}(x, \tau)\, d\tau\, dx\, ds \qquad \forall t \in [0, T]. \quad (4.23)$$

Using (4.1) and Lemma 2.3 with ε sufficiently small, we see that the left-hand side of (4.23) is bounded from below by

$$\int_0^1 \{\tfrac{1}{4}\underline{\nu}\underline{\psi}^2 U_{xx}^2 + \tfrac{1}{2}\underline{\psi} U_{xt}^2\}(x, t) - C \int_{-\infty}^t \int_0^1 [\psi'(w_x)\, U_{xx}^2](x, s)\, dx\, ds$$

$$\forall t \in [0, T], \qquad (4.24)$$

where C is a constant that can be chosen independently of M and T.

It follows from (4.18) that

$$\int_0^1 U_{tt}^2(x, t)\, dt \leqslant 3 \int_0^1 \{\phi'(w_x)^2\, U_{xx}^2 + [\phi'(w_x) - \phi'(\bar{w}_x)]^2\, \bar{u}_{xx}^2\}(x, t)\, dx$$

$$+ 6a(0)^2 \max_{s \in [0, t]} \int_0^1 \{\psi'(w_x)^2 U_{xx}^2$$

$$+ [\psi'(w_x) - \psi'(\bar{w}_x)]^2 \bar{u}_{xx}^2(x, s)\}\, dx \qquad \forall t \in [0, T]. \quad (4.25)$$

We combine (4.23) and (4.25) and proceed as in the derivation of (3.33). Exploiting the fact that $W \equiv 0$ on $[0, 1] \times (-\infty, 0]$, we obtain (after a rather long computation) and estimate of the form

$$\rho(Sw, S\bar{w}) \leqslant P(M, T) \exp(T \cdot Q(M, T))\, \rho(w, \bar{w}) \quad \forall w, \bar{w} \in Z(M, T) \qquad (4.26)$$

for every $M, T > 0$, where P, Q: $[0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous functions with $P(M, 0) = 0\ \forall M > 0$.

The derivation of (4.26) from (4.23) and (4.25) is in much the same spirit as the derivation of (3.33). We show the detailed estimation of the first term on the right-hand side of (4.23). For each $\eta > 0$, we have

$$\left| \int_0^1 [\phi'(w_x) - \phi'(\bar{w}_x)]\, \psi'(w_x)\, \bar{u}_{xx}\, U_{xx}(x, t)\, dx \right|$$

$$\leqslant \eta \int_0^1 U_{xx}^2(x, t)\, dx + (4\eta)^{-1} \int_0^1 [\phi'(w_x)$$

$$- \phi'(\bar{w}_x)]^2\, \psi'(w_x)^2\, \bar{u}_{xx}^2(x, t)\, dx \qquad \forall t \in [0, T]. \qquad (4.27)$$

If we choose η sufficiently small, the first integral on the right-hand side of (4.27) can be absorbed by the first integral in (4.24). To estimate the last integral in (4.27), we first observe that by (4.17) and the mean value theorem

$$[\phi'(w_x)-\phi'(\bar{w}_x)]^2(x,t) \leqslant \Phi(M,T)\, W_x^2(x,t) \qquad \forall x \in [0,1], \quad t \in [0,T], \tag{4.28}$$

where $\Phi(M,T) := \max \phi''(\xi)^2$ and the max is taken over all ξ with $\xi^2 \leqslant 2V(1+2T^2)+(2T^2+4T^4)\,M$. Using the fact that $W \equiv 0$ on $[0,1] \times (-\infty, 0]$, the type of argument used to derive (4.17) yields

$$W_x^2(x,t) \leqslant 4M(T^2+T^4) \qquad \forall x \in [0,1], \quad t \in [0,T]. \tag{4.29}$$

Next, we set $\Psi(M,T) := \max \psi'(\xi)^2$, where the max is taken over all ξ with $\xi^2 \leqslant 2V(1+2T^2)+(2T^2+4T^4)\,M$. Then, using (4.13), (4.28), and the fact that $\bar{u} \in Z(M,T)$, we find

$$\int_0^1 [\phi'(w_x)-\phi'(\bar{w}_x)]^2\, \psi'(w_x)^2\, \bar{u}_{xx}^2(x,t)\,dx$$
$$\leqslant 8M(T^2+T^4)\,\Phi(M,T)\,\Psi(M,T)(V+MT^2) \qquad \forall t \in [0,T]. \tag{4.30}$$

The remaining steps in the derivation of (4.26) can be carried out in a similar fashion.

The contraction mapping principle and (4.26) imply that S has a unique fixed point $u \in Z(M,T)$ for a sufficiently small choice of $T>0$. It is obvious that u satisfies (1.1), (1.2), (1.3) on $[0,1] \times (-\infty, T]$. The uniqueness statement in Theorem 4.1 is immediate. If (2.18) and (4.2) hold, the additional regularity (3.12) follows from Theorem 3.1 and the fact that u satisfies (4.8), (1.2), (1.3) with $w=u$. The continuation of u to a maximal time interval $(-\infty, T_0)$ with the property that (4.3) implies $T_0=\infty$ follows from essentially the same argument as in [4].

It is easy to remove the extraneous assumption (4.8). To do so, we construct a functions $\tilde{\phi}$, $\tilde{\psi} \in C^3(R)$ which satisfy

$$\tilde{\phi}(\xi)=\phi(\xi), \qquad \tilde{\psi}(\xi)=\psi(\xi) \qquad \forall \xi \in [-2\sqrt{V}, 2\sqrt{V}], \tag{4.31}$$

$$\inf_{\xi \in R} \tilde{\phi}'(\xi)>0, \qquad \inf_{\xi \in R} \tilde{\psi}'(\xi)>0, \qquad \sup_{\xi \in R} \tilde{\psi}'(\xi)<\infty, \tag{4.32}$$

and we consider Eq. (1.1) with ϕ and ψ replaced by $\tilde{\phi}$ and $\tilde{\psi}$, respectively. The preceding argument shows that the modified history value problem has a unique solution u on $(-\infty, T]$ for some $T>0$. The Sobolev embedding theorem implies that

$$\sup_{\substack{x \in [0,1] \\ t \in (-\infty, 0]}} v_x^2(x,t) \leqslant V. \tag{4.33}$$

By virtue of (4.31), (4.33), and the continuity properties of u_x, u is a solution of the original problem on some smaller interval $(-\infty, \bar{T}]$ with $\bar{T}>0$. The additional properties of u as a solution of the original problem all follow easily. ∎

5. Global Existence

The following result is a precise analog of Theorem 4.1 in Dafermos and Nohel [4]. Recall that $\chi(\xi) := \phi(\xi) - a(0)\,\psi(\xi)\ \forall \xi \in R$.

THEOREM 5.1. *Let the following assumptions hold*:

(i) $a, a' \in L^1(0, \infty)$, $a \geqslant 0$, $a' \leqslant 0$, $a'' \geqslant 0$ (*in the sense of measures*); *the measure* a'' *has a nontrivial absolutely continuous component*;

(ii) $\phi, \psi \in C^3$, $\phi(0) = \psi(0) = 0$, $\phi'(0) > 0$, $\psi'(0) > 0$, $\chi'(0) > 0$;

(iii) f, f_x, $f_t \in L^\infty((-\infty, \infty);\ L^2(0, 1)) \cap L^2((-\infty, \infty);\ L^2(0, 1))$, $f_{tt} \in L^2((-\infty, \infty);\ L^2(0, 1))$, *and the norms of* f, f_x, f_t, f_{tt} *in the indicated spaces are sufficiently small*;

(iv) *the given history for* v *satisfies the equation and boundary conditions for* $t \leqslant 0$, v *and its derivatives through third order lie in* $L^\infty((-\infty, 0];\ L^2(0, 1)) \cap L^2((-\infty, 0];\ L^2(0, 1))$, $v_{xxtt} \in L^2((-\infty, 0];\ L^2(0, 1))$.

Then, (1.1), (1.2), (1.3) *has a unique solution* u *existing for all* $t \in (-\infty, \infty)$ *such that* u *and its derivatives through third order lie in* $L^\infty((-\infty, \infty);\ L^2(0, 1)) \cap L^2((-\infty, \infty);\ L^2(0, 1))$. *Moreover*, u *and its derivatives through second order converge to zero uniformly as* $t \to \infty$. *If, in addition, the* (A_2)-*condition* (2.18) *holds and* $f_x \in C([0, \infty), L^2(0, 1))$, *then third derivatives of* u *belong to* $C([0, \infty), L^2(0, 1))$.

The proof is essentially a line-by-line copy of the argument of Dafermos and Nohel. We need only note that in deriving their estimate (3.26) they use Lemma 2.5 with $\varepsilon = 0$, while we have to use Lemma 2.5 with $\varepsilon \neq 0$ but small. Apart from this simple change, their proof goes through unaltered.

Remarks. **5.1.** In assumption (iv), we did not require smallness of the norms. However, assumption (iii) and the fact that v satisfies the equation and boundary conditions for $t \leqslant 0$ imply that v is "small."

5.2. Theorem 5.1 applies without essential changes if Dirichlet conditions are replaced by Neumann or mixed conditions. In the case of Neumann conditions, the boundedness and decay statements apply to u

minus its spatial mean value $\langle u \rangle$ which evolves according to the trivial equation

$$(d^2/dt^2)\langle u \rangle(t) = \langle f \rangle(t).$$

5.3. The question of global existence for the all-space problem is more difficult. Hrusa and Nohel [13] gave a proof for regular kernels. This proof, however, makes essential use of the assumption $a'' \in L^1(0, \infty)$ and does not appear generalizable to singular kernels.

5.4. It would be interesting if a global existence result could be established assuming only $\chi' \geqslant 0$ in a neighborhood of 0 rather than $\chi'(0) > 0$. Even for regular kernels, this has been accomplished only for the case $\chi' \equiv 0$ which arises in modelling shear flows of viscoelastic fluids and in models for heat flow in materials with memory. (See [3, 17, 26].) The global estimates of Dafermos and Nohel [4], which, as remarked, can be carried out without assuming $a'' \in L^1$, can also be adapted to $\chi' \equiv 0$ without assuming $a'' \in L^1(0, \infty)$. However, the hypotheses on f in this case must be different than those above.

5.5. It is conceivable that for an appropriate class of singular kernels, global smooth solutions exist even for large data. However, we have not been able to verify this.

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Note added in proof. The authors now wish John Nohel a happy sixty-second birthday.

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