Estimates on enstrophy, palinstrophy, and invariant measures for 2-D turbulence

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ABSTRACT

We construct semi-integral curves which bound the projection of the global attractor of the 2-D Navier–Stokes equations in the plane spanned by enstrophy and palinstrophy. Of particular interest are certain regions of the plane where palinstrophy dominates enstrophy. Previous work shows that if solutions on the global attractor spend a significant amount of time in such a region, then there is a cascade of enstrophy to smaller length scales, one of the main features of 2-D turbulence theory. The semi-integral curves divide the plane into regions having limited ranges for the direction of the flow. This allows us to estimate the average time it would take for an intermittent solution to burst into a region of large palinstrophy. We also derive a sharp, universal upper bound on the average palinstrophy and show that it is achieved only for forces that admit statistical steady states where the nonlinear term is zero.

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1. Introduction

The energy (\(e\)), enstrophy (\(E\)), and palinstrophy (\(P\)) are essential physical quantities for the study of turbulence. The heuristic theory of Kolmogorov in 3-D \cite{1} followed by that of Batchelor, Kraichnan, Leith in 2-D \cite{2–4} lay out a remarkably robust set of laws for turbulent flow. Some are precise power laws dictating how energy varies with length scale. Others concern how energy injected at certain energy levels through external forcing is transferred to different energy levels. In 2-D turbu-
ence enstrophy is expected to transfer in a direct cascade to smaller length scales, while energy is often observed in both experiments and numerical simulations to display an inverse cascade to larger length scales (see e.g. [5,6]).

Sufficient conditions for direct cascades were established in [7] for the periodic, 2-D Navier–Stokes equations (NSE). There it is shown that if \( \kappa^2_\sigma = \langle P \rangle / \langle E \rangle \) is large, where \( \langle \cdot \rangle \) denotes an average, then the direct enstrophy cascade holds. Moreover, if the so-called \( \kappa^{-3} \) energy spectrum holds over the inertial range, then \( \kappa^2_\sigma \) is indeed sufficiently large. Similarly, if \( \kappa^2_{\sigma} = \langle E^2 \rangle / \langle \sigma \rangle \) is large, there is a direct cascade of energy. This cascade is expected to be weaker, if is present at all. Consistent with this, is the fact that \( \kappa_{\tau} \leq \kappa_{\sigma} \), yet on a finite domain, we still do not know if even \( \kappa^{2}_{\sigma} \) can be sufficiently large (without assuming the \( \kappa^{-3} \) energy spectrum). Shell models lend strong support to the \( \kappa^{-3} \) energy spectrum [8], but this is more elusive in direct numerical simulation of the NSE.

In the case where the force is in a single eigenspace of the Stokes operator the flow is known not to be turbulent [9,10]. Other than that little is known about how to characterize those external forces which yield either the inverse square power law, and consequently a large value for \( \kappa_{\sigma} \), or for that matter, the latter alone. In this paper, to better understand the range of \( \kappa_{\sigma} \), we examine the projection of the global attractor in the \( E, P \)-plane.

We have already studied the projection in the \( e, E \)-plane [11,12]. In general, the attractor projects into the region bounded by the parabola \( E = \sqrt{e} \) and the Poincaré line \( E = e \) (see Fig. 1). The parabolic bound can be achieved only at a steady state, and only if the force is in a single eigenspace of the Stokes operator. These universal bounds do not rule out a large value for \( \kappa_{\tau} \), except in certain cases.

Perhaps the most striking difference with our estimates in the \( E, P \)-plane presented here is one of scale. In our normalized variables, we have \( e < 1 \) and \( E < 1 \), but our bound for \( P \), which may not be sharp, is \( O(G^2) \), where \( G \) is the Grashof number (see (2.12)). Another difference is that the analog of the extreme case \( E = \sqrt{e} \) is in terms of averages. We have \( \langle P \rangle \leq 1 \) in general, and \( \langle P \rangle = 1 \) only at steady states where the nonlinear term is zero.

Intermittency, where high frequency activity makes bursts between relatively long quiescent periods is often observed for the Navier–Stokes equations and related models [13–15]. The estimates in [16] quantify this phenomenon in terms of moments of the energy and their long-time averages, but leave open the matter of their existence. We construct semi-integral curves of the form \( P = \varphi(E) \), on which \( \dot{P} - (d\varphi/dE)E \) does not change sign, thereby determining a range for the direction of the flow in this plane. This allows us to estimate the average time of a burst into the region where \( P = O(G^2) \), though it remains open whether this region is recurrent.

Patterns in turbulence are discernible only upon taking an average. Traditionally this has been done in practice with large time averages, and formally with infinite time averages. Though the latter lacks mathematical rigor, this can be overcome by considering an ensemble average with respect to an invariant probability measure. Such a measure is automatically concentrated on the global attractor. The averages in the quotients defining \( \kappa_{\sigma} \) and \( \kappa_{\tau} \) are made in this sense. Similar bounds can made for the more concrete, finite time averages (see [17]), but extra labor is needed to estimate the length in truncation time.

Toward a better gauge of \( \kappa_{\sigma} \), we compare the probabilities of projecting into various regions in the \( E, P \)-plane. Using again the semi-integral curves, we are able to relate the maximal and minimal time spent in certain regions to their invariant probability measures. While the regions do not occupy enough of the possible range of the attractor to provide a true estimate of \( \kappa_{\sigma} \) they narrow down the types of solutions that can make this wave number large.

This paper is structured as follows. After providing the basic preliminaries in Section 2, for motivational purposes we recall in Section 3 the estimates involving \( \kappa_{\sigma} \), \( \kappa_{\tau} \) and the cascades from [7]. In Section 4 we recall for the purpose of comparison some of the bounds in [11] on the attractor in the \( e, E \)-plane. We treat in Section 5 the case where the average palinstrophy achieves its maximum, which like the extreme case in the \( e, E \)-plane, occurs only under special circumstances. In Section 6 we partition the \( E, P \)-plane by four parabolas (one concave), and derive different estimates for the time derivatives of \( E \) and \( P \) in each region. Then in Section 7 we form differential inequalities involving \( dP/dE \) from quotients of these time derivative bounds. Solving the associated differential equations gives us the semi-integral curves. The semi-integral curves impose bounds on directions of flow in the plane, which we use in Section 8 to estimate bursting times for intermittent solutions.
2. Preliminaries

We write the incompressible Navier–Stokes equations

\[ \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = F, \]
\[ \text{div} u = 0, \]
\[ \int_\Omega u \, dx = 0, \quad \int_\Omega F \, dx = 0, \]
\[ u(x, 0) = u_0(x) \]

with periodic boundary conditions in \( \Omega = [0, L]^2 \) as a differential equation in a certain Hilbert space \( H \) (see [18] or [19]),

\[ \frac{du}{dt} + \nu Au + B(u, u) = f, \quad u \in H. \quad (2.1) \]

The phase space \( H \) is the closure in \( L^2(\Omega)^2 \) of all \( \mathbb{R}^2 \)-valued trigonometric polynomials \( u \) such that

\[ \nabla \cdot u = 0, \quad \text{and} \quad \int_\Omega u(x) \, dx = 0. \]

The bilinear operator \( B \) is defined as

\[ B(u, v) = \mathcal{P}(u \cdot \nabla) v, \]

where \( \mathcal{P} \) is the Helmholtz–Leray orthogonal projector of \( L^2(\Omega)^2 \) onto \( H \).

The scalar product in \( H \) is taken to be

\[ (u, v) = \int_\Omega u(x) \cdot v(x) \, dx, \quad \text{where} \quad a \cdot b = a_1 b_1 + a_2 b_2, \]

with associated norm

\[ |u| = (u, u)^{1/2} = \left( \int_\Omega u(x) \cdot u(x) \, dx \right)^{1/2}. \]

The operator \( A = -\Delta \) is self-adjoint, and its eigenvalues are of the form

\[ \left( \frac{2\pi}{L} \right)^2 k \cdot k, \quad \text{where} \ k \in \mathbb{Z}^2 \setminus \{0\}. \]

We denote these eigenvalues by \( 0 < \lambda_0 = (2\pi/L)^2 \leq \lambda_1 \leq \lambda_2 \leq \cdots \) arranged in increasing order and counted according to their multiplicities, and write \( w_0, w_1, w_2, \ldots \) for the corresponding normalized eigenvectors (i.e. \( |w_j| = 1 \) for \( j = 0, 1, 2, \ldots \)).

The positive roots of \( A \) are defined by linearity from

\[ A^{\alpha} w_j = \lambda_j^{\alpha} w_j, \quad \text{for} \ j = 0, 1, 2, \ldots \]
on the set
\[ D(A^0) = \left\{ u \in H : \sum_{j=0}^{\infty} \lambda_j 2^d (u, w_j)^2 < \infty \right\}. \]

We write \( V = D(A^{1/2}) \) and take the natural norm on \( V \) to be
\[ \|u\| = \left( \int \sum_{j=1}^{2} \frac{\partial}{\partial x_j} u(x) \cdot \frac{\partial}{\partial x_j} u(x) \, dx \right)^{1/2} = \left( \sum_{j=0}^{\infty} \lambda_j (u, w_j)^2 \right)^{1/2}. \]

Since the boundary conditions are periodic, we may express an element in \( H \) as a Fourier series
\[ u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{ik_0 k \cdot x}, \quad (2.2) \]
where
\[ \kappa_0 = \lambda_0^{1/2} = \frac{2\pi}{L}, \quad (2.3) \]
\( \hat{u}_0 = 0, \) \( \hat{u}_k^* = \hat{u}_{-k}, \) and due to incompressibility, \( k \cdot \hat{u}_k = 0. \) We associate to each term in (2.2) a wave number \( \kappa_0 |k|. \) Parseval’s identity reads as
\[ |u|^2 = L^2 \sum_{k \in \mathbb{Z}^2} |\hat{u}_k|^2 \]
(we assume it will be clear from the context when \( |\cdot| \) refers to the modulus of a vector in \( \mathbb{C}^2 \)) as well as
\[ (u, v) = L^2 \sum_{k \in \mathbb{Z}^2} \hat{u}_k \cdot \hat{v}_{-k}, \]
for \( v = \sum \hat{v}_k e^{ik_0 k \cdot x}. \) We define projectors \( R_\kappa : H \to \text{span}\{w_j \mid \lambda_j \leq \kappa^2\} \) by
\[ R_\kappa u = \sum_{\kappa_0 |k| \leq \kappa} \hat{u}_k e^{ik_0 k \cdot x}, \]
where \( u \) has the expansion in (2.2), along with \( Q_\kappa = I - R_\kappa. \) The projector \( P \) can be expressed in terms of Fourier coefficients as
\[ (P u)_\kappa = \hat{u}_k - \frac{\kappa \cdot \hat{u}_k}{|\kappa|^2} \kappa. \]

Recall the orthogonality relations of the bilinear term (see for instance [19])
\[ (B(u, v), w) = -(B(u, w), v), \quad (2.4) \]
and in two space dimensions only,
\[(B(u, u), Au) = 0. \quad (2.5)\]

We will use the strong form of enstrophy invariance
\[(B(Av, v), u) = (B(u, v), Av), \quad (2.6)\]
and the relation (cf. e.g. [7])
\[(B(v, v), Au) + (B(v, u), Av) + (B(u, v), Av) = 0. \quad (2.7)\]

For completeness, a proof of (2.6) is provided in Appendix A. We will also need several inequalities in 2-D (see [18,19]), one often referred to as Agmon’s
\[\|u\|_\infty \leq c_A |u|^{1/2} |Au|^{1/2}, \quad \text{for all } u \in D(A), \quad (2.8)\]
and one known as Ladyzhenskaya’s
\[|u|_{L^4(\Omega)}^2 \leq c_L |u||u|, \quad \text{for all } u \in D(A^{1/2}). \quad (2.9)\]
The constants $c_A$ and $c_L$ are universal.

We denote by $S$ the solution operator defined by $S(t)u_0 = u(t)$, where $u(t)$ is the unique solution to (2.1) such that $u(0) = u_0$. The global attractor $\mathcal{A}$ is defined by
\[\mathcal{A} = \bigcap_{t \geq 0} S(t) B, \]
where $B$ is a bounded absorbing set. Equivalently, $\mathcal{A}$ is the largest bounded, invariant set for $S(t)$ (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$). Since (2.1) always admits stationary solutions, $\mathcal{A} \neq \emptyset$ (cf. [18]). Moreover, it is known that $\mathcal{A}$ is a compact set in $D(A)$ of finite fractal and Hausdorff dimensions (see e.g. [18] or [19]).

Multiply (2.1) by $u$ (respectively $Au$), and integrate over $\Omega$ and apply (2.4), (2.5), to obtain
\[\frac{1}{2} \frac{d}{dt} |u|^2 + \nu \|u\|^2 = (f, u), \quad (2.10)\]
\[\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |Au|^2 = (f, Au). \quad (2.11)\]

In the scientific literature,
\[\frac{1}{L^2} |u|^2 = 2 \text{ times the total energy per unit mass}\]
and
\[\frac{1}{L^2} \|u\|^2 = \text{the total enstrophy per unit mass}.\]

We will consider the dimensionless parameter known as the Grashof number, defined by
\[G = \frac{|f|}{\nu^2 \kappa_0^2}. \quad (2.12)\]
3. Motivation from turbulence

One reason to examine the attractor in the enstrophy-palinstrophy-plane is that we can infer the existence of an enstrophy cascade from the averages of these quantities. Throughout this section we assume the following bound on the spectral range of the force

\[ f = (R_\kappa - R_\kappa) \, f \quad \text{with} \quad \kappa \leq C_0 \kappa_0. \]

Let \( r_\kappa = R_\kappa u, q_\kappa = Q_\kappa u \). Take the scalar product of the NSE with \( Aq_\kappa \), and apply (2.5) and (2.7) to arrive to the enstrophy invariance relation

\[
\frac{1}{2} \frac{d}{dt} \|q_\kappa\|^2 + \nu |Aq_\kappa|^2 = L^2 \mathcal{E}_\kappa + (f, Aq_\kappa)
\]

for \( \kappa \geq \kappa \), (3.1)

\[
= L^2 \mathcal{E}_\kappa, \quad \text{for} \quad \kappa \geq \kappa.
\]

where \( \mathcal{E}_\kappa = [\mathcal{E}_{\kappa}^+ - \mathcal{E}_{\kappa}^-] \) is the net rate of enstrophy transfer (flux) given in terms of the rate of enstrophy transfer (low to high)

\[
\mathcal{E}_{\kappa}^+(u) = -\frac{1}{L^2} (B(r_\kappa, r_\kappa), Aq_\kappa).
\]

and (high to low)

\[
\mathcal{E}_{\kappa}^-(u) = -\frac{1}{L^2} (B(q_\kappa, q_\kappa), Ar_\kappa).
\]

It follows from (3.2) that at wave numbers greater than that of the force, the average direction of enstrophy flux is from low to high

\[ 0 \leq \langle \mathcal{E}_\kappa(u) \rangle \quad \text{for} \quad \kappa > \kappa \]

where \( \langle \cdot \rangle \) is the average with respect to any fixed invariant probability measure \( \mu \). In the Kraichnan theory of 2-D turbulence [3] the enstrophy flux is roughly constant over the inertial range of wave numbers beyond those of the force. A condition for this to hold is expressed in the following.

**Theorem 3.1.** (See [7].) If

\[
\kappa \leq \kappa \leq \kappa_{\sigma} = \left( \frac{\langle |Au|^2 \rangle}{\langle |A^{1/2} u|^2 \rangle} \right)^{1/2},
\]

then

\[
1 - \left( \frac{\kappa}{\kappa_{\sigma}} \right)^2 \leq \frac{\langle \mathcal{E}_\kappa \rangle}{\eta} \leq 1,
\]

where

\[
\eta = \frac{\nu}{L^2 |Au|^2}.
\]
It follows that if

$$\kappa_\sigma \gg \kappa,$$

then there exists an enstrophy cascade:

$$\langle \mathcal{E}_\kappa \rangle \approx \eta, \quad \text{for } \kappa \lesssim \kappa_\sigma.$$

The quantity $\eta$ represents the average dissipation rate of enstrophy per unit mass. It is not known if (3.3) is achievable on a bounded domain. One of our main reasons for investigating where the global attractor projects in the enstrophy, palinstrophy-plane is to gain some understanding of the type of solutions that can make $\kappa_\sigma$ large.

A similar result holds for the transfer of energy $e_\kappa = [e_\kappa^- - e_\kappa^+]$, where

$$e_\kappa^- (u) = -\frac{1}{L^2} \langle B(p_\kappa, p_\kappa), q_\kappa \rangle \quad \text{and} \quad e_\kappa^+ (u) = -\frac{1}{L^2} \langle B(q_\kappa, q_\kappa), p_\kappa \rangle.$$

**Theorem 3.2.** (See [7].) If

$$\kappa \leq \kappa \leq \kappa_\tau = \left( \frac{\langle |u|^2 \rangle}{\langle |u| \rangle} \right)^{1/2},$$

then

$$1 - \left( \frac{\kappa}{\kappa_\tau} \right)^2 \leq \frac{\langle e_\kappa \rangle}{\epsilon} \leq 1,$$

where $\epsilon = \frac{\nu}{L^2} \langle |A^{1/2}u|^2 \rangle$.

Thus, if

$$\kappa_\tau \gg \kappa,$$

there is a direct energy cascade:

$$\langle e_\kappa \rangle \approx \epsilon, \quad \text{for } \kappa \lesssim \kappa \ll \kappa_\tau.$$

Similarly it is not known if (3.4) is achievable. Consistently, however, with the fact that the Kraichnan theory posits a direct enstrophy cascade, rather than a direct energy cascade, we have $\kappa_\tau \lesssim \kappa_\sigma$ [7]. Next we recall from [11] some of our previous results in the energy, enstrophy-plane.

**4. Bounds in energy, enstrophy-plane**

By rescaling we may assume that

$$\kappa_0 = 1, \quad \nu = 1,$$

so $G = |f|$ and $h = f/G$ satisfies $|h| = 1$. We then normalize the velocity to $v = u/G$ which satisfies

$$\frac{dv}{dt} + Av + GB(v, v) = h.$$  
(4.1)
Henceforth, we shall use $S(t)$ to denote the solution operator, and $\mathcal{A}$ the attractor of (4.1) as opposed to (2.1). In terms of $v$ the energy, enstrophy and palinstrophy can be written respectively as

$$e = |v|^2, \quad E = \|v\|^2 = |A^{1/2}v|^2, \quad P = |Av|^2.$$ 

**Theorem 4.1.** (See [11].) For all $v \in \mathcal{A}$

$$\|v\|^2 \leq |v|. \quad (4.2)$$

If there exists $v_0 \in \mathcal{A} \setminus \{0\}$ such that $\|v_0\|^2 = |v_0|$, then there exists a wave number $\kappa$ such that

$$h = \sum_{|k| = \kappa} \hat{h}_k e^{ik \cdot x} \quad (4.3)$$

and

$$v_0 = \frac{h}{\kappa^2}. \quad (4.4)$$

Moreover, in this case for all $v \in \mathcal{A} \setminus \{v_0\}$

$$\|v\| < \kappa |v|. \quad (4.5)$$

Theorem 4.1 contains Marchioro’s result [9], namely that if $\kappa = 1$, then $\mathcal{A} = \{h\}$, regardless of how large $G$ is. When combined with the Poincaré inequality, (4.2) says that the attractor is in the closed bounded region in the space $(e, E)$ given by $e \leq E \leq \sqrt{e}$ (see Fig. 1).
5. Averaged estimates

In this section we seek an analog in the E, P-plane to the extreme case result in the e, E-plane, where the parabolic bound (4.2) can be achieved if and only if \( h \) satisfies (4.3). In the enstrophy, palinstrophy-plane a similar result is found to hold for a wider class of forces \( h \), but only in an averaged sense. First we state and prove a well-known general bound.

**Proposition 5.1.** For \( v_0 \in A \) we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |AS(t)v_0|^2 \, d\tau \leq 1.
\]

**Proof.** From (2.11) and the Cauchy–Schwarz inequality we have

\[
E(t) - E(0) \leq 2 \int_0^t P(\tau)^{1/2} \, d\tau - 2 \int_0^t P(\tau) \, d\tau
\]

\[
\leq 2t^{1/2} \left( \int_0^t P(\tau) \, d\tau \right)^{1/2} - 2 \int_0^t P(\tau) \, d\tau.
\]

On \( A \) we have \(-1 \leq E(t) - E(0) \leq 1\), and hence by Young’s inequality

\[
\int_0^t P(\tau) \, d\tau \leq t^{1/2} \left( \int_0^t P(\tau) \, d\tau \right)^{1/2} + \frac{1}{2}
\]

\[
\leq \frac{t}{2} + \frac{1}{2} \int_0^t P(\tau) \, d\tau + \frac{1}{2}.
\]

Taking a time average, we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t P(\tau) \, d\tau \leq \lim_{t \to \infty} \frac{t + 1}{t} = 1. \quad \square
\]

It is not known whether infinite time averages of solution norms of the NSE exist. To circumvent this, we employ a generalized limit, denoted \( \text{Lim} \), derived via the Hahn–Banach extension, which satisfies \( \text{Lim}_{t \to \infty} = \text{lim}_{t \to \infty} \) whenever the latter exists. The following version of the Bogoliouboff–Kryloff theory [20] shows that there are natural ensemble averages associated these extensions of limits.

**Proposition 5.2.** *(See [21].)* Given \( v_0 \in D(A) \) and a generalized limit \( \text{Lim} \), there exists an invariant probability measure \( \mu_{v_0} \) such that

\[
\text{Lim}_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)v_0) \, d\tau = \int_A \Phi(v) \mu_{v_0}(dv)
\]

for all real-valued continuous (with respect to the H-norm) functions \( \Phi \) on \( D_A \).
Note that the probability measures $\mu_{v_0}$ are not unique. Moreover, they depend on the choice of the generalized limit, as well as on the initial condition $v_0$. For some fixed invariant measure $\mu$ (not necessarily of the type described in Proposition 5.2), we will write

$$\langle \Phi(v) \rangle = \int_A \Phi(v) \mu(dv)$$

for the average of $\Phi$ with respect to $\mu$. It is important to note that the results to follow will not depend on the choice of $\mu$. The next theorem allows us to work with ensemble averages with respect to arbitrary invariant measures.

**Theorem 5.1.** (See Dunford and Schwartz [22].) Let $\mu$ be an invariant probability measure on $H$, and let $\Phi \in L^1(H, \mu)$. Then the limit

$$\Phi^* (v) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)v) \, d\tau$$

exists for $\mu$-almost any $v \in H$. Moreover, $\Phi^*$ is $\mu$-integrable, $S$-invariant, and satisfies

$$\int_H \Phi^*(v) \, \mu(dv) = \int_H \Phi(v) \, \mu(dv).$$

In particular, Theorem 5.1 allows us to rewrite (5.1) as

$$\langle |Av|^2 \rangle \leq 1.$$  \hfill (5.4)

We next show that the maximal value for averaged palinstrophy occurs only for solutions whose averages are stationary solutions $\langle v \rangle$ where $B(\langle v \rangle, \langle v \rangle) = 0$. The force need not correspond to a single eigenvalue of the Stokes operator. We will use the following facts proved in [7].

**Proposition 5.3.** (See (2.14), (2.19) and (2.21) in [7].)

$$\langle Av \rangle = A\langle v \rangle,$$  \hfill (5.5)

$$\langle |Av|^2 \rangle = (h, A\langle v \rangle),$$  \hfill (5.6)

$$\langle |A(v - \langle v \rangle)|^2 \rangle = \langle |Av|^2 \rangle - |A\langle v \rangle|^2.$$  \hfill (5.7)

**Proposition 5.4.**

$$\langle |Av|^2 \rangle = 1$$  \hfill (5.8)

if and only if $\mu$ is an atomic measure with the support of $\{A^{-1}h\}$. In this case we have

$$B(A^{-1}h, A^{-1}h) = 0$$  \hfill (5.9)
and

\[ \langle v \rangle = A^{-1}h. \]  

(5.10)

**Proof.** Using (5.5) and (5.8) we have

\[ |A \langle v \rangle|^2 = |\langle Av \rangle|^2 = \left( \int_A A v \, \mu(dv) \right)^2 \leq \int_A |Av|^2 \, \mu(dv) = |\langle Av \rangle|^2 = 1. \]  

(5.11)

Using (5.6), (5.8), and the Cauchy–Schwarz inequality yields

\[ 1 = \langle |Av|^2 \rangle = (h, A \langle v \rangle) \leq |h||A \langle v \rangle| = |A \langle v \rangle|, \]  

(5.12)

so we have

\[ |A \langle v \rangle| = 1. \]  

(5.13)

Combining (5.12) with (5.13), we see that equality must hold where the Cauchy–Schwarz inequality was used:

\[ A \langle v \rangle = h, \]  

(5.14)

and hence \( \langle v \rangle = A^{-1}h. \) By (5.13), (5.8), and (5.7) we have

\[ \left( \int_A (v - \langle v \rangle) \, \mu(dv) \right)^2 \leq \int_A \left| (v - \langle v \rangle) \right|^2 \, \mu(dv) \leq \int_A |A(v - \langle v \rangle)|^2 \, \mu(dv) \]

\[ = (|A(v - \langle v \rangle)|^2) = |Av|^2 - |A \langle v \rangle|^2 = 1 - 1 = 0, \]

and thus \( v - \langle v \rangle = 0 \) almost everywhere with respect to \( \mu. \) In particular \( v - \langle v \rangle = 0 \) on the support of \( \mu. \) It follows that \( \mu(\{\langle v \rangle\}) = 1, \) and hence

\[ \{\langle v \rangle\} = \text{support } \mu. \]

Since \( \mu \) is invariant, we must have that \( \langle v \rangle \) is a stationary solution for (4.1), i.e. \( \langle v \rangle \) satisfies

\[ A \langle v \rangle + B(\langle v \rangle, \langle v \rangle) = h. \]  

(5.15)

Since \( \langle v \rangle = A^{-1}h, \) the equation above implies (5.9).

The converse statement follows – if an invariant probability measure \( \mu \) is the atomic measure supported on \( \{A^{-1}h\}, \) then \( \langle v \rangle = A^{-1}h \) and Eq. (5.15) is satisfied. Thus \( B(\langle v \rangle, \langle v \rangle) = 0 \) and \( \langle |Av|^2 \rangle = 1. \) \( \Box \)

The proposition above has the following consequence in terms of time averages.
Proposition 5.5. If

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |A S(\tau) v_0|^2 \, d\tau = 1
\]  

(5.16)

then

\[
B(A^{-1} h, A^{-1} h) = 0
\]  

(5.17)

and

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(\tau) v_0 \, d\tau = A^{-1} h.
\]  

(5.18)

Proof. Since $|A S(t) v_0|$ is bounded for $t$ large enough, we can extract from any sequence of times going to infinity a subsequence $\{t_j\}$ such that

\[
\text{weak}\, \lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} A S(\tau) v_0 \, d\tau = w_0.
\]

Now if (5.16) holds, then from (4.1)

\[
\lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} |A S(\tau) v_0|^2 \, d\tau = (h, w_0) \leq |h| |w_0|.
\]  

(5.19)

Since the left-hand side of the inequality, as well as $|h|$ are equal to 1, we obtain

\[
|w_0| \geq 1.
\]

But we also have

\[
|w_0| \leq \limsup_{t \to \infty} \left| \frac{1}{t} \int_0^t A S(\tau) v_0 \, d\tau \right| \leq \left( \lim_{t \to \infty} \frac{1}{t} \int_0^t |A S(\tau) v_0|^2 \, d\tau \right)^{1/2} = 1.
\]

Returning to (5.19), we can now write

\[
1 = (h, w_0) \leq |h| |w_0| = 1.
\]

Thus we need to have equality in the Cauchy–Schwarz inequality involved above, which means that

\[
w_0 = h.
\]

Since $A^{-1}$ is a compact operator,

\[
\lim_{t_j \to \infty} \frac{1}{t_j} A^{-1} \int_0^{t_j} A S(\tau) v_0 \, d\tau = A^{-1} \text{weak}\, \lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} A S(\tau) v_0 \, d\tau = A^{-1} h.
\]
We have just proven that from any sequence of times going to infinity we can extract a subsequence \( \{t_j\} \) such that
\[
\lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} S(\tau)v_0 \, d\tau = A^{-1}h.
\]

Clearly this implies that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^{t} S(\tau)v_0 \, d\tau = A^{-1}h.
\]

Now, by Proposition 5.2, there exists an invariant measure \( \mu_{v_0} \), such that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^{t} S(\tau)v_0 \, d\tau = (v) = A^{-1}h.
\]

That means that with respect to this measure
\[
\langle Av \rangle = A\langle v \rangle = h,
\]
and thus
\[
\left| \langle Av \rangle \right| = 1.
\]

Since \( \langle |Av| \rangle^2 \leq \langle |Av|^2 \rangle \) and by Proposition 5.1 \( \langle |Av|^2 \rangle \leq 1 \), we have
\[
\langle |Av|^2 \rangle = 1
\]
and can apply Proposition 5.4 to obtain (5.17) and (5.18).

We remark that Proposition 5.5 shows that the extremal case, namely when (5.16) holds for some initial data \( v_0 \), occurs only for the forces satisfying the condition \( B(A^{-1}h, A^{-1}h) = 0 \). Moreover, \( S(t)v_0 = A^{-1}h \) must be a stationary solution of the (4.1) and thus any invariant probability measure \( \mu_{v_0} \) associated with the infinite time average of \( S(t)v_0 \), is the atomic measure concentrated on \( \{A^{-1}h\} \).

6. Estimates for time derivatives

Taking the scalar product of (4.1) with \( Av \), and applying the Cauchy–Schwarz inequality and orthogonality relation (2.5), we arrive at the following bounds for \( dE/dt \):
\[
-2(A_1^{1/2}E^{1/2} + P) \leq \frac{dE}{dt} \leq 2(A_1^{1/2}E^{1/2} - P),
\]
where
\[
A_1 = |A^{1/2}h|
\]
is a shape factor which indicates how much of the force is in the higher modes. It follows that throughout

$$\mathcal{R} = \{(E, P) : P \geq 2\Lambda_1^{1/2}E^{1/2}\}$$

we have

$$-3P \leq \frac{dE}{dt} \leq -P. \quad (6.3)$$

Note that (2.6), (2.7) imply

$$(B(v, v), A^2v) = -(B(Av, v), Av).$$

For a bound on $dP/dt$ we take $(\text{NSE}, A^2v)$, apply the identity above as well as (2.8) and the Cauchy–Schwarz inequality, to obtain

$$\frac{1}{2} \frac{dP}{dt} = (h, A^2v) - (Av, A^2v) - G(B(v, v), A^2v)
\leq \Lambda_1^{1/2} \zeta^{1/2} - \zeta + cGPE^{1/4} \zeta^{1/4} \quad (6.4)$$

where $\zeta = |A^{3/2}v|^2$ and $c = c_A$.

An immediate consequence of the Cauchy–Schwarz inequality is that

$$\zeta \geq \frac{P^2}{E}. \quad (6.5)$$

It follows that in $\mathcal{R}$ we have $\zeta^{1/2} \geq 2\Lambda_1^{1/2}$ and hence

$$\frac{dP}{dt} \leq \Psi(\zeta) := -\zeta + 2cGPE^{1/4} \zeta^{1/4} \text{ for all } (E, P) \in \mathcal{R}. \quad (6.6)$$

The graph of the function $\Psi$ is shown in Fig. 2.

Since $\Psi(\zeta) < 0$ for

$$\zeta > \zeta^* = (2cGPE^{1/4})^{4/3}$$

we have by (6.5) that

$$\frac{dP}{dt} < 0 \text{ for } \frac{P^2}{E} > \zeta^*, \text{ i.e. for } P > (2cGE)^2. \quad (6.7)$$

Note that for

$$\zeta_{\text{max}} = \left( \frac{cGPE^{1/4}}{2} \right)^{4/3}, \quad \Psi(\zeta_{\text{max}}) = \max \Psi(\zeta) = 3\zeta_{\text{max}}.$$
and
\[ \frac{P^2}{E} \geq \zeta_{\text{max}} \quad \text{if and only if} \quad P \geq \left( \frac{cG}{2} \right)^2, \]
so that by (6.5) we have in \( \mathcal{R} \) that
\[ \frac{dP}{dt} \left\{ -\frac{P^2}{E} + 2cGP^{3/2} \quad \text{if} \quad P \geq \left( \frac{cG}{2} \right)^2, \right. \]
\[ \left. \frac{3(cGPE_{1/4})^{4/3}}{2} \quad \text{elsewhere}. \right\} \quad (6.8) \]

The estimates for these time derivatives are depicted in Fig. 3, which, like all figures in the \( E, P \)-plane to follow, is not drawn to scale, but is qualitatively faithful.

7. Bounds provided by \( dP/dE \)

We bound the enstrophy, palinstrophy along a solution to (2.1) by trapping the direction of the projection of the vector \( dv/dt \) in that plane. The situation is depicted in four cases by the shading in Fig. 3, which, like all figures in the \( E, P \)-plane to follow, is not drawn to scale, but is qualitatively faithful. Since we lack a lower bound on \( dP/dt \), the lower bound on \( dE/dt \) is irrelevant for \( P < (2cGE)^2 \), and the upper bound on \( dE/dt \) is irrelevant for \( P > (2cGE)^2 \). The projection of the solution to (2.1) starting at an initial condition projecting onto \((E(0), P(0))\) in any one of the three regions is then bounded by the integral curve of the equation defined by the quotient of the time derivatives in that region. In particular, if we start at \((E(0), P(0)) = (E_1, P_1)\) on the graph of \( P = \sqrt{4\Lambda_1E} \), we know the projection of the trajectory for (2.1) moves to the left (as long as \( P \geq \sqrt{4\Lambda_1E} \)) and that within the region
\[ \sqrt{4\Lambda_1E} \leq P \leq \left( \frac{cG}{2} \right)^2 \]
it is bounded from above by the integral curve of the separable differential equation
\[ \frac{dP}{dE} = -3 \left( \frac{cG}{2} \right)^{4/3} (PE)^{1/3}, \quad E_2 \leq E \leq E_1. \quad (7.1) \]
Fig. 3. Bounds on time derivatives. The bounds on $dE/dt$ apply throughout $\mathcal{R}$. Those on $dP/dt$ are used only in the subset between the parabolas where they are written. The gray regions and vectors indicate how the time derivatives are combined to give upper bounds $dP/dE$.

This curve is the graph of the convex function $\varphi_1$ given by

$$
\varphi_1(E, E_1) = \left\{ (4 \Lambda_1 E_1)^{1/3} + \frac{3}{2} \left( \frac{cG}{2} \right)^{4/3} (E_1^{4/3} - E^{4/3}) \right\}^{3/2}.
$$

The range for (7.1) is determined by the intersection of the curve $P = \varphi_1(E, E_1)$ with the parabola $P = (cGE/2)^2$, which gives

$$
E_2 = E_1^{1/4} \left\{ \frac{3}{5} E_1 + \frac{8}{5} \left( \frac{\Lambda_1}{cG^{4/3}} \right)^{1/3} \right\}^{3/4}, \quad P_2 = \left( \frac{cGE_2}{2} \right)^2.
$$

Continuing from the point $(E_2, P_2)$, we bound the projected trajectory within the subset of $\mathcal{R}$ where

$$
\left( \frac{cGE}{2} \right)^2 \leq P \leq (2cGE)^2.
$$
by the integral curve to the Bernoulli equation

\[
\frac{dP}{dE} = \frac{P}{E} - 2cGP^{1/2}, \quad E_3 \leq E \leq E_2.
\]

This second curve is the graph of the concave function \( \varphi_2 \) given by

\[
\varphi_2(E, E_1) = \left( \frac{cG}{2} \right)^2 \left[ 5(EE_2)^{1/2} - 4E \right]^2,
\]

where the notation reflects the fact that \( E_2 \) is determined by \( E_1 \). The endpoint \( E_3 \) is given by the intersection of the curve \( P = \varphi_2(E, E_1) \) and the parabola \( P = (2cGE)^2 \):

\[
E_3 = \frac{25}{64} E_2 = \frac{25}{64} E_1^{1/4} \left\{ \frac{3}{5} E_1 + \frac{8}{5} \frac{A_1^{1/3}}{(cG)^{4/5}} \right\}^{3/4}, \quad P_3 = (2cGE_2)^2. \tag{7.3}
\]

Finally, in \( E \geq (2cGE)^2 \), starting from \( (E_3, P_3) \), we have as a bound the integral curve to the Bernoulli equation

\[
\frac{dP}{dE} = \frac{P}{3E} - \frac{2cG}{3} P^{1/2}, \quad 0 \leq E \leq E_3,
\]

which is graph of the concave function \( \varphi_3 \) given by

\[
\varphi_3(E, E_1) = \left[ \frac{2cG}{5} (6(E_2^5E)^{1/6} - E) \right]^2. \tag{7.4}
\]

Points of intersection of the curve given by (7.4) with that given by \( P = \sqrt{4A_1E} \) satisfy

\[
(4A_1)^{1/4} E^{1/4} = \frac{2cG}{5} (6(E_2^5E)^{1/6} - E) \tag{7.5}
\]

which with \( z = E^{1/2} \) can be expressed as

\[
\alpha z^{12} + \beta z^3 - \gamma z^2 = z^2 g(z) = 0,
\]

where

\[
\alpha = \frac{2cG}{5}, \quad \beta = (4A_1)^{1/4}, \quad \gamma = \frac{12}{5} cGE_2^{5/6}.
\]

Since the only zero of \( g'(z) = 10\alpha z^9 + \beta \) is negative, we have by Rolle’s theorem that there exists at most one positive zero of \( g(z) \). Note that \( \varphi_4(6^{5/6}E_3) = 0 \), so that the continuation of the graph \( P = \varphi_3(E) \) for \( E > E_3 \) intersects the graph of \( P = \sqrt{4A_1E} \) at some \( E^* \in (E_3, 6^{5/6}E_3) \). Thus the curve \( P = \varphi_3(E) \) does not intersect the curve \( P = \sqrt{4A_1E} \) for \( E \in (0, E_3) \). This is consistent with the fact that

\[
\varphi_3(E) = O(E^{1/3}), \quad \text{as } E \to 0.
\]

Qualitative sketches of the integral curves given by \( \varphi_1 - \varphi_3 \) are shown in Fig. 3. It is assumed that

\[
2A_1^{1/2} \leq G^2. \tag{7.6}
\]
The discussion above provides the following bounds on the global attractor, which are illustrated in Fig. 4.

**Theorem 7.1.** The global attractor projects into the E, P-plane within a region bounded by the line P = E, and the sequence of integral curves given by \( \varphi_1 - \varphi_3 \), starting with \( E_1 = 1, P_1 = \sqrt{\Lambda_1} \).

**Remark 7.1.** The points \((\bar{E}_2, \bar{P}_2)\) and \((\bar{E}_3, \bar{P}_3)\), where the curves \( P = \varphi_1(E, \bar{E}_1) \), \( P = \varphi_2(E, \bar{E}_1) \) and \( P = \varphi_3(E, \bar{E}_1) \) starting at \((E_1, \bar{P}_1) = (1, \sqrt{\Lambda_1})\) intersect respectively the parabolas \( P = (cG/2)^2 \) and \( P = (2G)^2 \), satisfy the following:

\[
\bar{P}_i = O(G^2), \quad i = 2, 3;
\]

\[
\bar{E}_2 = \delta_2 + O\left(\Lambda_1^{1/3} G^{-4/3}\right), \quad \text{with } \delta_2 = (3/5)^{3/4} \approx 0.68;
\]

\[
\bar{E}_3 = \delta_3 + O\left(\Lambda_1^{1/3} G^{-4/3}\right), \quad \text{with } \delta_3 = (25/64)\delta_2 \approx 0.27.
\]

**Remark 7.2.** Recall that by Theorem 3.1, if \( \kappa_\sigma \) is large enough, we have an enstrophy cascade, which is a defining feature of 2-D turbulence. The bounds in Theorem 7.1 allow us to conclude that in order for \( \kappa_\sigma \) to be large, the solutions on the global attractor must recurrently have large palinstrophy, or all three quantities energy, enstrophy, and palinstrophy must spend a considerable amount of time in the neighborhood of zero.
Fig. 5. The region \( \mathcal{R}_{E_1} \). The three parabolas (partially hidden by \( \mathcal{R}_{E_1} \)) are as in Fig. 3.

8. Intermittency and estimates on invariant measures

By the estimates on the time derivatives in Section 6 we have that in the three subregions in Fig. 4 the direction of the flow must project as indicated. Let \((E_c, P_c)\) be the point where the parabolas \( P = \sqrt{4\Lambda_1 E} \) and \( P = (2cGE)^2 \) intersect. Consider for any \( E_1 \) in \((E_c, 1)\) the set \( \mathcal{R}_{E_1} \) that is bounded below by the line \( P = E \), on the right by the line \( E = E_1 \), and above by the curves \( P = \varphi_j(E, E_1) \), \( j = 1, 2, 3 \).

Due to the restrictions on the direction of the flow shown in Fig. 4, the projected solutions may exit \( \mathcal{R}_{E_1} \) only through the line segment \( \{ (E_1, P) : E_1 \leq P \leq 1 \} \) (see Fig. 5). Of greater interest, perhaps, is the complement of this region. Note that for a solution on the global attractor to, at any time \( t_2 \), project above \( \mathcal{R}_{E_1} \) recurrently, then it would be intermittent to a degree we can measure.

Our next goal is to estimate probabilities, with respect to invariant probability measures, of various regions above the parabola \( P = 2\Lambda_1^{1/2}E^{1/2} \). Note that in order for the probability associated with a region \( S \) to be positive, \( S \) must be recurrent, in the sense that there exists \( v_0 \in D(A) \), and a sequence of times \( \{ t_n \} \) with \( t_n \to \infty \) such that \( S(t_n)v_0 \in S \).

**Proposition 8.1.** Let \( \mu \) be an invariant probability measure on \( H \) and \( X, Y \) be \( \mu \)-measurable sets in \( H \), with \( X \cap Y = \emptyset \). Assume that there exist \( T_X^{\text{min}}, T_Y^{\text{max}} > 0 \) such that for any \( v_0 \in D(A) \) the following hold.

(i) Before each re-entry into \( Y \), the time \( S(t)v_0 \) spends in \( X \) must be total at least \( T_X^{\text{min}} \).

(ii) Once in \( Y \), the time it takes \( S(t)v_0 \) to exit \( Y \) should not exceed \( T_Y^{\text{max}} \).

Then

\[
\frac{\mu(Y)}{T_Y^{\text{max}}} \leq \frac{\mu(X)}{T_X^{\text{min}}} \quad (8.1)
\]
Proof. For \( v \in H \) and \( T > 0 \), denote by \( N_T(v) \) the number of times that \( S(t)v, \ t \in [0, T] \), visits \( Y \) (including possibly \( v \in Y \) and \( S(T)v \in Y \)). Conditions (i) and (ii) imply that \( 0 \leq N_T(v) < \infty \) for all \( v \in H \) and \( T > 0 \). Also, if \( N_T(v) > 0 \), the time spent in \( X \) by \( S(t)v \) must be at least \( (N_T(v) - 1)T_X^{\min} \).

We have

\[
\frac{1}{T} \int_0^T \chi_Y(S(\tau)v) \, d\tau \leq \frac{N_T(v)T_Y^{\max}}{T},
\]

and

\[
\frac{1}{T} \int_0^T \chi_X(S(\tau)v) \, d\tau \geq \frac{(N_T(v) - 1)T_X^{\min}}{T},
\]

where \( \chi_M \) is the characteristic function of the set \( M \). Denote

\[
\alpha_T(v) = \begin{cases} 
\frac{N_T(v) - 1}{N_T(v)}, & N_T(v) > 0, \\
0, & N_T(v) = 0.
\end{cases}
\]

Note that \( 0 \leq \alpha_T(v) < 1 \). Estimates (8.2) and (8.3) imply

\[
\frac{1}{T} \int_0^T \alpha_T(v) \left( \frac{1}{T} \int_0^T \chi_Y(S(\tau)v) \, d\tau \right) \leq \frac{1}{T_X^{\min}} \left( \frac{1}{T} \int_0^T \chi_X(S(\tau)v) \, d\tau \right).
\]

In order to get the estimate for the measures we proceed as follows. Denote

\[
\Phi_T(v) = \frac{1}{T} \int_0^T \chi_Y(S(\tau)v) \, d\tau.
\]

By Theorem 5.1 applied to the function \( \chi_Y \), there exists \( \Phi^* \in L^1(H, \mu) \) such that

\[
\Phi^*(v) = \lim_{T \to \infty} \Phi_T(v) \quad \mu\text{-almost surely},
\]

and

\[
\int_H \Phi^*(v) \mu(dv) = \int_H \chi_Y(v) \mu(dv) = \mu(Y).
\]

Note that if \( v \) is such that \( Y \) is not recurrent for \( S(t)v \), i.e., \( N_T(v) \) is bounded in \( T \), then by (8.2) \( \Phi_T(v) \to 0 \) as \( T \to \infty \), and so \( \Phi^*(v) = 0 \). Otherwise (if \( Y \) is recurrent for \( S(T)v \), \( N_T(v) \to \infty \) as \( T \to \infty \), and thus, \( \alpha_T(v) \to 1 \) as \( T \to \infty \). It follows that

\[
\alpha_T(v)\Phi_T(v) \to \Phi^*(v) \quad \text{as} \ T \to \infty \quad \mu\text{-almost surely}.
\]

Since, in addition to the relation above, \( 0 \leq \alpha_T(v)\Phi_T(v) \leq 1 \) for every \( T > 0 \) and \( v \in H \), by Lebesgue’s dominated convergence theorem we have
Similarly, by another application of Theorem 5.1 and Lebesgue's dominated convergence theorem, we have that the integral of the right-hand side of (8.5) over $H$ with respect to $\mu$ converges to the right-hand side of (8.1) as $T \to \infty$. Thus (8.1) is obtained from (8.5) by integrating over $H$ with respect to $\mu$ and passing to the limit as $T \to \infty$. \hfill \Box

Now consider in the $E, P$-plane for any $E_1$ to the right of the intersection of

$$P = (2cGE)^2 \quad \text{and} \quad P = \sqrt{4A_1E}, \quad \text{i.e.} \ E_1 > E_c = \frac{(A_1/4)^{1/3}}{(cG)^{4/3}},$$

the open set $Y_{E_1}$ bounded everywhere from above by $\varphi_3(E, 1)$, to the right by $E = E_3$, and from below by the parabola $P - A_1^{1/2}E^{1/2} = P_3 - A_1^{1/2}E_3^{1/2}$ and the curve $\varphi_3(E, E_1)$ (see Fig. 6). Consider also the open set $X_{E_1}$ bounded above by $P = \min(A_1^{1/2}E^{1/2}, 1)$, from below by $P = E$, from the left by $E = E_3$, and from the right by $E = E_1$. Let $X_{E_1}^1$, $Y_{E_1}^1$ be the liftings of $X_{E_1}$, $Y_{E_1}$ to $D(A)$.

**Theorem 8.1.** Let $E_1 \in ((A_1/4)^{1/3}/(cG)^{4/3}, 1)$, and the sets $X_{E_1}$ and $Y_{E_1}$ be defined as above. Then, for any invariant probability measure $\mu$ we have

$$\mu(Y_{E_1}^1) \leq E_3 \frac{\mu(X_{E_1}^1)}{E_1 - E_3 \frac{4((2cGE_3)^2 - A_1^{1/2}E_3^{1/2})}{}}. \quad (8.6)$$
Proof. We will show that the conditions of Proposition 8.1 are satisfied for the sets \( X_{E_1}^\uparrow \) and \( Y_{E_1}^\uparrow \) and any \( v_0 \in D(A) \).

By the directions of the projected flow indicated in Fig. 5, when it leaves \( Y_{E_1} \) through \( \mathcal{R}_{E_1} \). Moreover, the enstrophy balance (2.11) implies

\[
\frac{1}{2} \frac{dE}{dt} \leq -P + A_1^{1/2} E^{1/2}
\]

(8.7)

and

\[
\frac{1}{2} \frac{dE}{dt} \leq -P + P^{1/2}.
\]

(8.8)

So in fact, for any solution the enstrophy \( E \) must decrease whenever \( P > A_1^{1/2} E^{1/2} \) or \( P > 1 \). As a consequence, in order for any projected solution to re-enter \( Y_{E_1} \), it must cross the region \( X_{E_1} \) from \( E = E_3 \) to \( E = E_1 \) (at least once). We note that this crossing of \( X_{E_1} \) may be interrupted by an excursion out of \( X_{E_1} \). Upon re-entering \( X_{E_1} \), however, the enstrophy of the solution would have to be less than when it left \( X_{E_1} \). To apply Proposition 8.1 it is the total time spent in \( X_{E_1} \) before re-entering \( Y_{E_1} \) that should be estimated from below.

To estimate the time that the solution can spend inside \( Y_{E_1}^\uparrow \) integrate (8.7) as follows. Assume \( S(t)v_0 \in Y_{E_1}^\uparrow \) for \( t \in (t_0, t_1) \). Then,

\[
\frac{1}{2}(E(t_1) - E(t_0)) \leq (-P + A_1^{1/2} E_3^{1/2})(t_1 - t_0),
\]

so that

\[
t_1 - t_0 \leq \frac{E(t_0) - E(t_1)}{2(P_3 - A_1^{1/2} E_3^{1/2})} \leq \frac{E_3}{2(P_3 - A_1^{1/2} E_3^{1/2})}.
\]

Since \( P_3 = (2cG E_3)^2 \), we obtain that we can choose

\[ T_{y_{E_1}}^{\max} = \frac{E_3}{2((2cG E_3)^2 - A_1^{1/2} E_3^{1/2})}. \]

Now, to estimate the time that the solution must spend in \( X_{E_1}^\uparrow \) before re-entering \( Y_{E_1}^\uparrow \) we use (8.8):

\[
\frac{dE}{dt} \leq 2(-P + P^{1/2}) \leq \frac{1}{2}.
\]

Thus, if for \( t_0 < t_1 \), \( E(t_0) = E_3 \) and \( E(t_1) = E_1 \), we must have:

\[
E_1 - E_3 \leq \frac{1}{2}(t_1 - t_0),
\]

which means that we may choose

\[ T_{X_{E_1}}^{\min} = 2(E_1 - E_3). \]

Consequently, the conditions of Proposition 8.1 hold and (8.1) implies (8.6). \square
Remark 8.1. If $E_1$ is between the intersections of $P = \sqrt{4A_1}E$ and the two convex parabolas (see Fig. 3), the estimate on measure of $Y^t_{E_1}$ becomes $O(G^{2/3})\mu(X^t_{E_1})$, which is not useful for big $G$ (unless $\mu(X_{E_1}) < O(G^{-2/3})$). Clearly, Theorem 8.1 yields a nontrivial estimate if $E_1 > O(G^{-1})$, which is to the right of the intersection of $P = (4A_1E)$ and $P = (cGE^2/2)^2$.

Remark 8.2. If we keep $A_1$ and $E_1$ fixed and increase $G$, we have

$$\mu(Y^t_{E_1}) = O\left(\frac{1}{G^2}\right).$$

This shows that the probability of being in a region with high palinstrophy should be small for large values of $G$.

Indeed if we consider the set

$$X_{\delta} = \{v \in D(A): |Av| \geq \delta G\}, \quad (8.9)$$

then, using (5.4), and the fact that any invariant probability measure is supported on the global attractor we obtain

$$1 \geq \int_{X_{\delta}} |Av|^2 \mu(dv) \geq \delta^2 G^2 \mu(X_{\delta}),$$

and thus,

$$\mu(X_{\delta}) \leq \frac{1}{\delta^2 G^2}. \quad (8.10)$$

To obtain a more sophisticated bound we observe that the times a solution spends above the parabola $P = A_1^{1/2}E^{1/2}$ are closely related to the excursions of the solution into the set of points where $E$ is increasing:

$$J_+ = \{v \in D(A): (h, Av) - |Av|^2 > 0\}. \quad (8.11)$$

Note that (8.7) and (8.8) imply that

$$J_+ \subset \{v \in D(A): |Av|^2 < 1, |Av|^2 < A_1^{1/2}\|v\|\}.$$

Remark 8.3. Assume that $J_+$ is not recurrent for some solution $v(t)$, in other words, there exists $T_0$ such that $E(t)$ is nonincreasing for all $t > T_0$. Then there exist limits

$$E_\infty = \lim_{t \to \infty} E(t)$$

and

$$P_\infty = \lim_{t \to \infty} P(t)$$

satisfying $E_\infty < 1, P_\infty \leq \min\{1, A_1^{1/2}E_\infty^{1/2}\}$. 
For any $\delta > 0$ consider the region where $E$ decreases at least at the rate $\delta$:

$$D_\delta = \{ v \in D(A): (h, Av) - |Av|^2 \leq -\delta \}.$$  \hspace{1cm} (8.12)

**Theorem 8.2.** For any invariant probability measure $\mu$,

$$\mu(D_\delta) \leq \frac{1}{4\delta} \mu(J_+).$$  \hspace{1cm} (8.13)

**Proof.** If $\mu(D_\delta) = 0$, we have nothing to prove. Otherwise, let $v(t) = S(t)v_0$ be an orbit which recurrently visits $D_\delta$. Denote

$$T = \{ t > 0: (h, Av(t)) - |Av|^2 > 0 \}.$$  

Note that in order for $D_\delta$ to be recurrent, $J_+$ must be recurrent as well. Then, since $Av(t)$ is analytic,

$$T = \bigcup_{n=1}^{\infty} (s_n, t_n),$$

$s_{n+1} \geq t_n$ and $s_n \to \infty$ as $n \to \infty$. On each interval $(s_n, t_n)$ use (8.8):

$$0 < \frac{1}{2} \frac{dE}{dt} \leq \frac{P^{1/2}}{P} \leq \frac{1}{4},$$

which implies that for any $n \in \mathbb{N}$,

$$t_n - s_n \geq 2(E(t_n) - E(s_n)).$$  \hspace{1cm} (8.14)

Let

$$S_\delta = \{ t > 0: v(t) \in D_\delta \}.$$  

Note that

$$T \cap S_\delta = \emptyset.$$  

Also, if $[\alpha, \beta]$ is a maximal interval in $S_\delta$, then

$$E(\beta) - E(\alpha) \leq -2\delta(\beta - \alpha).$$

Consequently, for each $n \in \mathbb{N}$,

$$E(t_n) - E(s_{n+1}) \geq 2\delta l(S_\delta \cap [t_n, s_{n+1}]),$$  \hspace{1cm} (8.15)

where $l(M)$ is the Lebesgue measure of a set $M \subset \mathbb{R}$. If $t \in [t_N, s_{N+1}]$ then

$$2\delta l(S_\delta \cap [0, t]) \leq \sum_{n=0}^{N-1} (E(t_n) - E(s_{n+1})) + E(t_N) - E(t) \leq \sum_{n=0}^{N} (E(t_n) - E(s_{n+1})).$$
where we denote \( t_0 = 0 \). If \( t \in (s_{N+1}, t_{N+1}) \), then

\[
2\delta l(S_\delta \cap [0, t]) \leq \sum_{n=0}^{N} (E(t_n) - E(s_{n+1})).
\]

Since for any \( t > 0 \) there exists \( N = N(t) \) such that \( t_N \leq t < t_{N+1} \), and

\[
2\delta l(S_\delta \cap [0, t]) \leq \sum_{n=0}^{N} (E(t_n) - E(s_{n+1})).
\]

we have

\[
\frac{1}{t} \int_0^t \chi_{D_\delta}(v(\tau)) d\tau \leq \frac{1}{2\delta} \sum_{n=0}^{N} (E(t_n) - E(s_{n+1})).
\]

Rearranging the sum

\[
\sum_{n=0}^{N} (E(t_n) - E(t_{n+1})) = (E(t_0) - E(s_{N+1})) + \sum_{n=1}^{N} (E(t_n) - E(s_n)),
\]

and observing from (8.14) that

\[
\sum_{n=1}^{N} (E(t_n) - E(s_{n+1})) \leq \frac{1}{2} \int_0^t \chi_{J_\delta}(v(\tau)) d\tau,
\]

we conclude that

\[
\frac{1}{t} \int_0^t \chi_{D_\delta}(v(\tau)) d\tau \leq \frac{1}{2\delta} + \frac{1}{4\delta} \frac{1}{t} \int_0^t \chi_{J_\delta}(v(\tau)) d\tau.
\]

Proceeding as at the end of the proof of Proposition 8.1, we apply Theorem 5.1 to the functions \( \chi_{D_\delta} \) and \( \chi_{J_\delta} \), and the probability measure \( \mu \), as well as Lebesgue’s dominated convergence theorem to obtain (8.13). □

**Remark 8.4.** Because \( \mu(D_\delta) \leq 1 - \mu(J_\delta) \) for any \( \delta > 0 \), Theorem 8.2 has nontrivial meaning for \( \delta > 0 \) satisfying \((1 + (4\delta)^{-1})\mu(J_\delta) < 1\). In the case \( \mu(J_\delta) = 0 \), (8.13) implies that \( \mu(D_\delta) = 0 \) for all \( \delta > 0 \). This corresponds to the support of \( \mu \) projecting into an interval \( \{(E_\gamma, P): P_- \leq P \leq P_+\} \) for some \( E_\gamma \), \( P_- \), and \( P_+ \).

To compare to (8.10), we consider for \( \beta > 0 \) the set

\[
W_\beta = \{v \in D(A): P - \Lambda_1^{1/2}E^{1/2} \geq \beta^2G^2\}
\]  

(8.16)

(see Fig. 7).
Corollary 8.1. For any invariant measure $\mu$,

$$
\mu(W_{\beta}) \leq \frac{1}{4\beta^2G^2} \mu(J_+) .
$$

(8.17)

Proof. The corollary follows immediately from Theorem 8.2 if we note that

$$(h, Av) - |Av|^2 \leq \Lambda_1^{1/2} E^{1/2} - P$$

and choose $\delta = \beta^2 G^2$. \[\square\]

Appendix A. Proof of (2.6)

Proof. Denote

$$u(x) = (u_1(x_1, x_2), u_2(x_1, x_2)) \quad \text{and} \quad v(x) = (v_1(x_1, x_2), v_2(x_1, x_2)),$$

for $x = (x_1, x_2) \in \Omega = [0, L]^2$. Also, for convenience denote

$$\partial_i u_j = \frac{\partial u_j}{\partial x_i}(x_1, x_2), \quad \partial_i \partial_j u_k = \frac{\partial^2 u_k}{\partial x_i \partial x_j}(x_1, x_2) \quad \text{and} \quad \partial_i^2 u_j = \frac{\partial^2 u_j}{\partial x_i^2}(x_1, x_2),$$

$i, j, k = 1, 2$, and similarly for $v_i$. With this notation
\[(B(Au, u), v) = \int_\Omega (\Delta u \cdot \nabla)u \cdot v \, dx = \sum_{i,j,k=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_k v_k \, dx \]
\[= \sum_{i,j=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_j v_j \, dx + \sum_{k \neq j=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_k v_k \, dx + \sum_{j \neq i=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_i v_i \, dx.\]

Note that
\[\sum_{k \neq j=1}^2 \int_\Omega \partial_j^2 u_j \partial_j u_k v_k \, dx = \sum_{k \neq j=1}^{2} \int_\Omega \partial_j (\partial_j u_k)^2 v_k \, dx = \sum_{k \neq j=1}^2 \int_\Omega (\partial_j u_k)^2 \partial_k v_k \, dx\]
\[= \sum_{k \neq j=1}^2 \int_\Omega (\partial_j u_k)^2 \partial_j v_j \, dx = \sum_{k \neq j=1}^2 \int_\Omega \partial_j^2 u_k \partial_j u_j v_j \, dx.\]

(Above we have used the divergence-free condition \(\partial_k v_k = -\partial_j v_j\).) Also,
\[\sum_{j \neq i=1}^{2} \int_\Omega \partial_i^2 u_j \partial_j u_i v_i \, dx = \sum_{j \neq i=1}^2 \left[ \int_\Omega \partial_i u_j \partial_i \partial_j u_i v_i \, dx + \int_\Omega \partial_i u_j \partial_i u_j \partial_i v_i \, dx \right] \]
\[= \sum_{j \neq i=1}^2 \int_\Omega \partial_i u_j \partial_i^2 u_j v_i \, dx - \int_\Omega \partial_1 u_2 \partial_2 u_1 (\partial_1 v_1 + \partial_2 v_2) \, dx \]
\[= \sum_{j \neq i=1}^2 \int_\Omega \partial_i u_j \partial_i^2 u_j v_i \, dx.\]

(Above we have used the divergence-free conditions \(\partial_i u_i = -\partial_j u_j\) and \(\partial_1 v_1 + \partial_2 v_2 = 0\).)

Thus
\[(B(Au, u), v) = \sum_{i,j,k=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_j v_j \, dx + \sum_{k \neq j=1}^2 \int_\Omega \partial_i^2 u_j \partial_j u_k v_k \, dx + \sum_{j \neq i=1}^2 \int_\Omega \partial_i u_j \partial_i^2 u_j v_i \, dx \]
\[= \sum_{i,j,k=1}^2 \int_\Omega v_i \partial_i u_j \partial_i^2 u_j \, dx = (B(v, u), Au). \quad \Box\]

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References


